



Research article

On lot sizing with capacity acquisition

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Abstract: Determining the capacity level is a long-term strategic decision that closely interacts with multi-period lot-sizing choices. We consider a single-echelon lot-sizing problem with capacity acquisition (CALS), where the capacity level, along with production and inventory plans, needs to be determined optimally. In this study, CALS is formulated as a mixed-integer programming problem and we conduct a polyhedral study. We propose two exponential families of valid inequalities, which are shown to be polynomially separable, and we then study the facet conditions for these valid inequalities. Computational studies demonstrate that the proposed valid inequalities significantly strengthen the formulation and improve branch-and-cut performance, outperforming default CPLEX cuts in most test instances through tighter relaxations, reduced runtimes, and improved solution quality.

Keywords: capacitated lot sizing; capacity acquisition; valid inequalities; separation; facet conditions

Mathematics Subject Classification: 90C11, 90B30

1. Introduction

Capacity is a long-term strategic decision. It determines an enterprise's production or service ability to meet customer demands over a certain period. The determination of an optimal capacity level is a fundamental but critical decision that is often made early in the planning process. Insufficient capacity will lead to demand loss and frequent setups, especially when outsourcing and backlogging are not allowed. Excessive capacity will bring costly idle time and investment waste. Our study considers a single-echelon lot-sizing problem with capacity acquisition (CALS). A firm must determine its production capacity, as well as the production plan, to meet a series of known but dynamic demands on a finite planning horizon. Decades of research have been conducted about the capacitated lot-sizing problem with many variants. For example, Florian and Klein [1] consider the single-echelon lot-sizing problem with

constant capacity and propose an $O(n^4)$ dynamic-programming-based solution method, which is improved by van Hoesel and Wagelmans [2] to $O(n^3)$. Then, Pochet and Wolsey [3] develop the so-called (k, l, S, I) inequalities, which are generalized for the two-echelon case by Zhao and Zhang [4]. Florian et al. [5] and Bitran and Yanasse [6] show that the lot-sizing problem with varying capacity is NP-hard, for which van Hoesel and Wagelmans [7] and Chubanov et al. [8] propose polynomial approximation schemes and Pochet [9], Loparic et al. [10], and Atamtürk and Muñoz [11] develop various valid inequalities. Recent studies have also considered richer capacitated lot-sizing environments with complex operational features. For example, Deeratanasrikul and Mizuno [12] study a multi-stage capacitated lot-sizing and scheduling problem with sequence-dependent setups and propose mixed-integer programming formulations and solution approaches. Tang et al. [13] study a dynamic lot-sizing problem with production-or-outsourcing decisions under minimum production quantity constraints, and develop a dynamic programming approach based on structural properties of the optimal solution. Gao et al. [14] study the capacitated multi-item lot-sizing problem and develop a Lagrangian relaxation approach that decomposes it into single-item subproblems solved via dynamic programming, achieving high-quality solutions and improved computational efficiency. However, in these classic capacitated lot-sizing problems, the capacity level is given as opposed to being a conscious decision, as in our study. Considering capacity level as part of the decision, Bradley and Arntzen [15] build a mixed-integer programming model to make simultaneous decisions about capacity and inventory, which is shown to be financially beneficial. Pochet and Wolsey [3] also consider a lot-sizing problem with constant batches, where the capacity level is an integer multiple of the basic batch size. Determining the number of batches in each period is similar to determining the capacity level to acquire, except that the capacity level in their problem is discrete.

Modern manufacturing systems are increasingly characterized by flexible and reconfigurable capacity, such as the use of overtime, outsourcing, and modular production technologies. In these environments, capacity is no longer a fixed parameter but can be adjusted in response to demand variability and operational conditions. Relevant research in this area includes, among others, Rao [16], Chand et al. [17], Rajagopalan and Swaminathan [18], Hsu [19], Taş et al. [20], and Tang et al. [13]. The prevalence of multi-product, high-mix production, and the adoption of data-driven manufacturing systems further increase the complexity of production planning, such as in Darendeliler et al. [21]. From a managerial perspective, these features highlight the importance of optimization models that explicitly capture capacity decisions. The ability to make informed capacity decisions can significantly improve the trade-off between production efficiency and adaptability to demand variability. Our proposed formulation and solution approach enable decision makers to better evaluate capacity investment and utilization strategies, while the computational improvements make it practical to solve large-scale planning problems encountered in real-world settings.

Although our study focuses on the deterministic CALS model to derive polyhedral strengthening results, uncertainty frequently arises in practical capacity-related decisions. The capacity level is typically chosen before demand and cost information is fully revealed, while production and inventory decisions are adjusted over time. This has motivated several uncertainty-aware modeling approaches in the related literature. Scenario-based stochastic programming is a widely used approach when probabilistic information is available. For example, Ahmed et al. [22] propose the first multi-stage stochastic integer programming formulation and develop a branch-and-bound algorithm as well as a modified heuristic strategy. Similarly, Ahmed and Garcia [23] develop a decomposition-based branch-and-bound strategy to solve a dynamic capacity acquisition and assignment problem. Liu and

Küçükyavuz [24] study the polyhedral structure of the static probabilistic lot-sizing problem with chance constraints and propose strong valid inequalities that integrate the chance constraint with the lot-sizing structure, leading to tighter formulations and improved computational performance.

When distributional information is limited, robust optimization provides an alternative by protecting against adverse realizations within prescribed uncertainty sets. For example, Agra [25] proposes a robust model for a lot-sizing problem with uncertain demands where the adversary follows a simple policy. Attila et al. [26] study a lot-sizing problem with remanufacturing under uncertain demand and returns, and develop a min–max decomposition approach along with extended reformulations that significantly improve computational performance. Schlenkrich and Parragh [27] study capacitated multi-item multi-echelon lot-sizing under demand uncertainty using both stochastic and robust optimization approaches, highlighting the impact of modeling choices on solution performance. In fact, robust optimization frameworks and uncertainty-set based modeling have been widely studied in optimization theory (e.g., Özmen et al. [28]). More generally, optimization models that explicitly incorporate uncertainty through stochastic control or regime-switching dynamics have also been investigated in the literature [29, 30], and broader discussions of optimization, dynamics, and uncertainty in complex systems can be found in Weber et al. [31]. A further modeling approach represents uncertainty through fuzzy or possibilistic constructs, such as in Kropat et al. [32]. From an algorithmic perspective, many of these frameworks ultimately involve iteratively solving deterministic subproblems, for example, across scenario realizations in stochastic programming, worst-case realizations in robust optimization, and α -cut reformulations in fuzzy models. Consequently, tightening the deterministic formulation, such as using the valid inequalities and separation routines developed in this paper, can serve as building blocks for uncertainty-aware extensions of CALS.

To the best of our knowledge, not much polyhedral study has been done on CALS problem with a few notable exceptions. In Atamtürk and Hochbaum [33], for example, the authors study a lot-sizing problem with capacity acquisition and subcontracting under a variety of practical cost structures. They show that the number of possible distinct capacity levels in the extreme solutions is bounded from above by $O(n^3)$, based on which they solve a sequence of constant capacitated lot-sizing problems to determine the optimal capacity level and production plan. Li and Meissner [34] formulate the CALS problem as a nonlinear programming model and consider a smooth and convex capacity acquisition cost function. They propose a heuristic algorithm to solve the problem in polynomial time. In this paper, we formulate CALS as a mixed-integer programming problem and study the structure of its feasible region from a polyhedral perspective. In particular, we propose two exponential families of valid inequalities to strengthen the formulation and develop polynomial-time separation algorithms for both families, running in $O(n^2)$ and $O(n^3)$ time, respectively. We further identify explicit sufficient conditions under which the proposed valid inequalities are facet-defining. Finally, we integrate the separation procedures into a branch-and-cut framework and demonstrate computational improvements over default MIP cutting planes on test instances.

The remainder of this paper is organized as follows. In Section 2, we introduce the formulation of CALS. In Section 3, we propose two groups of valid inequalities with polynomial-time separation algorithms, and then we study the strength of the valid inequalities. In Section 4, we conduct computational experiments to study the effectiveness of the proposed valid inequalities. In Section 5, we discuss how the polyhedral results and the proposed branch-and-cut algorithm developed in this paper can be adapted to CALS problems under uncertainty. We conclude the paper in Section 6 with

future research directions.

2. Problem description and mathematical model

At the beginning of a finite planning horizon with n periods, we acquire a capacity $C \in \mathbb{R}_+$ with a given capacity acquisition cost $a \in \mathbb{R}_+$. For notational convenience, we let $[i, j]$ represent the interval of periods $\{i, i+1, \dots, j\}$ for $i \leq j$ and $[i, j] = \emptyset$ for $i > j$, and we let $N = [1, n]$. Over these n periods, the period-wise production capacity C is assumed to remain constant. Subject to the limited capacity C and given a series of dynamic demand $d_t > 0$ for $t \in N$, we make decisions on production quantity $x_t \in \mathbb{R}_+$ and inventory quantity $s_t \in \mathbb{R}_+$. We denote the unit production cost as $p'_t \in \mathbb{R}_+$ and the holding cost as $h_t \in \mathbb{R}_+$ for $t \in N$. Throughout this paper, we define $d_{kl} := \sum_{t=k}^l d_t$ with $k \leq l$ and $d_{kl} = 0$ if $k > l$. If we have a positive production quantity in a period $t \in N$, a fixed setup cost $f_t \in \mathbb{R}_+$ is incurred, and we denote the setup variable as $y_t \in \{0, 1\}$, i.e., $y_t = 1$ if $x_t > 0$. To satisfy the demand on time in each period over the planning horizon with minimum cost, we formulate the CALS problem as

$$\min \sum_{t=1}^n (aC + p'_t x_t + f_t y_t + h_t s_t) \quad (2.1)$$

$$\text{s.t.} \quad s_{t-1} + x_t = d_t + s_t \quad t \in N \quad (2.2)$$

$$x_t \leq d_{1t} y_t \quad t \in N \quad (2.3)$$

$$0 \leq x_t \leq C \quad t \in N \quad (2.4)$$

$$y_t \in \{0, 1\} \quad t \in N \quad (2.5)$$

$$s_0 = s_n = 0 \quad (2.6)$$

$$s_t \in \mathbb{R}_+ \quad t \in N \quad (2.7)$$

$$C \in \mathbb{R}_+. \quad (2.8)$$

Objective function (2.1) is to minimize total cost, including capacity acquisition cost, unit production cost, setup cost, and holding cost. Constraints (2.2) represent flow conservation in period t . Constraints (2.3) and (2.5) restrict y_t to be 1 if $x_t > 0$. Constraints (2.4) ensure that the amount produced in period t is nonnegative and within the production capacity C , which is also a decision variable in this model. Constraint (2.6) assumes that there is no initial or ending inventory. Because s_t can be projected out as $s_t = \sum_{i=1}^t x_i - d_{1t}$ for $t \in N$, the CALS problem can be reformulated without inventory variables as

$$\min \sum_{t=1}^n (aC + p_t x_t + f_t y_t) \quad (2.9)$$

$$\text{s.t.} \quad \sum_{i=1}^t x_i \geq d_{1t} \quad t \in [1, n-1] \quad (2.10)$$

$$\sum_{i=1}^n x_i = d_{1n} \quad (2.11)$$

$$(2.3), (2.4), (2.5), (2.8),$$

where $p_t = p'_t + \sum_{i=t}^n h_i$ for $t \in N$ and a constant value, $-\sum_{t=1}^n h_t d_{1t}$, is omitted from objective function (2.9). We denote

$$X^{CALs} = \{(C, \mathbf{x}, \mathbf{y}) \in \mathbb{R}_+ \times \mathbb{R}_+^n \times \{0, 1\}^n \mid (C, \mathbf{x}, \mathbf{y}) \text{ satisfying (2.3), (2.4), (2.5), (2.8), (2.10), (2.11)}\}$$

as the set of feasible solutions to CALS.

3. Valid inequalities for CALS

In this section, we propose two exponential classes of valid inequalities, and develop polynomial separation algorithms as well as the facet conditions.

Theorem 1. *Let a period $\ell \in [2, n]$ and a set $L = [2, \ell]$; sets A , B , and S be three subsets of L that partition L ; and $d_{1\ell} \geq \frac{d_{2n}}{2}$ if $|S| \geq 2$. Then, the first-group (ℓ, S, A, B) inequality*

$$C + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t + d_{1\ell} \sum_{t \in S} y_t \geq d_{1\ell} + \sum_{t \in S} x_t \quad (3.1)$$

is valid for CALS.

Before proving the validity of the first-group (ℓ, S, A, B) inequality, we provide its operational interpretation. For a given period ℓ , define the set $L = [2, \ell]$. The first-group (ℓ, S, A, B) inequality can be viewed as a demand-satisfying construction over this interval. The sets A , B , and S represent three different mechanisms for satisfying cumulative demand. Periods in set A contribute through realized production quantities x_t , periods in set B contribute through setup-based covering terms $d_{t\ell} y_t$, and periods in set S are shifted to the right-hand side and compensated by the cumulative-demand term $d_{1\ell} y_t$. Thus, the inequality ensures that the acquired capacity C , together with these three types of contributions, is sufficient to satisfy the cumulative demand $d_{1\ell}$.

The condition $d_{1\ell} \geq d_{2n}/2$ is only required when $|S| \geq 2$. When multiple periods in set S are simultaneously active, the inequality must account for the worst-case total production that can be distributed over those periods. Accordingly, this condition should be viewed as a sufficient condition for this particular cut construction and proof technique, rather than as a structural assumption of the CALS problem itself. Below, we present the detailed proof.

Proof. We prove the validity of the first-group (ℓ, S, A, B) inequality (3.1) in the following three cases.

1. Suppose that $y_t = 0$ for all $t \in S$, then inequality (3.1) reduces to

$$C + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t \geq d_{1\ell}.$$

Because $C \geq x_1$, we have $C + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t \geq x_1 + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t$. If $\{|i \in B : y_i = 1\}| \neq \emptyset$, we let $t^* = \min\{i \in B : y_i = 1\}$. Then, $\sum_{t \in \{1\} \cup A} x_t + \sum_{t \in B} d_{t\ell} y_t \geq \sum_{t \in \{1\} \cup A} x_t + d_{t^*\ell} \geq d_{1(t^*-1)} + d_{t^*\ell} \geq d_{1\ell}$. If $\{|i \in B : y_i = 1\}| = \emptyset$, $\sum_{t \in \{1\} \cup A} x_t + \sum_{t \in B} d_{t\ell} y_t = \sum_{t \in \{1\} \cup A} x_t \geq d_{1\ell}$. Therefore, inequality (3.1) is valid in this case.

2. Suppose that $|\{t \in S : y_t = 1\}| = 1$ and we denote $\{t^*\} = \{t \in S : y_t = 1\}$, that is $y_{t^*} = 1$. Then, $C + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t + d_{1\ell} y_{t^*} \geq C + d_{1\ell} \geq x_{t^*} + d_{1\ell}$. Thus, inequality (3.1) is valid in this case.

3. Suppose that $|\{t \in S : y_t = 1\}| = p \geq 2$, which implies $|S| \geq 2$. Thus, we have $d_{1\ell} \geq \frac{d_{2n}}{2}$. From constraints (2.10) and (2.11), we have $\sum_{t \in S} x_t \leq d_{2n}$ because $1 \notin S$. Let $t^* = \arg \max_{t \in S} \{x_t\}$. Then, $\sum_{t \in S \setminus \{t^*\}} x_t \leq \frac{p-1}{p} d_{2n}$. Thus, $\sum_{t \in S} x_t \leq x_{t^*} + \frac{p-1}{p} d_{2n}$. Consequently, $C + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t \geq C \geq x_{t^*} + (\frac{p-1}{p} - \frac{p-1}{2}) d_{2n} \geq \sum_{t \in S} x_t - (p-1) d_{1\ell} = d_{1\ell} + \sum_{t \in S} x_t - p d_{1\ell} = d_{1\ell} + \sum_{t \in S} x_t - d_{1\ell} \sum_{t \in S} y_t$. Therefore, inequality (3.1) is also valid in this case. $\square$

Remark When set $S = \emptyset$, the first-group (ℓ, S, A, B) inequality (3.1) reduces to $C + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t \geq d_{1\ell}$, which is a relaxation of the well-known (ℓ, S) [35] inequality for the uncapacitated single-echelon lot-sizing problem because $C \geq x_1$. In fact, the entire class of the (ℓ, S) inequalities is valid for $\text{conv}(X^{\text{CALs}})$.

Example We illustrate Theorem 1 using a five-period instance with demand vector $\mathbf{d} = (11, 7, 20, 5, 3)$. Let $\ell = 4$, so that $L = [2, 4] = \{2, 3, 4\}$. Choose $S = \{2\}$, $A = \{3\}$, and $B = \{4\}$.

Since $|S| = 1$, no additional demand condition is required. Moreover, $d_{1\ell} = d_{14} = 11 + 7 + 20 + 5 = 43$, and $d_{4\ell} = d_{44} = 5$. Substituting these values into inequality (3.1), we obtain

$$C + x_3 + 5y_4 + 43y_2 \geq 43 + x_2.$$

This example illustrates the roles of the three sets. Period $3 \in A$ contributes through realized production x_3 , period $4 \in B$ contributes through the setup-based covering term $d_{44}y_4 = 5y_4$, and period $2 \in S$ is shifted to the right-hand side and compensated by the cumulative-demand term $d_{14}y_2 = 43y_2$. Thus, the inequality ensures that the acquired capacity, together with these contributions, is sufficient to satisfy the cumulative demand over periods 1 to 4.

Proposition 2. Given a fractional solution $(\bar{C}, \bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathbb{R}_+^{2n+1}$, the most violated first-group (ℓ, S, A, B) inequality (3.1) can be found within $O(n^2)$ computational time, if any.

Proof. For each $t \in [2, \ell]$ with a given $\ell \in [2, n]$, if $\bar{x}_t \leq d_{t\ell} \bar{y}_t$ and $\bar{x}_t \leq d_{1\ell} \bar{y}_t - \bar{x}_t$, let t be in set A . If $d_{t\ell} \bar{y}_t \leq \bar{x}_t$ and $d_{t\ell} \bar{y}_t \leq d_{1\ell} \bar{y}_t - \bar{x}_t$, let t be in set B . If $d_{1\ell} \bar{y}_t - \bar{x}_t \leq d_{t\ell} \bar{y}_t$ and $d_{1\ell} \bar{y}_t - \bar{x}_t \leq \bar{x}_t$, let t be in set S . With the resulting sets A , B and S , if $\bar{C} + \sum_{t \in A} \bar{x}_t + \sum_{t \in B} d_{t\ell} \bar{y}_t + d_{1\ell} \sum_{t \in S} \bar{y}_t < d_{1\ell} + \sum_{t \in S} \bar{x}_t$, then sets A , B and S provide the most violated inequality (3.1) for a given period ℓ . Because $\ell \in [2, n]$ and $t \in [2, \ell]$, a straightforward implementation of this separation algorithm takes $O(n^2)$ computational time. $\square$

From here, we proceed with the facet-defining proof. First, we establish that

$$\dim(\text{conv}(X^{\text{CALs}})) = 2n - 1.$$

We then exhibit $2n - 1$ affinely independent feasible points lying on the face induced by inequality (3.1). Because the geometry of this face differs depending on whether S is empty, we consider the cases $S = \emptyset$ and $S \neq \emptyset$ separately. In both cases, the point constructions are chosen so that feasibility is maintained while the coefficients of the variables in the inequality can be identified independently, which proves that the inequality is facet-defining.

Proposition 3. The first-group (ℓ, S, A, B) inequality (3.1) is facet-defining for $\text{conv}(X^{\text{CALs}})$

- if $S = \emptyset$, $B = \emptyset$ and $\ell = n$; or
- if $S \neq \emptyset$, $d_{1(i-1)} < d_{in}$ for $i \in S$, $(j - \ell)d_{1\ell} > d_{(\ell+1)j}$ for $j \in [\ell + 1, n]$, and

- $\ell \leq n - 1$, $d_{1(j-1)} > d_{j\ell}$ and $(k - \ell)d_{1(j-1)} \geq d_{(\ell+1)k}$ for $j \in B$ and $k \in [\ell + 1, n]$ in the case of $B \neq \emptyset$.

Proof. First, we define $\mathbf{e}_i$ as the n -dimensional unit vector where the i -th component is 1 and all other components are 0, and we denote $\mathbf{1}_T = \sum_{i \in T} \mathbf{e}_i$ for a set T . Let ϵ be a very small positive scalar.

Because there are $2n + 1$ variables and two linearly independent equalities (2.11) and $y_1 = 1$, the dimension of $\text{conv}(X^{\text{CALS}})$ is at most $2n - 1$. Then, we consider $2n$ points in $\text{conv}(X^{\text{CALS}})$: $(C^0, \mathbf{x}^0, \mathbf{y}^0) = (d_{1n}, d_{1n}\mathbf{e}_1, \mathbf{e}_1)$, $(C^0, \mathbf{x}^0, \mathbf{y}^0) + (-\epsilon, -\epsilon\mathbf{e}_1 + \epsilon\mathbf{e}_i, \mathbf{e}_i)$ for $i \in [2, n]$, $(C^0, \mathbf{x}^0, \mathbf{y}^0) + (0, \mathbf{0}, \mathbf{e}_i)$ for $i \in [2, n]$, and $(C^0, \mathbf{x}^0, \mathbf{y}^0) + (\epsilon, \mathbf{0}, \mathbf{0})$, where $\mathbf{0}$ is an n -dimensional vector of 0s. It can be verified that these $2n$ points are affinely independent, so the dimension of $\text{conv}(X^{\text{CALS}})$ is at least $2n - 1$. Therefore, $\dim(\text{conv}(X^{\text{CALS}})) = 2n - 1$. We now exhibit $2n - 1$ affinely independent points on the face defined by inequality (3.1) for cases of $S = \emptyset$ and $S \neq \emptyset$, respectively.

1. If $S = \emptyset$, $B = \emptyset$ and $\ell = n$, then inequality (3.1) reduces to $C + \sum_{t \in [2, n]} x_t \geq d_{1n}$. We consider the following $2n - 1$ points:

- $(C^1, \mathbf{x}^1, \mathbf{y}^1) = (d_{1n} - (n - 1)\epsilon, (d_{1n} - (n - 1)\epsilon)\mathbf{e}_1 + \epsilon\mathbf{1}_{N \setminus \{1\}}, \mathbf{1}_N)$.
- $(C^2, \mathbf{x}^2, \mathbf{y}^2) = (d_{1n} - (n - 2)\epsilon, (d_{1n} - (n - 2)\epsilon)\mathbf{e}_1 + \epsilon\mathbf{1}_{N \setminus \{1, j\}}, \mathbf{1}_N \mid j \in [2, n])$.
- $(C^3, \mathbf{x}^3, \mathbf{y}^3) = (d_{1n} - (n - 2)\epsilon, (d_{1n} - (n - 2)\epsilon)\mathbf{e}_1 + \epsilon\mathbf{1}_{N \setminus \{1, j\}}, \mathbf{1}_{N \setminus \{j\}} \mid j \in [2, n])$.

It is straightforward to show that these $2n - 1$ points are all feasible and satisfy inequality (3.1) as an equality. To prove affine independence, we subtract point $(C^1, \mathbf{x}^1, \mathbf{y}^1)$ from points $(C^2, \mathbf{x}^2, \mathbf{y}^2)$ and $(C^3, \mathbf{x}^3, \mathbf{y}^3)$ to obtain the following $2n - 2$ points in the format of a matrix

$$\begin{pmatrix} C & x_1 & \{x_i\}_{i \in [2, n]} & y_1 & \{y_i\}_{i \in [2, n]} \\ \epsilon\mathbf{1} & \epsilon\mathbf{1} & -\epsilon\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \epsilon\mathbf{1} & \epsilon\mathbf{1} & -\epsilon\mathbf{I} & \mathbf{0} & -\mathbf{I} \end{pmatrix},$$

where $\mathbf{I}$ represents an appropriately sized identity matrix and $\mathbf{0}$ and $\mathbf{1}$ represent appropriately sized matrices of 0s and 1s. From the structure of the matrix, these points are linearly independent, which implies affine independence of points $(C^1, \mathbf{x}^1, \mathbf{y}^1)$, $(C^2, \mathbf{x}^2, \mathbf{y}^2)$, and $(C^3, \mathbf{x}^3, \mathbf{y}^3)$. Therefore, inequality (3.1) is facet-defining in this case.

2. If $S \neq \emptyset$, we denote set $S = \{p_1, p_2, \dots, p_{|S|}\}$ with $2 \leq p_1 < p_2 < \dots < p_{|S|} \leq \ell$. Additionally, for a period $m \in N$, we define a series of functions $g_{im}(C_i) = \min(d_{im} - \sum_{k=i+1}^m g_{km}(C_k), C_i)$ with the given capacity $C_i \in \mathbb{R}_+$ for $i \in [1, m]$ and let $g_{mm}(C_m) = \min(d_m, C_m)$. We consider the following $2n - 1$ points:

- $(C^1, \mathbf{x}^1, \mathbf{y}^1) = (d_{p_1n}, d_{1(p_1-1)}\mathbf{e}_1 + d_{p_1n}\mathbf{e}_{p_1}, \mathbf{1}_{\{1, p_1\}})$.
- $(C^2, \mathbf{x}^2, \mathbf{y}^2) = (d_{p_1n}, d_{1(p_1-1)}\mathbf{e}_1 + d_{p_1n}\mathbf{e}_{p_1}, \mathbf{1}_{\{1, p_1, j\}} \mid j \in [\ell + 1, n])$.
- $(C^3, \mathbf{x}^3, \mathbf{y}^3) = (d_{p_1n} - \epsilon, d_{1(p_1-1)}\mathbf{e}_1 + (d_{p_1n} - \epsilon)\mathbf{e}_{p_1} + \epsilon\mathbf{e}_j, \mathbf{1}_{\{1, p_1, j\}} \mid j \in [\ell + 1, n])$.
- $(C^4, \mathbf{x}^4, \mathbf{y}^4) = (d_{p_1n} - (n - \ell + 1)\epsilon, (d_{1(p_1-1)} + \epsilon)\mathbf{e}_1 + (d_{p_1n} - (n - \ell + 1)\epsilon)\mathbf{e}_{p_1} + \epsilon\mathbf{1}_{[\ell+1, n]}, \mathbf{1}_{\{1, p_1, [\ell+1, n]}\})$.
- $(C^5, \mathbf{x}^5, \mathbf{y}^5) = (d_{p_jn}, d_{1(p_j-1)}\mathbf{e}_1 + d_{p_jn}\mathbf{e}_{p_j}, \mathbf{1}_{\{1, p_j\}} \mid j \in [2, |S|])$.
- $(C^6, \mathbf{x}^6, \mathbf{y}^6) = (d_{p_jn} - \epsilon, (d_{1(p_j-1)} + \epsilon)\mathbf{e}_1 + (d_{p_jn} - \epsilon)\mathbf{e}_{p_j}, \mathbf{1}_{\{1, p_j\}} \mid j \in [2, |S|])$.
- $(C^7, \mathbf{x}^7, \mathbf{y}^7) = (d_{1\ell}, d_{1\ell}\mathbf{e}_1 + \sum_{k=\ell+1}^n g_{kn}(C_k)\mathbf{e}_k, \mathbf{1}_{\{1, [\ell+1, n]\}})$, where $C_k = d_{1\ell}$ for $k \in [\ell + 1, n]$.
- $(C^8, \mathbf{x}^8, \mathbf{y}^8) = (d_{1\ell}, d_{1\ell}\mathbf{e}_1 + \sum_{k=\ell+1}^n g_{kn}(C_k)\mathbf{e}_k, \mathbf{1}_{\{1, j, [\ell+1, n]\}} \mid j \in A)$, where $C_k = d_{1\ell}$ for $k \in [\ell + 1, n]$.
- $(C^9, \mathbf{x}^9, \mathbf{y}^9) = (d_{1\ell} - \epsilon, (d_{1\ell} - \epsilon)\mathbf{e}_1 + \epsilon\mathbf{e}_j + \sum_{k=\ell+1}^n g_{kn}(C_k)\mathbf{e}_k, \mathbf{1}_{\{1, j, [\ell+1, n]\}} \mid j \in A)$, where $C_k = d_{1\ell} - \epsilon$ for $k \in [\ell + 1, n]$.

- $(C^{10}, \mathbf{x}^{10}, \mathbf{y}^{10}) = (d_{1(j-1)}, d_{1(j-1)}\mathbf{e}_1 + d_{j\ell}\mathbf{e}_j + \sum_{k=\ell+1}^n g_{kn}(C_k)\mathbf{e}_k, \mathbf{1}_{\{1,j,[\ell+1,n]\}} \mid j \in B)$, where $C_k = d_{1(j-1)}$ for $k \in [\ell+1, n]$.
- $(C^{11}, \mathbf{x}^{11}, \mathbf{y}^{11}) = (d_{1(j-1)}, d_{1(j-1)}\mathbf{e}_1 + (d_{j\ell} + \epsilon)\mathbf{e}_j + (d_{\ell+1} - \epsilon)\mathbf{e}_{\ell+1} + \sum_{k=\ell+2}^n g_{kn}(C_k)\mathbf{e}_k, \mathbf{1}_{\{1,j,[\ell+1,n]\}} \mid j \in B)$, where $C_k = d_{1(j-1)}$ for $k \in [\ell+2, n]$.

Because $d_{1(i-1)} < d_{in}$ for any $i \in S$, points $\{(C^j, \mathbf{x}^j, \mathbf{y}^j)\}_{j \in [1,6]}$ are feasible. Because $(j-\ell)d_{1\ell} > d_{(\ell+1)j}$ for any $j \in [\ell+1, n]$, points $\{(C^j, \mathbf{x}^j, \mathbf{y}^j)\}_{j \in [7,9]}$ are feasible. Because $\ell \leq n-1$, $d_{1(j-1)} > d_{j\ell}$ and $(k-\ell)d_{1(j-1)} \geq d_{(\ell+1)k}$ for any $j \in B \neq \emptyset$ and $k \in [\ell+1, n]$, points $(C^{10}, \mathbf{x}^{10}, \mathbf{y}^{10})$ and $(C^{11}, \mathbf{x}^{11}, \mathbf{y}^{11})$ are feasible. In addition, it is straightforward to verify that all the points listed above are on the face defined by inequality (3.1).

To prove affine independence, we subtract point $(C^1, \mathbf{x}^1, \mathbf{y}^1)$ from all other points to obtain $2n-2$ points, which are arranged in the following matrix after conducting necessary elementary row operations.

$$\begin{matrix} C & x_1 & \{x_i\}_{i \in A \cup B} & x_{p_1} & \{x_i\}_{i \in S \setminus \{p_1\}} & \{x_i\}_{i \in [\ell+1, n]} & y_1 & \{y_i\}_{i \in A \cup B} & y_{p_1} & \{y_i\}_{i \in S \setminus \{p_1\}} & \{y_i\}_{i \in [\ell+1, n]} \\ \left(\begin{array}{cccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ * & \mathbf{0} & \mathbf{0} & * & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & 1 & \mathbf{0} & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} & * & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -1 & \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{0} & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \mathbf{0} & \mathbf{I} & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right), \end{matrix}$$

where $\mathbf{I}$ represents an appropriately sized identity matrix, $\mathbf{0}$ represents an appropriately sized matrix of 0s, and $*$ represents an appropriately sized unspecified matrix. From the structure of the matrix, these points are linearly independent, which implies affine independence of the $2n-1$ points listed above. Therefore, inequality (3.1) is also facet-defining in this case.

□

Example We illustrate Proposition 3 using the same five-period CALS instance with $\mathbf{d} = (11, 7, 20, 5, 3)$.

First, the inequality introduced after Theorem 1 with $\ell = 4$, $S = \{2\}$, $A = \{3\}$, and $B = \{4\}$ is $C + x_3 + 5y_4 + 43y_2 \geq 43 + x_2$, which is facet-defining according to Proposition 3. Indeed, for the only period $i = 2 \in S$, $d_{1(i-1)} = d_{11} = 11 < d_{2n} = 35$. Also, for the only period $j = 5 \in [\ell+1, n]$, $(j-\ell)d_{1\ell} = (5-4)d_{14} = 43 > d_{55} = 3$. Finally, since $B = \{4\}$, we check $d_{1(j-1)} = d_{13} = 38 > d_{44} = 5$, and $(k-\ell)d_{1(j-1)} = (5-4)d_{13} = 38 \geq d_{55} = 3$. Hence, all conditions of Proposition 3 are satisfied.

Second, consider the choice $\ell = 5$, $S = \emptyset$, $A = \{2, 3, 4, 5\}$, and $B = \emptyset$. Then, inequality 3.1 reduces to $C + x_2 + x_3 + x_4 + x_5 \geq 46$. This inequality is facet-defining by the first case of Proposition 3, since $S = \emptyset$, $B = \emptyset$, and $\ell = n$.

We next present a stronger aggregated class of demand-satisfying inequalities. Compared with the first-group (ℓ, S, A, B) inequalities, this class replaces multiple production terms with repeated use of the capacity variable C , while only a small number of periods are treated individually. More precisely, the sets A and B identify the exceptional periods handled separately, whereas the remaining periods are aggregated through the capacity term. The resulting inequality therefore captures the idea that aggregate

capacity coverage, together with a few explicitly modeled periods, must still be sufficient to satisfy the cumulative demand over $[1, \ell]$.

Theorem 4. Let a period $\ell \in [2, n]$ and a set $L = [2, \ell]$; sets S, A , and B be three pairwise disjoint subsets of set L ; and $|A \cup B| \leq 2$. The second-group (ℓ, S, A, B) inequality

$$|S|(|L| - |A \cup B|)C + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t \geq |S|d_{1\ell} - d_{1\ell} \sum_{t \in S} y_t + (|L| - |A \cup B|) \sum_{t \in S} x_t \quad (3.2)$$

is valid for CALS.

The restriction $|A \cup B| \leq 2$ should be interpreted as defining a tractable subclass rather than as a business rule. Once more than two periods are treated as exceptional, the validity argument becomes substantially more complicated and the resulting separation structure is less manageable. Thus, this restriction reflects a deliberate trade-off between strength and tractability in the cut construction.

Remark Because inequality (3.2) is trivial when $S = \emptyset$, we assume $S \neq \emptyset$ for the second-group (ℓ, S, A, B) inequalities in the rest of this paper.

Proof. For any given feasible solution $(C, \mathbf{x}, \mathbf{y}) \in X^{CALS}$, we denote $S^{=0} := \{t \in S : y_t = 0\}$.

1. If $|S^{=0}| = 0$, then inequality (3.2) reduces to $|S|(|L| - |A \cup B|)C + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t \geq (|L| - |A \cup B|) \sum_{t \in S} x_t$, which is valid because $C \geq x_t$ for all $t \in S$.
2. If $|S^{=0}| = 1$, then inequality (3.2) reduces to $|S|(|L| - |A \cup B|)C + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t \geq d_{1\ell} + (|L| - |A \cup B|) \sum_{t \in S \setminus S^{=0}} x_t$. Let $t^* = \min\{i \in B : y_i = 1\}$. Then $(|L| - |A \cup B|)C + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t \geq C + (|L| - |A \cup B \cup S^{=0}|)C + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t \geq x_1 + \sum_{t \in L \setminus \{A \cup B \cup S^{=0}\}} x_t + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t \geq d_{1(t^*-1)} + d_{t^*\ell} \geq d_{1\ell}$. In addition, because $(|S| - 1)(|L| - |A \cup B|)C \geq (|L| - |A \cup B|) \sum_{t \in S \setminus S^{=0}} x_t$, inequality (3.2) is valid.
3. If $|S^{=0}| \geq 2$, then inequality (3.2) reduces to

$$|S|(|L| - |A \cup B|)C + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t \geq |S^{=0}|d_{1\ell} + (|L| - |A \cup B|) \sum_{t \in S \setminus S^{=0}} x_t. \quad (3.3)$$

Because $|A \cup B| \leq 2$, then we have $(|L| - |A \cup B| + 1)C \geq (|L| - 2 + 1)C \geq (|L| - |S^{=0}| + 1)C \geq \sum_{t \in [1, \ell] \setminus S^{=0}} x_t \geq d_{1\ell}$, which gives

$$(|S^{=0}| - 1)(|L| - |A \cup B| + 1)C \geq (|S^{=0}| - 1)d_{1\ell}. \quad (3.4)$$

Also, because $\sum_{t \in [1, \ell] \setminus (A \cup B \cup S^{=0})} x_t + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t \geq d_{1\ell}$ and $C \geq x_t$ for any $t \in [1, n]$, we have

$$(|L| - |A \cup B| - |S^{=0}| + 1)C + \sum_{t \in A} x_t + \sum_{t \in B} d_{t\ell} y_t \geq d_{1\ell}. \quad (3.5)$$

Additionally, because $C \geq x_t$ for any $t \in [1, n]$, we have

$$|S \setminus S^{=0}|(|L| - |A \cup B|)C \geq (|L| - |A \cup B|) \sum_{t \in S \setminus S^{=0}} x_t. \quad (3.6)$$

Summing inequalities (3.4)–(3.6), the resulting inequality is (3.3). Therefore, inequality (3.2) is valid in this case as well, and this completes the proof. $\square$

Example We next illustrate Theorem 4 using the same demand vector $\mathbf{d} = (11, 7, 20, 5, 3)$. Let $\ell = 4$, so that $L = [2, 4] = \{2, 3, 4\}$, and choose $S = \{2, 3\}$, $A = \{4\}$, $B = \emptyset$.

Here, $|S| = 2$, $|L| = 3$, $|A \cup B| = 1 \leq 2$, so the construction satisfies the requirement of Theorem 4. Also, $d_{1\ell} = d_{14} = 11 + 7 + 20 + 5 = 43$. Substituting these values into inequality (3.2), we obtain

$$|S|(|L| - |A \cup B|)C + x_4 \geq |S|d_{14} - d_{14}(y_2 + y_3) + (|L| - |A \cup B|)(x_2 + x_3),$$

which is,

$$4C + x_4 \geq 86 - 43(y_2 + y_3) + 2(x_2 + x_3).$$

This example shows the aggregated nature of the second group. Periods 2 and 3 are treated symmetrically through the set S , so their production terms are shifted to the right-hand side and balanced by cumulative-demand coefficients. Period $4 \in A$ is kept explicitly through x_4 , while the remaining demand cover is represented by repeated use of the capacity variable C . Hence, the inequality captures an aggregated demand-satisfying requirement over periods 1 to 4.

Proposition 5. *Given a fractional solution $(\bar{C}, \bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathbb{R}_+^{2n+1}$, the most violated second-group (ℓ, S, A, B) inequality (3.2) can be found within $O(n^3)$ computational time, if any.*

Proof. For a given $\ell \in [2, n]$, we first consider the case of $|S| = 1$. For each $t \in L$ and $i \in L \setminus \{t\}$, let $\delta_{ti} = \min(0, -\bar{C} + \bar{x}_i + \bar{x}_t, -\bar{C} + d_{i\ell}\bar{y}_i + \bar{x}_t)$ and $\phi_t = |L|\bar{C} - d_{1\ell} + d_{1\ell}\bar{y}_t - |L|\bar{x}_t + \sum_{i \in L \setminus \{t\}} \delta_{ti}$. If there exists a period $t \in L$ such that $\phi_t < 0$, then let $t^* = \operatorname{argmin}\{\phi_t, t \in L\}$. For the given $\ell \in L$, in the case of $|S| = 1$, we have found the most violated second-group (ℓ, S, A, B) inequality, where $S = \{t^*\}$, $A = \{i \in L \setminus \{t^*\} : -\bar{C} + \bar{x}_i + \bar{x}_{t^*} < \min(0, -\bar{C} + d_{i\ell}\bar{y}_i + \bar{x}_{t^*})\}$, and $B = \{i \in L \setminus \{t^*\} : -\bar{C} + d_{i\ell}\bar{y}_i + \bar{x}_{t^*} \leq \min(0, -\bar{C} + \bar{x}_i + \bar{x}_{t^*})\}$. Because $t \in L$ and $i \in L \setminus \{t\}$, the complexity for the case of $|S| = 1$ is $O(n^2)$.

Then, suppose $|S| \geq 2$. For each $t \in L$, we define $\beta_t = \min\{\bar{x}_t, d_{t\ell}\bar{y}_t\}$ and $\gamma_t = \min(0, (|L| - |A \cup B|)\bar{C} - d_{1\ell} + d_{1\ell}\bar{y}_t - (|L| - |A \cup B|)\bar{x}_t)$. Then, we consider three cases of $|A \cup B| \in \{0, 1, 2\}$. When $|A \cup B| = 0$, we let $\omega_0 = \sum_{t \in L} \gamma_t$. When $|A \cup B| = 1$, we define $t' = \operatorname{argmin}\{\beta_t - \gamma_t, t \in L\}$ and let $\omega_1 = \sum_{t \in L \setminus \{t'\}} \gamma_t + \beta_{t'}$. When $|A \cup B| = 2$, we define $t' = \operatorname{argmin}\{\beta_t - \gamma_t, t \in L\}$ and $t'' = \operatorname{argmin}\{\beta_t - \gamma_t, t \in L \setminus \{t'\}\}$. Then, we let $\omega_2 = \sum_{t \in L \setminus \{t', t''\}} \gamma_t + \beta_{t'} + \beta_{t''}$. If $\omega_0 \leq \min(\omega_1, \omega_2)$ and $\omega_0 < 0$, then the most violated second-group (ℓ, S, A, B) inequality is with $S = \{t \in L : \gamma_t < 0\}$ and $A = B = \emptyset$. If $\omega_1 \leq \min(\omega_0, \omega_2)$ and $\omega_1 < 0$, then the most violated second-group (ℓ, S, A, B) inequality is with $S = \{t \in L \setminus \{t'\} : \gamma_t < 0\}$, $A = \{t'\}$ and $B = \emptyset$ if $\bar{x}_{t'} < d_{t'\ell}\bar{y}_{t'}$; otherwise, $A = \emptyset$ and $B = \{t'\}$. If $\omega_2 \leq \min(\omega_0, \omega_1)$ and $\omega_2 < 0$, then the most violated second-group (ℓ, S, A, B) inequality is with $S = \{t \in L \setminus \{t', t''\} : \gamma_t < 0\}$, $A = \{t \in \{t', t''\} : \bar{x}_t < d_{t\ell}\bar{y}_t\}$, and $B = \{t', t''\} \setminus A$. The complexity for the case of $|S| \geq 2$ with a given $\ell \in [2, n]$ is also $O(n^2)$.

The inequality with a stronger violation between those obtained in the cases of $|S| = 1$ and $|S| \geq 2$ for all $\ell \in [2, n]$ is the most violated second-group (ℓ, S, A, B) inequality. The overall complexity of the separation algorithm for the second-group (ℓ, S, A, B) inequality is $O(n^3)$. $\square$

The proof again uses a facet-verification argument. Since Proposition 3 already establishes that $\dim(\operatorname{conv}(X^{\text{CALIS}})) = 2n - 1$, it remains to construct $2n - 1$ affinely independent feasible points on the face defined by inequality (3.2). We do this separately for the cases $|A \cup B| = 0$ and $|A \cup B| \in \{1, 2\}$, because the structure of the inequality and the corresponding point constructions are substantially different in these two settings. The goal in each case is to generate points that remain feasible, satisfy the inequality at equality, and isolate the relevant coefficients so as to prove that the inequality is facet-defining.

Proposition 6. The second-group (ℓ, S, A, B) inequality (3.2) is facet-defining for $\text{conv}(X^{\text{CALs}})$ if the following conditions are satisfied:

- If $|A \cup B| = 0$, then S is a singleton denoted as $S = \{p_1\}$, $\frac{td_{1\ell}}{\ell-1} \geq d_{1t}$ for $t \in [1, p_1 - 1]$, $\frac{(t-1)d_{1\ell}}{\ell-1} \geq d_{1t}$ for $t \in [p_1, \ell]$, $\frac{(t-\ell)d_{1\ell}}{\ell-1} \geq d_{(\ell+1)t}$ for $t = [\ell + 1, n]$, $d_{p_1n} > d_1$ and $(t-1)d_{p_1n} > d_{1t}$ for $t \in [2, p_1 - 1]$.
- If $|A \cup B| := r \in \{1, 2\}$, we denote $A \cup B = \{q_1, q_r\}$ with $q_r = q_1$ if $r = 1$ and $q_1 < q_r$ if $r = 2$, and $S = \{p_i\}_{i=1}^{|S|}$ with $2 \leq p_1 < \dots < p_{|S|} \leq \ell$. We define $o_1 = \min\{q_1, q_r, p_1\}$, $o_2 = \min\{\{q_1, q_r, p_1\} \setminus \{o_1\}\}$, $o_3 = \{q_1, q_r, p_1\} \setminus \{o_1, o_2\}$ if $\{q_1, q_r, p_1\} \setminus \{o_1, o_2\} \neq \emptyset$, and $o_3 = \ell + 1$ otherwise. Then,
 - i. $\frac{id_{p_1n}}{|S|} > d_{p_1(p_{i+1}-1)}$ for $i \in [1, |S|]$, where $p_{|S|+1} = n + 1$;
 - ii. $\frac{(t-1)d_{p_1n}}{|S|} > d_{1t}$ for $t \in [2, o_1 - 1]$, $\frac{(t-2)d_{p_1n}}{|S|} > d_{1t}$ for $t \in [o_1, \min(p_1, o_2) - 1]$, and $\frac{(t-3)d_{p_1n}}{|S|} > d_{1t}$ for $t \in [\min(p_1, o_2), \min(p_1, o_3) - 1]$;
 - iii. $\frac{td_{1(\ell-|B|)}}{|L \setminus (A \cup B)|} > d_{1t}$ for $t \in [1, o_1 - 1]$, $\frac{(t-1)d_{1(\ell-|B|)}}{|L \setminus (A \cup B)|} > d_{1t}$ for $t \in [o_1, o_2 - 1]$, $\frac{(t-2)d_{1(\ell-|B|)}}{|L \setminus (A \cup B)|} > d_{1t}$ for $t \in [o_2, o_3 - 1]$, $\frac{(t-3)d_{1(\ell-|B|)}}{|L \setminus (A \cup B)|} > d_{1t}$ for $t \in [o_3, \ell]$, and $\frac{(t-\ell)d_{1(\ell-|B|)}}{|L \setminus (A \cup B)|} > d_{(\ell+1)t}$ for $t \in [\ell + 1, n]$;
 - iv. if $B \neq \emptyset$, then $\ell \leq n - 1$, $B = \{\ell - |B| + 1, \ell\}$, and $\frac{d_{1(\ell-|B|)}}{|L \setminus (A \cup B)|} > d_{(\ell-|B|+1)\ell}$.

Proof. We have shown $\dim(\text{conv}(X^{\text{CALs}})) = 2n - 1$ in the proof of Proposition 3. We now exhibit $2n - 1$ affinely independent points on the face defined by inequality (3.2) for the cases of $|A \cup B| = 0$ and $|A \cup B| \in \{1, 2\}$, respectively.

1. If $|A \cup B| = 0$, then $S = \{p_1\}$ and the second-group (ℓ, S, A, B) inequality reduces to

$$|L|C \geq d_{1\ell}(1 - y_{p_1}) + |L|x_{p_1}. \quad (3.7)$$

We consider the following $2n - 1$ points:

- $(C^1, \mathbf{x}^1, \mathbf{y}^1) = (\frac{d_{1\ell}}{\ell-1}, \frac{d_{1\ell}}{\ell-1} \mathbf{1}_{[1, \ell] \setminus \{p_1\}} + \sum_{k=\ell+1}^n g_{kn}(C_k) \mathbf{e}_k, \mathbf{1}_{[1, n] \setminus \{p_1\}})$, where $C_k = \frac{d_{1\ell}}{\ell-1}$ for $k \in [\ell + 1, n]$.
- $(C^2, \mathbf{x}^2, \mathbf{y}^2) = (d_{p_1n}, \sum_{k=1}^{p_1-1} g_{k(p_1-1)}(C_k) \mathbf{e}_k + d_{p_1n} \mathbf{e}_{p_1}, \mathbf{1}_{[1, p_1] \cup \{j\}} \mid j \in [p_1, n])$, where $C_k = d_{p_1n}$ for $k \in [1, p_1 - 1]$.
- $(C^3, \mathbf{x}^3, \mathbf{y}^3) = (d_{p_1n}, \sum_{k=1}^{p_1-1} g_{k(p_1-1)}(C_k) \mathbf{e}_k + d_{p_1n} \mathbf{e}_{p_1}, \mathbf{1}_{[1, p_1] \setminus \{j\}} \mid j \in [2, p_1 - 1])$, where $C_k = d_{p_1n}$ for $k \in [1, p_1 - 1] \setminus \{j\}$ and $C_j = 0$.
- $(C^4, \mathbf{x}^4, \mathbf{y}^4) = (d_{p_1n}, \sum_{k=1}^{p_1-1} g_{k(p_1-1)}(C_k) \mathbf{e}_k + d_{p_1n} \mathbf{e}_{p_1}, \mathbf{1}_{[1, p_1]} \mid j \in [2, p_1 - 1])$, where $C_k = d_{p_1n}$ for $k \in [1, p_1 - 1] \setminus \{j\}$ and $C_j = 0$.
- $(C^5, \mathbf{x}^5, \mathbf{y}^5) = (d_{p_1n} - \epsilon, d_1 \mathbf{e}_1 + \sum_{k=2}^{p_1-1} g_{k(p_1-1)}(C_k) \mathbf{e}_k + (d_{p_1n} - \epsilon) \mathbf{e}_{p_1} + \epsilon \mathbf{e}_j, \mathbf{1}_{[1, p_1] \cup \{j\}} \mid j \in [p_1 + 1, n])$, where $C_k = d_{p_1n} - \epsilon$ for $k \in [2, p_1 - 1]$.
- $(C^6, \mathbf{x}^6, \mathbf{y}^6) = (d_{p_1n} - \epsilon, (d_1 + \epsilon) \mathbf{e}_1 + \sum_{k=2}^{p_1-1} g_{k(p_1-1)}(C_k) \mathbf{e}_k + (d_{p_1n} - \epsilon) \mathbf{e}_{p_1}, \mathbf{1}_{[1, p_1]})$, where $C_k = d_{p_1n} - \epsilon$ for $k \in [2, p_1 - 1]$.

Because $\frac{td_{1\ell}}{\ell-1} \geq d_{1t}$ for $t \in [1, p_1 - 1]$, $\frac{(t-1)d_{1\ell}}{\ell-1} \geq d_{1t}$ for $t \in [p_1, \ell]$, and $\frac{(t-\ell)d_{1\ell}}{\ell-1} \geq d_{(\ell+1)t}$ for $t = [\ell + 1, n]$, point $(C^1, \mathbf{x}^1, \mathbf{y}^1)$ is a feasible solution and satisfies inequality (3.7) as an equality. Because $d_{p_1n} > d_1$ and $(t-1)d_{p_1n} > d_{1t}$ for $t \in [2, p_1 - 1]$, points $\{(C^i, \mathbf{x}^i, \mathbf{y}^i)\}_{i \in [2, 6]}$ are feasible solutions and satisfy inequality (3.7) as an equality. To prove affine independence, we subtract point $(C^1, \mathbf{x}^1, \mathbf{y}^1)$ from all other points to obtain $2n - 2$ points, which are arranged in the format of the following matrix

after conducting necessary elementary row operations.

$$C \begin{matrix} x_1 & \{x_i\}_{i \in [2, p_1-1]} & x_{p_1} & \{x_i\}_{i \in [p_1+1, n]} & y_1 & \{y_i\}_{i \in [2, p_1-1]} & \{y_i\}_{i \in [p_1, n]} \end{matrix} \begin{pmatrix} * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ * & * & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ * & * & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where $\mathbf{I}$ represents an appropriately sized identity matrix, $\mathbf{0}$ represents an appropriately sized matrix of 0s, and $*$ represents an appropriately sized unspecified matrix. Because these points are linearly independent, the $2n - 1$ points listed above are affinely independent. Therefore, inequality (3.7) is facet-defining.

2. If $|A \cup B| \in \{1, 2\}$, we consider the following $2n - 1$ points:

- $(C^1, \mathbf{x}^1, \mathbf{y}^1) = (\frac{d_{p_1 n}}{|S|}, \sum_{k=1}^{p_1-1} g_{k(p_1-1)}(C_k)\mathbf{e}_k + \frac{d_{p_1 n}}{|S|}\mathbf{1}_S, \mathbf{1}_{([1, p_1-1] \cup S) \setminus (A \cup B)})$, where $C_k = \frac{d_{p_1 n}}{|S|}$ for $k \in [1, p_1 - 1] \setminus (A \cup B)$ and $C_k = 0$ for $k \in [1, p_1 - 1] \cap (A \cup B)$.
- $(C^2, \mathbf{x}^2, \mathbf{y}^2) = (\frac{d_{p_1 n}}{|S|}, \sum_{k=1}^{p_1-1} g_{k(p_1-1)}(C_k)\mathbf{e}_k + \frac{d_{p_1 n}}{|S|}\mathbf{1}_S, \mathbf{1}_{([1, p_1-1] \cup S) \setminus (A \cup B) \cup \{j\}} \mid j \in A \cup ([p_1 + 1, n] \setminus (S \cup A \cup B)))$, where $C_k = \frac{d_{p_1 n}}{|S|}$ for $k \in [1, p_1 - 1] \setminus (A \cup B)$ and $C_k = 0$ for $k \in [1, p_1 - 1] \cap (A \cup B)$.
- $(C^3, \mathbf{x}^3, \mathbf{y}^3) = (\frac{d_{p_1 n} - \epsilon}{|S|}, \sum_{k=1}^{p_1-1} g_{k(p_1-1)}(C_k)\mathbf{e}_k + \frac{d_{p_1 n} - \epsilon}{|S|}\mathbf{1}_S + \epsilon \mathbf{e}_j, \mathbf{1}_{([1, p_1-1] \cup S \cup \{j\}) \setminus (A \cup B)} \mid j \in [p_1 + 1, n] \setminus (S \cup A \cup B))$, where $C_k = \frac{d_{p_1 n} - \epsilon}{|S|}$ for $k \in [1, p_1 - 1] \setminus (A \cup B)$ and $C_k = 0$ for $k \in [1, p_1 - 1] \cap (A \cup B)$.
- $(C^4, \mathbf{x}^4, \mathbf{y}^4) = (\frac{d_{p_1 n}}{|S|}, \sum_{k=1}^{p_1-1} g_{k(p_1-1)}(C_k)\mathbf{e}_k + \frac{d_{p_1 n}}{|S|}\mathbf{1}_S, \mathbf{1}_{([1, p_1-1] \cup S) \setminus (A \cup B \cup \{j\})} \mid j \in [2, p_1 - 1] \setminus (A \cup B))$, where $C_k = \frac{d_{p_1 n}}{|S|}$ for $k \in [1, p_1 - 1] \setminus (A \cup B \cup \{j\})$ and $C_k = 0$ for $k \in [1, p_1 - 1] \cap (A \cup B \cup \{j\})$.
- $(C^5, \mathbf{x}^5, \mathbf{y}^5) = (\frac{d_{p_1 n}}{|S|}, \sum_{k=1}^{p_1-1} g_{k(p_1-1)}(C_k)\mathbf{e}_k + \frac{d_{p_1 n}}{|S|}\mathbf{1}_S, \mathbf{1}_{([1, p_1-1] \cup S) \setminus (A \cup B)} \mid j \in [2, p_1 - 1] \setminus (A \cup B))$, where $C_k = \frac{d_{p_1 n}}{|S|}$ for $k \in [1, p_1 - 1] \setminus (A \cup B \cup \{j\})$ and $C_k = 0$ for $k \in [1, p_1 - 1] \cap (A \cup B \cup \{j\})$.
- $(C^6, \mathbf{x}^6, \mathbf{y}^6) = (\frac{d_{1\ell} - \epsilon}{|L \setminus (A \cup B)|}, \frac{d_{1\ell} - \epsilon}{|L \setminus (A \cup B)|}\mathbf{1}_{[1, \ell] \setminus (A \cup B \cup \{p_1\})} + \epsilon \mathbf{e}_j + \sum_{k=\ell+1}^n g_{kn}(C_k)\mathbf{e}_k, \mathbf{1}_{[1, n] \setminus (A \cup B \cup \{p_1\}) \cup \{j\}} \mid j \in A)$, where $C_k = \frac{d_{1\ell} - \epsilon}{|L \setminus (A \cup B)|}$ for $k \in [\ell + 1, n]$.
- $(C^7, \mathbf{x}^7, \mathbf{y}^7) = (\frac{d_{1(j-1)}}{|L \setminus (A \cup B)|}, \frac{d_{1(j-1)}}{|L \setminus (A \cup B)|}\mathbf{1}_{[1, \ell] \setminus (A \cup B \cup \{p_1\})} + d_{j\ell}\mathbf{e}_j + \sum_{k=\ell+1}^n g_{kn}(C_k)\mathbf{e}_k, \mathbf{1}_{[1, n] \setminus (A \cup B \cup \{p_1\}) \cup \{j\}} \mid j \in B)$, where $C_k = \frac{d_{1(j-1)}}{|L \setminus (A \cup B)|}$ for $k \in [\ell + 1, n]$.
- $(C^8, \mathbf{x}^8, \mathbf{y}^8) = (\frac{d_{1(j-1)}}{|L \setminus (A \cup B)|}, \frac{d_{1(j-1)}}{|L \setminus (A \cup B)|}\mathbf{1}_{[1, \ell] \setminus (A \cup B \cup \{p_1\})} + (d_{j\ell} + \epsilon)\mathbf{e}_j + (d_{\ell+1} - \epsilon)\mathbf{e}_{\ell+1} + \sum_{k=\ell+2}^n g_{kn}(C_k)\mathbf{e}_k, \mathbf{1}_{[1, n] \setminus (A \cup B \cup \{p_1\}) \cup \{j\}} \mid j \in B)$, where $C_k = \frac{d_{1(j-1)}}{|L \setminus (A \cup B)|}$ for $k \in [\ell + 2, n]$.
- $(C^9, \mathbf{x}^9, \mathbf{y}^9) = (\frac{d_{1\ell}}{|L \setminus (A \cup B)|}, \frac{d_{1\ell}}{|L \setminus (A \cup B)|}\mathbf{1}_{[1, \ell] \setminus (A \cup B \cup \{j\})} + \sum_{k=\ell+1}^n g_{kn}(C_k)\mathbf{e}_k, \mathbf{1}_{[1, n] \setminus (A \cup B \cup \{j\})} \mid j \in S)$, where $C_k = \frac{d_{1\ell}}{|L \setminus (A \cup B)|}$ for $k \in [\ell + 1, n]$.
- $(C^{10}, \mathbf{x}^{10}, \mathbf{y}^{10}) = (\frac{d_{1\ell} - |A|\epsilon}{|L \setminus (A \cup B)|}, \frac{d_{1\ell} - |A|\epsilon}{|L \setminus (A \cup B)|}\mathbf{1}_{[1, \ell] \setminus (A \cup B \cup \{j\})} + \epsilon \mathbf{1}_A + \sum_{k=\ell+1}^n g_{kn}(C_k)\mathbf{e}_k, \mathbf{1}_{[1, n] \setminus (B \cup \{j\})} \mid j \in S \setminus \{p_1\})$ if $A \neq \emptyset$, where $C_k = \frac{d_{1\ell} - |A|\epsilon}{|L \setminus (A \cup B)|}$ for $k \in [\ell + 1, n]$.
- $(C^{11}, \mathbf{x}^{11}, \mathbf{y}^{11}) = (\frac{d_{1\ell} - 2|A|\epsilon}{|L \setminus (A \cup B)|}, \frac{d_{1\ell} - 2|A|\epsilon}{|L \setminus (A \cup B)|}\mathbf{1}_{[1, \ell] \setminus (A \cup B \cup \{p_1\})} + 2\epsilon \mathbf{1}_A + \sum_{k=\ell+1}^n g_{kn}(C_k)\mathbf{e}_k, \mathbf{1}_{[1, n] \setminus (B \cup \{p_1\})})$ if $A \neq \emptyset$.
- $(C^{12}, \mathbf{x}^{12}, \mathbf{y}^{12}) = (\frac{d_{1(\ell-1)}}{|L \setminus (A \cup B)|}, \frac{d_{1(\ell-1)}}{|L \setminus (A \cup B)|}\mathbf{1}_{[1, \ell-1] \setminus (A \cup B \cup \{j\})} + d_{\ell}\mathbf{e}_{\ell} + \sum_{k=\ell+1}^n g_{kn}(C_k)\mathbf{e}_k, \mathbf{1}_{[1, n] \setminus (A \cup B \cup \{j\})} \mid j \in S \setminus \{p_1\})$ if $A = \emptyset$, where $C_k = \frac{d_{1(\ell-1)}}{|L \setminus (A \cup B)|}$ for $k \in [\ell + 1, n]$.
- $(C^{13}, \mathbf{x}^{13}, \mathbf{y}^{13}) = (\frac{d_{p_1 n} - 2\epsilon}{|S|}, \sum_{k=1}^{p_1-1} g_{k(p_1-1)}(C_k)\mathbf{e}_k + \frac{d_{p_1 n} - 2\epsilon}{|S|}\mathbf{1}_S + 2\epsilon \mathbf{e}_n, \mathbf{1}_{[1, p_1-1] \cup S \cup \{n\}})$ if $A = \emptyset$, where $C_k = \frac{d_{p_1 n} - 2\epsilon}{|S|}$ for $k \in [1, p_1 - 1]$.

Because of conditions i and ii, points $\{(C^i, \mathbf{x}^i, \mathbf{y}^i)\}_{i \in [1,5]}$ and $(C^{13}, \mathbf{x}^{13}, \mathbf{y}^{13})$ are feasible. Because of conditions iii and iv, points $\{(C^i, \mathbf{x}^i, \mathbf{y}^i)\}_{i \in [6,12]}$ are feasible. Also, all of the points listed above satisfy inequality (3.2) as an equality. To prove affine independence of the points listed above, we subtract point $(C^1, \mathbf{x}^1, \mathbf{y}^1)$ from all other points and conduct necessary elementary row operations to obtain $2n - 2$ points, which are arranged in the format of the following matrix:

$$\begin{pmatrix} C & x_1 & \{x_i\}_{i \in A} & \{x_i\}_{i \in B} & \{x_i\}_{i \in [2, p_1-1] \setminus (A \cup B)} & \{x_i\}_{i \in S} & \{x_i\}_{i \in [p_1+1, n] \setminus (S \cup A \cup B)} & y_1 & \{y_i\}_{i \in A \cup ([p_1+1, n] \setminus (S \cup A \cup B))} & \{y_i\}_{i \in [2, p_1-1] \setminus (A \cup B)} & \{y_i\}_{i \in B} & \{y_i\}_{i \in S} \\ * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ * & * & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where each row represents a (group of) point(s) and the last row represents points $(C^{10}, \mathbf{x}^{10}, \mathbf{y}^{10})$ and $(C^{11}, \mathbf{x}^{11}, \mathbf{y}^{11})$ when $A \neq \emptyset$ or points $(C^{12}, \mathbf{x}^{12}, \mathbf{y}^{12})$ and $(C^{13}, \mathbf{x}^{13}, \mathbf{y}^{13})$ when $A = \emptyset$. Also, $\mathbf{I}$ represents an appropriately sized identity matrix, $\mathbf{0}$ represents an appropriately sized matrix of 0s, and $*$ represents an appropriately sized unspecified matrix. Based on the structure of the matrix, the $2n - 1$ points listed above are affinely independent. Therefore, inequality (3.2) is facet-defining. $\square$

Example To illustrate Proposition 6, we revisit the five-period CALS instance with $\mathbf{d} = (11, 7, 20, 5, 3)$. First, for $\ell = 4$, $S = \{2, 3\}$, $A = \{4\}$, and $B = \emptyset$, inequality (3.2) reduces to

$$4C + x_4 \geq 86 - 43(y_2 + y_3) + 2(x_2 + x_3).$$

This inequality is facet-defining according to Proposition 6. In particular, with $p_1 = 2$, $p_2 = 3$, $q_1 = 4$, $o_1 = 2$, $o_2 = 3$, and $o_3 = 4$, the following key conditions hold:

$$\frac{d_{2n}}{2} = \frac{35}{2} > d_{22} = 7, \quad \frac{2d_{2n}}{2} = 35 > d_{23} = 27,$$

and

$$\frac{d_{14}}{2} = \frac{43}{2} > d_{12} = 18, \quad d_{14} = 43 > d_{13} = 38, \quad \frac{(5-4)d_{14}}{2} = \frac{43}{2} > d_{55} = 3.$$

Hence, Proposition 6 verifies that the inequality is facet-defining.

Second, for $\ell = 4$, $S = \{2\}$, $A = \emptyset$, and $B = \{4\}$, inequality 3.2 reduces to

$$2C + 5y_4 \geq 43 - 43y_2 + 2x_2.$$

This inequality is also facet-defining by Proposition 6. The key additional condition for the case $B \neq \emptyset$ is satisfied because

$$\ell = 4 \leq n - 1, \quad B = \{\ell\}, \quad \frac{d_{1(\ell-1)}}{|L \setminus B|} = \frac{d_{13}}{2} = 19 > d_{44} = 5.$$

Therefore, both inequalities are facet-defining examples of the second group.

While the previous examples illustrate the construction and facet-defining nature of the proposed inequalities, the next example shows numerically how they strengthen the LP relaxation by excluding fractional feasible points.

Example (Numerical illustration of cut strength) Consider again the five-period instance with $\mathbf{d} = (11, 7, 20, 5, 3)$ and the fractional point

$$(\bar{C}, \bar{\mathbf{x}}, \bar{\mathbf{y}}) = \left(20, (20, 18, 8, 0, 0), \left(1, \frac{18}{35}, \frac{2}{7}, 0, 0 \right) \right),$$

which satisfies the LP relaxation of CALS.

For the first-group inequality with $\ell = 4$, $S = \{2\}$, $A = \{3\}$, $B = \{4\}$, we obtain $C + x_3 + 5y_4 + 43y_2 \geq 43 + x_2$, and substitution yields

$$20 + 8 + 43 \times \frac{18}{35} = \frac{1754}{35} < 61.$$

Hence, this inequality cuts off the fractional point.

For the second-group inequality with $\ell = 4$, $S = \{2, 3\}$, $A = \{4\}$, $B = \emptyset$, we obtain $4C + x_4 \geq 86 - 43(y_2 + y_3) + 2(x_2 + x_3)$, and substitution gives

$$80 < 86 - 43 \left(\frac{18}{35} + \frac{2}{7} \right) + 2(26) = \frac{518}{5}.$$

Thus, the second-group inequality also cuts off the same LP-relaxation point. This example numerically illustrates the strengthening effect of the proposed facet-defining inequalities.

4. Computational study

This section reports two complementary computational evaluations of the proposed valid inequalities. The first evaluation uses a C++/Concert branch-and-cut implementation with IBM ILOG CPLEX 12.6.0. The second evaluation uses a rebuilt Python implementation with IBM ILOG CPLEX Optimization Studio 22.1.2 and a pooled adaptive root-loop cut-management strategy. The purpose of reporting both evaluations is to assess the behavior of the proposed inequalities under different solver environments and cut-adding mechanisms. Because the two evaluations differ in solver version, modeling API, cut-management strategy, and time limit, CPU times should be interpreted within each evaluation rather than as direct cross-environment comparisons.

4.1. Branch-and-cut evaluation with CPLEX 12.6.0

Following the approach of Li and Meissner [34], we assume a planning horizon of 54 periods with varying demand patterns. The demand is defined as: $d_t = \beta_t \times \hat{d}$, where β_t represents the seasonality factor and $\hat{d} = 50$ is the mean demand. We consider five distinct demand patterns based on varying β_t for $t = 1, \dots, 54$: (I) Time-invariant: $\beta_t = 1$; (II) Linear growth: $\beta_t = 0.25 + \frac{1.5(t-1)}{53}$; (III) Linear decline: $\beta_t = 1.75 - \frac{1.5(t-1)}{53}$; (IV) Holiday season at the beginning of the planning horizon: $\beta_t = \frac{54}{114} + \frac{54(t-1)}{57}$ for $t = 1, \dots, 6$, $\frac{594}{114} - \frac{54(t-7)}{57}$ for $t = 7, \dots, 12$, and $\frac{54}{114}$ for $t = 13, \dots, 54$; (V) Holiday season

at the end of the planning horizon: $\beta_t = \frac{54}{114}$ for $t = 1, \dots, 42$, $\frac{54}{114} + \frac{54(t-43)}{57}$ for $t = 43, \dots, 48$, and $\frac{594}{114} - \frac{54(t-49)}{57}$ for $t = 49, \dots, 54$.

The fixed setup cost f_t is determined based on the time between orders (TBO), calculated as the EOQ-cycle time $\sqrt{\frac{2f_t}{hd_t}}$. Assuming a holding cost $h_t = 5$, the setup costs are set at $f_t = 1200$ for low TBO, $f_t = 3100$ for medium TBO, and $f_t = 8000$ for high TBO. Using the combination of the five demand patterns and three TBO levels, we generate 15 test cases. For each test case, we evaluate four methods: the default CPLEX MIP solver, CPLEX MIP solver with first-group (ℓ, S, A, B) cuts, CPLEX MIP solver with second-group (ℓ, S, A, B) cuts, and CPLEX MIP solver with both cuts. We assess these methods based on the CPU time required to find the optimal solution. Given the extensive computation time for most problems, a time limit of 3600 seconds is imposed. For cases where the optimal solution is not found within the time limit, we report the duality gap for the final incumbent solution. The results are summarized in Table 1.

Table 1. Comparison of CPLEX default cuts and user added cuts.

TBO	Demand Pattern	CPLEX Cuts		First-Group Cuts		Second-Group Cuts		Both Cuts	
		Gap	Time	Gap	Time	Gap	Time	Gap	Time
Low	DP1	0.00%	929.044	0.00%	380.164	0.00%	1895.39	0.00%	446.522
	DP2	0.00%	713.565	0.00%	1455.15	0.00%	446.651	0.00%	1746.74
	DP3	0.00%	7.8	0.00%	6.608	0.00%	4.894	0.00%	11.92
	DP4	0.00%	95.858	0.00%	55.454	0.00%	86.714	0.00%	11.541
	DP5	0.00%	1015.69	0.00%	219.228	0.00%	88.592	0.00%	257.269
Medium	DP1	2.45%	3600	1.98%	3600	2.62%	3600	1.88%	3600
	DP2	1.17%	3600	1.46%	3600	1.48%	3600	1.60%	3600
	DP3	0.00%	580.964	0.00%	278.574	0.00%	921.317	0.00%	456.295
	DP4	0.00%	2812.63	0.52%	3600	0.22%	3600	0.30%	3600
	DP5	1.74%	3600	0.83%	3600	1.34%	3600	1.12%	3600
High	DP1	3.27%	3600	3.03%	3600	3.78%	3600	3.74%	3600
	DP2	1.77%	3600	2.80%	3600	1.75%	3600	1.96%	3600
	DP3	0.00%	457.935	0.00%	444.374	0.00%	234.434	0.00%	411.469
	DP4	1.33%	3600	0.18%	3600	0.76%	3600	1.03%	3600
	DP5	3.33%	3600	2.12%	3600	3.55%	3600	2.12%	3600

In Table 1, we observed that our added cuts outperformed the default CPLEX cuts in 13 out of 15 test cases, demonstrating their effectiveness in solving CALS problems. Adding only the first group of cuts led to either shorter computational time or better solution quality within the time limit for 11 out of 15 test cases, while adding only the second group led to improvements in 8 out of 15 cases, and adding both groups led to improvements in 9 cases. The first group of (ℓ, S, A, B) cuts proved to be more effective and efficient than the second group, presumably due to the high separation complexity of the second group of (ℓ, S, A, B) cuts.

In cases where problems could not be solved to optimality within the time limit, the CPLEX solver achieved a better duality gap after incorporating our proposed cuts. For example, in the test cases with a medium TBO and demand pattern 1, after one hour of solving time, the CPLEX cuts resulted in a

2.45% duality gap, whereas our cuts reduced the gap to 1.88%. Although this improvement might seem modest, Figure 1 illustrates that our cuts enabled the solver to achieve better solution quality in less than 100 seconds, compared to 1000 seconds when using only CPLEX cuts.

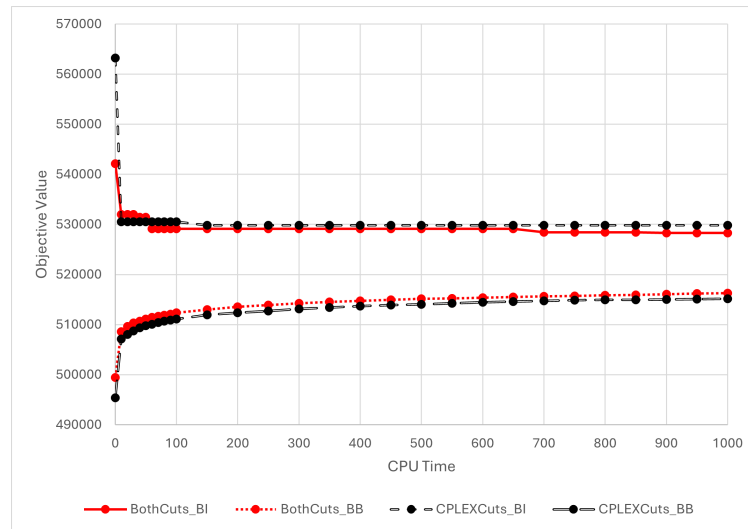


Figure 1. Solution process for TBO2D1.

In Table 2, we present the time required for our proposed cuts to achieve the same solution quality as the final solution obtained using only CPLEX cuts for the six test cases that could not be solved to optimality within 3600 seconds. In three out of the six test cases, our proposed cuts enabled the solver to reach the same solution in less than a quarter of the time limit. This result further confirms the effectiveness of our proposed cuts in solving CALS problems.

Table 2. Time used to reach the same solution for using user cuts.

TBO	Demand	No Cuts		First-Group Cuts		Second-Group Cuts	Both Cuts
	Pattern	Gap	Time	Time	Time	Time	Time
Medium	DP1	2.45%	3600	748.72			666.84
	DP2	1.17%	3600				
	DP5	1.74%	3600	705.48			839.55
High	DP1	3.27%	3600	2269.75			
	DP2	1.77%	3600		3437		
	DP5	3.33%	3600	584.7			663.27

4.2. Root-loop evaluation with CPLEX 22.1.2

We next report an additional evaluation using IBM ILOG CPLEX Optimization Studio 22.1.2. The same 15 core instances are used in order to preserve the structured benchmark design described above.

In this evaluation, cuts are generated through a pooled adaptive root-loop procedure. At each root-loop round, the LP relaxation is solved with the accumulated cuts. The separator then generates pools of violated first-group and/or second-group cuts, ranks them by violation, and adds a limited number of sufficiently strong cuts from each selected group. The root loop terminates when no new violated cuts

are found, when candidate violations are too weak, when the root bound stalls, or when the maximum number of root-loop rounds is reached. The final formulation is then solved with the accumulated root-loop cuts.

In addition to CPU time and final optimality status, we report the root gap before and after adding user cuts, the root-gap reduction, branch-and-bound nodes, and the number of user cuts added. The root gap is computed as

$$\text{Root gap} = 100 \times \frac{z^{\text{inc}} - z^{\text{root}}}{\max\{|z^{\text{inc}}|, 10^{-9}\}},$$

where z^{inc} is the incumbent objective function value and z^{root} is the root LP bound.

Table 3. Root-loop evaluation with CPLEX 22.1.2 on the 15 core instances.

Method	Solved	Time limit	Median time	Median nodes
Baseline	9	6	77.89	237,365
First	9	6	174.49	305,271
Second	9	6	53.63	142,490
Both	9	6	95.56	243,199

Method	Root gap before (%)	Root gap after (%)	Root-gap red. (%)	Mean cuts
Baseline	14.54	14.54	0.00	0.0
First	14.51	13.69	5.65	9.0
Second	14.53	11.27	22.43	37.6
Both	14.53	10.66	26.67	79.2

Table 3 shows that the proposed inequalities continue to strengthen the root relaxation in the CPLEX 22.1.2 evaluation. The second-group cuts reduce the average root gap from 14.53% to 11.27%, while using both cut families further reduces it to 10.66%. The second-group cuts also provide the best computational trade-off in this experiment: the median node count decreases from 237,365 to 142,490, and the median runtime decreases from 77.89 seconds to 53.63 seconds. The combined method gives the tightest root relaxation, but it adds substantially more cuts and does not dominate the second-group method in runtime.

The number of solved instances remains 9 out of 15 for all four methods. This indicates that the proposed cuts improve formulation strength and can reduce search effort, but they do not uniformly solve the hardest instances within the time limit. The difficult cases are concentrated in the medium- and high-setup-cost instances with time-invariant, linear-growth, and linear-decline demand patterns, while the holiday-pattern instances are comparatively easier.

These results also help explain the mixed computational performance observed across the two evaluations. A tighter root relaxation does not necessarily imply faster end-to-end MIP solution time. Adding user cuts requires separation time, increases the size of the LP relaxations, and can change the branch-and-cut trajectory of the solver. Therefore, the computational value of a cut group depends on the balance between bound improvement and cut-management overhead. In the CPLEX 22.1.2 root-loop implementation, the second-group cuts provide the most favorable balance, whereas the combined method produces the strongest root bound but requires many more cuts.

To further examine the role of cut-management choices, Appendix A.1 reports an implementation-sensitivity analysis comparing three root-loop cut-delivery strategies: a fixed single-cut root loop, an adaptive single-cut root loop, and the pooled adaptive root loop used in this section. Appendix A.2 reports a supplementary perturbed-demand robustness check. These additional results show that the proposed cuts consistently strengthen the root relaxation, although the resulting improvements in final MIP solution time are instance-dependent.

The computational results also provide several managerial insights for capacity planning in modern manufacturing systems. In particular, the improved solution quality and reduced computational time enabled by the proposed valid inequalities allow decision makers to evaluate capacity decisions more effectively, especially in environments with flexible or reconfigurable capacity. For example, faster convergence and tighter bounds make it possible to assess alternative capacity investment or utilization strategies within practical time limits. This is particularly important in large-scale planning problems with high product variety and demand uncertainty, where traditional solution approaches may struggle to provide timely and reliable decisions.

5. Extensions to stochastic and robust optimization

Although the CALS model studied in Section 2 is deterministic, it provides the foundational mathematical structure required to solve large-scale CALS problems under uncertainty, where demand uncertainty is incorporated through forecast updates, sampled demand scenarios, or explicitly defined uncertainty sets, and the resulting deterministic problem is repeatedly solved. Below, we discuss how the proposed branch-and-cut algorithm with the proposed valid inequalities can be extended to two-stage stochastic programming and robust optimization for stochastic CALS models.

5.1. Two-stage stochastic programming

To explicitly account for demand uncertainty, the CALS model can be extended to a two-stage stochastic program. Let Ω denote a finite set of scenarios, where each scenario $\omega \in \Omega$ specifies a demand vector $\mathbf{d}^\omega = (d_1^\omega, d_2^\omega, \dots, d_n^\omega)$ and a cost vector $(\mathbf{p}^\omega, \mathbf{f}^\omega) = ((p_1^\omega, f_1^\omega), (p_2^\omega, f_2^\omega), \dots, (p_n^\omega, f_n^\omega))$ over the planning horizon, with associated probability π_ω . In this setting, the capacity decision C is made in the first stage, prior to the realization of demand, reflecting its long-term and strategic nature. After the demand realization is observed, second-stage decisions are made for each scenario $\omega \in \Omega$, including production quantities x_t^ω and setup decisions y_t^ω for all periods $t \in N = [1, n]$.

The resulting deterministic equivalent formulation is as follows:

$$\min \quad \sum_{t=1}^n aC + \sum_{\omega \in \Omega} \pi_\omega \left(\sum_{t=1}^n (p_t^\omega x_t^\omega + f_t^\omega y_t^\omega) \right) \quad (5.1)$$

$$\text{s.t.} \quad \sum_{i=1}^t x_i^\omega \geq d_{1t}^\omega \quad t \in [1, n-1], \omega \in \Omega \quad (5.2)$$

$$\sum_{i=1}^n x_i^\omega = d_{1n}^\omega \quad \omega \in \Omega \quad (5.3)$$

$$x_t^\omega \leq d_{1n}^\omega y_t^\omega \quad t \in N, \omega \in \Omega \quad (5.4)$$

$$0 \leq x_t^\omega \leq C \quad t \in N, \omega \in \Omega \quad (5.5)$$

$$y_t^\omega \in \{0, 1\} \quad t \in N, \omega \in \Omega \quad (5.6)$$

$$C \in \mathbb{R}_+. \quad (5.7)$$

Model (5.1)–(5.7) consists of $|\Omega|$ scenario-specific copies of the original CALS model, in which the original constraints (2.3), (2.4), (2.5), (2.8), (2.10), and (2.11) are replicated for each scenario $\omega \in \Omega$,

while the capacity variable C is shared across all scenarios. The objective function (5.1) minimizes the expected total cost over all scenarios.

The branch-and-cut framework developed in Section 4 can therefore be naturally extended to this stochastic setting by separating violated (ℓ, S, A, B) cuts within each scenario. Specifically, the valid inequalities are generated by incorporating the capacity variable C and scenario-dependent counterparts of the original production variables (x_t, y_t) . With the same assumptions on ℓ, S, A , and B as in Theorems 1 and 4, the first- and second-group (ℓ, S, A, B) inequalities (3.1) and (3.2) are adapted as

$$C + \sum_{t \in A} x_t^\omega + \sum_{t \in B} d_{t\ell}^\omega y_t^\omega + d_{1\ell}^\omega \sum_{t \in S} y_t^\omega \geq d_{1\ell}^\omega + \sum_{t \in S} x_t^\omega$$

and

$$|S|(|L| - |A \cup B|)C + \sum_{t \in A} x_t^\omega + \sum_{t \in B} d_{t\ell}^\omega y_t^\omega \geq |S|d_{1\ell}^\omega - d_{1\ell}^\omega \sum_{t \in S} y_t^\omega + (|L| - |A \cup B|) \sum_{t \in S} x_t^\omega$$

for each scenario $\omega \in \Omega$ for the stochastic CALS model. Given that scenario-based formulations can become significantly larger and more computationally challenging, strengthening the linear programming relaxation through such inequalities is especially critical for improving computational efficiency.

The formulation presented above captures a baseline stochastic model. More general extensions may incorporate additional constraints and modeling features, such as chance constraints and service-level requirements. Importantly, the valid inequalities developed in this paper are based on the underlying polyhedral structure of the deterministic CALS model and do not depend on specific assumptions about the objective or uncertainty representation. As a result, they remain valid for these extended formulations and can be directly leveraged to strengthen their linear relaxations.

5.2. Robust optimization

Demand uncertainty can also be modeled within a robust optimization framework by assuming that the demand vector $\mathbf{d} = (d_1, d_2, \dots, d_n)$ belongs to a prescribed uncertainty set $\mathcal{U}^d$, and that cost vector $(\mathbf{p}, \mathbf{f}) = ((p_1, f_1), (p_2, f_2), \dots, (p_n, f_n))$ belongs to a prescribed uncertainty set $\mathcal{U}^{(p,f)}$. In this setting, the capacity decision C and the associated production and setup decisions must remain feasible for all realizations $\mathbf{d} \in \mathcal{U}^d$. The robust counterpart of the model is therefore obtained by requiring that constraints depending on demand be reformulated by replacing the corresponding demand-dependent expressions with their worst-case values over $\mathcal{U}^d$.

$$\min \max_{(\mathbf{p}, \mathbf{f}) \in \mathcal{U}^{(p,f)}} \sum_{t=1}^n (aC + p_t x_t + f_t y_t) \quad (5.8)$$

$$\text{s.t.} \quad \sum_{i=1}^t x_i \geq \max_{\mathbf{d} \in \mathcal{U}^d} d_{1t} \quad t \in [1, n-1] \quad (5.9)$$

$$\sum_{i=1}^n x_i = \max_{\mathbf{d} \in \mathcal{U}^d} d_{1n} \quad (5.10)$$

$$x_t \leq \max_{\mathbf{d} \in \mathcal{U}^d} d_{tn} y_t \quad t \in N \quad (5.11)$$

$$(2.4), (2.5), (2.8).$$

Similar to Section 5.1, the first- and second-group (ℓ, S, A, B) inequalities (3.1) and (3.2) can be adapted as follows by replacing demand with its worst-case value over the uncertainty set $\mathcal{U}^d$, under the same assumptions on ℓ, S, A and B as in Theorems 3 and 4.

$$C + \sum_{t \in A} x_t + \sum_{t \in B} \max_{\mathbf{d} \in \mathcal{U}^d} d_{t\ell} y_t + \max_{\mathbf{d} \in \mathcal{U}^d} d_{1\ell} \sum_{t \in S} y_t \geq \max_{\mathbf{d} \in \mathcal{U}^d} d_{1\ell} + \sum_{t \in S} x_t,$$

and

$$|S|(|L| - |A \cup B|)C + \sum_{t \in A} x_t + \sum_{t \in B} \max_{\mathbf{d} \in \mathcal{U}^d} d_{t\ell} y_t \geq |S| \max_{\mathbf{d} \in \mathcal{U}^d} d_{1\ell} - \max_{\mathbf{d} \in \mathcal{U}^d} d_{1\ell} \sum_{t \in S} y_t + (|L| - |A \cup B|) \sum_{t \in S} x_t.$$

In this basic robust formulation, uncertainty is incorporated in a simple manner by replacing demand parameters with their worst-case realizations over the uncertainty set. This leads to a tractable counterpart in which the original lot-sizing constraints are preserved, with only the demand terms adjusted to ensure feasibility over the uncertainty set $\mathcal{U}^d$. More generally, richer uncertainty representations may be considered by introducing additional modeling features, such as budgeted uncertainty sets, temporal correlation across periods, lead-time effects, service-level requirements, and capacity adjustment restrictions. These extensions may result in more complex expressions for worst-case demand and additional constraints in the formulation. However, the underlying structural relationships between capacity and production decisions remain unchanged. Since the valid inequalities developed in this paper are derived from these fundamental structural properties, rather than from specific assumptions on demand realizations, they remain valid for such extended robust formulations. In particular, the valid inequalities can be incorporated by evaluating the worst-case values of the demand-dependent expressions over the corresponding uncertainty set, thereby preserving their strengthening effect on the model.

6. Concluding remarks

In this paper, we proposed two classes of valid inequalities for the single-echelon lot-sizing problem with capacity acquisition, studied their strengths, developed polynomial separation algorithms, and presented computational results demonstrating the effectiveness of the proposed inequalities. Several directions for future research remain. For example, the complete linear description of $\text{conv}(X^{\text{CALS}})$ is still unknown. Other model variants may also be explored, including extensions with multiple echelons, varying capacity acquisition costs, and adjustable capacity levels.

The computational results show that the proposed inequalities materially strengthen the root relaxation and can reduce branch-and-cut effort. The effect on total solution time, however, is not uniform across cut families, instances, and cut-management strategies. Adding cuts requires separation time, increases LP size, and can alter the branch-and-cut trajectory of a modern MIP solver. Thus, the proposed inequalities should be viewed as formulation-strengthening devices whose practical performance depends on the implementation strategy. Designing more adaptive cut-selection and coordination strategies for the two cut families remains an important direction for future research.

Another important direction is to further investigate uncertainty-aware variants of CALS. Examples include multi-stage stochastic formulations, budgeted or correlated uncertainty sets, and models with service-level or chance constraints. Such extensions often lead to large-scale mixed-integer programs

or decomposition-based solution approaches. It would therefore be valuable to investigate how the polyhedral structure identified in this paper can be exploited in these more general settings, including through strengthened formulations, decomposition-enhanced cutting-plane methods, and separation procedures tailored to uncertainty-aware CALS models.

Data availability

The data that support the findings of this study are available on request from the corresponding author.

Declarations

The authors have no relevant financial or non-financial interests to disclose.

Conflict of interest

All authors declare no conflicts of interest in this paper.

Use of Generative-AI tools declaration

During the preparation of this work, the authors used generative-AI tools to assist with language polishing and formatting. The authors reviewed and edited all AI-assisted outputs and take full responsibility for the content of this publication.

Author contributions

Jia Guo: Conceptualization, methodology, formal analysis, software, investigation, and writing–review and editing.

Minjiao Zhang: Conceptualization, methodology, formal analysis, investigation, writing–original draft, and writing–review and editing.

All authors have read and approved the final manuscript.

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