



Research article

Dynamic learning for a DC pension plan with random risk aversion

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Abstract: This paper investigated the equilibrium investment strategy for a defined contribution (DC) pension plan that incorporates random risk aversion and stock return predictability. The financial market consists of a risk-free asset, an inflation-indexed bond, and a stock, where the expected stock return is driven by an unobservable mean-reverting predictor. By filtering techniques, a DC member dynamically learns and updates the estimation of this unobservable predictor based on the observations. Different from previous studies that consider only random risk aversion and resort to a system of ordinary differential equations (ODEs), this paper further incorporated stock return predictability and characterized the equilibrium strategy as a system of partial differential equations (PDEs). In the case of power utility, we derived the semi-closed form of the equilibrium investment strategy as the solution of a PDE system, and verified the existence of the solution by the Banach fixed-point theorem. Rigorous analysis showed that ignoring either random risk aversion or stock return predictability leads to welfare loss. Numerical analysis confirmed the welfare loss.

Keywords: DC pension plan; random risk aversion; certainty equivalents; stock return predictability; learning

Mathematics Subject Classification: Primary: 93E20, 91B30; Secondary: 93E11, 91B16

1. Introduction

Due to the rapid aging of the global population, ensuring the long-term sustainability of pension plans while maintaining members' living standards after retirement has become a major concern in actuarial science and financial economics. As a vital component of modern pension systems, DC pension plans have gained widespread global adoption in recent years, owing to their advantage of transferring investment and longevity risks from the plan sponsor to the plan member. For DC plan members, developing efficient asset allocation strategies over accumulation periods lasting decades, achieving a stable financial condition after retirement remains a significant challenge.

The literature on DC pension investment strategies branches into two directions. The first direction aims to maximize the utility of terminal wealth. Practical factors, such as inflation risk, stochastic interest rates, contribution rate volatility, and minimum return guarantees, have been gradually incorporated (see, e.g., [1–3]). The second direction aims to minimize the variance of terminal wealth and addresses the time-inconsistency issue arising from the variance term by solving the extended Hamilton–Jacobi–Bellman (HJB) equations [4, 5]. Typical examples of the second direction include [6–10].

Table 1. Comparison of this paper with [8, 11, 12].

Attributes	This paper	[8]	[11, 12]
Information	Partial information	Partial information	Full information
Risk aversion	Random risk aversion	Constant risk aversion	Random risk aversion
Solution form	PDE system	Closed-form	ODE system

In [1–3], the utility function is assumed to be known and constant; accordingly, the DC member is characterized by a constant and known level of risk aversion. In practice, however, individuals often struggle to accurately quantify their own risk preferences and may even lack clear awareness of them [13, 14]. Alserda et al. [15] measured the risk preferences of pension plan members, highlighting the importance of such preferences in optimizing asset allocation. To address the limitation of assuming constant risk aversion, a body of literature modeled risk aversion as a time-varying parameter related to current wealth. For instance, Björk et al. [4], Hu et al. [16], Li et al. [17], Sun et al. [18], and Wang et al. [19] assumed that risk aversion is inversely proportional to a linear function of wealth.

To address the portfolio problem with random risk aversion, Desmettre and Steffensen [11] established a stochastic control framework based on certainty equivalence. The construction is executed sequentially: utilities under various risk-preference scenarios are first transformed into a standardized monetary measure, after which the optimization objective is given by the expected transformed utility, with the expectation taken with respect to the distribution of random risk aversion. This effectively circumvents the challenges associated with summing quantities expressed in distinct units that lack a clear common scale. However, this formulation inevitably introduces time inconsistency. Therefore, Desmettre and Steffensen [11] derived the equilibrium strategy by solving the extended HJB equations. Recently, Wang and Jia [12] incorporated random risk aversion into the equilibrium investment strategy for DC pension plans.

In each of the aforementioned models, it was assumed that the parameters of stock returns are either fixed or deterministic functions of time. This suggests that members of DC pension plans have accurate and complete information about stock returns when they are making investment decisions. In real financial markets, however, members cannot directly observe the instantaneous expected return of stocks. Instead, they must rely on publicly available information, such as historical stock prices, to form real-time estimates. Numerous empirical and theoretical studies [20–26] confirmed that stock returns are predictable and that members can improve their decisions by learning from observable predictors. This learning process makes the optimal investment strategy non-myopic and horizon-dependent. As retirement approaches, DC pension plan members typically adopt a defensive strategy and reduce stock allocation, which is consistent with the investment decision-making practices of DC pension plans [8]. Therefore, incorporating stock return predictability into DC pension planning has important practical significance.

Given this importance, a large number of studies have investigated the impact of learning in DC pension plans. For instance, Wang et al. [8], Wang et al. [27], and Wang et al. [28] solved the problem of stock return predictability in DC pension plan investment under the criteria of maximizing the utility of terminal wealth and minimizing the variance of terminal wealth; Peng and Luo [29] studied an asset allocation problem for DC pensions under partial observation and a mean weighted variance-CVaR criterion; Peng and Fan [30] explored the optimal investment and dynamic attention allocation strategy for a DC pension plan with stock return predictability; and Gao et al. [31] investigated an optimally DC pension fund problem with partial information and obtained the optimal investment strategy by using the Bayesian approach and the martingale approach. These studies collectively demonstrate that ignoring learning behaviors can lead to significant welfare loss. However, it is noteworthy that all of these learning-based studies assumed that the member has a known constant risk aversion. To the best of our knowledge, no paper has yet investigated the joint effect of random risk aversion and stock return predictability on the equilibrium investment strategy in DC pension plans.

This paper integrates random risk aversion and stock return predictability into DC pension plans. Table 1 compares our work with the related literature. Specifically, we analyze a financial market that includes a risk-free asset, an inflation-indexed bond, and a stock, where the expected stock return is driven by an unobservable predictor that follows a mean-reverting stochastic process. Pension plan members cannot directly observe the predictors influencing stock returns, but can dynamically learn and update their estimates through filtering techniques based on the observable stock price process. Furthermore, the model incorporates stochastic contribution rates, inflation risk, and minimum annuity guarantees post-retirement. Using the certainty equivalent measure proposed by [11], we address the time inconsistency issues arising from the inclusion of random risk aversion. By solving the extended HJB system of equations, we reduce the solution for the equilibrium investment strategy to a PDE system and apply the Banach fixed-point theorem to verify the existence of the local solution. This paper also demonstrates that ignoring either random risk aversion or stock return predictability leads to welfare losses. Furthermore, numerical analysis confirms these results.

The remainder of this paper is organized as follows. Section 2 reviews [12], which studied the equilibrium strategy with random risk aversion and a completely observable stock return. Section 3 extends the analysis by incorporating stock return predictability into the optimization and provides a semi-closed-form solution for the equilibrium investment strategy. Section 4 theoretically demonstrates that ignoring either random risk aversion or the learning process leads to welfare losses. Section 5 verifies these results through numerical simulation. Section 6 concludes the paper.

2. Equilibrium investment strategy with random risk aversion and a completely observable stock return

This section reviews [12], which studied the equilibrium investment problem for a DC-pension-plan member facing random risk preferences. The problem is first transformed into a self-financing investment problem, and the semi-closed form of the equilibrium investment strategy is derived under a power utility function, up to the solution of an ODE system.

2.1. Financial market setting

Consider a complete filtrated probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0 \leq t \leq T})$, where $T \in (0, \infty)$ is the retirement time of the DC pension member. All stochastic processes discussed in this paper are defined on this probability space and are adapted to the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$. It is assumed that the pension fund can be invested in the financial market continuously without transaction cost or tax.

The financial market consists of a risk-free asset, an inflation-indexed bond, and a stock. The price process of the risk-free asset is described by

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) > 0,$$

where $r > 0$ is the risk-free return rate.

The price process of the stock follows

$$\frac{dS(t)}{S(t)} = rdt + \sigma_S (\mu(t)dt + dZ_S(t)), \quad S(0) > 0,$$

where $\mu(t)$ denotes the market price of the stock price risk, $\sigma_S > 0$ is a constant volatility, and $\{Z_S(t), t \in [0, T]\}$ is a standard Brownian motion.

The nominal price level of a representative bundle of commodity goods is denoted by $P(t)$, where

$$\frac{dP(t)}{P(t)} = \eta dt + \sigma_{p1} dZ_S(t) + \sigma_{p2} dZ_P(t), \quad P(0) > 0, \quad (1)$$

where $\eta > 0$ is the expected inflation rate, $\sigma_{p1} > 0$ is the stock volatility on $P(t)$, $\sigma_{p2} > 0$ is the volatility of $P(t)$ determined by factors other than the stock, and $\{Z_P(t), t \in [0, T]\}$ is a standard Brownian motion independent of $\{Z_S(t), t \in [0, T]\}$. Suppose that the market price of inflation risk is λ_p and the price process of the inflation-indexed bond $I(t, K)$ is

$$\frac{dI(t, K)}{I(t, K)} = (r + \sigma_{p1}\mu(t) + \sigma_{p2}\lambda_p)dt + \sigma_{p1}dZ_S(t) + \sigma_{p2}dZ_P(t). \quad (2)$$

2.2. Stochastic contribution rate and annuity guarantee

Regular contributions are made by members to their individual accounts in the DC pension plan during the accumulation phase, continuing until retirement at time T . Let $C(t)$ denote the contribution rate to the pension account at time t . Therefore, Wang and Jia [12] assumes that $C(t)$ satisfies the following stochastic differential equation (SDE):

$$\frac{dC(t)}{C(t)} = \lambda dt + \sigma_{C1} dZ_S(t) + \sigma_{C2} dZ_P(t), \quad C(0) > 0,$$

where $\lambda > 0$ is the expected growth rate, and $\sigma_{C1} > 0$ and $\sigma_{C2} > 0$ are positive volatility factors that represent the respective sensitivities of the contribution rate to stock market risk and inflation risk.

As emphasized in [1, 8, 12], DC-pension-plan members aim to accumulate terminal wealth that is adequate to cover a target retirement benefit, hence protecting their living standard after retirement. The annuity target for the member over $t \in [T, T']$ is $a(t)$, where T' denotes the random time of death. Thus, the actuarial present value of the target at time T is

$$G(T) = \int_T^\omega a(s)I(T, s)_{s-T}p_T ds,$$

where ω is the maximum survival age. $a(s) = a(T)\exp\{a_0(s - T)\}$, where $a(T)$ is the target annuity stream at the moment of retirement, and $a_0 > 0$ represents the inflation rate associated with the cost of living. ${}_{s-T}p_T$ is the probability that the member will survive to s , and can be calculated by

$${}_{s-T}p_T = \exp\left(-\int_T^s \lambda(u)du\right) = \frac{\omega - s}{\omega - t},$$

where $\lambda(t) = 1/(\omega - t)$, which is modeled by the Abraham De Moivre model.

2.3. Optimization problem

Let x_0 be the initial wealth, $\pi_I(t)$ and $\pi_S(t)$ be the proportions of money invested in the inflation-indexed bond and the stock at time t , respectively, and denote the portfolio strategy as $\pi(t) = (\pi_I(t), \pi_S(t))'$. Then, the wealth process $X^\pi(t)$ of the DC pension account is

$$\begin{aligned} dX^\pi(t) &= (1 - \pi_I(t) - \pi_S(t))X^\pi(t)\frac{dS_0(t)}{S_0(t)} + \pi_I(t)X^\pi(t)\frac{dI(t, K)}{I(t, K)} + \pi_S(t)X^\pi(t)\frac{dS(t)}{S(t)} + C(t)dt \\ &= \left[r + \pi_I(t)(\sigma_{p1}\mu(t) + \sigma_{p2}\lambda_p) + \pi_S(t)\sigma_S\mu(t)\right]X^\pi(t)dt + C(t)dt \\ &\quad + (\pi_S(t)\sigma_S + \pi_I(t)\sigma_{p1})X^\pi(t)dZ_S(t) + \pi_I(t)\sigma_{p2}X^\pi(t)dZ_P(t). \end{aligned} \quad (3)$$

Wang and Jia [12] presented the optimization problem in the following form:

$$\begin{aligned} V(t, x, p, c) &= \sup_{\pi} \int (u^\gamma)^{-1} \left(\mathbb{E}_{t,x,p,c} \left[u^\gamma \left(\frac{X^\pi(T) - G(T)}{P(T)} \right) \right] \right) d\Gamma(\gamma) \\ \text{s.t. } &X^\pi(T) > G(T), \end{aligned} \quad (4)$$

where u^γ is the member's utility function with parameter γ and γ is random with the cumulative distribution function (CDF) Γ . The random parameter γ is independent of $\{Z_S(t), t \in [0, T]\}$ and $\{Z_P(t), t \in [0, T]\}$. Here, $\mathbb{E}_{t,x,p,c}[\cdot] = \mathbb{E}[\cdot \mid X^\pi(t) = x, P(t) = p, C(t) = c]$ denotes the conditional expectation at time t .

3. Equilibrium investment strategy with random risk aversion and a partially observable stock return

In [12], the expected stock return is a deterministic function of time, implying that DC-pension-plan members possess accurate and complete information when making investment decisions. However, this assumption is unrealistic in real-world financial markets. In practice, members can only access partial information from the market, while the current instantaneous expected stock return rate is typically unobservable [8, 29]. Numerous empirical and theoretical studies have confirmed that investors can utilize observable predictors such as book-to-market ratios and historical returns to make their predictions [24, 25, 32, 33]. Members can gather as much information as possible and continuously estimate the current instantaneous expected stock return rate. Therefore, it is essential to incorporate learning about return predictability into investment decisions for DC pension plans. In this section, the problem is first transformed into a self-financing investment problem, and the semi-closed form of the equilibrium investment strategy is derived under the power utility function up to the solution of a PDE system.

3.1. Financial market setting

Existing research on investment optimization under partial information primarily employs three types of stock return models. The first type models stock returns as a linear combination of observable and unobservable predictors, both of which follow mean-reverting processes. Investors infer the real-time status of unobservable predictors from observable ones using filtering techniques [33–37]. The second type assumes that the expected stock return is driven by an unobservable predictor that follows a mean-reverting stochastic process. Investors dynamically update their estimates of this predictor by observing the historical trajectory of stock prices [8, 38, 39]. The third type models stock returns as a function of hidden Markov chains [29, 40–43].

Following [8], we adopt the second type, which supposes that the predictor $\mu(t)$ follows

$$d\mu(t) = \kappa(\bar{\mu} - \mu(t))dt + \sigma_\mu \rho dZ_S(t) + \sigma_\mu \sqrt{1 - \rho^2} dZ_\mu(t),$$

where $\bar{\mu} > 0$ is the long-run mean value of $\mu(t)$, $\kappa > 0$ represents the speed of mean reversion, $\sigma_\mu > 0$ is the volatility of the predictor $\mu(t)$, and $\rho \in [-1, 1]$ is the correlation coefficient between the stock price $S(t)$ and the predictor $\mu(t)$. Here $\{Z_\mu(t), t \in [0, T]\}$ is a standard Brownian motion and is independent of $\{Z_S(t), t \in [0, T]\}$ and $\{Z_P(t), t \in [0, T]\}$.

In the financial market, a DC-pension-plan member attempts to utilize a Kalman filter to estimate the partially observable predictor $\mu(t)$. Mathematically, let $\mathcal{F}_t^{S,P,C} \subset \mathcal{F}_t$ be the filtration generated by observing the stock price $S(t)$, the price level $P(t)$, and the contribution rate $C(t)$, and define

$$\hat{\mu}(t) = \mathbb{E}[\mu(t) | \mathcal{F}_t^{S,P,C}], \quad m(t) = \mathbb{E}[(\mu(t) - \hat{\mu}(t))^2 | \mathcal{F}_t^{S,P,C}].$$

Hence, $\hat{\mu}(t)$ is the estimation of $\mu(t)$ and $m(t)$ represents the variance of the estimation error. Applying Theorem 12.7 from [44], the dynamics of the stock price, the estimated predictor, and the inflation-indexed bond can be rewritten in terms of $\mathcal{F}_t^{S,P,C}$ -adapted innovation processes as follows:

$$\frac{dS(t)}{S(t)} = rdt + \sigma_S (\hat{\mu}(t)dt + dW_S(t)), \quad (5)$$

$$d\hat{\mu}(t) = \kappa(\bar{\mu} - \hat{\mu}(t))dt + \Sigma(t)dW_S(t), \quad (6)$$

$$\frac{dI(t, K)}{I(t, K)} = (r + \sigma_{p1}\hat{\mu}(t) + \sigma_{p2}\lambda_p)dt + \sigma_{p1}dW_S(t) + \sigma_{p2}dW_P(t), \quad (7)$$

where

$$\Sigma(t) = \sigma_\mu \rho + m(t), \quad \frac{dm(t)}{dt} = -2\kappa m(t) + \sigma_\mu^2 - \Sigma^2(t).$$

For completeness, we also rewrite the dynamics of the price level $P(t)$ and the contribution rate $C(t)$ in terms of the innovation processes.

$$\frac{dP(t)}{P(t)} = \eta dt + \sigma_{p1}dW_S(t) + \sigma_{p2}dW_P(t), \quad P(0) > 0, \quad (8)$$

$$\frac{dC(t)}{C(t)} = \lambda dt + \sigma_{C1}dW_S(t) + \sigma_{C2}dW_P(t), \quad C(0) > 0. \quad (9)$$

Here, $W_S(t)$ and $W_P(t)$ are the innovation processes from the filtering theory, which form a two-dimensional standard Brownian motion with respect to the observation filtration $\mathcal{F}_t^{S,P,C}$, and $W_S(t)$ and $W_P(t)$ are independent. All subsequent analysis will be based on these filtered dynamics.

Through the process of learning, the dynamics of the stock price $S(t)$ and the predictor $\mu(t)$ under partial observation are transformed into equivalent processes under complete observation from the member's perspective. The detailed proof can be found in [8, 34], and is therefore omitted here. Following [33, 35, 38], we consider that the filtering process has reached a steady state for tractability, implying that the estimation variance $m(t)$ has converged to its long-term mean m , which yields

$$\frac{dm(t)}{dt} = -2\kappa m + \sigma_\mu^2 - (\sigma_\mu \rho + m)^2 = 0.$$

Solving for m and taking the positive root (since the variance must be non-negative) gives

$$m = -(\kappa + \sigma_\mu \rho) + \sqrt{(\kappa + \sigma_\mu \rho)^2 + \sigma_\mu^2(1 - \rho^2)} > 0.$$

Consequently, in the subsequent discussion, we consider

$$\Sigma = \sigma_\mu \rho + m = -\kappa + \sqrt{(\kappa + \sigma_\mu \rho)^2 + \sigma_\mu^2(1 - \rho^2)} > 0$$

as a given constant.

The wealth process $X^\pi(t)$ of the DC pension account becomes

$$\begin{aligned} dX^\pi(t) &= (1 - \pi_I(t) - \pi_S(t)) X^\pi(t) \frac{dS_0(t)}{S_0(t)} + \pi_I(t) X^\pi(t) \frac{dI(t, K)}{I(t, K)} + \pi_S(t) X^\pi(t) \frac{dS(t)}{S(t)} + C(t)dt \\ &= \left[r + \pi_I(t) (\sigma_{p1} \hat{\mu}(t) + \sigma_{p2} \lambda_p) + \pi_S(t) \sigma_S \hat{\mu}(t) \right] X^\pi(t) dt + C(t) dt \\ &\quad + (\pi_S(t) \sigma_S + \pi_I(t) \sigma_{p1}) X^\pi(t) dW_S(t) + \pi_I(t) \sigma_{p2} X^\pi(t) dW_P(t). \end{aligned} \quad (10)$$

Definition 1. We define Π as the set of admissible strategies $\pi(t)$ that satisfy the following conditions:

1. Equation (10) admits a unique strong solution for the initial point $(t, x, \hat{\mu}) \in [0, T] \times \mathbb{R} \times \mathbb{R}$;
2. $\int (u^\gamma)^{-1} \left(\mathbb{E}_{t,x,p,\hat{\mu},c} \left[u^\gamma \left(\frac{X^\pi(T) - G(T)}{P(T)} \right) \right] \right) d\Gamma(\gamma) < \infty$, where $\mathbb{E}_{t,x,p,\hat{\mu},c}[\cdot] = \mathbb{E}[\cdot | X^\pi(t) = x, P(t) = p, \hat{\mu}(t) = \hat{\mu}, C(t) = c]$ denotes the conditional expectation at time t ;
3. $\pi(t)$ is continuous and $\mathcal{F}_t^{S,P,C}$ -progressively measurable.

3.2. Optimization problem

In this section, we first transform the wealth process $X^\pi(t)$ of the DC pension account into a self-financing case. Specifically, $F(t, T)$ is the present value of $C(s)$ from t to T and follows the process

$$dF(t, T) = -C(t)dt + (r + \sigma_{C1} \hat{\mu}(t) + \sigma_{C2} \lambda_p) F(t, T)dt + \sigma_{C1} F(t, T) dW_S(t) + \sigma_{C2} F(t, T) dW_P(t).$$

$G(t)$ is the present value of the target retirement benefit $G(T)$ at time t as

$$G(t) = \int_T^\omega a(s) I(t, s)_{s-T} p_T ds.$$

The following lemma demonstrates that $F(t, T)$ and $G(t)$ can be replicated using a portfolio of traded assets, as shown in [12]. Therefore, the detailed proof is omitted.

Lemma 2. $F(t, T)$ and the continuous contribution can be reproduced by the risk-free asset, inflation-indexed bond, and stock as follows:

$$\frac{dF(t, T) + C(t)dt}{F(t, T)} = \left(1 - \pi_I^F(t) - \pi_S^F(t)\right) \frac{dS_0(t)}{S_0(t)} + \pi_I^F(t) \frac{dI(t, K)}{I(t, K)} + \pi_S^F(t) \frac{dS(t)}{S(t)}, \quad (11)$$

where

$$\pi_I^F(t) = \frac{\sigma_{C1}\sigma_{p2} - \sigma_{C2}\sigma_{p1}}{\sigma_S\sigma_{p2}}, \quad \pi_S^F(t) = \frac{\sigma_{C2}}{\sigma_{p2}}. \quad (12)$$

$G(t)$ follows a self-financing process:

$$\frac{dG(t)}{G(t)} = \left(r + \sigma_{p1}\hat{\mu}(t) + \sigma_{p2}\lambda_p\right) dt + \sigma_{p1}dW_S(t) + \sigma_{p2}dW_P(t) = \frac{dI(t, K)}{I(t, K)}.$$

Now, define the auxiliary process:

$$Y^\phi(t) = X^\pi(t) + F(t, T) - G(t).$$

We can derive the dynamics of $Y^\phi(t)$ from Lemma 2 and Ito's formula,

$$\frac{dY^\phi(t)}{Y^\phi(t)} = rdt + \left(\phi_I(t)\sigma_{p1} + \phi_S(t)\sigma_S\right) (\hat{\mu}(t)dt + dW_S(t)) + \phi_I(t)\sigma_{p2} (\lambda_p dt + dW_P(t)), \quad (13)$$

where

$$\phi_I(t) = \frac{\pi_I(t)X^\pi(t) + \pi_I^F(t)F(t, T) - G(t)}{Y^\phi(t)}, \quad (14)$$

$$\phi_S(t) = \frac{\pi_S(t)X^\pi(t) + \pi_S^F(t)F(t, T)}{Y^\phi(t)}, \quad (15)$$

and $\phi(t) = (\phi_I(t), \phi_S(t))'$ is also admissible when the original investment strategy $\pi(t)$ is admissible. By definition, $Y^\phi(T) = X^\pi(T) - G(T)$. Therefore, the original optimization (4) can be transformed into a self-financing optimization,

$$\sup_{\phi \in \Pi} J^\phi(t, y, p, \hat{\mu}), \quad (16)$$

with

$$J^\phi(t, y, p, \hat{\mu}) = \int (u^\gamma)^{-1} \left(\mathbb{E}_{t,y,p,\hat{\mu}} \left[u^\gamma \left(\frac{Y^\phi(T)}{P(T)} \right) \right] \right) d\Gamma(\gamma), \quad (17)$$

subject to $Y^\phi(T) > 0$, (8), and (13). Here, $\mathbb{E}_{t,y,p,\hat{\mu}}[\cdot] = \mathbb{E}[\cdot \mid Y^\phi(t) = y, P(t) = p, \hat{\mu}(t) = \hat{\mu}]$ denotes the conditional expectation at time t . Moreover, we assume $Y^\phi(0) = X^\pi(0) + F(0, T) - G(0) > 0$ and consider $Y^\phi(t) > 0$ for the existence of admissible strategies. Given that the process $P(t)$ follows a geometric Brownian motion with $P(0) > 0$, it stays positive on the entire interval $[0, T]$. Therefore, the domain of $J^\phi(t, y, p, \hat{\mu})$ in (16) is $[0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$. In fact, the equilibrium strategy obtained in Proposition 6 is clearly independent of Y^ϕ itself. Consequently, under the equilibrium investment strategy, $Y^{\phi^*}(t) > 0$ holds for all $t \in [0, T]$. A similar discussion can be found in [12].

Desmettre and Steffensen [11] pointed out that incorporating random risk aversion leads to time inconsistency. Thus, we define the equilibrium control for (16), which is consistent with [5, 8, 11, 12].

Definition 3. We call an admissible strategy ϕ^* an equilibrium strategy if, for any $\phi \in \Pi$ and a fixed real number $h > 0$, the strategy ϕ_h defined as

$$\phi_h(s) = \begin{cases} \phi(s) & \text{for } s \in [t, t+h), \\ \phi^*(s) & \text{for } s \in [t+h, T), \end{cases}$$

satisfies the condition

$$\liminf_{h \rightarrow 0} \frac{J^{\phi^*}(t, y, p, \hat{\mu}) - J^{\phi_h}(t, y, p, \hat{\mu})}{h} \geq 0.$$

In this case, the equilibrium value function $V_\mu(t, y, p)$ is written as

$$V_\mu(t, y, p) = J^{\phi^*}(t, y, p, \hat{\mu}).$$

The equilibrium strategy outlined in Definition 3 is consistent over time. Consequently, we operate under the assumption that the objective of a member, at any initial state, is to identify an equilibrium investment approach along with its associated equilibrium value function for the optimization problem (16). From this point forward, we will refer to the equilibrium investment strategy as the time-consistent investment strategy.

3.3. Verification theorem

This section gives the verification theorem of problem (16). Let

$$z^{\phi, \gamma}(t, y, p, \hat{\mu}) = \mathbb{E}_{t, y, p, \hat{\mu}} \left[u^\gamma \left(\frac{Y^\phi(T)}{P(T)} \right) \right],$$

and then $J^\phi(t, y, p, \hat{\mu})$ follows:

$$J^\phi(t, y, p, \hat{\mu}) = \int (u^\gamma)^{-1} (z^{\phi, \gamma}(t, y, p, \hat{\mu})) d\Gamma(\gamma).$$

We introduce the variational operator $\mathcal{A}^\phi$ via

$$\begin{aligned} \mathcal{A}^\phi(\Psi) = & \Psi_{yy} \left[r + \phi_I(t) \sigma_{p2} \lambda_p + (\phi_S(t) \sigma_S + \phi_I(t) \sigma_{p1}) \hat{\mu}(t) \right] + \Psi_p p \eta + \Psi_{\hat{\mu}} \kappa (\bar{\mu} - \hat{\mu}) \\ & + \frac{1}{2} \Psi_{yy} y^2 \left((\phi_S(t) \sigma_S + \phi_I(t) \sigma_{p1})^2 + \phi_I^2(t) \sigma_{p2}^2 \right) + \frac{1}{2} \Psi_{pp} p^2 (\sigma_{p1}^2 + \sigma_{p2}^2) + \frac{1}{2} \Sigma^2 \Psi_{\hat{\mu}\hat{\mu}} \\ & + \Psi_{yp} y p (\phi_I(t) \sigma_{p2}^2 + \sigma_{p1} (\phi_S(t) \sigma_S + \phi_I(t) \sigma_{p1})) + \Psi_{y\hat{\mu}} y \Sigma (\phi_S(t) \sigma_S + \phi_I(t) \sigma_{p1}) + \Psi_{p\hat{\mu}} p \Sigma \sigma_{p1}, \end{aligned}$$

where $\Psi_y, \Psi_p, \Psi_{\hat{\mu}}, \Psi_{yy}, \Psi_{pp}, \Psi_{\hat{\mu}\hat{\mu}}, \Psi_{yp}, \Psi_{y\hat{\mu}}$ are the partial derivatives of Ψ . For the sake of simplicity in writing, we define

$$\iota^\gamma(x) = \frac{d}{dx} (u^\gamma)^{-1}(x).$$

Similar to Definition 5.1 in [5], Definition 3.4 in [11], and Definition 3 in [12], we need to introduce the definition of the space $L^2(Y^\phi)$.

Definition 4. Considering an arbitrary admissible control ϕ , a function $h : [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is said to belong to the space $L^2(Y^\phi)$ if it satisfies

$$\mathbb{E}_{t,y,p,\hat{\mu}} \left[\int_t^T \|h_y(s, Y^\phi(s), P(s), \hat{\mu}(s)) Y^\phi(s) (\phi_S(s)\sigma_S + \phi_I(s)\sigma_{p1}, \phi_I(s)\sigma_{p2}) + h_p(s, Y^\phi(s), P(s), \hat{\mu}(s)) P(s) (\sigma_{p1}, \sigma_{p2}) + h_{\hat{\mu}}(s, Y^\phi(s), P(s), \hat{\mu}(s)) (\Sigma, 0)\|^2 ds \right] < +\infty$$

for any $(t, y, p, \hat{\mu})$, and the derivatives of h with respect to y , p , and $\hat{\mu}$ are denoted by h_y , h_p , and $h_{\hat{\mu}}$, respectively.

Theorem 5. Assume that there are functions U and Z^γ in $C^{1,2,2,2}([0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R})$ for all γ , so that

(i) the following conditions hold:

$$U_t(t, y, p, \hat{\mu}) = \inf_{\phi} \left\{ -\mathcal{A}^\phi U(t, y, p, \hat{\mu}) + H_t(t, y, p, \hat{\mu}) + \mathcal{A}^\phi H(t, y, p, \hat{\mu}) - \int \iota^\gamma(Z^\gamma(t, y, p, \hat{\mu})) (Z_t(t, y, p, \hat{\mu}) + \mathcal{A}^\phi Z^\gamma(t, y, p, \hat{\mu})) d\Gamma(\gamma) \right\} \quad (18)$$

and

$$Z_t^\gamma(t, y, p, \hat{\mu}) = -\mathcal{A}^{\phi^*} Z^\gamma(t, y, p, \hat{\mu}), \quad (19)$$

where

$$H(t, y, p, \hat{\mu}) = \int (u^\gamma)^{-1}(Z^\gamma(t, y, p, \hat{\mu})) d\Gamma(\gamma) \in C^{1,2,2,2}([0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}), \quad (20)$$

$$\phi^* = \operatorname{arginf}_{\phi} \left\{ -\mathcal{A}^\phi U(t, y, p, \hat{\mu}) + H_t(t, y, p, \hat{\mu}) + \mathcal{A}^\phi H(t, y, p, \hat{\mu}) - \int \iota^\gamma(Z^\gamma(t, y, p, \hat{\mu})) (Z_t(t, y, p, \hat{\mu}) + \mathcal{A}^\phi Z^\gamma(t, y, p, \hat{\mu})) d\Gamma(\gamma) \right\}, \quad (21)$$

subject to boundary conditions

$$U(T, y, p, \hat{\mu}) = y/p \text{ and } Z^\gamma(T, y, p, \hat{\mu}) = u^\gamma(y/p), \text{ hold for all } \gamma;$$

(ii) U , H , and Z^γ for all γ are in the space $L^2(Y^{\phi^*})$;

hence, ϕ^* is an equilibrium strategy and

$$V_\mu(t, y, p) = U(t, y, p, \hat{\mu}),$$

$$z^{\phi^*, \gamma}(t, y, p, \hat{\mu}) = Z^\gamma(t, y, p, \hat{\mu}) \text{ for all } \gamma.$$

Proof. The proof follows the same line as in [11, 12], and is therefore omitted here. $\square$

Listing the first-order and second-order derivatives of $H(t, y, p, \hat{\mu})$ for each component, and substituting them into the HJB equation (18) yields

$$U_t(t, y, p, \hat{\mu}) = \inf_{\phi} \left\{ -\mathcal{A}^\phi U(t, y, p, \hat{\mu}) \right.$$

$$\begin{aligned}
& + \frac{1}{2}y^2 \left((\phi_S(t)\sigma_S + \phi_I(t)\sigma_{p1})^2 + \phi_I^2(t)\sigma_{p2}^2 \right) \int (\iota^\gamma)' (Z^\gamma(t, y, p, \hat{\mu})) (Z_y^\gamma(t, y, p, \hat{\mu}))^2 d\Gamma(\gamma) \\
& + \frac{1}{2}p^2 (\sigma_{p1}^2 + \sigma_{p2}^2) \int (\iota^\gamma)' (Z^\gamma(t, y, p, \hat{\mu})) (Z_p^\gamma(t, y, p, \hat{\mu}))^2 d\Gamma(\gamma) \\
& + \frac{1}{2}\Sigma^2 \int (\iota^\gamma)' (Z^\gamma(t, y, p, \hat{\mu})) (Z_{\hat{\mu}}^\gamma(t, y, p, \hat{\mu}))^2 d\Gamma(\gamma) \\
& + yp \left(\phi_I(t)\sigma_{p2}^2 + \sigma_{p1} (\phi_S(t)\sigma_S + \phi_I(t)\sigma_{p1}) \right) \int (\iota^\gamma)' (Z^\gamma(t, y, p, \hat{\mu})) Z_y^\gamma(t, y, p, \hat{\mu}) Z_p^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma) \\
& + y\Sigma (\phi_S(t)\sigma_S + \phi_I(t)\sigma_{p1}) \int (\iota^\gamma)' (Z^\gamma(t, y, p, \hat{\mu})) Z_y^\gamma(t, y, p, \hat{\mu}) Z_{\hat{\mu}}^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma) \\
& + p\Sigma\sigma_{p1} \int (\iota^\gamma)' (Z^\gamma(t, y, p, \hat{\mu})) Z_p^\gamma(t, y, p, \hat{\mu}) Z_{\hat{\mu}}^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma) \Big\}.
\end{aligned}$$

It should be emphasized that the role of the superscript γ is to mark the functions' reliance on γ , rather than to raise the functions to the power of γ . Using the first-order conditions, it is easy to obtain that $\phi_I^*(t)$ and $\phi_S^*(t)$ take the following forms:

$$\begin{aligned}
\phi_I^*(t) &= -\frac{1}{y} \left(\frac{\lambda_p}{\sigma_{p2}} \frac{\int \iota^\gamma (Z^\gamma(t, y, p, \hat{\mu})) Z_y^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)}{\int \iota^\gamma (Z^\gamma(t, y, p, \hat{\mu})) Z_{yy}^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)} \right. \\
&\quad \left. + p \frac{\int \iota^\gamma (Z^\gamma(t, y, p, \hat{\mu})) Z_{yp}^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)}{\int \iota^\gamma (Z^\gamma(t, y, p, \hat{\mu})) Z_{yy}^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)} \right), \\
\phi_S^*(t) &= -\frac{1}{\sigma_S y} \left(\left(\hat{\mu}(t) - \frac{\lambda_p \sigma_{p1}}{\sigma_{p2}} \right) \frac{\int \iota^\gamma (Z^\gamma(t, y, p, \hat{\mu})) Z_y^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)}{\int \iota^\gamma (Z^\gamma(t, y, p, \hat{\mu})) Z_{yy}^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)} \right. \\
&\quad \left. + \Sigma \frac{\int \iota^\gamma (Z^\gamma(t, y, p, \hat{\mu})) Z_{y\hat{\mu}}^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)}{\int \iota^\gamma (Z^\gamma(t, y, p, \hat{\mu})) Z_{yy}^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)} \right),
\end{aligned}$$

which are independent of the function U . Thus, solving the problem reduces to determining $Z^\gamma(t, y, p, \hat{\mu})$ for all γ .

3.4. Equilibrium strategy under the power utility function

Typical examples for u^γ include the power utility function and the exponential utility function. For the exponential utility, Desmettre and Steffensen [11] showed that the equilibrium strategy is the same as the classical one, just replacing the random γ by its expectation. Thus, we focus on the power utility function

$$u^\gamma(w) = \frac{1}{1-\gamma} w^{1-\gamma}. \quad (22)$$

Its inverse and the corresponding derivative of (22) are

$$(u^\gamma)^{-1}(v) = (1-\gamma)^{\frac{1}{1-\gamma}} v^{\frac{1}{1-\gamma}}, \quad \frac{d}{dv} (u^\gamma)^{-1}(v) = (1-\gamma)^{\frac{\gamma}{1-\gamma}} v^{\frac{\gamma}{1-\gamma}}.$$

Then, the equilibrium strategy can be rewritten as

$$\begin{aligned} \phi_I^*(t) = & -\frac{1}{y} \left(\frac{\lambda_p}{\sigma_{p2}} \frac{\int (1-\gamma)^{\frac{\gamma}{1-\gamma}} (Z^\gamma(t, y, p, \hat{\mu}))^{\frac{\gamma}{1-\gamma}} Z_y^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)}{\int (1-\gamma)^{\frac{\gamma}{1-\gamma}} (Z^\gamma(t, y, p, \hat{\mu}))^{\frac{\gamma}{1-\gamma}} Z_{yy}^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)} \right. \\ & \left. + p \frac{\int (1-\gamma)^{\frac{\gamma}{1-\gamma}} (Z^\gamma(t, y, p, \hat{\mu}))^{\frac{\gamma}{1-\gamma}} Z_{yp}^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)}{\int (1-\gamma)^{\frac{\gamma}{1-\gamma}} (Z^\gamma(t, y, p, \hat{\mu}))^{\frac{\gamma}{1-\gamma}} Z_{yy}^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)} \right), \end{aligned} \quad (23)$$

$$\begin{aligned} \phi_S^*(t) = & -\frac{1}{\sigma_S y} \left(\left(\hat{\mu}(t) - \frac{\lambda_p \sigma_{p1}}{\sigma_{p2}} \right) \frac{\int (1-\gamma)^{\frac{\gamma}{1-\gamma}} (Z^\gamma(t, y, p, \hat{\mu}))^{\frac{\gamma}{1-\gamma}} Z_y^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)}{\int (1-\gamma)^{\frac{\gamma}{1-\gamma}} (Z^\gamma(t, y, p, \hat{\mu}))^{\frac{\gamma}{1-\gamma}} Z_{yy}^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)} \right. \\ & \left. + \Sigma \frac{\int (1-\gamma)^{\frac{\gamma}{1-\gamma}} (Z^\gamma(t, y, p, \hat{\mu}))^{\frac{\gamma}{1-\gamma}} Z_{y\hat{\mu}}^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)}{\int (1-\gamma)^{\frac{\gamma}{1-\gamma}} (Z^\gamma(t, y, p, \hat{\mu}))^{\frac{\gamma}{1-\gamma}} Z_{yy}^\gamma(t, y, p, \hat{\mu}) d\Gamma(\gamma)} \right). \end{aligned} \quad (24)$$

Based on the above forms of $\phi_I^*(t)$ and $\phi_S^*(t)$, the following proposition provides a semi-closed solution of the equilibrium strategy up to the solution of a PDE system.

Proposition 6. Provided that the following PDE system

$$g^\gamma(T, \hat{\mu}) = 1, \quad (25)$$

$$g_I^\gamma(t, \hat{\mu}) = \frac{1-\gamma}{\gamma} \quad (26)$$

$$\begin{aligned} & \times \left\{ -\left(r + \sigma_{p1} \hat{\mu}(t) + \sigma_{p2} \lambda_p - \eta \right) - \frac{(\lambda_p - \sigma_{p2})^2 + (\hat{\mu}(t) - \sigma_{p1})^2}{\varepsilon(t, \hat{\mu}, \Gamma)} \right. \\ & \left. + \frac{\gamma (\lambda_p - \sigma_{p2})^2 + (\hat{\mu}(t) - \sigma_{p1})^2}{2 \varepsilon^2(t, \hat{\mu}, \Gamma)} - \Sigma \varphi \hat{\mu} \left(1 - \frac{\gamma}{\varepsilon(t, \hat{\mu}, \Gamma)} \right) + \frac{\gamma}{2} \Sigma^2 \varphi^2 \right\} g^\gamma \\ & - \left\{ \kappa(\bar{\mu} - \hat{\mu}) + (1-\gamma) \frac{\Sigma \hat{\mu}}{\varepsilon(t, \hat{\mu}, \Gamma)} + (1-\gamma) \Sigma^2 \varphi \right\} g_{\hat{\mu}}^\gamma - \frac{1}{2} \Sigma^2 g_{\hat{\mu}\hat{\mu}}^\gamma + \frac{1-\gamma}{2} \Sigma^2 \frac{(g_{\hat{\mu}}^\gamma)^2}{g^\gamma}, \end{aligned}$$

with

$$\varepsilon(t, \hat{\mu}, \Gamma) = \frac{\int \gamma (g^\gamma(t, \hat{\mu}))^{\frac{\gamma}{1-\gamma}} d\Gamma(\gamma)}{\int (g^\gamma(t, \hat{\mu}))^{\frac{\gamma}{1-\gamma}} d\Gamma(\gamma)}, \quad (27)$$

$$\varphi(t, \hat{\mu}, \Gamma) = \frac{\int \gamma (g^\gamma(t, \hat{\mu}))^{\frac{\gamma}{1-\gamma}} \frac{g_{\hat{\mu}}^\gamma(t, \hat{\mu})}{g^\gamma(t, \hat{\mu})} d\Gamma(\gamma)}{\int \gamma (g^\gamma(t, \hat{\mu}))^{\frac{\gamma}{1-\gamma}} d\Gamma(\gamma)}, \quad (28)$$

has a continuous solution on the interval $[0, T]$, then the equilibrium investment strategy that solves problem (16) is

$$\phi_I^*(t) = \left(\lambda_p / \sigma_{p2} - 1 \right) \frac{1}{\varepsilon(t, \hat{\mu}, \Gamma)} + 1, \quad (29)$$

$$\phi_S^*(t) = \frac{\hat{\mu}(t) - \lambda_p \sigma_{p1} / \sigma_{p2}}{\sigma_S \varepsilon(t, \hat{\mu}, \Gamma)} + \frac{\Sigma \varphi}{\sigma_S}. \quad (30)$$

Furthermore,

$$z^{\phi^*, \gamma}(t, y, p, \hat{\mu}) = Z^{\gamma}(t, y, p, \hat{\mu}) = \frac{1}{1 - \gamma} (g^{\gamma}(t, \hat{\mu}))^{\gamma} \left(\frac{y}{p}\right)^{1-\gamma}.$$

Proof. We substitute the ansatz

$$Z^{\gamma}(t, y, p, \hat{\mu}) = \frac{1}{1 - \gamma} (g^{\gamma}(t, \hat{\mu}))^{\gamma} \left(\frac{y}{p}\right)^{1-\gamma}$$

into (23) and (24), which yields (29) and (30). By substituting the ansatz into (19) with the boundary control $Z^{\gamma}(T, y, p, \hat{\mu}) = \frac{1}{1-\gamma} \left(\frac{y}{p}\right)^{1-\gamma}$, we obtain (26). $\square$

The equilibrium investment strategy of the stock $\phi_S^*(t)$ consists of two terms

$$\phi_{S1}^*(t) = \frac{\hat{\mu}(t) - \lambda_p \sigma_{p1} / \sigma_{p2}}{\sigma_S \varepsilon(t, \hat{\mu}, \Gamma)}, \quad \phi_{S2}^*(t) = \Sigma \phi / \sigma_S. \quad (31)$$

The first term $\phi_{S1}^*(t)$ is the speculative demand and the second term $\phi_{S2}^*(t)$ captures the hedging demand against changes in the estimated drift that arise from learning.

3.5. Special cases

Solving the infinite-dimensional PDE system in Proposition 6 and ensuring the existence of its solution are generally challenging. To make the problem tractable, we focus on three special cases in the following. First, we consider the case of two-point risk aversion and learning. In this case, the infinite-dimensional PDE system can be reduced to two coupled PDEs, allowing us to apply the Banach fixed-point theorem to prove the existence of a local solution. Second, we consider the case of constant risk aversion and learning, where the member ignores the randomness of risk aversion and uses the expectation of the random risk aversion as a substitute. Third, we analyze the case of random risk aversion and no-learning, where the member treats the unobservable predictor as fixed at its long-run mean $\bar{\mu}$, thereby regarding the market as completely observable.

3.5.1. Two-point risk aversion and learning

Following [11, 12], we assume that the risk aversion distribution of γ is two-point, taking values γ_1 and γ_2 with probabilities p_1 and p_2 , respectively, where $p_i = \mathbb{P}(\gamma = \gamma_i)$ for $i = 1, 2$ and $\gamma_2 > \gamma_1 > 0$. In this case,

$$\varepsilon(t, \hat{\mu}) = \frac{p_1 \gamma_1 (g^{\gamma_1}(t, \hat{\mu}))^{\frac{\gamma_1}{1-\gamma_1}} + p_2 \gamma_2 (g^{\gamma_2}(t, \hat{\mu}))^{\frac{\gamma_2}{1-\gamma_2}}}{p_1 (g^{\gamma_1}(t, \hat{\mu}))^{\frac{\gamma_1}{1-\gamma_1}} + p_2 (g^{\gamma_2}(t, \hat{\mu}))^{\frac{\gamma_2}{1-\gamma_2}}}, \quad (32)$$

$$\varphi(t, \hat{\mu}) = \frac{p_1 (g^{\gamma_1}(t, \hat{\mu}))^{\frac{\gamma_1}{1-\gamma_1}} \frac{g^{\gamma_1}(t, \hat{\mu})}{g^{\gamma_1}(t, \hat{\mu})} + p_2 (g^{\gamma_2}(t, \hat{\mu}))^{\frac{\gamma_2}{1-\gamma_2}} \frac{g^{\gamma_2}(t, \hat{\mu})}{g^{\gamma_2}(t, \hat{\mu})}}{p_1 (g^{\gamma_1}(t, \hat{\mu}))^{\frac{\gamma_1}{1-\gamma_1}} + p_2 (g^{\gamma_2}(t, \hat{\mu}))^{\frac{\gamma_2}{1-\gamma_2}}}. \quad (33)$$

Moreover, we can obtain the following PDE:

$$g_t^{\gamma_1}(t, \hat{\mu}) = \frac{1 - \gamma_1}{\gamma_1} \quad (34)$$

$$\begin{aligned}
& \left\{ -\left(r + \sigma_{p1}\hat{\mu}(t) + \sigma_{p2}\lambda_p - \eta\right) - \frac{(\lambda_p - \sigma_{p2})^2 + (\hat{\mu}(t) - \sigma_{p1})^2}{\varepsilon(t, \hat{\mu})} \right. \\
& \quad \left. + \frac{\gamma_1}{2} \frac{(\lambda_p - \sigma_{p2})^2 + (\hat{\mu}(t) - \sigma_{p1})^2}{\varepsilon^2(t, \hat{\mu})} - \Sigma\varphi\hat{\mu}\left(1 - \frac{\gamma_1}{\varepsilon(t, \hat{\mu})}\right) + \frac{\gamma_1}{2}\Sigma^2\varphi^2 \right\} g^{\gamma_1} \\
& - \left\{ \kappa(\bar{\mu} - \hat{\mu}) + (1 - \gamma_1)\frac{\Sigma\hat{\mu}}{\varepsilon(t, \hat{\mu})} + (1 - \gamma_1)\Sigma^2\varphi \right\} g^{\gamma_1}_{\hat{\mu}} - \frac{1}{2}\Sigma^2 g^{\gamma_1}_{\hat{\mu}\hat{\mu}} + \frac{1 - \gamma_1}{2}\Sigma^2 \frac{(g^{\gamma_1}_{\hat{\mu}})^2}{g^{\gamma_1}}, \\
& g_t^{\gamma_2}(t, \hat{\mu}) = \frac{1 - \gamma_2}{\gamma_2} \\
& \left\{ -\left(r + \sigma_{p1}\hat{\mu}(t) + \sigma_{p2}\lambda_p - \eta\right) - \frac{(\lambda_p - \sigma_{p2})^2 + (\hat{\mu}(t) - \sigma_{p1})^2}{\varepsilon(t, \hat{\mu})} \right. \\
& \quad \left. + \frac{\gamma_2}{2} \frac{(\lambda_p - \sigma_{p2})^2 + (\hat{\mu}(t) - \sigma_{p1})^2}{\varepsilon^2(t, \hat{\mu})} - \Sigma\varphi\hat{\mu}\left(1 - \frac{\gamma_2}{\varepsilon(t, \hat{\mu})}\right) + \frac{\gamma_2}{2}\Sigma^2\varphi^2 \right\} g^{\gamma_2} \\
& - \left\{ \kappa(\bar{\mu} - \hat{\mu}) + (1 - \gamma_2)\frac{\Sigma\hat{\mu}}{\varepsilon(t, \hat{\mu})} + (1 - \gamma_2)\Sigma^2\varphi \right\} g^{\gamma_2}_{\hat{\mu}} - \frac{1}{2}\Sigma^2 g^{\gamma_2}_{\hat{\mu}\hat{\mu}} + \frac{1 - \gamma_2}{2}\Sigma^2 \frac{(g^{\gamma_2}_{\hat{\mu}})^2}{g^{\gamma_2}},
\end{aligned} \tag{35}$$

with $g^{\gamma_i}(T, \cdot) = 1$. Wang and Jia [12] considered the random risk aversion case without stock return predictability, i.e., $\hat{\mu}(t) \equiv \bar{\mu}$. Then, Equations (34) and (35) degenerate into ODE form, and the Peano existence theorem guarantees the existence of at least a local solution [11]. However, when it comes to the PDE case, we have to introduce the following theorem to prove that a local solution exists. The proof of Theorem 7 is based on [45–47].

Theorem 7. (Existence for a two-point distribution) Assume that $\gamma_2 > \gamma_1 > 0$ and $\gamma_i \neq 1$, $i = 1, 2$. Then for any sufficiently large $L > 0$, there exists a $\delta > 0$ such that the PDE system (32)–(35) admits a unique solution $(g^{\gamma_1}, g^{\gamma_2})$ on $[T - \delta, T] \times [-L, L]$ with the boundary conditions $g^{\gamma_i}(t, \pm L) = 1$ for $t \in [T - \delta, T]$.

Proof. See Appendix A. □

Remark 8. The Ornstein–Uhlenbeck (O–U) process $\hat{\mu}(t)$ has exponentially decaying tails, so the probability that $|\hat{\mu}(t)|$ exceeds a large threshold L decays exponentially in L over any finite time horizon. Thus, for sufficiently large L , the truncation error is negligible, and it suffices to work on the bounded domain $[-L, L]$ with homogeneous Dirichlet boundary conditions. A solution on the whole real line is not required.

It is essential to verify whether the two-point distributed PDE system (32)–(35) established in Theorem 7 satisfies the conditions in Theorem 5. This verification ensures that the equilibrium strategy and corresponding value function are clearly defined.

Proposition 9. With the two-point distribution for γ , the conditions in Theorem 5 are satisfied.

Proof. By Proposition 6, we have

$$Z^\gamma(t, y, p, \hat{\mu}) = \frac{1}{1-\gamma} (g^\gamma(t, \hat{\mu}))^\gamma \left(\frac{y}{p}\right)^{1-\gamma}, \quad \gamma = \gamma_1, \gamma_2.$$

The function g^γ is the unique local solution of the PDE system provided by Theorem 7. The Sobolev embedding in its proof gives $g^\gamma \in C^{1,2}([T-\delta, T] \times [-L, L])$ for some $\delta > 0$ and $L > 0$. Consequently, g^γ and $g_{\hat{\mu}}^\gamma$ are bounded, and $Z^\gamma \in C^{1,2,2,2}([T-\delta, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times [-L, L])$.

Substituting the partial derivatives of Z^γ into the expression required by Definition 4 yields

$$\left(\frac{Y^{\phi^*}(s)}{P(s)}\right)^{1-\gamma} \left[g^\gamma(\phi_I^* \sigma_{p1} + \phi_S^* \sigma_S, \phi_I^* \sigma_{p2}) - g^\gamma(\sigma_{p1}, \sigma_{p2}) + \frac{\gamma}{1-\gamma} (g^\gamma)^{\gamma-1} g_{\hat{\mu}}^\gamma(\Sigma, 0) \right].$$

All terms inside the parentheses are bounded on $[T-\delta, T] \times [-L, L]$. Indeed, g^γ and $g_{\hat{\mu}}^\gamma$ are bounded by Theorem 7. Since ε and φ are continuous functions of $g^{\gamma_1}, g^{\gamma_2}$, and their $\hat{\mu}$ -derivatives, they inherit the boundedness. The strategy components ϕ_I^* and ϕ_S^* , given by (29) and (30), depend only on $t, \hat{\mu}$ and the bounded ε, φ , so they are bounded as well. Thus it suffices to prove that $M(s) := (Y^{\phi^*}(s)/P(s))^{1-\gamma}$ is square-integrable on $[T-\delta, T]$.

From (13) and (8), a direct application of Itô's formula shows that $M(s)$ satisfies a linear SDE whose drift and diffusion coefficients are continuous in $(s, \hat{\mu}(s))$. On the compact $\hat{\mu}$ -domain $[-L, L]$, any continuous function is uniformly bounded and uniformly continuous, hence Lipschitz and of linear growth. By Theorem 2.3.1 in [48], $\mathbb{E}[\int_t^T (M(s))^2 ds] < \infty$. Therefore $Z^\gamma \in L^2(Y^{\phi^*})$ for $\gamma = \gamma_1, \gamma_2$. The verification for U and H is analogous and is omitted. $\square$

Remark 10. As noted in [11, 12], the existence argument for the two-point distribution case carries over to discrete distributions with bounded support. This is because the PDE system (32)–(35) generalizes naturally to a finite set of coupled equations when γ takes finitely many values, and the Cole–Hopf transformation and the Banach fixed-point theorem remain applicable.

However, for a continuous or general distribution Γ , the situation is fundamentally different. System (26) becomes an infinite family of PDEs parameterized by γ , coupled through the nonlocal integral expressions (27) and (28). This infinite-dimensional, nonlocal coupling prevents a direct extension of the finite-dimensional fixed-point argument. While one may approximate a general Γ by a sequence of discrete distributions and solve the corresponding finite systems, passing to the limit requires uniform estimates and convergence analysis that are not covered by the current proof.

3.5.2. Constant risk aversion and learning

For a degenerate risk aversion distribution, the member ignores random risk aversion and uses the expected value of the random risk aversion, i.e., $\varepsilon(t, \hat{\mu}) = \gamma$. Then, the derivative of g^γ is given by

$$\begin{aligned} g_t^\gamma &= \frac{1-\gamma}{\gamma} \left\{ -\left(r + \sigma_{p1}\hat{\mu}(t) + \sigma_{p2}\lambda_p - \eta\right) - \frac{(\lambda_p - \sigma_{p2})^2 + (\hat{\mu}(t) - \sigma_{p1})^2}{2\gamma} \right\} g^\gamma \\ &\quad - \left\{ \kappa(\bar{\mu} - \hat{\mu}) + (1-\gamma)\frac{\Sigma\hat{\mu}}{\gamma} \right\} g_{\hat{\mu}}^\gamma - \frac{1}{2}\Sigma^2 g_{\hat{\mu}\hat{\mu}}^\gamma + \frac{\gamma-1}{2}\Sigma^2 \frac{(g_{\hat{\mu}}^\gamma)^2}{g^\gamma}, \quad g^\gamma(T, \hat{\mu}) = 1. \end{aligned} \quad (36)$$

The following proposition derives the explicit solutions for the equilibrium strategy and value function.

Proposition 11. For the optimization (16) when ignoring random risk aversion, the equilibrium investment strategy satisfies

$$\phi_I^*(t) = \left(\lambda_p/\sigma_{p2} - 1\right) \frac{1}{\gamma} + 1, \quad (37)$$

$$\phi_S^*(t) = \frac{\hat{\mu}(t) - \lambda_p\sigma_{p1}/\sigma_{p2}}{\sigma_S\gamma} + \frac{\Sigma}{\sigma_S} (2E_1(t)\hat{\mu} + E_2(t)), \quad (38)$$

and the equilibrium value function satisfies

$$V_{\hat{\gamma}, \hat{\mu}}(t, y, p) = \frac{y}{p} \exp \left\{ \frac{\gamma}{1-\gamma} \left[E_1(t)\hat{\mu}^2 + E_2(t)\hat{\mu} + E_3(t) \right] \right\}, \quad (39)$$

where $E_1(t)$, $E_2(t)$, and $E_3(t)$ are given by the following ODE system:

$$\begin{aligned} E_1'(t) &= -\frac{1-\gamma}{2\gamma^2} + 2 \left(\kappa - \frac{1-\gamma}{\gamma} \Sigma \right) E_1(t) + 2(\gamma-2)\Sigma^2 E_1^2(t), \\ E_2'(t) &= \sigma_{p1} \frac{(1-\gamma)^2}{\gamma^2} - 2\kappa\bar{\mu}E_1(t) + \left(\kappa - \frac{1-\gamma}{\gamma} \Sigma \right) E_2(t) + 2(\gamma-2)\Sigma^2 E_1(t)E_2(t), \\ E_3'(t) &= \frac{1-\gamma}{\gamma} \left\{ -r + \eta - \sigma_{p2}\lambda_p - \frac{(\lambda_p - \sigma_{p2})^2}{2\gamma} - \frac{\sigma_{p1}^2}{2\gamma} \right\} - \kappa\bar{\mu}E_2(t) - \Sigma^2 E_1(t) + \frac{\gamma-2}{2} \Sigma^2 E_2^2(t), \end{aligned} \quad (40)$$

with $E_1(T) = E_2(T) = E_3(T) = 0$.

Proof. We conjecture the form of g^γ as

$$g^\gamma = \exp \left\{ E_1(t)\hat{\mu}^2 + E_2(t)\hat{\mu} + E_3(t) \right\}, \quad (41)$$

and putting (41) into (36) yields (40). Furthermore, (39) can also be obtained. $\square$

The proof of the degenerate case follows the same lines as that of Proposition 9 and thus ensures that the conditions in Theorem 5 hold.

3.5.3. Random risk aversion and no-learning

When the member ignores that the expected return rate of the stock is partially observable, i.e., $\hat{\mu}(t) \equiv \bar{\mu}$, which implies that the member regards the market as completely observable, in this case, the PDE system reduces to the ODE case:

$$\begin{aligned} g_I^\gamma(t) &= \frac{1-\gamma}{\gamma} \left\{ - \left(r + \sigma_{p1}\bar{\mu} + \sigma_{p2}\lambda_p - \eta \right) - \frac{(\lambda_p - \sigma_{p2})^2 + (\bar{\mu} - \sigma_{p1})^2}{\varepsilon(t, \Gamma)} \right. \\ &\quad \left. + \frac{\gamma}{2} \frac{(\lambda_p - \sigma_{p2})^2 + (\bar{\mu} - \sigma_{p1})^2}{\varepsilon^2(t, \Gamma)} \right\} g^\gamma \end{aligned} \quad (42)$$

and the equilibrium investment strategy is

$$\phi_I^*(t) = \left(\lambda_p/\sigma_{p2} - 1\right) \frac{1}{\varepsilon(t, \Gamma)} + 1, \quad (43)$$

$$\phi_S^*(t) = \frac{\bar{\mu} - \lambda_p \sigma_{p1} / \sigma_{p2}}{\sigma_S \varepsilon(t, \Gamma)}, \quad (44)$$

with

$$\varepsilon(t, \Gamma) = \frac{\int \gamma (g^\gamma(t))^{\frac{\gamma}{1-\gamma}} d\Gamma(\gamma)}{\int (g^\gamma(t))^{\frac{\gamma}{1-\gamma}} d\Gamma(\gamma)}. \quad (45)$$

Moreover, we have

$$\varepsilon_t(t, \Gamma) = \frac{1}{2} \left((\lambda_p - \sigma_{p2})^2 + (\bar{\mu} - \sigma_{p1})^2 \right) \frac{1}{\varepsilon^2(t, \Gamma)} \left(\frac{\int \gamma^2 (g^\gamma(t))^{\frac{\gamma}{1-\gamma}} d\Gamma(\gamma)}{\int (g^\gamma(t))^{\frac{\gamma}{1-\gamma}} d\Gamma(\gamma)} - \varepsilon^2(t, \Gamma) \right). \quad (46)$$

(42)–(46) align with the findings of [12]. Furthermore, Wang and Jia [12] provided the special cases corresponding to two-point and degenerate distributions of γ . Obviously, the results for these two special cases proposed by [12] are specializations of Sections (3.5.1) and (3.5.2), thus are omitted here.

3.6. Solution to the original problem

Given the relationship between $\phi(t) = (\phi_I(t), \phi_S(t))'$ and $\pi(t) = (\pi_I(t), \pi_S(t))'$ from (14) and (15), the original equilibrium strategy is given by

$$\pi_I^*(t) = \frac{Y^{\phi^*}(t) \phi_I^*(t) - \pi_I^F(t) F(t, T) + G(t)}{X^{\pi^*}(t)}, \quad (47)$$

$$\pi_S^*(t) = \frac{\phi_S^*(t) Y^{\phi^*}(t) - \pi_S^F(t) F(t, T)}{X^{\pi^*}(t)}. \quad (48)$$

Proposition 12. Under the following two special cases concerning the risk aversion parameter γ and the predictor process $\hat{\mu}(t)$, the equilibrium investment strategy $\pi(t) = (\pi_I(t), \pi_S(t))'$ for the original problem, given by Equations (47) and (48), is admissible:

1. Two-point distribution: γ follows a two-point distribution (γ_1, γ_2) with learning (without learning);
2. Degenerate distribution: γ is a constant with learning (without learning).

Proof. The proof is similar to Proposition 3 in [12], so we omit it here. $\square$

4. Welfare loss

This section quantifies the welfare loss resulting from ignoring random risk aversion and from ignoring learning by comparing the equilibrium value functions. The random risk aversion $\tilde{\gamma}$ follows a two-point distribution, and its expectation is denoted by $\bar{\gamma} = \mathbb{E}[\tilde{\gamma}]$. For the unobservable predictor, the member can either dynamically update its estimate $\hat{\mu}(t)$ via filtering or fix it at the long-run mean $\bar{\mu}$. The corresponding four equilibrium value functions are defined as follows:

1. $V_{\tilde{\gamma}, \hat{\mu}}(t, y, p)$: the corresponding equilibrium value function for random risk aversion $\tilde{\gamma}$ and partially observable stock return $\tilde{\mu}(t)$.

$$V_{\tilde{\gamma}, \hat{\mu}}(t, y, p) = \frac{y}{p} \left(p_1 g_t^{\gamma_1}(t, \hat{\mu})^{\frac{\gamma_1}{1-\gamma_1}} + p_2 g_t^{\gamma_2}(t, \hat{\mu})^{\frac{\gamma_2}{1-\gamma_2}} \right), \quad (49)$$

where $g_t^{\gamma_1}(t, \hat{\mu})$ and $g_t^{\gamma_2}(t, \hat{\mu})$ satisfy (32)–(35).

2. $V_{\tilde{\gamma}, \tilde{\mu}}(t, y, p)$: the corresponding equilibrium value function for random risk aversion $\tilde{\gamma}$ and completely observable stock return $\mu(t) \equiv \tilde{\mu}$.

$$V_{\tilde{\gamma}, \tilde{\mu}}(t, y, p) = \frac{y}{p} \left(p_1 g^{\gamma_1}(t)^{\frac{\gamma_1}{1-\gamma_1}} + p_2 g^{\gamma_2}(t)^{\frac{\gamma_2}{1-\gamma_2}} \right), \quad (50)$$

where $g^{\gamma_1}(t)$ and $g^{\gamma_2}(t)$ satisfy

$$g_t^{\gamma_1}(t) = \frac{1-\gamma_1}{\gamma_1} \left\{ - (r + \sigma_{p1}\tilde{\mu} + \sigma_{p2}\lambda_p - \eta) - \frac{(\lambda_p - \sigma_{p2})^2 + (\tilde{\mu} - \sigma_{p1})^2}{\varepsilon(t)} + \frac{\gamma_1}{2} \frac{(\lambda_p - \sigma_{p2})^2 + (\tilde{\mu} - \sigma_{p1})^2}{\varepsilon^2(t)} \right\} g^{\gamma_1}, \quad (51)$$

$$g_t^{\gamma_2}(t) = \frac{1-\gamma_2}{\gamma_2} \left\{ - (r + \sigma_{p1}\tilde{\mu} + \sigma_{p2}\lambda_p - \eta) - \frac{(\lambda_p - \sigma_{p2})^2 + (\tilde{\mu} - \sigma_{p1})^2}{\varepsilon(t)} + \frac{\gamma_2}{2} \frac{(\lambda_p - \sigma_{p2})^2 + (\tilde{\mu} - \sigma_{p1})^2}{\varepsilon^2(t)} \right\} g^{\gamma_2}, \quad (52)$$

with

$$\varepsilon(t, \Gamma) = \frac{p_1 \gamma_1 g^{\gamma_1}(t)^{\frac{\gamma_1}{1-\gamma_1}} + p_2 \gamma_2 g^{\gamma_2}(t)^{\frac{\gamma_2}{1-\gamma_2}}}{p_1 g^{\gamma_1}(t)^{\frac{\gamma_1}{1-\gamma_1}} + p_2 g^{\gamma_2}(t)^{\frac{\gamma_2}{1-\gamma_2}}}.$$

3. $V_{\tilde{\gamma}, \tilde{\mu}}(t, y, p)$: the corresponding equilibrium value function for constant risk aversion $\tilde{\gamma} = \mathbb{E}[\tilde{\gamma}]$ and partially observable stock return $\tilde{\mu}$, which can be found in (39)–(40).
 4. $V_{\tilde{\gamma}, \tilde{\mu}}(t, y, p)$: the corresponding equilibrium value function for constant risk aversion $\tilde{\gamma} = \mathbb{E}[\tilde{\gamma}]$ and completely observable stock return $\mu(t) \equiv \tilde{\mu}$.

$$V_{\tilde{\gamma}, \tilde{\mu}}(t, y, p) = \frac{y}{p} \exp \left\{ - \left(r - \eta + \sigma_{p1}\tilde{\mu} + \sigma_{p2}\lambda_p + \frac{(\lambda_p - \sigma_{p2})^2 + (\tilde{\mu} - \sigma_{p1})^2}{2\gamma} \right) (T - t) \right\}. \quad (53)$$

4.1. Welfare loss of ignoring random risk aversion

This subsection focuses on the welfare difference between random risk aversion and constant risk aversion. The welfare loss can be expressed as $V_{\tilde{\gamma}, \hat{\mu}}(t, y, p) - V_{\tilde{\gamma}, \hat{\mu}}(t, y, p)$. By proving that $g^\gamma(t, \hat{\mu})^{\frac{\gamma}{1-\gamma}}$ is a convex function, we can deduce from the following proposition that $V_{\tilde{\gamma}, \hat{\mu}}(t, y, p) \geq V_{\tilde{\gamma}, \hat{\mu}}(t, y, p)$.

Proposition 13. Ignoring the randomness of risk aversion leads to welfare loss, i.e., $V_{\tilde{\gamma}, \hat{\mu}}(t, y, p) \geq V_{\tilde{\gamma}, \hat{\mu}}(t, y, p)$.

Proof. See Appendix B. □

Remark 14. The key to the proof lies in the convexity of the function $g^\gamma(t, \hat{\mu})^{\frac{\gamma}{1-\gamma}}$. Once the convexity of $g^\gamma(t, \hat{\mu})^{\frac{\gamma}{1-\gamma}}$ is ensured, Proposition 13 can be generalized to the case of discrete distributions with bounded support, rather than just two-point distributions. In particular, for a discrete distribution taking values $\gamma_1, \gamma_2, \dots, \gamma_n$ with probabilities $p_1, p_2, \dots, p_n$, Jensen's inequality gives

$$V_{\tilde{\gamma}, \hat{\mu}}(t, y, p) = \frac{y}{p} \sum_{i=1}^n p_i g^{\gamma_i}(t, \hat{\mu})^{\frac{\gamma_i}{1-\gamma_i}} \geq \frac{y}{p} g^{\tilde{\gamma}}(t, \hat{\mu})^{\frac{\tilde{\gamma}}{1-\tilde{\gamma}}} = V_{\tilde{\gamma}, \hat{\mu}}(t, y, p),$$

where $\bar{\gamma} = \sum_{i=1}^n p_i \gamma_i$. The same convexity argument can extend to a general distribution $\Gamma(\gamma)$:

$$V_{\bar{\gamma}, \hat{\mu}}(t, y, p) = \frac{y}{p} \int g^{\gamma}(t, \hat{\mu})^{\frac{\gamma}{1-\gamma}} d\Gamma(\gamma) \geq \frac{y}{p} g^{\bar{\gamma}}(t, \hat{\mu})^{\frac{\bar{\gamma}}{1-\bar{\gamma}}} = V_{\bar{\gamma}, \hat{\mu}}(t, y, p).$$

Thus, ignoring the randomness of risk aversion leads to a strict welfare loss once the function $g^{\gamma}(t, \hat{\mu})^{\frac{\gamma}{1-\gamma}}$ is proved to be convex.

4.2. Welfare loss of ignoring stock return predictability

This subsection focuses on the welfare difference between partially observable and completely observable stock returns. The welfare loss can be expressed as $V_{\bar{\gamma}, \hat{\mu}}(t, y, p) - V_{\bar{\gamma}, \bar{\mu}}(t, y, p)$. The following proposition shows that $V_{\bar{\gamma}, \hat{\mu}}(t, y, p) \geq V_{\bar{\gamma}, \bar{\mu}}(t, y, p)$ always holds true.

Proposition 15. Ignoring learning leads to welfare loss, i.e., $V_{\bar{\gamma}, \hat{\mu}}(t, y, p) \geq V_{\bar{\gamma}, \bar{\mu}}(t, y, p)$.

Proof. See Appendix C. □

Remark 16. As long as the predictive factors are random (i.e., $\kappa > 0$, $\Sigma > 0$), the optimal strategy for members with learning always depends on updated estimates $\hat{\mu}(t)$, whereas the strategy without learning is fixed as a constant. Due to the fact that the fixed strategy belongs to a subset of the learner's feasible strategy set and is different from the optimal strategy with learning, the conclusion of Proposition 15 always holds true.

5. Numerical examples

This section simulates 10,000 sample paths of the surplus process and adopts the average of the equilibrium investment strategy. First, we conduct a sensitivity analysis to examine the impact of key model parameters on the equilibrium investment strategy. Second, we compare the equilibrium strategies that ignore random risk aversion and ignore stock return predictability. The numerical values used in our study are as follows. The risk-free return rate is $r = 0.02$. For the stock market, the parameters are $\sigma_S = 0.2$, $\kappa = 0.043$, $\sigma_{\mu} = 0.07$, $\rho = -0.4$, and $\bar{\mu} = 0.24$. Inflation dynamics are governed by $\eta = 0.014$, $\sigma_{p1} = 0.15$, $\sigma_{p2} = 0.015$, and $\lambda_p = -0.02$. Turning to the contribution process, we take $\lambda = 0.01$, $\sigma_{C1} = 0.18$, and $\sigma_{C2} = 0.01$. The state variables start from $X(0) = 1$, $P(0) = 1$, $C(0) = 1$. We set the retirement time $T = 40$ years as the accumulation horizon, with the maximum survival age $\omega = 120$, and choose a_0 along with the terminal annuity target $a(T) = 0.9$. The random risk aversion is modeled by a two-point distribution: $\gamma_1 = 0.5$ with probability $p_1 = 0.5$ and $\gamma_2 = 4$ with probability $p_2 = 0.5$. The above parameter setting is the same as that in [8, 12].

5.1. Sensitivity analysis

This subsection focuses on the sensitivity analysis of the parameter's effect on the equilibrium strategy.

5.1.1. Effect of the distribution of risk aversion

As shown in Figure 1(a), the proportion invested in the stock is positive and decreases as retirement approaches. In contrast, as shown in Figure 1(b), the position in the inflation-indexed bond is negative in

the early stage, indicating that the member hedges inflation risk through short-selling. This phenomenon reflects that, in the presence of inflation risk and stochastic contributions, the member is more inclined to hedge long-term inflation uncertainty by leveraging in the early phase. As age increases, they gradually shift toward conservative assets.

Figure 1 illustrates how the equilibrium investment strategy depends on the distribution of risk aversion. In this section, the risk aversion parameter γ follows a two-point distribution ($\gamma_1 = 0.5$ and $\gamma_2 = 4$). We consider three distributions of γ : $(p_1, p_2) = (0.45, 0.55)$, $(0.5, 0.5)$, and $(0.55, 0.45)$. It is clear that a lower overall level of risk aversion corresponds to a higher allocation to the stock and more short selling of the inflation-indexed bond. Conversely, a higher overall level of risk aversion results in a lower allocation to the stock and less short selling of the inflation-indexed bond.

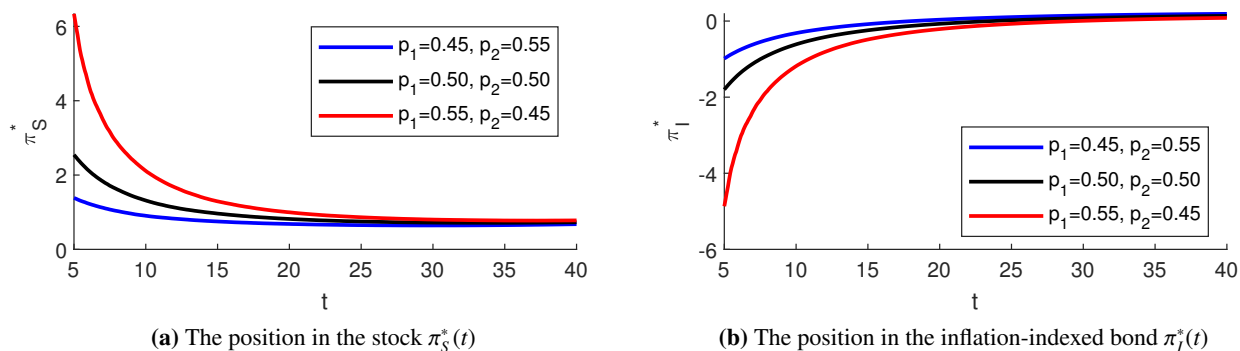


Figure 1. Equilibrium strategy under different risk aversion distributions with $\gamma_1 = 0.5$ and $\gamma_2 = 4$.

5.1.2. Effect of the parameters σ_{p1} and σ_{p2}

Figure 2 examines the effect of the inflation level volatility parameter σ_{p1} on the equilibrium investment strategy. The parameter σ_{p1} represents the volatility of $Z_S(t)$, and this Brownian motion simultaneously governs the price dynamics of both the inflation-indexed bond and the stock. Figure 2 shows that when σ_{p1} increases, the equilibrium proportion allocated to the stock increases, while the equilibrium short position in the inflation-indexed bond decreases during the early accumulation phase and increases during the latter part of the accumulation phase. Moreover, the intersections of the equilibrium investment proportion curves for the inflation-indexed bond are consistent with [12].

In contrast, the parameter σ_{p2} captures the volatility contribution of the independent Brownian motion $Z_P(t)$ to the inflation-indexed bond, which is uncorrelated with stock price fluctuations. As shown in Figure 3, a higher σ_{p2} leads to a larger equilibrium weight on the inflation-indexed bond and a concomitant reduction in the stock allocation.

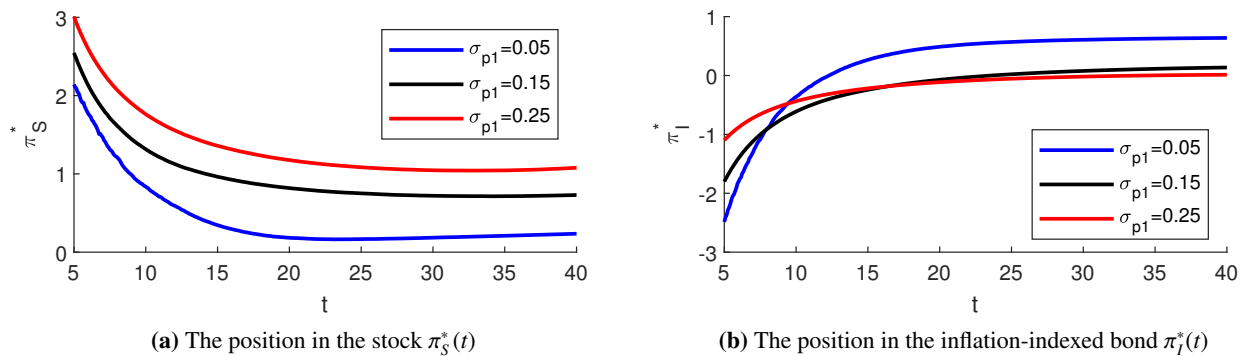


Figure 2. Equilibrium strategy under different values of σ_{p1} .

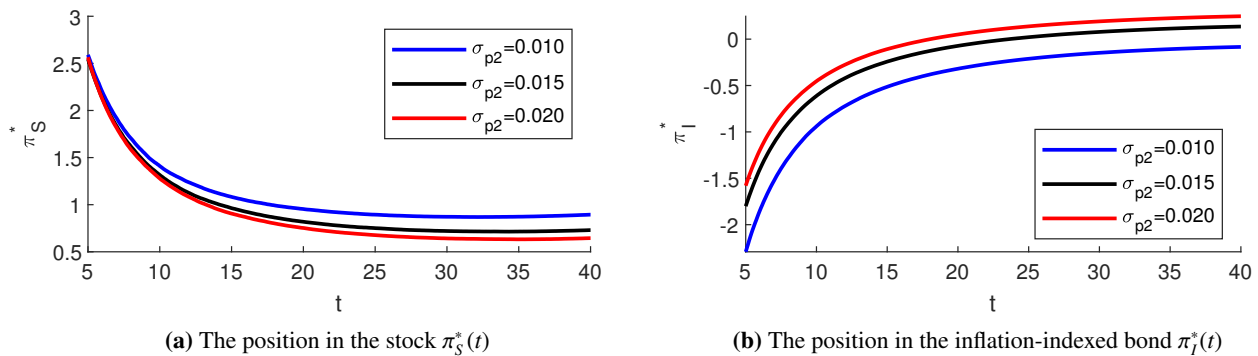


Figure 3. Equilibrium strategy under different values of σ_{p2} .

5.1.3. Effect of the parameter ρ

Figure 4 examines the effect of the correlation coefficient between the stock price $S(t)$ and the predictor $\mu(t)$ on the equilibrium investment strategy. When $|\rho|$ is larger, the learning effect improves, as stock prices convey more precise information about the unobservable predictor. However, a negative ρ means that the learning process indicates a negative correlation between current stock returns and future expected returns: an increase in stock price means a decrease in future expected returns. Therefore, the member adopts a more conservative investment strategy in response.

5.1.4. Effect of the parameter κ

Figure 5 examines the effect of the speed of mean reversion on the equilibrium investment strategy. A high speed of mean reversion implies that the investor's current estimate of the risk premium, $\hat{\mu}(t)$, is expected to revert quickly to its long-term mean. This reduces the value of learning. Even if the information suggests a high current risk premium, the investor knows it will be short-lived, and thus the potential gain from aggressively adjusting the portfolio based on this information is limited. Consequently, the optimal strategy becomes more conservative. This conclusion is consistent with Corollary 3 in [49]. In fact, as κ gradually increases, the curves of $\pi_I^*(t)$ and $\pi_S^*(t)$ will gradually converge to a suboptimal investment strategy that ignores learning.

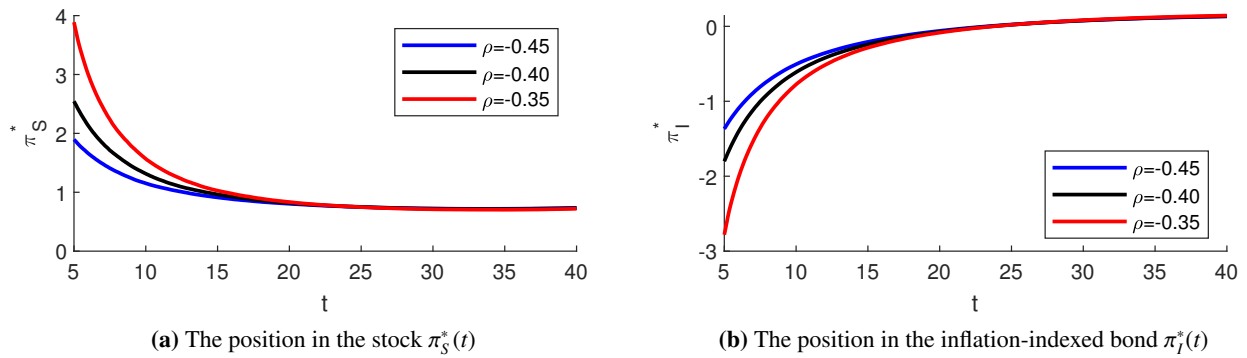


Figure 4. Equilibrium strategy under different values of ρ .

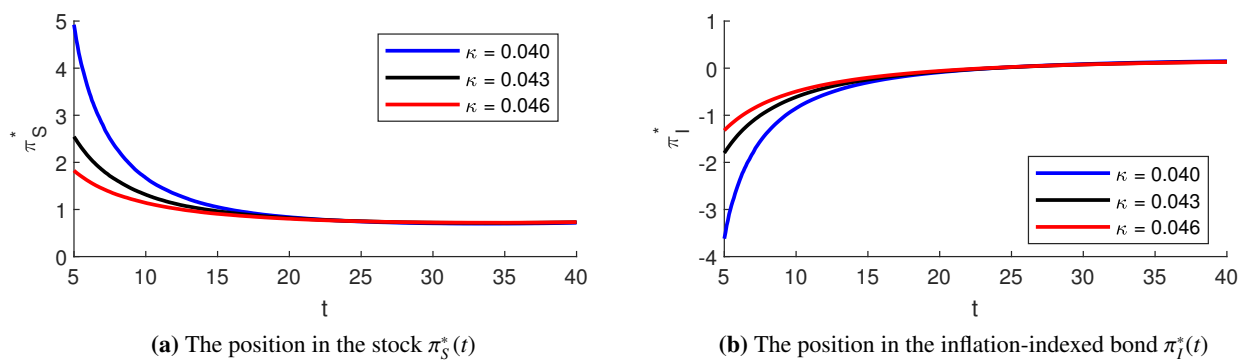


Figure 5. Equilibrium strategy under different values of κ .

5.2. Comparative analysis

This section explores the impact of ignoring both random risk aversion and learning on the equilibrium strategy.

5.2.1. Comparison between random risk aversion and constant risk aversion

Figure 6(a) illustrates the difference between the equilibrium investment strategy under random risk aversion and that under constant risk aversion. In the constant risk aversion case, $\bar{\gamma} = p_1\gamma_1 + p_2\gamma_2 = 2.25$. As shown in the figure, the equilibrium strategy for constant risk aversion is more conservative than that for random risk aversion, allocating more to the stock and short-selling less of the inflation-indexed bond.

5.2.2. Comparison between learning and ignoring learning

Figure 6(b) illustrates the equilibrium investment strategy for the partially observable case, i.e., the member ignores the predictability of $\mu(t)$ and fixes it as $\mu(t) \equiv \bar{\mu}$. As shown in the figure, the investment strategy that ignores learning is more conservative than the one that considers learning. This is because members who ignore learning utilize less information when formulating their investment strategy; they fail to exploit observable stock price information, resulting in less accurate estimates of stock returns and consequently adopting a more conservative investment strategy.

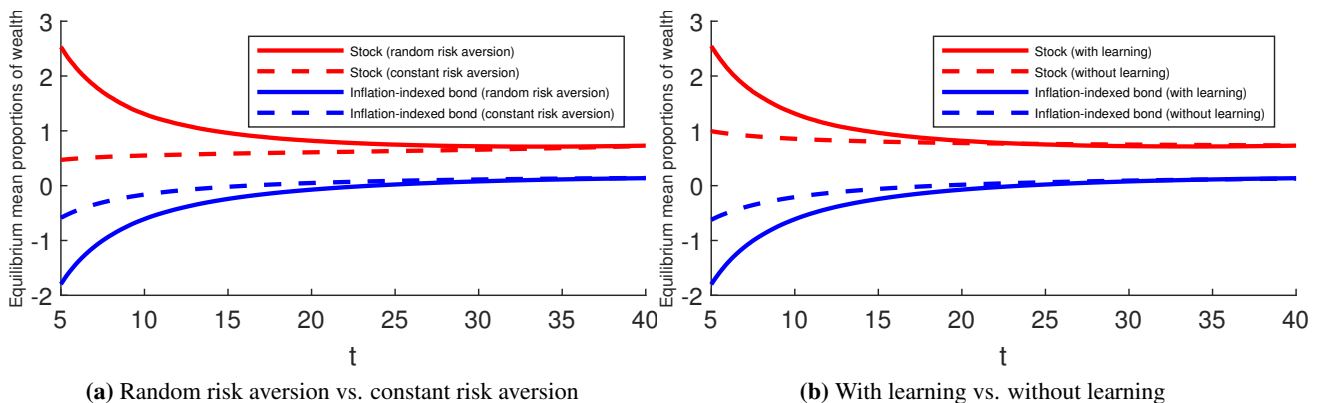


Figure 6. Equilibrium strategy comparisons. (a) Random risk aversion vs. constant risk aversion; (b) with learning vs. without learning.

5.3. Analysis of welfare loss

This subsection compares the welfare of four cases represented in Section 4. For each case, the corresponding equilibrium value is under the same parameter setting with $y/p = 1$ and $p_1 = p_2 = 0.5$, and the four value functions are shown in Figure 7.

To interpret Figure 7 meaningfully, we restrict attention to pairwise comparisons that isolate a single source of ignorance, leaving the other dimension fixed. First, we fix the learning behavior, i.e., the member dynamically updates $\hat{\mu}(t)$, and compare the effect of random risk aversion. The curve $V_{\bar{\gamma}, \hat{\mu}}(t, y, p)$ lies strictly above $V_{\bar{\gamma}, \bar{\mu}}(t, y, p)$, showing that ignoring the randomness of risk aversion leads to

a welfare loss, in line with Proposition 13. Second, we fix the treatment of risk aversion (the two-point distribution) and compare the effect of learning. The curve $V_{\tilde{\gamma},\tilde{\mu}}(t, y, p)$ lies strictly above $V_{\tilde{\gamma},\hat{\mu}}(t, y, p)$, confirming that incorporating learning improves welfare. This is consistent with Proposition 15. The value function $V_{\tilde{\gamma},\hat{\mu}}(t, y, p)$ when considering both random risk aversion and learning attains the highest value, while the most naive case $V_{\tilde{\gamma},\tilde{\mu}}(t, y, p)$ attains the lowest, which is intuitive.

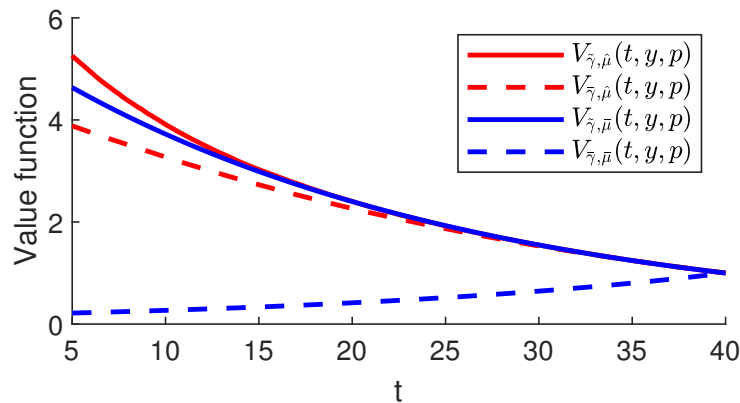


Figure 7. The relative differences in the value functions of two pension-plan members using random risk aversion and expected risk aversion for $y/p = 1$ when $p_1 = p_2 = 0.5$.

6. Conclusion

This paper incorporates random risk aversion and stock return predictability into the investment decision of DC pension plans. Our main findings are summarized as follows.

1. Using the certainty-equivalent framework of [11], we address the time inconsistency issues and obtain the semi-closed form of the equilibrium investment strategy. This strategy is characterized as the solution of a PDE system.
2. The existence of a local solution to this PDE system is rigorously proved via the Banach fixed-point theorem. This mathematical result ensures that the equilibrium strategies derived in this paper are well-defined on a short time interval before retirement.
3. Special cases where members ignore random risk aversion or ignore learning about return predictability are analyzed. We prove theoretically that each of these omissions leads to welfare loss. Numerical simulations further confirm the conclusion.

Two directions are worth exploring in future research. First, this paper focuses on the utility function of terminal wealth. A natural extension is to study optimization problems under a mean-variance objective. However, this extension is technically challenging because the certainty-equivalent approach relies on the monotonicity of the utility function, whereas the quadratic function arising in mean-variance analysis is not monotonic, rendering its inverse function ill-defined.

Second, the framework can be extended to more general market settings, such as incorporating life insurance considerations, incomplete markets with short-selling constraints, or portfolio optimization problems with time delays. These directions deserve further investigation.

Author contributions

Jiaqi Qian: Writing – original draft, Methodology, Formal analysis; Jiaming Su: Writing – review and editing, Methodology, Formal analysis; Yongqiao Wang: Writing – review and editing, Methodology, Formal analysis, Funding acquisition.

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The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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