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Research article

On the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{3p}$

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Abstract: In the present paper we obtained all positive integer solutions of some diophantine equations related to unit fraction.

Keywords: diophantine equation; greatest common divisor; prime number; positive integer solution; fraction

1. Introduction

In 1950, Erdös [1] conjectured for any positive integer $n > 1$, that the following diophantine equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{4}{n} \tag{1.1}$$

has positive integer solutions x, y, z. Later, Strauss [1] made a more powerful conjecture: let $n > 2$, then diophantine equation (1.1) has positive integer solution x, y, z with $x \neq y, x \neq z, y \neq z$. He proved that the conjecture is true with $n < 5000$. In 1964, Zhao Ke, Qi Sun and Xianjue Zhang [3] proved that Strauss conjecture is equivalent to Erdös conjecture. In 1979, Ke and Sun [4] proved that Erdös-Strauss conjecture is true with $n < 4 \cdot 10^5$. In 1965, Yamamoto [10] proved that Erdös-Strauss conjecture is also true with $n < 10^7$. In 1978, Franceschine [2] proved that Erdös-Strauss conjecture is true with $n < 10^8$. Sierpiǹski made a similar conjecture: for any positive integer $n > 1$, that the following diophantine equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{5}{n} \tag{1.2}$$

has the solutions x, y, z of positive integer. Palama [6, 7] proved that Sierpiǹski conjecture is true with $n < 922321$. Stewart [9] also obtained above the result with $n \leq 105743881$ and $n \not\equiv 1 \pmod{278460}$.

In 1984, Liu [5] obtained all the solution of positive integers of the following diophantine equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{5}{121}. \tag{1.3}$$

Write $n = 12p$ in (1.1) or write $n = 15p$ in (1.2), where p is an odd prime. We obtain diophantine equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{3p}, x \le y \le z. \tag{1.4}$$

One can easily get all the solutions of positive integer of (1.4) when $p|x$. If $3|x, p \nmid x$, then one lets

$$x = 3ax_1, y = ay_1, z = az_1, a = \gcd(x/3, y, z),$$

and so (1.4) is changed to

$$3p(y_1 + z_1)x_1 = (ax_1 - p)y_1z_1. \tag{1.5}$$

Moreover, we write $y_1 = dy_2, z_1 = dz_2, d = \gcd(y_1, z_1)$. It is easy to see that $\gcd(d, x_1) = 1$.

If $3p|d$, then we get that $x_1|y_2z_2$ and $y_2z_2|x_1$. It follows that $x_1 = y_2z_2$. Write $\frac{d}{3p} = d_1$, then (1.5) is changed to

$$\frac{y_2 + z_2}{d_1} = ay_2z_2 - p, \gcd(y_2, z_2) = 1, y_2 \le z_2.$$

If $3 \nmid d, p|d$ and $3|y_1$, then we get that $x_1|y_3z_2$ and $y_3z_2|x_1$ where $y_2 = 3y_3$. It follows that $x_1 = y_3z_2$. Write $\frac{d}{p} = d_1$, then (1.5) is changed to

$$\frac{3y_3 + z_2}{d_1} = ay_3z_2 - p, \gcd(3y_3, z_2) = 1, 3y_3 \le z_2.$$

If $3 \nmid d, p|d$ and $3|z_1$, then we get that $x_1|y_2z_3$ and $y_2z_3|x_1$ where $z_2 = 3z_3$. It follows that $x_1 = y_2z_3$. Write $\frac{d}{p} = d_1$, then (1.5) is changed to

$$\frac{y_2 + 3z_3}{d_1} = ay_2z_3 - p, \gcd(y_2, 3z_3) = 1.$$

If $p \nmid d, 3|d$ and $p|y_1$, then we get that $x_1|y_3z_2$ and $y_3z_2|x_1$ where $y_2 = py_3$. It follows that $x_1 = y_3z_2$. Write $\frac{d}{3} = d_1$, then (1.5) is changed to

$$\frac{py_3 + z_2}{d_1} = ay_3z_2 - p, \gcd(py_3, z_2) = 1, py_3 \le z_2.$$

If $p \nmid d, 3|d$ and $p|z_1$, then we get that $x_1|y_2z_3$ and $y_2z_3|x_1$ where $z_2 = pz_3$. It follows that $x_1 = y_2z_3$. Write $\frac{d}{3} = d_1$, then (1.5) is changed to

$$\frac{y_2 + pz_3}{d_1} = ay_2z_3 - p, \gcd(y_2, pz_3) = 1.$$

If $3p \nmid d$ and $3|y_1, p|z_1$, then we get that $x_1|y_3z_3$ and $y_3z_3|x_1$ where $y_2 = 3y_3, z_2 = pz_3$. It follows that $x_1 = y_3z_3$. Then (1.5) is changed to

$$\frac{3y_3 + pz_3}{d} = ay_3z_3 - p, \gcd(3y_3, pz_3) = 1.$$

If $3p \nmid d$ and $p|y_1, 3|z_1$, then we get that $x_1|y_3z_3$ and $y_3z_3|x_1$ where $y_2 = py_3, z_2 = 3z_3$. It follows that $x_1 = y_3z_3$. Then (1.5) is changed to

$$\frac{py_3 + 3z_3}{d} = ay_3z_3 - p, \gcd(py_3, 3z_3) = 1.$$

If $3 \nmid y_1z_1, p \nmid d$ and $p|y_1$, then we get that $x_1|y_3z_2$ and $y_3z_2|x_1$ where $y_2 = py_3$. It follows that $x_1 = y_3z_2$. Then (1.5) is changed to

$$\frac{py_3 + z_2}{d} = \frac{ay_3z_2 - p}{3}, \gcd(py_3, z_2) = 1.$$

If $3 \nmid y_1z_1, p \nmid d$ and $p|z_1$, then we get that $x_1|y_2z_3$ and $y_2z_3|x_1$ where $z_2 = pz_3$. It follows that $x_1 = y_2z_3$. Then (1.5) is changed to

$$\frac{y_2 + pz_3}{d} = \frac{ay_2z_3 - p}{3}, \gcd(y_2, pz_3) = 1.$$

If $3 \nmid y_1z_1, p|d$, then we get that $x_1|y_2z_2$ and $y_2z_2|x_1$. It follows that $x_1 = y_2z_2$. Write $\frac{d}{p} = d_1$, then (1.5) is changed to

$$\frac{y_2 + z_2}{d_1} = \frac{ay_2z_2 - p}{3}, \gcd(y_2, z_2) = 1.$$

Thus we have proved that solving the equation (1.5) is equivalent to solving the following diophantine equations:

$$\frac{y + z}{d} = ayz - p, \gcd(y, z) = 1, y \le z, \tag{1.6}$$

$$\frac{3y + z}{d} = ayz - p, \gcd(3y, z) = 1, \tag{1.7}$$

$$\frac{py + z}{d} = ayz - p, \gcd(py, z) = 1, \tag{1.8}$$

$$\frac{3y + pz}{d} = ayz - p, \gcd(3y, pz) = 1, \tag{1.9}$$

$$\frac{py + z}{d} = \frac{ayz - p}{3}, \gcd(py, z) = 1, 3 \nmid yz, \tag{1.10}$$

$$\frac{y + z}{d} = \frac{ayz - p}{3}, \gcd(y, z) = 1, y \le z, 3 \nmid yz. \tag{1.11}$$

In this paper, we investigate the equations (1.6), (1.7) and (1.8) with $p = 661$. Actually, we get all the solutions of positive integer of them. That is, we have the following results:

Theorem 1.1. *If $ayz - 661 \ge 80$, then all the solutions of positive integers of (1.6) are given by* $(a, d, y, z) =$

$$(1, 1, 2, 663), (1, 1, 3, 332), (1, 2, 1, 1323), (1, 2, 3, 265), (1, 3, 1, 992), (1, 3, 2, 397),$$

$$(2, 1, 1, 662), (2, 1, 2, 221), (2, 1, 4, 95), (2, 2, 1, 441), (3, 1, 1, 331),$$

$$(3, 1, 3, 83), (3, 3, 1, 248)(4, 2, 1, 189).$$

Theorem 1.2. *If* $ayz - 661 \geq 90$*, then all the solutions of positive integers of (1.7) are given by* $(a, d, y, z) =$

$$(1,1,2,667),(1,1,3,335),(1,1,5,169),(1,1,9,86),(1,1,84,11),(1,1,167,7),(1,1,333,5),$$
$$(1,1,665,4),(1,2,1,1325),(1,4,2645,1),(1,5,1,827),(1,5,1653,1),(1,7,1157,1),(1,11,909,1),$$
$$(1,19,785,1),(2,1,1,664),(2,1,3,134),(2,1,27,14),(2,1,133,4),(2,1,663,2),(2,2,1323,1),$$
$$(2,4,529,1),(3,1,1,332),(3,1,221,2),(3,2,1,265),(3,2,441,1),(4,1,662,1),(5,1,1,166),$$
$$(5,1,331,1),(5,2,189,1),(6,2,147,1),(8,1,51,2),(10,1,39,2),(12,2,63,1).$$

Theorem 1.3. *Each of the following is true:*

1. *If* $ayz - 661 < 80$*, then all the solutions of positive integers of (1.6) are given by* $(a, d, y, z) =$

$$(1,3,5,142),(1,3,17,40),(1,7,6,113),(1,11,18,37),(1,28,17,39),(1,32,13,51),$$
$$(1,67,2,333),(1,112,3,221),(1,332,1,663),(1,333,2,331),(1,663,1,662),(2,1,5,74),$$
$$(2,1,11,32),(2,1,14,25),(2,2,11,31),(2,7,1,356),(2,8,5,67),(2,20,1,339),(2,26,1,337),$$
$$(2,29,4,83),(2,112,1,332),(2,332,1,331),(3,1,11,21),(3,14,13,17),(3,28,1,223),$$
$$(3,111,1,221),(4,1,6,29),(4,5,1,174),(4,24,1,167),(5,1,4,35),(5,1,6,23),(5,2,1,147),$$
$$(5,15,1,134),(6,4,1,115),(6,8,3,37),(7,1,2,51),(7,6,5,19),(7,24,1,95),(8,28,1,83),$$
$$(9,1,2,39),(9,1,4,19),(9,2,3,25),(9,15,1,74),(11,2,1,63),(11,3,1,62),(13,10,3,17),$$
$$(13,26,1,51),(14,2,1,49),(15,1,5,9),(17,8,3,13),(17,20,1,39),(19,3,5,7),(19,9,1,35),$$
$$(20,1,2,17),(21,3,1,32),(23,5,1,29),(24,1,4,7),(25,2,1,27),(26,1,2,13),(28,1,3,8),$$
$$(29,4,1,23),(32,2,1,21),(35,5,1,19),(39,9,1,17),(51,7,1,13),(74,2,1,9),(83,3,1,8),$$
$$(95,2,1,7),(111,1,2,3),(221,2,1,3),(331,3,1,2),(332,1,1,2),(662,2,1,1),(663,1,1,1).$$

2. *If* $ayz - 661 < 90$*, then all the solutions of positive integers of (1.7) are given by* $(a, d, y, z) =$

$$(1,9,1,744),(1,29,3,223),(1,31,35,19),(1,37,5,133),(1,67,39,17),(1,73,95,7),$$
$$(1,83,51,13),(1,101,133,5),(1,115,3,221),(1,127,677,1),(1,167,1,665),(1,333,1,663),$$
$$(1,337,2,331),(1,665,1,662),(1,995,331,2),(1,995,663,1),(1,1987,662,1),(2,1,13,28),$$
$$(2,2,3,121),(2,200,333,1),(2,334,1,331),(2,994,331,1),(3,2,4,58),(3,7,2,113),$$
$$(3,11,6,37),(3,28,13,17),(3,32,17,13),(3,67,111,2),(3,112,1,221),(3,332,221,1),$$
$$(4,2,9,19),(5,10,7,19),(5,16,19,7),(5,34,1,133),(5,100,133,1),(6,5,16,7),(6,20,113,1),$$
$$(8,1,18,5),(8,23,84,1),(9,1,1,83),(9,1,7,11),(12,1,2,29),(12,5,58,1),(13,2,1,53),$$
$$(13,13,3,17),(15,1,2,23),(15,2,49,1),(17,11,3,13),(17,59,39,1),(18,8,1,37),(26,2,27,1),$$
$$(27,1,13,2),(27,2,1,25),(33,2,21,1),(34,1,5,4),(37,11,18,5),(39,10,1,17),(39,26,17,1),$$
$$(45,1,3,5),(51,8,1,13),(51,20,13,1),(75,2,9,1),(84,1,1,8),(96,2,7,1),(112,1,3,2),$$
$$(133,2,1,5),(167,1,1,4),(221,5,3,1),(222,2,3,1),(331,5,1,2),(331,7,2,1),(333,1,1,2),$$
$$(334,1,2,1),(662,4,1,1),(663,2,1,1),(665,1,1,1).$$

Theorem 1.4. *All the solutions of positive integers of (1.8) with $py \le z$ are given by* $(a, d, y, z) =$

$$(1,1,2,1983),(1,2,1,1983),(1,3,1,1322),(1,3,2,661),(1,331,2,333),(1,662,1,663),$$

$$(1,1323,1,662),(1,1653,2,331),(2,1,1,1322),(2,2,1,661),(2,331,1,332),(2,992,1,331).$$

We organize this paper as follows. In Section 2, we present some lemmas which are needed in the proof of our main results. Consequently, in Sections 3 to 6, we give the proofs of Theorem 1.1 to 1.4 respectively.

2. Some lemmas

To prove the main theorems, we need the following lemmas.

Lemma 2.1. *Let $661 + b = p$, where p is an odd prime. If one of the following conditions is satisfied*

$$p \equiv 1 \pmod q, q = 3, 5 \text{ or } 11 \text{ or } p \equiv 3 \pmod 7,$$

$$\text{or } p \equiv 23 \pmod{29}, \text{ or } p \equiv 5 \pmod{41},$$

then both equation (1.6) and equation (1.7) have no solution (a, d, y, z) of positive integers with $ayz - 661 = b$.

Proof. We only prove the case $p \equiv 23 \pmod{29}$. The proofs of other cases are similarly. By the assumption we have $b \equiv p - 661 \equiv 0 \pmod{29}$. Assume either (1.6) or (1.7) has a positive integer solution (a, d, y, z) with $ayz - 661 = b$. It follows $(y, z) \in \{(1, 1), (1, p), (p, 1)\}$ and either

$$y + z \equiv 0 \pmod{29} \tag{2.1}$$

or

$$3y + z \equiv 0 \pmod{29}. \tag{2.2}$$

One can easily to see neither (2.1) nor (2.2) is satisfied for any element $(y, z) \in \{(1, 1), (1, p), (p, 1)\}$. This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let $661 + b = pq$, where p, q are prime numbers.*
(i)If one of the following conditions is satisfied

$$p \equiv q \equiv 1 \pmod 3, \text{ or } p \equiv q \equiv 1 \pmod 5, \text{ or } p \equiv 1 \pmod 7, q \equiv 3 \pmod 7,$$

$$\text{or } p \equiv 3 \pmod{13}, q \equiv -5 \pmod{13}, \text{ or } p \equiv 3 \pmod{17}, q \equiv 5 \pmod{17},$$

$$\text{or } p \equiv 3 \pmod{19}, q \equiv 5 \pmod{19}, \text{ or } p \equiv 3 \pmod{20}, q \equiv 7 \pmod{20},$$

$$\text{or } p \equiv 7 \pmod{23}, q \equiv 9 \pmod{23}, \text{ or } p \equiv 3 \pmod{31}, q \equiv -7 \pmod{31},$$

$$\text{or } p \equiv 2 \pmod{37}, q \equiv 14 \pmod{37}, \text{ or } p \equiv 23 \pmod{52}, q \equiv 31 \pmod{52},$$

$$\text{or } p \equiv 3 \pmod{56}, q \equiv 15 \pmod{56}, \text{ or } p \equiv 2 \pmod{57}, q \equiv 17 \pmod{57},$$

$$\text{or } p \equiv q \equiv 2 \pmod{73}, \text{ or } p \equiv 2 \pmod{85}, q \equiv 33 \pmod{85},$$

$$or\ p \equiv 7 \pmod{88}, q \equiv 19 \pmod{88},$$

then both equation (1.6) and equation (1.7) have no positive integer solution (a, d, y, z) *with* $ayz - 661 = b$*.*

(ii). If one of the following conditions is satisfied

$$p \equiv q \equiv 1 \pmod 4,\ or\ p \equiv q \equiv \pm 2 \pmod 9,$$

$$or\ p \equiv 2 \pmod{11}, q \equiv 6 \pmod{11},$$

then equation (1.6) has no positive integer solution (a, d, y, z) *with* $ayz - 661 = b$*.*

Proof. (i)We only prove the case $p \equiv 7 \pmod{23}, q \equiv 9 \pmod{23}$. The proofs of other cases are similarly. By the assumption we have $b \equiv pq - 661 \equiv 0 \pmod{23}$. Assume either (1.6) or (1.7) has a positive integer solution (a, d, y, z) with $ayz - 661 = b$. It follows $(y, z) \in \{(1, p^u q^v), (p, q^v), (q, p^u), (pq, 1), u, v = 0, 1\}$ and either

$$y + z \equiv 0 \pmod{23} \tag{2.3}$$

or

$$3y + z \equiv 0 \pmod{23}. \tag{2.4}$$

One can easily to see neither (2.3) nor (2.4) is satisfied for any element $(y, z) \in \{(1, p^u q^v), (p, q^v), (q, p^u), (pq, 1), u, v = 0, 1\}$. This completes the proof of part (i) of Lemma 2.2.

The proof of part (ii) is similar. □

Lemma 2.3. *Let* $661 + b = pqr$*, where* p, q, r *are prime. If one of the following conditions is satisfied*

$$p \equiv q \equiv 3 \pmod{11}, r \equiv 5 \pmod{11},\ or\ p \equiv q \equiv 3 \pmod{25}, r \equiv 4 \pmod{25},$$

$$or\ p \equiv q \equiv 2 \pmod{31}, r \equiv 18 \pmod{31},\ or\ p \equiv q \equiv 2 \pmod{55}, r \equiv 14 \pmod{55},$$

$$or\ p \equiv q \equiv 19 \pmod{61}, r \equiv 2 \pmod{61},\ or\ p \equiv q \equiv 5 \pmod{64}, r \equiv 29 \pmod{64},$$

$$or\ p \equiv 3 \pmod{80}, q \equiv 13 \pmod{80}, r \equiv 19 \pmod{80},$$

then both equation (1.6) and equation (1.7) have no positive integer solution (a, d, y, z) *with* $ayz - 661 = b$*.*

Proof. We only prove the case $p \equiv q \equiv 19 \pmod{61}, r \equiv 2 \pmod{61}$. The proofs of other cases are similarly. By the assumption we have $b \equiv pqr - 661 \equiv 0 \pmod{61}$. Assume either (1.6) or (1.7) has a positive integer solution (a, d, y, z) with $ayz - 661 = b$. It follows

$$(y, z) \in \{(1, p^u q^v r^t), (p, q^v r^t), (q, p^u r^t), (r, p^u q^v), (pq, r^t), (pr, q^v), (qr, p^t), (pqr, 1), u, v, t = 0, 1\}$$

and either

$$y + z \equiv 0 \pmod{61} \tag{2.5}$$

or

$$3y + z \equiv 0 \pmod{23}. \tag{2.6}$$

One can easily to see neither (2.5) nor (2.6) is satisfied for any element $(y, z) \in \{(1, p^u q^v r^t), (p, q^v r^t), (q, p^u r^t), (r, p^u q^v), (pq, r^t), (pr, q^v), (qr, p^t), (pqr, 1), u, v, t = 0, 1\}$. This completes the proof of Lemma 2.3. □

Lemma 2.4. *If* $3|b$*, then equation (1.7) has no positive integer solution* (a, d, y, z) *with* $ayz - 661 = b$*.*

Proof. Assume (1.7) has a positive integer solution (a, d, y, z) with $ayz - 661 = b$. Then we get $b|3y + z$. It implies that $3|z$ since $3|b$, which contradicts with $\gcd(3, z) = 1$. This completes the proof of Lemma 2.4. □

Lemma 2.5. *Let* $661 + b = 2^k p$*, where* k *is an integer more than* 2 *and* p *is an odd prime.*

If $1 + 2^u p^v \not\equiv 0 \pmod{b}$*,* $p + 2^u \not\equiv 0 \pmod{b}$*, and* $2^t + p^v \not\equiv 0 \pmod{b}$*, where* $v \in \{0, 1\}, 0 \le u \le k, 1 \le t \le k$*, then equation (1.6) has no positive integer solution* (a, d, y, z) *with* $ayz - 661 = b$*.*

Proof. Assume (1.6) has a positive integer solution (a, d, y, z) with $ayz - 661 = b$. It follows

$$(y, z) \in \{(1, 2^u p^v), (p, 2^u), (2^t, p^v), (2^t p, 1), 0 \le u \le k, v = 0, 1, 1 \le t \le k\}$$

and

$$y + z \equiv 0 \pmod{b}, \tag{2.7}$$

which contradicts the assumption of Lemma 2.5.

This completes the proof of Lemma 2.5. □

Lemma 2.6. *Let* (a, d, y, z) *be a positive integer solution of equation (1.6) with* $ayz - 661 = b$*. If* $b \ge 80, ad \ge 2$*, then* $y < 6$*.*

Proof. We first prove that $b \le 1322$. Otherwise $b > 1322$, then we have $z > \frac{y+z}{2} \ge \frac{1322}{2} = 661$.

If $ay > 2$, then from $ayz - 661 = \frac{y+z}{d} \le 2z$, we get $z \le (ay - 2)z \le 661$ which contradicts with $z > 661$.

If $a = y = d = 1$,then we have $1 = -661$ which is impossible.

If $a = y = 1, d > 1$, then we have

$$z = \frac{661d + 1}{d - 1} = 661 + \frac{662}{d - 1} \le 661 + 662 = 1323,$$

which contradicts with $z > 1322 + 661 = 1983$.

If $a = 1, y = 2$, then we have

$$z = \frac{661}{2} + \frac{665}{2(2d - 1)} \le \frac{661 + 665}{2} = 663,$$

which contradicts with $z > 1322 + 661 = 1983$.

If $a = 2, y = 1$, then we have

$$z = \frac{661}{2} + \frac{663}{2(2d - 1)} \le \frac{661 + 663}{2} = 662,$$

which also contradicts with $z > 1322 + 661 = 1983$. Hence $b \le 1322$ as desired.

Assume now $y \ge 6$.

If $d \ge 2, b \ge 200$, then $y + z = db \ge 400$. Since quadratic function

$$f(y) = y(db - y), 0 < y \le \frac{db}{2},$$

is increasing function, so we have

$$yz = f(y) \geq f(6) = 6 \cdot (db - 6) \geq 6 \cdot 394 = 2364,$$

which contradicts with $ayz \leq 1983$. If $d \geq 2, 80 \leq b < 200$, then similarly we have

$$yz = f(y) \geq f(6) = 6 \cdot (db - 6) \geq 6 \cdot 154 = 924,$$

which contradicts with $ayz \leq 661 + 200 = 861$.

If $d = 1$, then we have $a \geq 2$. It follows that $yz \leq [\frac{1983}{a}] \leq [\frac{1983}{2}] = 991$. If $b \geq 200$, then similarly we have

$$yz = f(y) \geq f(6) = 6 \cdot (b - 6) \geq 6 \cdot 194 = 1164,$$

which contradicts with $yz \leq 991$. If $80 \leq b < 200$, then similarly we have

$$yz = f(y) \geq f(6) = 6 \cdot (b - 6) \geq 6 \cdot 74 = 444,$$

which contradicts with $yz \leq [\frac{861}{a}] \leq [\frac{861}{2}] = 430$.

This completes the proof of Lemma 2.6. □

Lemma 2.7. *Let (a, d, y, z) be a positive integer solution of equation (1.7) with $ayz - 661 = b$. If $80 \leq b \leq 2644$, and $a \geq 5$ or $d \geq 7$, then we have either $y < 6$ or $z < 6$.*

Proof. Since $\gcd(3y, z) = 1$, so we have either $3y < z$ or $3y > z$. We prove that $y < 6$ if $3y < z$ and $z < 6$ if $3y > z$.

We prove that $z < 6$ if $3y > z$. Otherwise $z \leq 6$.

Case 1: $d \geq 7$. Then $3y + z = db$, we get $z \leq (ay - 2)z \leq 661$ which contradicts with $z > 661$.

If $b \geq 300$, then $y + z = db \geq 2100$. Since quadratic function

$$g(z) = z(db - z), 0 < z \leq \frac{db}{2},$$

is increasing function, so we have

$$yz = \frac{g(z)}{3} \geq \frac{g(6)}{3} \geq 2 \cdot 2094 = 4188,$$

which contradicts with $ayz \leq 3305$.

If $90 \leq b < 300$, then similarly we have

$$yz = \frac{g(z)}{3} \geq \frac{g(6)}{3} \geq 2 \cdot 624 = 1248,$$

which contradicts with $ayz \leq 661 + 300 = 961$.

Case 2:$a \geq 5$, then we have $yz \leq [\frac{3305}{a}] \leq [\frac{3305}{5}] = 661$.

If $b \geq 400$, then similarly we have

$$yz = \frac{g(z)}{3} \geq \frac{g(6)}{3} \geq 2 \cdot 394 = 788,$$

which contradicts with $yz \leq 661$.

If $150 \le b < 400$, then similarly we have

$$yz = \frac{g(z)}{3} \ge \frac{g(6)}{3} \ge 2 \cdot 144 = 288,$$

which contradicts with $yz \le [\frac{1061}{a}] \le [\frac{1061}{5}] = 212$.

If $90 \le b < 150$, then similarly we have

$$yz = \frac{g(z)}{3} \ge \frac{g(6)}{3} \ge 2 \cdot 84 = 168,$$

which contradicts with $yz \le [\frac{811}{a}] \le [\frac{811}{5}] = 162$.

Similarly we can prove that $y < 6$ if $3y < z$.

This completes the proof of Lemma 2.7. □

3. Proof of Theorem 1.1

Assume that (a, d, y, z) is a positive integer solution of equation (1.6) with $ayz - 661 = b \ge 80$. Then we have $y \le 5$ by Lemma 2.6. We divide the proof into four cases.

Case 1: $d = 1$. From equation (1.6), we get

$$(ay - 1)z = 661 + y. \tag{3.1}$$

Replacing y by $1, 2, 3, 4, 5$ in (3.1) respectively, we obtain

$$(a - 1)z = 2 \cdot 331, (2a - 1)z = 3 \cdot 13 \cdot 17, (3a - 1)z = 8 \cdot 83, (4a - 1)z = 5 \cdot 7 \cdot 19, (5a - 1)z = 2 \cdot 9 \cdot 37.$$

Thus we get

$$(a, y, z) = (2, 1, 662), (3, 1, 331), (2, 2, 221), (3, 3, 83), (2, 4, 95).$$

Case 2: $d = 2$. From equation (1.6), we get

$$(2ay - 1)z = 1322 + y. \tag{3.2}$$

Replacing y by $1, 2, 3, 4, 5$ in (3.2) respectively, we obtain

$$(2a - 1)z = 9 \cdot 49, (4a - 1)z = 4 \cdot 331, (6a - 1)z = 25 \cdot 53, (8a - 1)z = 6 \cdot 13 \cdot 17, (10a - 1)z = 1327.$$

Thus we get

$$(a, y, z) = (1, 1, 1323), (2, 1, 441), (4, 1, 189), (1, 3, 265).$$

Case 3: $d = 3$. From equation (1.6), we get

$$(3ay - 1)z = 1983 + y. \tag{3.3}$$

Replacing y by $1, 2, 3, 4, 5$ in (3.3) respectively, we obtain

$$(3a - 1)z = 4^3 \cdot 31, (6a - 1)z = 5 \cdot 397, (9a - 1)z = 6 \cdot 331, (12a - 1)z = 1987, (15a - 1)z = 4 \cdot 7 \cdot 71.$$

Thus we get

$$(a, y, z) = (1, 1, 992), (3, 1, 248), (1, 2, 397).$$

Case 4: $d \geq 4$. If $b \geq 100$, then by the proof of Lemma 2.6 we know

$$yz = f(y) \geq f(6) \geq 6 \cdot 394 = 2364,$$

which contradicts with $ayz \leq 1983$. If $80 \leq b < 100$, then similarly we know

$$yz = f(y) \geq f(6) \geq 6 \cdot 314 = 1884,$$

which contradicts with $ayz \leq 761$.

So Theorem 1.1 is proved.

4. Proof of Theorem 1.2

Assume that (a, d, y, z) is a positive integer solution of equation (1.7) with $ayz - 661 = b \geq 90$. We divide the proof into three cases.

Case 1: $b > 2644$, assume that $y > z$. Then $y > \frac{3y+z}{4} = \frac{b}{4} > 661$. If $az > 4$, then from (1.7), we get $ayz - 661 = \frac{3y+z}{d} < 4y$. It implies that $y \leq (az - 4)y < 661$, which contradicts with $y > 661$. Hence $az \leq 4$. On the other hand, we have $(adz - 3)b = az^2 + 1983 \leq 16 + 1983 = 1999$ which contradicts with $b > 2644$. Similarly we can prove that $y \leq z$ is impossible.

Case 2: $90 \leq b \leq 2644, d < 7$ and $a < 5$, then we have $a \in \{1, 2, 3, 4\}, d \in \{1, 2, 4, 5\}$ since $d | 3y + z, \gcd(3y, z) = 1$.

If $a = 1$, from equation (1.7), we get

$$(dy - 1)(dz - 3) = 661d^2 + 3. \tag{4.1}$$

Replacing d by $1, 2, 4, 5$ in (4.1) respectively, we obtain

$$(y - 1)(z - 3) = 8 \cdot 83, (2y - 1)(2z - 3) = 2647, (4y - 1)(4z - 3) = 71 \cdot 149, (5y - 1)(5z - 3) = 16 \cdot 1033.$$

Thus we get

$$(d, y, z) = (1, 2, 667), (1, 3, 335), (1, 5, 169), (1, 9, 86), (1, 84, 11), (1, 167, 7), (1, 333, 5),$$

$$(1, 665, 4), (2, 1, 1325), (2, 1324, 2), (4, 2645, 1), (5, 1, 827), (5, 1653, 1).$$

If $a = 2$, from equation (1.7), we get

$$(2dy - 1)(2dz - 3) = 1322d^2 + 3. \tag{4.2}$$

Replacing d by $1, 2, 4, 5$ in (4.2) respectively, we obtain

$$(2y-1)(2z-3) = 25 \cdot 53, (4y-1)(4z-3) = 11 \cdot 13 \cdot 17, (8y-1)(8z-3) = 5 \cdot 4231, (5y-1)(5z-3) = 33053.$$

Thus we get

$$(d, y, z) = (1, 1, 664), (1, 3, 134), (1, 27, 14), (1, 133, 4), (1, 663, 2),$$

$$(2, 1323, 1), (4, 529, 1).$$

If $a = 3$, from equation (1.7), we get

$$(3dy - 1)(dz - 1) = 661d^2 + 1. \tag{4.3}$$

Replacing d by $1, 2, 4, 5$ in (4.3) respectively, we obtain

$$(3y-1)(z-1) = 2\cdot331, (6y-1)(2z-1) = 5\cdot23^2, (9y-1)(3z-1) = 2\cdot5^2\cdot7\cdot17, (15y-1)(5z-1) = 2\cdot8263.$$

By computing we get that

$$(d, y, z) = (1, 1, 332), (1, 221, 2), (2, 1, 265), (2, 441, 1).$$

If $a = 4$, we contend $d < 3$. Otherwise $d \geq 4$, we have $3y + z = db \geq 4b$. If $b \geq 300$, then we have $yz \geq 1197$, which contradicts with $yz \leq [\frac{3305}{4}] = 826$. If $90 \leq b < 300$, then we have $yz \geq 357$, which contradicts with $yz \leq [\frac{961}{4}] = 240$. Hence $d < 3$ as desired. From equation (1.7), we get

$$(4dy - 1)(4dz - 3) = 2644d^2 + 3. \tag{4.4}$$

Replacing d by $1, 2$ in (4.4) respectively, we obtain

$$(4y - 1)(4z - 3) = 2647, (8y - 1)(8z - 3) = 71 \cdot 149.$$

Thus we get

$$(d, y, z) = (1, 662, 1).$$

Case 3: $90 \leq b \leq 2644, d \geq 7$ or $a \geq 5$. Then we have either $y \leq 5$ or $z \leq 5$ by Lemma 2.7.

Subcase 1: $d = 1, a \geq 5$, from equation (1.7), we get

$$(ay - 1)z = 661 + 3y \tag{4.5}$$

and

$$(az - 3)y = 661 + z. \tag{4.6}$$

Replacing y by $1, 2, 3, 4, 5$ in (4.5) respectively, we obtain

$$(a - 1)z = 8 \cdot 83, (2a - 1)z = 23 \cdot 29, (3a - 1)z = 2 \cdot 5 \cdot 67, (4a - 1)z = 673, (5a - 1)z = 4 \cdot 13^2.$$

Thus we get

$$(a, y, z) = (5, 1, 166).$$

Replacing z by $1, 2, 4, 5$ in (4.6) respectively, we obtain

$$(a - 3)y = 2 \cdot 331, (2a - 3)y = 3 \cdot 13 \cdot 17, (4a - 3)y = 5 \cdot 7 \cdot 19, (5a - 3)y = 2 \cdot 3^2 \cdot 37.$$

By computing we get

$$(a, y, z) = (5, 331, 1), (8, 51, 2), (10, 39, 2).$$

Subcase 2: $d = 2, a \geq 5$, from equation (1.7), we get

$$(2ay - 1)z = 1322 + 3y \tag{4.7}$$

and

$$(2az - 3)y = 1322 + z. \tag{4.8}$$

Replacing y by $1, 2, 3, 4, 5$ in (4.7) respectively, we obtain

$$(2a-1)z = 5^2 \cdot 53, (4a-1)z = 2^4 \cdot 83, (6a-1)z = 11^3, (8a-1)z = 2 \cdot 23 \cdot 29, (10a-1)z = 7 \cdot 191.$$

By computing we know that the above equations have no positive integer solution (a, z) such that $a \geq 5, b \geq 90$.

Replacing z by $1, 2, 4, 5$ in (4.8) respectively, we obtain

$$(2a-3)y = 3^3 \cdot 7^2, (4a-3)y = 2^2 \cdot 331, (8a-3)y = 2 \cdot 3 \cdot 13 \cdot 17, (10a-3)y = 1327.$$

By computing we get

$$(a, y, z) = (6, 147, 1), (12, 63, 1), (5, 189, 1).$$

Subcase 3: $d \geq 4, a \geq 5$, then we have $3y + z = db \geq 4b$. If $b \geq 200$, then we get that $yz \geq 797$ which contradicts with $yz \leq [\frac{3305}{5}] = 661$. If $90 \leq b < 200$, then we get that $yz \geq 357$ which contradicts with $yz \leq [\frac{861}{5}] = 172$.

Subcase 4: $d \geq 7, a < 5$. We contend that $a = 1$. Otherwise $a \geq 2$. If $b \geq 300$, then we get that $yz \geq 2097$ which contradicts with $yz \leq [\frac{3305}{2}] = 1652$. If $90 \leq b < 300$, then we get that $yz \geq 627$ which contradicts with $yz \leq [\frac{861}{2}] = 430$. Thus $a = 1$ as desired. From equation (1.7), we get

$$(dy - 1)(yz - 661) = 661 + 3y^2 \tag{4.9}$$

and

$$(dz - 3)(yz - 661) = 1983 + z^2. \tag{4.10}$$

Replacing y by $1, 2, 3, 4, 5$ in (4.9) respectively, we obtain

$$(d-1)(z-661) = 8 \cdot 83, (2d-1)(2z-661) = 673, (3d-1)(3z-661) = 4^2 \cdot 43, (4d-1)(4z-661) = 709,$$

$$(5d-1)(5z-661) = 2^5 \cdot 23.$$

By computing we know that the above equations have no positive integer solution (a, z) such that $d \geq 7, b \geq 90$.

Replacing z by $1, 2, 4, 5$ in (4.8) respectively, we obtain

$$(d-3)(y-661) = 2^6 \cdot 31, (2d-3)(2y-661) = 1987, (4d-3)(4y-661) = 1999, (5d-3)(5y-661) = 8 \cdot 251.$$

By computing we get that

$$(d, y, z) = (7, 1157, 1), (11, 909, 1), (19, 785, 1).$$

Therefore the proof of Theorem 1.2 is complete.

5. Proof of Theorem 1.3

Since

$$661+12=673\equiv 1 \pmod 3, 661+22=683\equiv 1 \pmod{11}, 661+30=691\equiv 1 \pmod 3,$$

$$661+40=701\equiv 1 \pmod 5, 661+48=709\equiv 1 \pmod 3, 661+58=719\equiv 23 \pmod{29},$$

$$661+66=727\equiv 1 \pmod 3, 661+72=733\equiv 1 \pmod 3, 661+78=739\equiv 1 \pmod 3,$$

$$661+82=743\equiv 5 \pmod{41},$$

so both (1.6) and (1.7) have no positive integer solution (a,d,y,z) with $ayz-661=b$ for $b\in\{12,22,30,40,48,58,66,72,78,82\}$ by Lemma 2.1.

Since

$$661+10=671=11\cdot 61, 11\equiv 61\equiv 1 \pmod 5,$$

$$661+20=681=3\cdot 227, 227\equiv 7 \pmod{20},$$

$$661+26=687=3\cdot 229, 229\equiv -5 \pmod{13},$$

$$661+28=689=13\cdot 53,$$

$$661+34=695=5\cdot 139, 139\equiv 3 \pmod{17},$$

$$661+37=698=2\cdot 347, 347\equiv 14 \pmod{37},$$

$$661+38=699=3\cdot 233, 233\equiv 5 \pmod{19},$$

$$661+46=707=7\cdot 101, 101\equiv 9 \pmod{23},$$

$$661+52=713=23\cdot 31,$$

$$661+56=717=3\cdot 239, 239\equiv 1 \pmod 7,$$

$$661+57=718=2\cdot 359, 359\equiv 17 \pmod{57},$$

$$661+60=721=7\cdot 103, 7\equiv 103\equiv 1 \pmod 3,$$

$$661+62=723=3\cdot 241, 241\equiv -7 \pmod{31},$$

$$661+70=731=17\cdot 43, 17\equiv 3 \pmod 7, 43\equiv 1 \pmod 7,$$

$$661+56=717=3\cdot 239, 239\equiv 15 \pmod{56},$$

$$661+73=734=2\cdot 367, 367\equiv 2 \pmod{73},$$

$$661+76=737=11\cdot 67,$$

$$661+85=746=2\cdot 373, 373\equiv 33 \pmod{85},$$

$$661+88=749=7\cdot 107, 107\equiv 19 \pmod{88},$$

so both (1.6) and (1.7) have no positive integer solution (a,d,y,z) with $ayz-661=b$ for $b\in\{10,20,26,28,34,37,38,46,52,56,57,60,62,70,73,76,85,88\}$ by Lemma 2.2(i).

Since

$$661+31=692=2^2\cdot 173, 173\equiv 18 \pmod{31},$$

$$661 + 44 = 705 = 3 \cdot 5 \cdot 47, 47 \equiv 3 \pmod{11},$$

$$661 + 50 = 711 = 3^2 \cdot 79, 79 \equiv 4 \pmod{25},$$

$$661 + 55 = 716 = 2^2 \cdot 179, 179 \equiv 14 \pmod{55},$$

$$661 + 61 = 722 = 2 \cdot 19^2,$$

$$661 + 64 = 725 = 5^2 \cdot 29,$$

$$661 + 80 = 741 = 3 \cdot 13 \cdot 19,$$

so both (1.6) and (1.7) have no positive integer solution (a, d, y, z) with $ayz - 661 = b$ for $b \in \{31, 44, 50, 55, 61, 64\}$ by Lemma 2.3(i).

Since

$$661 + 47 = 708 = 2^2 \cdot 3 \cdot 59, 661 + 59 = 720 = 2^4 \cdot 3^2 \cdot 5, 661 + 68 = 729 = 3^6,$$

$$661 + 71 = 732 = 2^2 \cdot 3 \cdot 61, 661 + 77 = 738 = 2 \cdot 3^2 \cdot 41,$$

by computing we know both (1.6) and (1.7) have no positive integer solution (a, d, y, z) with $ayz - 661 = b$ for $b \in \{47, 59, 68, 71, 77\}$.

Since $661 + 27 = 688 = 2^4 \cdot 43, 27 \nmid x$, for any

$$x \in \{1 + 2^\alpha 43^\beta, 43 + 2^\alpha, 2^t + 43^\beta\},$$

where $\alpha \in \{0, 1, 2, 3, 4\}, \beta \in \{0, 1, \}, t \in \{1, 2, 3, 4\}$.$661 + 75 = 736 = 2^5 \cdot 23, 75 \nmid x$, for any

$$x \in \{1 + 2^\alpha 23^\beta, 23 + 2^\alpha, 2^t + 23^\beta\},$$

where $\alpha \in \{0, 1, 2, 3, 4, 5\}, \beta \in \{0, 1, \}, t \in \{1, 2, 3, 4, 5\}$ so (1.6) has no positive integer solution (a, d, y, z) with $ayz - 661 = b$ for $b \in \{27, 75\}$ by Lemma 2.5.

Since

$$661 + 18 = 679 = 7 \cdot 97, 7 \equiv 97 \equiv -2 \pmod{9},$$

$$661 + 24 = 685 = 5 \cdot 137, 5 \equiv 137 \equiv 1 \pmod{4},$$

$$661 + 33 = 694 = 2 \cdot 347, 347 \equiv 6 \pmod{11},$$

$$661 + 36 = 697 = 17 \cdot 41, 17 \equiv 41 \equiv 1 \pmod{4},$$

$$661 + 42 = 703 = 19 \cdot 37, 19 \equiv 37 \equiv 1 \pmod{3},$$

$$661 + 45 = 706 = 2 \cdot 353, 353 \equiv 2 \pmod{9},$$

so (1.6) has no positive integer solution (a, d, y, z) with $ayz - 661 = b$ for $b \in \{18, 24, 33, 36, 42, 45\}$ by Lemma 2.2(ii).

For any $b = 3t, 1 \leq t < 29$, (1.7) has no positive integer solution (a, d, y, z) with $ayz - 661 = b$ by Lemma 2.4.

Since

$$661 + 3 = 2^3 \cdot 83, 661 + 6 = 2 \cdot 3^2 \cdot 37, 661 + 9 = 2 \cdot 5 \cdot 67, 661 + 15 = 2^2 \cdot 13^2,$$

$$661+21=2\cdot 11\cdot 31, 661+39=2^2\cdot 5^2\cdot 7, 661+51=2^3\cdot 89,$$

so by computing we get that $(a,d,y,z)=$

$$(2,29,4,83),(2,112,1,332),(8,28,1,83),(83,3,1,8),(332,1,1,2),(23,5,1,29),$$

$$(29,4,1,23),(2,8,5,67),(5,15,1,134),(26,1,2,13),(2,2,11,31),(11,3,1,62),$$
$$(2,1,14,25),(5,1,4,35),(2,7,1,356),$$

are positive integer solutions of (1.6).

We are now in a position to discuss

$$b\in\{1,2,4,5,7,8,11,13,14,16,17,19,23,25,28,29,32,35,41,43,49,53,59,65,67,74,79\}.$$

Since

$$661+1=2\cdot 331, 661+2=3\cdot 13\cdot 17, 661+4=5\cdot 7\cdot 19, 661+5=2\cdot 3^2\cdot 37,$$

$$661+7=2^2\cdot 167, 661+8=3\cdot 223, 661+11=2^5\cdot 3\cdot 7, 661+13=2\cdot 337, 661+14=3^3\cdot 5^2$$
$$661+16=677, 661+17=2\cdot 3\cdot 113, 661+19=2^3\cdot 5\cdot 17, 661+23=2^2\cdot 3^2\cdot 19,$$
$$661+25=2\cdot 7^3, 661+28=13\cdot 53, 661+29=2\cdot 3\cdot 5\cdot 23, 661+32=3^2\cdot 7\cdot 11,$$
$$661+35=2^3\cdot 3\cdot 29, 661+41=2\cdot 3^3\cdot 13, 661+43=2^6\cdot 11, 661+49=2\cdot 5\cdot 71,$$
$$661+53=2\cdot 3\cdot 7\cdot 17, 661+59=2^4\cdot 5\cdot 3^2, 661+65=2\cdot 3\cdot 11^2,$$
$$661+67=2^3\cdot 7\cdot 13, 661+74=3\cdot 5\cdot 7^2, 661+79=2^2\cdot 5\cdot 37,$$

so by computing we get that $(a,d,y,z)=$

$$(662,2,1,1),(1,663,1,662),(1,333,2,331),(2,332,1,331),(331,3,1,2),(663,1,1,1),$$

$$(1,332,1,663),(1,112,3,221),(1,32,13,51),(1,28,17,39),(3,111,1,221),(3,14,13,17),$$
$$(13,26,1,51),(13,10,3,17),(17,20,1,39),(17,8,3,13),(39,9,1,17),(51,7,1,13),$$
$$(221,2,1,3),(7,6,5,19),(19,3,5,7),(7,24,1,95),(19,9,1,35),(35,5,1,19),$$
$$(95,2,1,7),(1,11,18,37),(1,67,2,333),(9,15,1,74),(74,2,1,9),(6,8,3,37),$$
$$(111,1,2,3),(4,24,1,167),(3,28,1,223),(32,2,1,21),(24,1,4,7),(28,1,3,8),$$
$$(21,3,1,32),(2,26,1,337),(15,1,5,9),(9,2,3,25),(25,2,1,27),(1,7,6,113),$$
$$(2,20,1,339),(1,3,17,40),(20,1,2,17),(9,1,4,19),(14,2,1,49),(5,1,6,23),$$
$$(6,4,1,115),(11,2,1,63),(3,1,11,21),(4,1,6,29),(4,5,1,174),(9,1,2,39),$$
$$(2,1,11,32),(1,3,5,142),(7,1,2,51),(5,2,1,147),(2,1,5,74)$$

are the solutions of (1.6) and $(a,d,y,z)=$

$$(662,4,1,1),(1,337,2,331),(1,665,1,662),(1,1987,662,1),(1,995,331,2),(2,334,1,331),$$

$$(2, 994, 331, 1), (331, 5, 1, 2), (331, 7, 2, 1), (663, 2, 1, 1), (1, 333, 1, 663), (1, 995, 663, 1),$$
$$(1, 115, 3, 221), (1, 83, 51, 13), (1, 67, 39, 17), (3, 112, 1, 221), (3, 332, 221, 1), (3, 28, 13, 17),$$
$$(3, 32, 17, 13), (13, 13, 3, 17), (17, 59, 39, 1), (17, 11, 3, 13), (39, 10, 1, 17), (39, 26, 17, 1),$$
$$(51, 8, 1, 13), (51, 20, 13, 1), (221, 5, 3, 1), (1, 167, 1, 665), (665, 1, 1, 1), (1, 37, 5, 133),$$
$$(1, 101, 133, 5), (1, 31, 35, 19), (1, 73, 95, 7), (5, 34, 1, 133), (5, 10, 7, 19), (5, 16, 19, 7),$$
$$(133, 2, 1, 5), (5, 100, 133, 1), (333, 1, 1, 2), (18, 8, 1, 37), (222, 2, 3, 1), (3, 11, 6, 37),$$
$$(3, 67, 111, 2), (37, 11, 18, 5), (2, 200, 333, 1), (334, 1, 2, 1), (167, 1, 1, 4), (1, 29, 3, 223),$$
$$(6, 5, 16, 7), (8, 23, 84, 1), (84, 1, 1, 8), (96, 2, 7, 1), (112, 1, 3, 2), (45, 1, 3, 5),$$
$$(27, 2, 1, 25), (75, 2, 9, 1), (1, 127, 677, 1), (3, 7, 2, 113), (6, 20, 113, 1), (34, 1, 5, 4),$$
$$(4, 2, 9, 19), (13, 2, 1, 53), (15, 1, 2, 23), (9, 1, 7, 11), (33, 2, 21, 1), (12, 1, 2, 29),$$
$$(12, 5, 58, 1), (3, 2, 4, 58), (27, 1, 13, 2), (26, 2, 27, 1), (8, 1, 18, 5), (2, 2, 3, 121),$$
$$(2, 1, 13, 28), (15, 2, 49, 1)$$

are the solutions of (1.7).

We are now in a position to discuss $b \in \{83, 86, 89\}$.

Since $661 + 83 = 2^3 \cdot 3 \cdot 31, 661 + 86 = 3^2 \cdot 83$ so by computing we get that $(a, d, y, z) = (1, 9, 1, 744), (9, 1, 1, 83)$ are solutions of (1.7).

Since $661 + 89 = 2 \cdot 3 \cdot 5^3$, so by computing we get that (1.7) has no positive integer solution.

This finishes the proof of Theorem 1.3.

6. Proof of Theorem 1.4

If $ay > 2$, then from $ayz - 661 = \frac{661y+z}{d} < 2z$, we get that $z \le (ay-2)z < 661$ which contradicts with $z > 661y$. Hence $a = y = 1$ or $a = 1, y = 2$ or $a = 2, y = 1$.

$a = y = 1$ implies that $(d-1)(z-661) = 2 \cdot 661$. So

$$(d, z) = (2, 1983), (3, 1322), (662, 663), (1323, 662).$$

$a = 1, y = 2$ implies that $(2d-1)(2z-661) = 5 \cdot 661$. So

$$(d, z) = (1, 1983), (3, 661), (331, 333), (1653, 331).$$

$a = 2, y = 1$ implies that $(2d-1)(2z-661) = 3 \cdot 661$. So

$$(d, z) = (1, 1322), (2, 661), (331, 332), (992, 331).$$

This concludes the proof of Theorem 1.4.

Conflict of Interest

All authors declare no conflicts of interest in this paper.

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