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Research article

Coefficient bounds for a subclass of multivalent functions of reciprocal order

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Abstract: The aim of this paper is to introduce a new subclass of multivalent functions of complex order and to study some interesting properties such as coefficient estimates, sufficiency criteria, Fekete-Szego inequality, inclusion result and integral preserving property for this newly defined class.

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1. Introduction

Let $\mathcal{A}_p$ denote the class of p-valent functions f which are analytic in the regions $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by

$$f(z) = z^p +\sum_{k=1}^{\infty} a_{p+k}z^{p+k}. \tag{1.1}$$

We note that $\mathcal{A}_1 = \mathcal{A}$. Also let $\mathcal{N}(\alpha)$ and $\mathcal{M}(\alpha)$ denote the usual classes of starlike and convex functions of reciprocal order α, $\alpha > 1$, and are defined by

$$\mathcal{N}(\alpha) = \left\{ f(z) \in \mathcal{A} : Re\frac{zf'(z)}{f(z)} < \alpha, \ \ (z \in \mathbb{U}) \right\}, \tag{1.2}$$

$$\mathcal{M}(\alpha) = \left\{ f(z) \in \mathcal{A} : 1 + Re\frac{zf''(z)}{f'(z)} < \alpha, \ \ (z \in \mathbb{U}) \right\}. \tag{1.3}$$

These classes were introduced by Uralegaddi et.al [21] in 1994 and then studied by the authors in [12]. After that Nunokawa and his coauthors [11] proved that for $f \in \mathcal{N}(\alpha)$, $0 < \alpha < \frac{1}{2}$, if and only if the following inequality holds

$$\left| \frac{zf'(z)}{f(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \ \ (z \in \mathbb{U}).$$

In 2002, Owa and Srivastava [13] generalized this idea for the classes of p-valent starlike and p-valent convex functions of reciprocal order α with $\alpha > p$, and further investigated by Polatoglu et.al [14]. Recently in 2011, Uyanik et.al [22] extended this idea to the classes of p-valently spirallike and p-valently Robertson functions and discussed coefficient inequalities and sufficient conditions for the functions of these classes.

The convolution (Hadamard product) of functions $f, g \in \mathcal{A}_p$ is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} b_{k+p} z^{n+p}, \quad z \in \mathbb{U},$$

where f is given by (1.1) and

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}, \quad z \in \mathbb{U}.$$

The incomplete beta function ϕ_p defined by

$$\phi_p(a, c; z) = z^p + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p}, \quad (a \in \mathbb{R}, \ c \in \mathbb{R} \setminus (0, -1, ...), \quad z \in \mathbb{U}),$$

where $(\alpha)_n$ is the pochhamer symbol defined in terms of Gamma function by

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha + 1)...(\alpha + n - 1), & \text{if } n \in \mathbb{N} \\ 1, & \text{if } n = 0. \end{cases}$$

With the help of incomplete beta function ϕ_p and concepts of convolution, Saitoh in [18, 19] introduced the operator $\mathcal{L}_p : \mathcal{A}_p \to \mathcal{A}_p$ and is defined by

$$\begin{aligned} \mathcal{L}_p(a, c) f(z) &= \phi_p(a, c; z) * f(z), \\ &= z^p + \sum_{n=1}^{\infty} \varphi_n(a) \, a_{n+p} \, z^{n+p} 1.4 \end{aligned} \tag{1.4}$$

with $a > -p$ and

$$\varphi_n(a) = \frac{\Gamma(a + n)\Gamma(c)}{\Gamma(a)\Gamma(c + n)}. \tag{1.5}$$

This operator is an extension of the familiar Carlson-Shaffer operator, which has been used widely on the space of analytic and univalent functions in $\mathbb{U}$, see [3, 20]. The following identity can be easily derived

$$z\left(\mathcal{L}_p(a, c) f(z)\right)' = a\mathcal{L}_p (a + 1, c) f(z) - (a - p)\mathcal{L}_p(a, c) f(z). \tag{1.6}$$

Motivated from the above mentioned work, we now introduce a new subclass of multivalent functions of reciprocal order using the operator defined in (1.4).

An analytic multivalent function f of the form (1.1) belongs to the class $\mathcal{SC}_p^{\lambda}(a,b,c,\beta)$, if and only if

$$Re\left\{\frac{2e^{i\lambda}}{b}\left(\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}-1\right)\right\}<(\beta-1)\cos\lambda,$$

where $b\in\mathbb{C}\diagdown\{0\}$, λ is real with $|\lambda|<\frac{\pi}{2},\beta>1$.

It is noticed that, by giving specific values to a,b,c,β and λ in $\mathcal{SC}_p^{\lambda}(a,b,c,\beta)$, we obtain many well-known as well as new subclasses of analytic, univalent and multivalent functions, for example;

(i). For $\lambda=0$, $a=1=c$ and $b=2$, we obtain $\mathcal{SC}_p^{0}(1,2,1,\beta)=\mathcal{M}_p(\beta)$ studied in [13] and further for $p=1$, we have the class $\mathcal{M}(\beta)$ introduced and studied in [7, 21].

(ii). For $p=c=b=1$, $\lambda=0$ and $a=2$, we have $\mathcal{SC}_1^{0}(1,2,1,\beta)=\mathcal{N}(\beta)$ studied in [7, 21].

(iii). For $a=1=c$ and $b=2$, we get the class $\mathcal{SC}_p^{\lambda}(1,2,1,\beta)=\mathcal{S}_p(\lambda,\beta)$ and for $a=2$, $b=c=1$, we obtain $\mathcal{SC}_p^{\lambda}(1,2,1,\beta)=C_p(\lambda,\beta)$, introduced and studied in [22].

(iv). For $\lambda=0$, $a=2$, $b=p=1$, and $c=2-\alpha$, we obtain the class $\mathcal{SC}_1^{0}(2,1,2-\alpha,\beta)=\mathcal{P}_\alpha(\beta)$ [4].

The q^{th} Hankel determinant $H_q(n)$, $q\geq 1$, $n\geq 1$, stated by Pommerenke [15] and furter investigated by Noonan and Thomas [10] as

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1}\\ a_{n+1} & a_{n+2} & \cdots & a_{n+q}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}\end{vmatrix}$$

This Hankel determinant is useful and has also been studied by several authors for details see [1, 2]. The growth rate of Hankel determinant $H_q(n)$ as $n\to\infty$ was investigated, respectively,when f is a member of certain subclass of analytic functions, such as the class of p-valent functions [15, 10], the class of starlike functions [15], the class of univalent functions [16], the class of close-to-convex functions [8] , a new class V_k [9]. It is well known that the Fekete-Szegö functional is $H_2(1)=\left|a_3-a_2^2\right|$. Fekete and Szegö further generalized the estimate $\left|a_3-\mu a_2^2\right|$ where μ is real and $f\in S$, the class of univalent functions. For our discussion in this paper we investigate the coeficient bound, the upper bounds of the Hankel determinant $H_3(1)$ for a subclass of multivalent functions.

We will need the following lemma's for our work.

Lemma 1.2. [17]. *If q is an analytic function with $Re q(z)>0$ and*

$$q(z)=1+_{n=2}^{\infty}d_nz^n,\ z\in\mathbb{U},\tag{1.7}$$

then for $n\geq 1$,

$$|d_n|\leq 2.$$

Lemma 1.3. [6]. *If q is of the form* (1.7) *with positive real part, then the following sharp estimate holds*

$$\left|d_2\_\upsilon d_1^2\right|\leq 2\max\{1,|2\upsilon-1|\},\ \text{for all }\upsilon\in\mathbb{C}.$$

Lemma 1.4. [5]. *If q is of the form* (1.7) *with positive real part, then*

$$\begin{aligned}2d_2&=d_1^2+x(4-d_1^2).\\ 4d_3&=d_1^3+2(4-d_1^2)d_1x-d_1(4-d_1^2)x^2+2(4-d_1^2)\left(1-|x|^2\right)z.\end{aligned}$$

for some x, z with $|x|\leq 1$ and $|z|\leq 1$.

2. Main Results

Theorem 2.1. *If $f \in \mathcal{SC}_p^{\lambda}(a,b,c,\beta)$, then*

$$\left|a_{p+1}\right| \leq \frac{|b|\,|\eta|\,c}{p},$$

and

$$\left|a_{n+p-1}\right| \leq \frac{a\,|b|\,|\eta|}{(n+p-2)\varphi_{n-1}(a)} \prod_{j=1}^{n-2}\left(1+\frac{a\,|b|\,|\eta|}{(p+j)}\right), \quad n \geq 3, \tag{2.1}$$

where $\varphi_{n-1}(\delta)$ is given by (1.5) *and*

$$\eta = (1-\beta)\cos\lambda + i\sin\lambda. \tag{2.2}$$

Proof. Let $f \in \mathcal{SC}_p^{\lambda}(a,b,c,\beta)$. Then we have

$$Re\left\{\frac{2e^{i\lambda}}{b}\left(\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}-1\right)\right\} < (\beta-1)\cos\lambda, \quad z \in \mathbb{U}.$$

Now let us define a function q by

$$e^{i\lambda}\left(1-\frac{2}{b}+\frac{2}{b}\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}\right) = ((1-\beta)q(z)+\beta)\cos\lambda + i\sin\lambda. \tag{2.3}$$

where q is analytic in $\mathbb{U}$ with $q(0)=1$ and $Req(z)>0$, $z \in \mathbb{U}$. Then (2.3) can be written as

$$1-\frac{2}{b}+\frac{2}{b}\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)} = 1+\frac{(1-\beta)\cos\lambda+i\sin\lambda}{e^{i\lambda}}\sum_{n=1}^{\infty}c_n z^n,$$

or equivalently

$$2e^{i\lambda}\left(\mathcal{L}_p(a+1,c)f(z)-\mathcal{L}_p(a,c)f(z)\right) = b\eta\mathcal{L}_p(a,c)f(z)\sum_{n=1}^{\infty}c_n z^n, \tag{2.4}$$

where η is given by (2.2). From (2.4) and (1.6) we have

$$2e^{i\lambda}\left[z\left(\mathcal{L}_p(a,c)f(z)\right)' - p\mathcal{L}_p(a,c)f(z)\right] = ab\eta\mathcal{L}_p(a,c)f(z)\sum_{n=1}^{\infty}c_n z^n,$$

that is,

$$2e^{i\lambda}\left[\sum_{k=1}^{\infty}(k+p-1)\varphi_k(a)a_{k+p}z^{k+p}\right] = ab\eta\left[z^p+\sum_{k=1}^{\infty}\varphi_k(a)a_{k+p}z^{k+p}\right]\left(\sum_{k=1}^{\infty}d_n z^n\right).$$

Comparing the coefficients of z^{n+p-1} on both sides, we obtain

$$2e^{i\lambda}(n+p-2)\varphi_{n-1}(a)a_{n+p-1} = ab\eta\left\{d_1 a_{n-2}\varphi_{n+p-2}(a)+...+d_{n-1}\right\}. \tag{2.5}$$

Taking absolute on both sides and then applying Lemma 1.1, we have

$$\left|a_{n+p-1}\right| \leq \frac{a\,|b|\,|\eta|}{(n+p-2)\varphi_{n-1}(a)}\left\{1+\varphi_1(a)\left|a_{p+1}\right|+...+\varphi_{n-2}(a)\left|a_{n+p-2}\right|\right\} \tag{2.6}$$

We now apply mathematical induction on (2.6). So for $n = 2$

$$\left|a_{p+1}\right| \leq \frac{|b|\,|\eta|\,c}{p}.$$

which shows that the result is true for $n = 2$. For $n = 3$

$$\left|a_{p+2}\right| \leq \frac{a\,|b|\,|\eta|}{(p+1)\varphi_2(a)}\left\{1+\varphi_1(a)\left|a_{p+1}\right|\right\} \tag{2.7}$$

and using the bound of $\left|a_{p+1}\right|$ in (2.7), we have

$$\left|a_{p+2}\right| \leq \frac{a\,|b|\,|\eta|}{(p+1)\varphi_2(a)}\left\{1+\frac{a\,|b|\,|\eta|}{p}\right\}.$$

Therefore (2.1) holds for $n = 3$.

Assume that (2.1) is true for $n = k$, that is,

$$\left|a_{k+p-1}\right| \leq \frac{a\,|b|\,|\eta|}{(k+p-2)\varphi_{k-1}(a)}\prod_{j=1}^{k-2}\left(1+\frac{a\,|b|\,|\eta|}{(p+j)}\right).$$

Consider

$$\begin{aligned}\left|a_{k+p}\right| &\leq \frac{a\,|b|\,|\eta|}{(k+p-1)\varphi_k(a)}\left\{\left(1+\frac{a\,|b|\,|\eta|}{p}\right)+\frac{a\,|b|\,|\eta|}{(p+1)}\left(1+\frac{a\,|b|\,|\eta|}{p}\right)\right. \\ &\qquad \left. +\ldots+\frac{a\,|b|\,|\eta|}{(p+k-1)}\prod_{j=1}^{k-2}\left(1+\frac{a\,|b|\,|\eta|}{(p+j)}\right)\right\} \\ &= \frac{a\,|b|\,|\eta|}{(k+p-1)\varphi_k(a)}\prod_{j=1}^{k-1}\left(1+\frac{a\,|b|\,|\eta|}{(p+j)}\right).\end{aligned}$$

Therefore, the result is true for $n = k + 1$ and hence by using mathematical induction, (2.1) holds true for all $n \geq 3$. □

The following corollaries which were proved by owa and Nishawski [12] comes as a special case from Theorem 2.1 by varying the parameter a, b, c, p and λ.

Corollary 2.2. *If $f \in \mathcal{M}(\beta)$, then*

$$|a_n| \leq_{l=2}^{n} \frac{(l+2\beta-4)}{(n-1)!}, \quad \text{for all } n \geq 2.$$

Corollary 2.3. *If $f \in \mathcal{N}(\beta)$, then*

$$|a_n| \leq_{l=2}^{n} \frac{(l+2\beta-4)}{n!}, \quad \text{for all } n \geq 2.$$

Theorem 2.4. *Let $f \in \mathcal{A}_p$ and satisfies*

$$\sum_{n=1}^{\infty} \left| \left(1 + \frac{2n}{ab}\right) e^{i\lambda} - \beta \cos\lambda \right| \varphi_n(a) \left| a_{n+p} \right| < \left| \beta \cos\lambda - e^{i\lambda} \right|. \tag{2.8}$$

Then $f \in \mathcal{SC}_p^{\lambda}(a, b, c, \beta)$.

Proof. To prove that f belongs to $\mathcal{SC}_p^{\lambda}(a, b, c, \beta)$ we need to prove that

$$\left| \frac{e^{i\lambda}\left(1 - \frac{2}{b} + \frac{2}{b}\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}\right) - 1}{e^{i\lambda}\left(1 - \frac{2}{b} + \frac{2}{b}\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}\right) - (2\beta\cos\lambda - 1)} \right| < 1. \tag{2.9}$$

For this consider the left hand side of (2.9), we have

$$\begin{aligned} LHS &= \left| \frac{e^{i\lambda}\left(1 - \frac{2}{b} + \frac{2}{b}\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}\right) - 1}{e^{i\lambda}\left(1 - \frac{2}{b} + \frac{2}{b}\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}\right) - (2\beta\cos\lambda - 1)} \right| \\ &= \left| \frac{\left(e^{i\lambda} - 1\right) b z^p + \sum_{n=1}^{\infty}\left(\left(b + \frac{2n}{a}\right) e^{i\lambda} - 1\right)\varphi_n(a) a_{n+p} z^{n+p}}{(be^{i\lambda} - 2b\beta\cos\lambda + b) + \sum_{n=1}^{\infty}\left(\left(b + \frac{2n}{a}\right) e^{i\lambda} - 2b\beta\cos\lambda + b\right)\varphi_n(a) a_{n+p} z^{n+p}} \right| \\ &\leq \frac{\left|\left(e^{i\lambda} - 1\right)\right| + \sum_{n=1}^{\infty}\left|\left(1 + \frac{2n}{ab}\right) e^{i\lambda} - 1\right| \varphi_n(a) \left|a_{n+p}\right|}{|2\beta\cos\lambda - e^{i\lambda} - 1| - \sum_{n=1}^{\infty}\left|\left(1 + \frac{2n}{ab}\right) e^{i\lambda} - 2\beta\cos\lambda + 1\right| \varphi_n(a) \left|a_{n+p}\right|}. \end{aligned}$$

The last expression is bounded above by 1 if

$$\begin{aligned} \left|e^{i\lambda} - 1\right| + \sum_{n=1}^{\infty}\left|\left(1 + \frac{2n}{ab}\right) e^{i\lambda} - 1\right| \varphi_n(a) \left|a_{n+p}\right| &\leq \left|2\beta\cos\lambda - e^{i\lambda} - 1\right| \\ &\quad - \sum_{n=1}^{\infty}\left|\left(1 + \frac{2n}{ab}\right) e^{i\lambda} - 2\beta\cos\lambda + 1\right| \varphi_n(a) \left|a_{n+p}\right| \end{aligned}$$

which is equivalent to the condition (2.8) and so $f \in \mathcal{SC}_p^{\lambda}(a, b, c, \beta)$. □

If we set $p = 1, \lambda = 0$, $a = 1$, $b = 2$ and $c = 1$, in above Theorem we have the following result by [12].

Corollary 2.5. *If $f \in A$ satisfies*

$$\sum_{n=2}^{\infty} \{(n - 1) + |n - 2\beta + 1|\} |a_n| \leq 2(\beta - 1)$$

for some $\beta(\beta > 1)$, then $f \in \mathcal{M}(\beta)$.

Corollary 2.6. [4]. *A function $f \in \mathcal{P}_\alpha(\beta)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}(n - \beta) |a_n| \leq (\beta - 1).$$

The result is sharp.

Theorem 2.7. *Let* $f \in \mathcal{SC}_p^0(a,b,c,\beta)$ *and of the form* (1.1). *Then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{|b|\, c(c+1)(1-\beta)}{(a+1)(p+1)} \max\{1, |2\upsilon - 1|\}, \tag{2.10}$$

where

$$\upsilon = \frac{\mu bc(a+1)(p+1)(1-\beta)}{2p^2(c+1)} - \frac{ab(1-\beta)}{2p}. \tag{2.11}$$

Proof. Let $f \in \mathcal{SC}_p^0(a,b,c,\beta)$. Then from (2.5) with $\lambda = 0$, we have

$$\begin{aligned} a_{p+1} &= \frac{bc(1-\beta)}{2p} d_1 \\ a_{p+2} &= \frac{bc(c+1)(1-\beta)}{2(a+1)(p+1)} \left\{ d_2 + \frac{ab(1-\beta)}{2p} d_1^2 \right\}. \end{aligned}$$

For any complex number μ, we have

$$\begin{aligned} a_{p+2} - \mu a_{p+1}^2 &= \frac{bc(c+1)(1-\beta)}{2(a+1)(p+1)} \left[d_2 - \frac{b(1-\beta)}{2p} \left\{ \frac{\mu c(a+1)(p+1)}{p(c+1)} - a \right\} d_1^2 \right] \\ &= \frac{bc(c+1)(1-\beta)}{2(a+1)(p+1)} \left[d_2 - \upsilon d_1^2 \right], \end{aligned}$$

where υ is given by (2.11)

Taking modulus on both sides and applying Lemma 1.2, we have

$$\begin{aligned} \left|a_{p+2} - \mu a_{p+1}^2\right| &= \left|\frac{bc(c+1)(1-\beta)}{2(a+1)(p+1)}\right| \left|d_2 - \upsilon d_1^2\right| \\ &\leq \frac{|b|\, c(c+1)(1-\beta)}{(a+1)(p+1)} \max\{1, |2\upsilon - 1|\}. \end{aligned}$$

This proves the required result. □

Taking $\mu = 1$, we obtain the following result.

Corollary 2.8. *Let* $f \in \mathcal{SC}_p^0(a,b,c,\beta)$ *and of the form* (1.1). *Then*

$$\left|a_{p+2} - a_{p+1}^2\right| \leq \frac{|b|\, c(c+1)(1-\beta)}{(a+1)(p+1)} \max\{1, |2\upsilon - 1|\},$$

where

$$\upsilon = \frac{bc(a+1)(p+1)(1-\beta)}{2p^2(c+1)} - \frac{ab(1-\beta)}{2p}.$$

Theorem 2.9. *Let* $f \in \mathcal{SC}_p^\lambda(a,b,c,\beta)$. *Then*

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \leq \left[\frac{4\,|b|\, c(c+1)(1-\beta)}{(p+1)(a+1)}\right]^2. \tag{2.12}$$

Proof. Let $f \in \mathcal{SC}_p^{\lambda}(a,b,c,\beta)$. Then from (2.5) we have

$$\begin{aligned}
a_{p+1} &= \frac{ab(1-\beta)}{2p\varphi_1(a)}d_1. && (2.13)\\
a_{p+2} &= \frac{ab(1-\beta)}{2(p+1)\varphi_2(a)}\left\{d_2+\frac{ab(1-\beta)}{2p}d_1^2\right\}. && (2.14)\\
a_{p+3} &= \frac{ab(1-\beta)}{2(p+2)\varphi_3(a)}\left\{d_3+\frac{ab(1-\beta)}{2(p+1)}d_2^2+\frac{\{ab(1-\beta)\}^2}{4p(p+1)}d_1^2d_2+\frac{ab(1-\beta)}{2p}d_1^2\right\}. && (2.15)
\end{aligned}$$

From (2.13), (2.14) and (2.15) we obtain

$$\begin{aligned}
\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| &= \frac{\{ab(1-\beta)\}^2}{4p(p+2)\varphi_1(a)\varphi_3(a)}\times\\
&\left\{d_3+\frac{ab(1-\beta)}{2(p+1)}d_2^2+\frac{\{ab(1-\beta)\}^2}{4p(p+1)}d_1^2d_2+\frac{ab(1-\beta)}{2p}d_1^2\right\}\\
&-\frac{\{ab(1-\beta)\}^2}{4(p+1)^2\varphi_2^2(a)}\left\{d_2^2+\frac{\{ab(1-\beta)\}^2}{4p^2}d_1^4+\frac{ab(1-\beta)}{p}d_1^2d_2\right\}.
\end{aligned}$$

After some simplification we have

$$\begin{aligned}
\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| = \frac{|A|^2}{4}\Big[&Bd_1d_3+Cd_1d_2^2+\\
&Ed_1^3d_2+Fd_1^3-Gd_2^2-Hd_1^4-Kd_1^2d_2,\Big] \qquad (2.16)
\end{aligned}$$

where

$$\begin{array}{lll}
A=ab(1-\beta), & B=\frac{1}{p(p+1)\varphi_1(a)\varphi_3(a)}, & C=\frac{A}{2p(p+1)(p+2)\varphi_1(a)\varphi_3(a)},\\
E=\frac{A^2}{4p^2(p+1)(p+2)\varphi_1(a)\varphi_3(a)}, & F=\frac{A}{2p^2(p+1)\varphi_1(a)\varphi_3(a)}, & G=\frac{1}{(p+1)^2\varphi_2^2(a)},\\
H=\frac{A^2}{4p^2(p+1)^2\varphi_2^2(a)}, & K=\frac{A}{p(p+1)^2\varphi_2^2(a)}. &
\end{array}$$

Substituting the values of d_2 and d_3 frome Lemma 1.3 in (2.16) we have

$$\left|Bd_1d_3+Cd_1d_2^2+Ed_1^3d_2+Fd_1^3-Gd_2^2-Hd_1^4-Kd_1^2d_2\right|$$

$$\begin{aligned}
= &\left|\frac{1}{4}Bd_1\left\{d_1^3+2d_1(4-d_1^2)x-d_1(4-d_1^2)x^2+2(4-d_1^2)\left(1-|x|^2\right)z\right\}\right.\\
&+\frac{1}{4}Cd_1\left\{d_1^4+2d_1^2(4-d_1^2)x+(4-d_1^2)^2x^2\right\}+\frac{1}{2}Ed_1^3\left\{d_1^2+(4-d_1^2)x\right\}\\
&+Fd_1^3-\frac{1}{4}G\left\{d_1^4+2d_1^2(4-d_1^2)x+(4-d_1^2)^2x^2\right\}-Hd_1^4\\
&\left.-\frac{1}{2}Kd_1^2\left\{d_1^2+(4-d_1^2)x\right\}\right|.
\end{aligned}$$

Simple computation gives

$$4\left|Bd_1d_3 + Cd_1d_2^2 + Ed_1^3d_2 + Fd_1^3 - Gd_2^2 - Hd_1^4 - Kd_1^2d_2\right| = \Big|(C + 2E)\, d_1^5 +$$

$$(B - G - H - 2K)\, d_1^4 + Fd_1^3 + (2B + 2Cd_1 + 2Ed_1 - 2G - 2K)\, d_1^2 \left(4 - d_1^2\right)|x|$$

$$+2Bd_1 \left(4 - d_1^2\right)\left(1 - |x|^2\right)|z| + \left\{Cd_1 \left(4 - d_1^2\right) - Bd_1^2 - G\left(4 - d_1^2\right)\right\}\left(4 - d_1^2\right)|x|^2\Big| \tag{2.17}$$

Applying triangle inequality and replacing $|x|$ by ρ in (2.17) we have

$$\begin{aligned} 4\left|Bd_1d_3 + Cd_1d_2^2 + Ed_1^3d_2 + Fd_1^3 - Gd_2^2 - Hd_1^4 - Kd_1^2d_2\right| &\leq (|C| + 2|E|)\, d_1^5 + |F|\, d_1^3 + \\ &(B - G + |H| + 2|K|)\, d_1^4 + \{2B + 2|C|\, d_1 + 2|E|\, d_1 - 2G + 2|K|\}\, d_1^2 \left(4 - d_1^2\right)\rho \\ &+2Bd_1 \left(4 - d_1^2\right)\left(1 - \rho^2\right) + \left\{|C|\, d_1 \left(4 - d_1^2\right) - Bd_1^2 - G\left(4 - d_1^2\right)\right\}\left(4 - d_1^2\right)\rho^2 \\ &= F(d_1, \rho). \end{aligned} \tag{2.18}$$

Taking partial derivative of $F(d_1, \rho)$ with respect to ρ, we have

$$\begin{aligned} \frac{\partial F(d_1, \rho)}{\partial \rho} &= \{2B + 2|C|\, d_1 + 2|E|\, d_1 - 2G + 2|F|\}\, d_1^2 \left(4 - d_1^2\right) - 4Bd_1 \left(4 - d_1^2\right)\rho \\ &+2\left\{|C|\, d_1 \left(4 - d_1^2\right) - Bd_1^2 - G\left(4 - d_1^2\right)\right\}\left(4 - d_1^2\right)\rho \end{aligned}$$

Clearly $\frac{\partial F(d_1,\rho)}{\partial \rho} > 0$, for $0 < \rho < 1$ and $0 < d_1 < 2$. Therefore, $F(d_1, \rho)$ is an increasing function of ρ. Also for a fixed $d_1 \in [0, 2]$, we have

$$\max_{0 \leq \rho \leq 1} F(d_1, \rho) = F(d_1, \rho) = J(d_1).$$

Therefore by putting $\rho = 1$ in (2.18) we have

$$\begin{aligned} J(d_1) &= \{|C| + 2|E|\}\, d_1^5 + \{B - G + |H| + 2|K|\}\, d_1^4 + |F|\, d_1^3 \\ &+ \{2B + 2|C|\, d_1 + 2|E|\, d_1 - 2G + 2|K|\}\, d_1^2 \left(4 - d_1^2\right) \\ &+ \left\{|C|\, d_1 \left(4 - d_1^2\right) - Bd_1^2 - G\left(4 - d_1^2\right)\right\}\left(4 - d_1^2\right) \end{aligned}$$

Differentiating with respect to d_1, we have

$$\begin{aligned} J'(d_1) &= 5\{|C| + 2|E|\}\, d_1^4 + 4\{B - G + |H| + 2|K|\}\, d_1^3 + 3|F|\, d_1^2 \\ &+4\{B - G + 2|K|\}\, d_1 \left(4 - d_1^2\right) - 4\{B - G + 2|K|\}\, d_1^3 \\ &+6\{|C| + |E|\}\, d_1^2 \left(4 - d_1^2\right) - 4\{|C| + |E|\}\, d_1^4 + |C|\, d_1^2 \left(4 - d_1^2\right)^2 \\ &-4|C|\, d_1^2 \left(4 - d_1^2\right) - 2Bd_1 \left(4 - d_1^2\right) + 2Bd_1^3 - 4Gd_1 \left(4 - d_1^2\right) \end{aligned}$$

Again differentiating with respect to d_1 we have

$$J''(d_1) = 20\{|C| + 2|E|\}\, d_1^3 + 12\{B - G + |H| + 2|K|\}\, d_1^2$$

$$+6|F|d_1 + 4\{B - G + 2|K|\}\left(4 - d_1^2\right) - 8\{B - G + 2|K|\}d_1^2$$
$$-12\{B - G + 2|K|\}d_1^2 + 126\{|C| + |E|\}\left(4 - d_1^2\right) - 126\{|C| + |E|\}d_1^3$$
$$-16\{|C| + |E|\}d_1^3 + 2|C|d_1\left(4 - d_1^2\right)^2 - 4|C|d_1^3\left(4 - d_1^2\right)^2 - 8|C|d_1\left(4 - d_1^2\right)$$
$$+8|C|d_1^3 - 2B\left(4 - d_1^2\right) + 4Bd_1^2 + 6Bd_1^2 - 4G\left(4 - d_1^2\right) + 8Gd_1^2.$$

For maximum value of $J(d_1)$, clearly $J'(d_1) = 0$ for $d_1 = 0$ and $J''(0) < 0$, so $J(d_1)$ has maximum value at $d_1 = 0$ hence

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \leq \left[\frac{4|b|c(c+1)(1-\beta)}{(p+1)(a+1)}\right]^2.$$

□

3. Subordination Results for the Function Class $\mathcal{SC}_p^\lambda(a, b, c, \beta)$

Given functions $f, g \in A$, f is said to subordinate to g denoted by $f \prec g$, $z \in \mathbb{U}$, if there exist a function $w \in V$, where

$$V = \{w \in A : w(0) = 0,\ |w(z)| < 1,\ z \in \mathbb{U}\}$$

such that $f(z) = g(w(z))$.

Lemma 3.1. [18]. *Let $q(z)$ be convex in $\mathbb{U}$ and $Re\,(\mu_1 q(z) + \mu_2) > 0$, where $\mu_1, \mu_2 \in \mathbb{C}\setminus\{0\}$, $z \in \mathbb{U}$. If $h(z)$ is analytic in $\mathbb{U}$ with $q(0) = h(0)$ and*

$$h(z) + \frac{zh'(z)}{\mu_1 h(z) + \mu_2} \prec q(z), \quad z \in \mathbb{U},$$

then $h(z) \prec q(z)$.

Lemma 3.2. *A function $f \in \mathcal{SC}_p^\lambda(a, b, c, \beta)$, if and only if*

$$e^{i\lambda}\left(1 - \frac{2}{b} + \frac{2}{b}\frac{\mathcal{L}_p(a+1, c)f(z)}{\mathcal{L}_p(a, c)f(z)}\right) \prec q(z), \ \ z \in \mathbb{U},$$

where

$$q(z) = \frac{\cos\lambda - \{2\beta\cos\lambda + i\sin\lambda - \cos\lambda\}z}{1 - z}. \tag{3.1}$$

for some real $\lambda(|\lambda| < \frac{\pi}{2})$ and $\beta > p$.

The proof of above lemma is similier to that of Theorem 1 in [22] so we omit the proof.

Theorem 3.3. *Let $\beta > p$, $b \in \mathbb{C}\setminus\{0, -1\}$. Then*

$$\mathcal{SC}_p^0(a+1, b, c, \beta) \subset \mathcal{SC}_p^0(a, b+1, c, \beta_1),$$

where

$$\beta_1 = \frac{b(a+1)}{a(b+1)}\beta - \frac{b-a}{a(b+1)}.$$

Proof. Suppose $f \in \mathcal{SC}_p^0(a+1, b, c, \beta)$ and set

$$1 - \frac{2}{b+1} + \frac{2}{b+1} \frac{\mathcal{L}_p(a+1, c) f(z)}{\mathcal{L}_p(a, c) f(z)} = h(z), \tag{3.2}$$

where h is analytic in $\mathbb{U}$ and $h(0) = 1$.

Logarithmic differentiation of (3.2) , gives

$$\frac{z\left(\mathcal{L}_p(a+1, c) f(z)\right)'}{\mathcal{L}_p(a+1, c) f(z)} - \frac{z\left(\mathcal{L}_p(a, c) f(z)\right)'}{\mathcal{L}_p(a, c) f(z)} = \frac{(b+1)\, z h'(z)}{(b+1)\{h(z) - 1\} + 2}.$$

Using the identity(1.6) we have

$$1 - \frac{2}{b} + \frac{2}{b} \frac{\mathcal{L}_p(a+2, c) f(z)}{\mathcal{L}_p(a+1, c) f(z)} = 1 - \frac{a}{a+1} \frac{b+1}{b} + \frac{a}{a+1} \frac{b+1}{b} h(z) + \frac{2}{b(a+1)} \frac{z h'(z)}{h(z) - 1 + \frac{2}{b+1}}. \tag{3.3}$$

Let

$$1 - \frac{a}{a+1} \frac{b+1}{b} + \frac{a}{a+1} \frac{b+1}{b} h(z) = H(z),$$

where H is analytic in $\mathbb{U}$ and $H(0) = 1$. From (3.3) we have

$$1 - \frac{2}{b} + \frac{2}{b} \frac{\mathcal{L}_p(a+2, c) f(z)}{\mathcal{L}_p(a+1, c) f(z)} = H(z) + \frac{z H'(z)}{\mu_1 H(z) + \mu_2},$$

where $\mu_1 = \frac{b(a+1)}{2}$ and $\mu_2 = \frac{2a-ab-b}{2}$. Since $f(z) \in VD_p^0(a+1, b, c, \beta)$, so from Lemma 3.2 we have

$$H(z) + \frac{z H'(z)}{\mu_1 H(z) + \mu_2} = 1 - \frac{2}{b} + \frac{2}{b} \frac{\mathcal{L}_p(a+2, c) f(z)}{\mathcal{L}_p(a+1, c) f(z)} \prec q(z),$$

where q is given by

$$q(z) = \frac{1 - (2\beta - 1)\, z}{1 - z}. \tag{3.4}$$

Applying Lemma 3.1 we have

$$H(z) \prec q(z)$$

or equivalently

$$h(z) \prec \frac{1 - (2\beta_1 - 1)\, z}{1 - z},$$

where

$$\beta_1 = \frac{b(a+1)}{a(b+1)} \beta - \frac{b-a}{a(b+1)}.$$

This complete the proof. □

Theorem 3.4. *Let $f \in \mathcal{SC}_p^0(a, b, c, \beta)$. Then $F \in \mathcal{SC}_p^0(a, b, c, \beta)$, where F is Bernardi integral operator defined by*

$$F(z) = \frac{p+1}{z^c} \int_0^z t^{c-1} f(t) dt, \quad c > -1. \tag{3.5}$$

Proof. Suppose

$$1-\frac{2}{b}+\frac{2}{b}\frac{\mathcal{L}_p(a+2,c)F(z)}{\mathcal{L}_p(a+1,c)F(z)}=h(z), \tag{3.6}$$

where h is analytic in $\mathbb{U}$ and $h(0)=1$.

Now differentiating (3.5) we have

$$(c+p)f(z)=cF(z)+zF^{'}(z)$$

Applying the operator $\mathcal{L}_p(a,c)$ we have

$$(c+p)\mathcal{L}_p(a,c)f(z)=c\mathcal{L}_p(a,c)F(z)+\mathcal{L}_p(a+1,c)F(z) \tag{3.7}$$

and

$$(c+p)\mathcal{L}_p(a+1,c)f(z)=c\mathcal{L}_p(a+1,c)F(z)+\mathcal{L}_p(a+2,c)F(z) \tag{3.8}$$

From (3.7) and (3.8) we have

$$\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}=\frac{c\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}+\frac{\mathcal{L}_p(a+2,c)f(z)}{\mathcal{L}_p(a+1,c)f(z)}\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}}{c+\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}}. \tag{3.9}$$

Logarithmic differentiation of (3.6) , totgether with (1.6) and (3.9) we have

$$1-\frac{2}{b}+\frac{2}{b}\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}=h(z)+\frac{zh^{'}(z)}{\mu_3 h(z)+\mu_4},$$

where $\mu_3=\frac{ab}{2}$ and $\mu_4=c+p-\frac{ab}{2}$

Since $f\in\mathcal{SC}_p^0(a,b,c,\beta)$, so from Lemma 3.2 we have

$$h(z)+\frac{zh^{'}(z)}{\mu_3 h(z)+\mu_4}=1-\frac{2}{b}+\frac{2}{b}\frac{\mathcal{L}_p(a+1,c)f(z)}{\mathcal{L}_p(a,c)f(z)}\prec q(z),$$

where q is given by (3.4)

Applying Lemma 3.1 we have

$$h(z)\prec q(z),$$

which implies that $F\in\mathcal{SC}_p^0(a,b,c,\beta)$. □

Conflict of Interest

All authors declare no conflicts of interest in this paper.

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