



Research article

Dividend and capital injection control with liquidation under two-sided jump-diffusion models

Ningning Feng, Yixuan Liu and Ran Xu*

Department of Financial and Actuarial Mathematics, Xi'an Jiaotong-Liverpool University, Suzhou, 215123, China

* **Correspondence:** Email: ran.xu@xjtlu.edu.cn; Tel: +86-512-8188-9181;
Fax: +86-512-8188-9181.

Abstract: This paper investigates the optimal dividend and capital injection problem under a two-sided jump-diffusion model, incorporating the flexibility of liquidation when the surplus becomes negative. Unlike prior studies, we relax the constraint on capital injection, allowing decision-makers to choose between rescuing the firm via capital injection or liquidating it upon financial distress. We derive the dynamic programming principle and the Hamilton–Jacobi–Bellman (HJB) equation for this control problem. To overcome the analytical challenges posed by two-sided jumps, we use the Markov chain approximation (MCA) method, constructing a discrete-time controlled Markov chain that approximates the original jump-diffusion process. We show the convergence of the value function under MCA to the original value function through the weak convergence theory. We then solve the approximated problem numerically based on a value iteration algorithm. Extensive numerical examples illustrate the optimal strategies, showcasing the interplay among dividend payments, capital injections, and liquidation decisions.

Keywords: dividend; capital injection; liquidation; Markov chain approximation; jump-diffusion

Mathematics Subject Classification: 91G05, 93E20, 49M25

1. Introduction

The problem of optimal dividend policy was first introduced by [1] in the literature, which established that the optimal dividend strategy should maximize the expected discounted dividends paid until the surplus drops below zero (that is, until ruin occurs). Under a simple discrete risk model, De Finetti demonstrated that the optimal strategy should follow a barrier dividend strategy: A non-negative constant barrier such that the excess amount of the surplus above the barrier should be paid out as dividends to the shareholders. Subsequent research has shown that, in many cases, a barrier-type

dividend strategy remains optimal for maximizing the expected discounted dividend payments until ruin, while a threshold-type strategy becomes optimal when dividends are paid continuously with a bounded dividend density; see, for example, [2–10], and the references therein.

However, a key criticism of De Finetti's framework is its exclusive focus on shareholder returns (via dividend maximization) without accounting for solvency risks. This omission implies that under a barrier strategy, ruin occurs with probability one. To address this, subsequent work incorporated risk management constraints, such as capital injections, into the dividend problem. Dickson and Waters [11] pioneered this extension under the Cramér–Lundberg model, proposing that shareholders should inject capital to prevent ruin, thereby optimizing the difference between the expected net present values (NPVs) of dividends and injected capital (see also [12]). Studies under the Cramér–Lundberg model were extended to have proportional administration costs associated with each capital injection by [13] and to have fixed transaction costs by [14]. The optimal dividend and capital injection problems under a Brownian motion model were studied by [15]; and a similar dividend and capital injection problem with reinsurance is investigated by [16] as well, where diffusion approximation was applied to the underlying Cramér–Lundberg risk model. For more research on optimal dividend and capital injection problems under spectrally negative Lévy processes can refer to [17, 18], and the references therein. Similar studies also apply to the dual risk models in the literature; some related works include, among others, [19–22].

An alternative modification introduces a penalty payment at ruin, penalizing aggressive dividend strategies that trigger early ruin, where the optimal strategy maximizes the expected NPV of dividends while accounting for penalty payments upon ruin. This framework can allow liquidation (bankruptcy) as an admissible strategy, which can be optimal in certain scenarios. This concept aligns with the liquidation policies studied by [23], who discussed the set of sufficient conditions when a simple form of band strategies for dividends, reinvestment, and liquidation are optimal. Other related works in the literature include [24–29]. Recent works by [30] (restricted dividend density) and [31] (unrestricted dividend payments), studied affine penalty functions under the Cramér–Lundberg model, where ruin is forced when the surplus jumps below zero, but capital injections remain viable for positive but low surplus levels. In addition, Zhang and Xiong [32] studied the optimal dividend and capital injection for a perturbed classical risk model, where liquidation occurs with zero payoff when the surplus falls below a critical level; Han et al. [33], on the other hand, considered the finite-horizon optimal dividend and capital injection problem under a jump-diffusion model, where the bailout time-dependent dividend barrier was identified as the optimal strategy.

Meanwhile, the optimal dividend problem for processes with two-sided jumps (reflecting both gains and losses) has received increasing attention in the literature. Such models capture random investment returns or insurance claims more realistically. Prior studies on two-sided jump models without capital injection include [34–37]. More recently, Noba [38] incorporated capital injection, proving the optimality of double-barrier strategies for certain (two-sided jump) Lévy processes.

Despite the extensive literature on optimal dividend and capital injection problems, several important gaps remain unaddressed. First, while existing studies have examined dividend optimization under various risk models, the interplay between capital injection decisions and liquidation options in the presence of two-sided jump-diffusion processes remains largely unexplored. Moreover, previous frameworks often impose restrictive assumptions, such as mandating capital injection to avoid ruin or forcing liquidation upon a negative surplus, without allowing decision-makers the flexibility to choose between these alternatives. Hence, this paper investigates an optimal dividend and capital injection

control problem under a two-sided jump-diffusion model that incorporates the strategic flexibility of liquidation.

The contributions of this paper are summarized as the following two aspects.

(i) We extend the literature by studying the optimal dividend and capital injection control under a two-sided jump-diffusion model, with the feature of allowing liquidation when the surplus becomes negative. That is, the decision-maker can choose between capital injection and stopping by liquidation when the surplus process hits the range $(-\infty, 0]$ by downward jumps or diffusion. Unlike [31] (liquidation is mandatory for negative surplus) and [38] (capital injections are forced at ruin under a two-sided Lévy process), our model allows the decision-maker to choose between capital injection and liquidation. This flexibility better reflects real-world trade-offs between rescuing a firm and cutting losses with liquidation. On the other hand, our work extends the study in [32] by introducing a two-sided jump-diffusion model and a flexible liquidation payoff function, allowing decision-makers to weigh the costs of capital injection against the benefits of liquidation. In addition, we introduce a type of affine liquidation payoff function that is designed to have an economic connection to the classical optimal dividend and capital injection problem without liquidation (see Remark 2.2). This design of liquidation payoff generates interesting trade-offs between optimal dividend policy and capital retention (see the Discussion of Example 6.3).

(ii) Analytical challenges arise because the presence of two-sided jumps precludes the use of the powerful analytical tool of scale functions for the class of Lévy processes. Consequently, it becomes necessary to resort to numerical methods. We adopt the Markov chain approximation (MCA) method (see [39]), which constructs a discrete-time Markov chain to approximate the original jump-diffusion process. Our convergence analysis, grounded in weak convergence theory, rigorously guarantees the validity of this approximation. The approximated problem, driven by a locally consistent controlled Markov chain, is then solved numerically using the value iteration method. The flexibility of this numerical approach allows us to investigate optimal strategies across various scenarios and systematically examine how liquidation thresholds, model parameters, and claim distributions shape these strategies. Moreover, the MCA framework—combined with diverse value and policy iteration algorithms—can be seamlessly extended to address more complex models, such as jump-diffusion processes incorporating regime-switching.

The rest of the paper is organized as follows: In Section 2, we introduce the model and the formulation of the optimal dividend and capital injection control problem with liquidation, and also present some properties of the optimal value function. Section 3 is devoted to deriving the Hamilton–Jacobi–Bellman (HJB) equation associated with the control problem and establishing the verification theorem for the optimality of a certain double-barrier strategy. Section 4 describes the numerical procedures for implementing the MCA method. In Section 5, we provide a formal analysis of the convergence of the proposed MCA method. Various numerical examples are presented in Section 6 to illustrate the results of the optimal dividend, capital injection, and liquidation strategies. The paper is concluded in Section 7.

2. The model and problem formulation

Let us consider a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is the corresponding filtration that satisfies the usual conditions. Let the surplus without control evolve as a

jump-diffusion process,

$$X_t = x + ct + \sigma B_t + \sum_{i=1}^{N_t} Y_i, \quad (2.1)$$

where $x \geq 0$ is the firm's initial surplus at time zero, $c > 0$ can be interpreted as the fixed income rate (e.g., the premium for an insurance company). Here, $B = (B_t)_{t \geq 0}$ is a $\mathcal{F}_t$ standard Brownian motion, which describes the “small” changes as diffusion of the surplus process, $\sigma > 0$ is the corresponding volatility, $N = (N_t)_{t \geq 0}$ is a homogeneous Poisson process with intensity $\lambda > 0$, $\{Y_i\}_{i \geq 1}$ represents independent and identically distributed $\mathbb{R} \setminus \{0\}$ -valued random variables with cumulative distribution function F . We further assume that F admits a density function f which has the following form:

$$f(x) = pf_+(x)\mathbf{1}_{\{x>0\}} + (1-p)f_-(-x)\mathbf{1}_{\{x<0\}}, \quad p \in [0, 1], \quad (2.2)$$

where f_+ and f_- are two proper density functions of some positive random variables with a finite mean, and $\mathbf{1}_{\{\cdot\}}$ is the indicator function. Hence, the surplus process given in (2.1) is a jump-diffusion process with two-sided jumps.

We further introduce two control processes associated with dividend payments and capital injections. Let $L = (L_t)_{t \geq 0}$ and $R = (R_t)_{t \geq 0}$ denote the cumulative dividend and capital injection processes. Hence, the controlled surplus process is now expressed as

$$U_t := X_t - L_t + R_t, \quad t \geq 0. \quad (2.3)$$

Meanwhile, we assume that transaction costs are involved with both dividend and capital injection payments. In particular, whenever the company pays a unit of dividend, the shareholders can only receive ϕ_1 units ($0 < \phi_1 \leq 1$ and $1 - \phi_1$ can be interpreted as the tax rate or proportional transaction costs associated with dividend payments). On the other hand, when the shareholders want to inject one unit of capital into the surplus, they are required to pay $\phi_2 - 1$ units as transaction costs for some $\phi_2 > 1$. Note that ϕ_2 may also be interpreted as the cost per unit of capital injection (see e.g. [13]) for the scenario of penalized capital injection, or $\phi_2 - 1$ can be interpreted as proportional transaction costs.

A dividend and capital injection strategy $\pi = ((L_t)_{t \geq 0}, (R_t)_{t \geq 0})$ is said to be admissible if both $(L_t)_{t \geq 0}$ and $(R_t)_{t \geq 0}$ are non-decreasing, $\mathbb{F}$ -adapted càdlàg process with $L_{0-} = 0$, $R_{0-} = 0$, $L_t - L_{t-} \leq U_{t-}$, and $U_t \geq 0$ for all $t \geq 0$, and the expected NPVs of dividend payments and capital injections are finite. We let Π denote the set of such admissible strategies.

The performance function under an arbitrary dividend and capital injection strategy $\pi = ((L_t)_{t \geq 0}, (R_t)_{t \geq 0}) \in \Pi$ now can be expressed as

$$J(x; \pi) = \mathbb{E}_x \left[\int_{[0, \infty)} e^{-rt} (\phi_1 dL_t - \phi_2 dR_t) \right], \quad (2.4)$$

where $r > 0$ is the discount rate and $\mathbb{E}_x$ is the conditional expectation given that $U_{0-} = x$. Then, the corresponding value function can be defined as

$$V(x) := \sup_{\pi \in \Pi} J(x; \pi), \quad x \in \mathbb{R}. \quad (2.5)$$

Remark 2.1. The two-sided jump-diffusion process given in (2.1), where with the jump density follows (2.2), satisfies Assumption 2.1 in [38]. Hence, a candidate optimal dividend and capital injection strategy

for Problem (2.5) is, in general, a double-barrier strategy. In addition, since we require X to have no monotone path, we assume $p < 1$ when $\sigma = 0$; we will not directly study this degenerate case in this paper, but our model can approximate the bounded variation case by letting σ sufficiently small when implementing the numerical algorithm.

As discussed in [32], shareholders exhibit rational behavior by refraining from capital injection when the deficit becomes sufficiently severe. Building upon this framework, we extend the dividend and capital injection problem (2.5) by incorporating liquidation options. Specifically, when the surplus process enters the negative domain $(-\infty, 0]$ (either through downward diffusion or negative jumps), we introduce two alternative corporate actions: (i) Capital injection to maintain solvency above a certain optimal threshold; (ii) immediate liquidation with receipt of a terminal payoff. We model this liquidation payoff as an increasing, measurable function $P : (-\infty, 0] \rightarrow \mathbb{R}$.

We consider two distinct forms of liquidation payoff functions.

- (1) Affine liquidation payoff: $P(x) = K + \varphi x$, where $\varphi > 0$ and $K \in \mathbb{R}$, with $x \leq 0$ representing the deficit position at liquidation. This formulation admits several economic interpretations: When $K \leq 0$, the function operates as a pure penalty mechanism (see e.g., [25, 30]). For $\varphi \in (0, 1]$, shareholders bear partial responsibility for the deficit coverage; when $\varphi > 1$, the penalty includes an additional surcharge proportional to the deficit.
- (2) Exponential liquidation payoff: $P(x) = -Ae^{-ax}$ ($A, a > 0$) (see e.g., [29]), which inherently represents a penalty structure.

Remark 2.2. The relationship between $V(0)$ (the value function at zero surplus without liquidation as defined in (2.5)) and $P(0)$ (equal to K or $-A$) merits careful consideration. Our model assumes $V(0) \geq P(0)$, ensuring that immediate liquidation at zero surplus cannot dominate continued operation with capital injections. This reflects the economic intuition that the salvage value $P(0)$ when there is no surplus should not exceed the value of continuing operations $V(0)$. To this end, we define the relationship as $P(0) = \alpha V(0)$, where $\alpha \geq 0$ is a constant. A value $\alpha \leq 1$ is natural when $V(0) \geq 0$, while a value $\alpha > 1$ is considered if $V(0) < 0$, representing a scenario where liquidation adds a cost over a negative continuing operation value.

Following a similar analytical approach to [32], we derive bounds for the optimal liquidation threshold, denoted $-\underline{x}$. The derivation rests on the economic rationale that for any deficit $-\underline{x} \leq x < 0$, the value of an immediate capital injection (bringing surplus to zero) given by $V(0) + \phi_2 x$ should no less than the immediate liquidation value $P(x)$.

Proposition 2.1.

- (i) For $P(x) = K + \varphi x$ with $\varphi < \phi_2$, we have an upper bound associated with $\underline{x}$,

$$\underline{x} \leq \frac{\frac{1}{\Phi(r)} - \frac{\psi'(0^+)}{r} - K}{\phi_2 - \varphi}. \quad (2.6)$$

where

$$\psi(s) = \frac{1}{2}\sigma^2 s^2 - cs + \lambda p(\mathbb{E}[e^{-sY^+}] - 1), \quad \Phi(r) = \sup\{s \geq 0 : \psi(s) = r\},$$

and Y^+ is a positive random variable with density $f_+(\cdot)$ given in (2.2).

(ii) For $P(x) = K + \varphi x$ with $\varphi \geq \phi_2$, we have $\underline{x} = \infty$ and hence no liquidation.

(iii) For $P(x) = -Ae^{-ax}$ ($A, a > 0$) with $Aa \geq \phi_2$, we have $\underline{x} = \infty$ and hence no liquidation.

Proof. (i). Obviously, it is worth injecting capital when the surplus drops to $-\underline{x}$ if $V(0) - \phi_2 \underline{x} \geq P(-\underline{x})$. Hence, for an affine liquidation function, we have

$$\underline{x} \leq \frac{V(0) - K}{\phi_2 - \varphi}. \quad (2.7)$$

Let the initial surplus level $x = 0$, and consider any dividend and capital injection strategy $\pi \in \Pi$, we have $L_t - R_t \leq \overline{X}_t^+$, where

$$\overline{X}_t^+ = \sup_{s \leq t} \left(ct + \sigma W_t + \sum_{i=1}^{N_t^+} Y_i^+ \right),$$

N_t^+ is a Poisson process with intensity λp , $\{Y_i^+\}_{i \geq 1}$ are independently and identically distributed (i.i.d.) positive random variables with a density $f_+(\cdot)$. Therefore,

$$\begin{aligned} V(0) &= \mathbb{E} \left[\int_{[0, \infty)} e^{-rt} (\phi_1 dL_t - \phi_2 dR_t) \right] \\ &\leq r \mathbb{E} \left[\int_0^\infty e^{-rt} (L_t - R_t) dt \right] \\ &\leq r \mathbb{E} \left[\int_0^\infty e^{-rt} \overline{X}_t^+ dt \right] \\ &= \mathbb{E}[\overline{X}_{e_r}^+] = -\frac{\psi'(0^+)}{r} + \frac{1}{\Phi(r)}, \end{aligned}$$

where e_r is an independent exponential random variable with rate r . If we combine this with (2.7), we arrive at (2.6) in (i). The result in (ii) is obvious, since when $\varphi \geq \phi_2$, $V(0) + \phi_2 x$ will dominate $P(x)$ for all $x < 0$; hence $\underline{x} = \infty$. For (iii), under the assumption that $V(0) \geq P(0)$ for the exponential liquidation function in the form $-Ae^{-ax}$ with $Aa \geq \phi_2$, one immediately has $V(0) + \phi_2 x \geq -Ae^{-ax}$ for all $x < 0$; hence, there is no liquidation. $\square$

We note that if $Aa < \phi_2$ for the exponential liquidation function, the inequality $V(0) + \phi_2 x < -Ae^{-ax}$ may hold for deficits x near zero. However, this would imply a preference for liquidation even for minor deficits, which is inconsistent with the fundamental motivation of our model (i.e., shareholders choose liquidation rather than capital injection when the deficit becomes sufficiently severe). A comprehensive investigation of this scenario is reserved for future research.

Remark 2.3. Note that a typical example of (2.2) is the so-called double exponential distribution (see e.g., [40, 41]), that is

$$f(x) = pa_+ e^{-a_+ x} \mathbf{1}_{\{x > 0\}} + (1 - p)a_- e^{a_- x} \mathbf{1}_{\{x < 0\}}, \quad (2.8)$$

for some $a_+, a_- > 0$, $p \in (0, 1)$. In the later numerical studies, we will focus on the examples with double exponential distribution. But we also illustrate the case when the downward jump follows a Pareto distribution with a finite mean in (2.8), which may contribute to the scenario when the downward jumps are heavy-tailed compared with upward jumps. Note that, under (2.8), one can obtain a more

explicit bound in (2.6). To be specific $-\psi'(0^+) = c + \lambda p/a_+$, and $\Phi(r)$ is the unique positive solution of the equation

$$\left(\frac{1}{2}\sigma^2 s^2 - cs - r\right)(a_+ + s) - \lambda ps = 0.$$

We now define the liquidation time for a modified problem of (2.5). For any $\pi \in \Pi$, let

$$\tau^\pi := \inf\{t \geq 0 : U_t < -\underline{x}\}. \quad (2.9)$$

We then modify the value function as

$$V(x) = \sup_{\pi \in \Pi} \mathbb{E}_x \left[\int_{[0, \tau^\pi)} e^{-rt} (\phi_1 dL_t - \phi_2 dR_t) + e^{-r\tau^\pi} P(U_{\tau^\pi}) \right], \quad x \in \mathbb{R}. \quad (2.10)$$

Moreover, the existing literature (e.g., [16]) has established that when the marginal cost of capital injection exceeds unity (i.e., $\phi_2 > 1$) and the discount rate remains positive ($r > 0$), injecting capital prior to insolvency is suboptimal. Consequently, in our framework, optimal capital injection only occurs once the surplus the process enters negative domain. Hence, we have

$$V(x) = V(0) + \phi_2 x, \quad x \in [-\underline{x}, 0), \quad \text{and} \quad V(x) = P(x), \quad x \in (-\infty, -\underline{x}). \quad (2.11)$$

To proceed, we introduce the following condition satisfied by the liquidation function such that the value function in (2.10) is well-defined. For all $x \in \mathbb{R}$, the admissible control strategy $\pi = ((L_t)_{t \geq 0}, (R_t)_{t \geq 0})$, and the $\mathbb{F}$ -stopping time τ , we have

$$\mathbb{E}_x \left[e^{-r\tau} |P(U_\tau)| \right] < \infty. \quad (2.12)$$

Note that for an affine liquidation function, the condition in (2.12) is satisfied obviously (see e.g., [42]). For an exponential liquidation, one may consider extreme control with no dividends and maximum capital injection such that all negative jumps are compensated and the controlled process is reflected at zero. With the help of the optional sampling theorem and the Laplace exponent of such a controlled process, one can arrive at an upper bound for the value function which is finite if the parameter a in the exponential function is less than a positive constant that depends on the model parameters and the discount rate r . For more detailed derivations, one can refer to [29].

Proposition 2.2. The value function given in (2.10) is increasing. Moreover, (i) $\phi_1(x-y) \leq V(x) - V(y) \leq \phi_2(x-y)$ for any $0 \leq y \leq x$; (ii) a constant $C > 0$ exists such that $V(x) \leq x + C$; (iii) $V(x) \geq P(0)$ for all $x \geq 0$, and $V(x) \geq P(x)$ for all $x < 0$.

Proof. The proof of (i) follows the standard arguments of constructing an admissible strategy with an immediate dividend or capital injection, followed by an (ϵ) -optimal strategy (see e.g., [31]). Then the increasing property is direct result of (i). The linear upper bound in (ii) follows from the the upper bound for $V(0)$ in the proof of Proposition 2.1(i) and Proposition 2.2(i). Finally, (iii) holds obviously by the definition of the value function. $\square$

To end this section, we show that a class of double-barrier strategies is admissible (i.e., (2.10) is finite). Consider a class of double-barrier strategies $\pi^{(-\underline{x}, w)}$ for some $-\underline{x} \leq 0, w > 0$ defined as follows:

We pay dividends whenever the surplus exceeds the barrier w , and when the surplus hits zero by diffusion, we reflect the process with capital injection; however, when the surplus jumps below zero with a deficit less than $\underline{x}$, we inject capital and bring the surplus back to Level 0; otherwise, if the deficit exceeds $\underline{x}$, we stop by applying liquidation. We then have the following lemma, which shows that such double-barrier strategies are admissible.

Lemma 2.1. For any $-\underline{x} \leq 0$, $w > 0$ and $x \in \mathbb{R}$, we have,

$$J(x; \pi^{(-\underline{x}, w)}) = \mathbb{E}_x \left[\int_{[0, \tau^{\pi^{(-\underline{x}, w)}})} e^{-rt} (\phi_1 dL_t - \phi_2 dR_t) + e^{-r\tau^{\pi^{(-\underline{x}, w)}}} P(U_{\tau^{\pi^{(-\underline{x}, w)}}}) \right] < \infty.$$

Proof. The liquidation term is obviously finite by the condition (2.12). Let us define two auxiliary performance functions associated with dividends and capital injection, respectively.

$$J_1(x) := \int_{[0, \tau^{\pi^{(-\underline{x}, w)}})} e^{-rt} dL_t, \quad J_2(x) := \int_{[0, \tau^{\pi^{(-\underline{x}, w)}})} e^{-rt} dR_t.$$

We just need to prove that for $J_2(x) < \infty$, the result for J_1 is similar. By noting that the capital injection process $R = (R_t)_{t \geq 0}$ under the strategy $\pi^{(-\underline{x}, w)}$ is bounded above by the capital injection process; say, $\bar{R} = (\bar{R}_t)_{t \geq 0}$, under the double barrier strategy $\pi^{(0, w)}$ (i.e., with reflection below at level zero). Therefore, we have

$$J_2(x) \leq \mathbb{E}_x \left[\int_{[0, \infty)} e^{-rt} d\bar{R}_t \right] < \infty, \quad (2.13)$$

where the proof of the second inequality in (2.13) refers to [38, Lemma 3.2]. $\square$

3. The Hamilton–Jacobi–Bellman equation and the verification theorem

We first give the dynamic programming principle of our control problem (2.10). If we let T be any $\mathbb{F}$ -stopping time, then

$$V(x) = \sup_{\pi \in \Pi} \mathbb{E}_x \left[\int_{[0, T \wedge \tau^\pi)} e^{-rt} (\phi_1 dL_t - \phi_2 dR_t) + e^{-rT} V(U_T) \mathbf{1}_{\{T < \tau^\pi\}} + e^{-r\tau^\pi} P(U_{\tau^\pi}) \mathbf{1}_{\{T \geq \tau^\pi\}} \right]. \quad (3.1)$$

Throughout this paper, a function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is said to be *sufficiently smooth* if φ is twice continuously differentiable (continuously differentiable if $\sigma = 0$ in (2.1)). Then, for any function φ that is sufficiently smooth, we define an operator $\mathcal{L}$ by

$$\mathcal{L}\varphi(x) := \frac{\sigma^2}{2} \varphi''(x) + c\varphi'(x) - (\lambda + r)\varphi(x) + \lambda \int_{\mathbb{R} \setminus \{0\}} \varphi(x + y) dF(y). \quad (3.2)$$

Then the Hamilton–Jacobi–Bellman (HJB) equation associated with the problem in (2.10) is given, at least heuristically, as

$$\begin{cases} \max \{ \mathcal{L}V(x), \phi_1 - V'(x), V'(x) - \phi_2 \} = 0, & x \in (0, \infty), \\ V'(0) = \phi_2, \end{cases} \quad (3.3)$$

with the boundary condition in (2.11) for $x < 0$.

The proof follows the standard derivations in the control theory; here, we provide a heuristic argument for (3.3). We assume that the value function given in (2.10) is sufficiently smooth and satisfies (3.3), which can be rewritten as variational inequalities

$$\begin{aligned} \mathcal{L}V(x) &\leq 0, \quad \phi_1 - V'(x) \leq 0, \quad V'(x) - \phi_2 \leq 0, \\ \text{and } \mathcal{L}V(x)(\phi_1 - V'(x))(V'(x) - \phi_2) &= 0, \quad \text{for } x \in (0, \infty). \end{aligned}$$

Starting from a surplus level $x > 0$, if it is optimal to choose not to intervene in the process (i.e., no control action), then we have $\mathcal{L}V(x) = 0$, which is a direct result of applying Itô's formula. If the optimal control is to pay dividends immediately (for example, with the amount dx), then, according to the dynamic programming principle, we have $V(x) = V(x - dx) + \phi_1 dx$; hence, we have $V'(x) = \phi_1$; otherwise, $V(x) > V(x - dx) + \phi_1 dx$, i.e., $V'(x) > \phi_1$, given that the value function is differentiable at x . Similarly, if the optimal control is to inject capital immediately (say, with the amount of dx), we have $V(x) = V(x + dx) - \phi_2 dx$; hence, we have $V'(x) = \phi_2$; otherwise, $V(x) > V(x + dx) - \phi_2 dx$, and then we have $V'(x) < \phi_2$.

In the following analysis, we assume that the value function is sufficiently smooth and concave on $(0, \infty)$, according to Proposition 2.2, one notes that $\phi_1 \leq V'(x) \leq \phi_2$. Let

$$z := \sup\{x : V'(x) = \phi_2\} \quad \text{and} \quad b := \inf\{x : V'(x) = \phi_1\}. \quad (3.4)$$

Hence, b is the dividend barrier such that for all $x \geq b$, we have $V'(x) = \phi_1$ and $V(x) = V(b) + \phi_1(x - b)$. In addition, z is the post-capital injection level, such that for any $x \in [-\underline{x}, z]$, $V'(x) = \phi_2$, and $V(x) = V(z) - \phi_2(z - x)$. However, we shall prove in Proposition 3.1 below that $z = 0$, which is consistent with the conventional result without liquidation in the literature. Note that we just need to discuss the case in Proposition 2.1(i); otherwise, we have no liquidation.

According to the boundary conditions (2.11), one must have

$$P(-\underline{x}) = V(z) - \phi_2(z + \underline{x}).$$

In particular, for the case with affine liquidation discussed in Proposition 2.1(i), one can derive an explicit expression for $\underline{x}$

$$\underline{x} = \frac{V(z) - \phi_2 z - K}{\phi_2 - \varphi}.$$

Note that according to the Proposition 2.2(i) and Remark 2.2, we have $\underline{x} \geq 0$ for any $z \geq 0$.

Proposition 3.1. If V , as a solution of the HJB equation (3.3), is sufficiently smooth and concave, then, the post capital injection level z given by (3.4) equals zero.

Proof. We prove this by contradiction. Let $z > 0$. By noting that $\mathcal{L}V(0) \leq 0$, $\lim_{x \downarrow z} \mathcal{L}V(x) = 0$, and $V'(0) = V'(z) = \phi_2$, one can obtain the following (in)equalities:

$$\begin{aligned} c\phi_2 - (\lambda + r)V(0) + \lambda p \int_0^\infty V(y)f_+(y)dy + \lambda(1 - p) \int_0^{\underline{x}} V(-y)f_-(y)dy \\ + \lambda(1 - p) \int_{\underline{x}}^\infty P(-y)f_-(y)dy \leq 0, \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} c\phi_2 - (\lambda + r)V(z) + \lambda p \int_0^\infty V(z+y)f_+(y)dy + \lambda(1-p) \int_0^{\underline{x}+z} V(z-y)f_-(y)dy \\ + \lambda(1-p) \int_{\underline{x}+z}^\infty P(z-y)f_-(y)dy = 0, \end{aligned} \quad (3.6)$$

where $\underline{x} = (V(z) - \phi_2 z - K)/(\phi_2 - \varphi)$. Since $V(0) = V(z) - \phi_2 z$, by (3.5) and (3.6), we have

$$\begin{aligned} 0 &\geq c\phi_2 - (\lambda + r)(V(z) - \phi_2 z) + \lambda p \int_0^\infty V(y)f_+(y)dy \\ &\quad + \lambda(1-p) \int_0^{\underline{x}} V(-y)f_-(y)dy + \lambda(1-p) \int_{\underline{x}}^\infty P(-y)f_-(y)dy \\ &= (\lambda + r)\phi_2 z + \lambda p \int_0^\infty (V(y) - V(z+y))f_+(y)dy + \lambda(1-p) \int_0^{\underline{x}} (V(z) - \phi_2(y+z))f_-(y)dy \\ &\quad - \lambda(1-p) \int_0^{\underline{x}+z} (V(z) - \phi_2 y)f_-(y)dy \\ &\quad + \lambda(1-p) \int_{\underline{x}}^\infty (K - \varphi y)f_-(y)dy - \lambda(1-p) \int_{\underline{x}+z}^\infty (K - \varphi(y-z))f_-(y)dy \\ &\geq (\lambda + r)\phi_2 z - \lambda p \phi_2 z + \lambda(1-p) \int_0^{\underline{x}} (V(0) - \phi_2 y)f_-(y)dy - \lambda(1-p) \int_0^{\underline{x}+z} (V(0) - \phi_2 y + \phi_2 z)f_-(y)dy \\ &\quad + \lambda(1-p) \int_{\underline{x}}^{\underline{x}+z} (K - \varphi z)f_-(y)dy - \lambda(1-p)\varphi z(1 - F_-(\underline{x} + z)) \\ &= r\phi_2 z + \lambda(1-p)(\phi_2 - \varphi)z(1 - F_-(\underline{x} + z)) - \lambda(1-p) \int_{\underline{x}}^{\underline{x}+z} (V(0) - K - (\phi_2 - \varphi)y)f_-(y)dy \\ &\geq r\phi_2 z > 0, \end{aligned}$$

which is a contradiction. Hence, we must have $z = 0$. Note that in the second-last inequality above, we have applied the fact that $V(0) - K - (\phi_2 - \varphi)y \leq 0$ for all $y \geq \underline{x}$. $\square$

To proceed with the numerical solution, we first provide a verification theorem for the optimality of a class of double-barrier strategies when we have a classical solution to the HJB equation (3.3), and we further characterize the value function as the smallest solution.

Proposition 3.2. Let v be a solution of the HJB equation (3.3) with the boundary condition (2.11), which is increasing, concave, linearly bounded, and $\phi_1 \leq v' \leq \phi_2$. Define

$$b := \inf\{x \geq 0 : v'(x) = \phi_1\}.$$

We consider the strategy $\pi^* = ((L_t^*))_{t \geq 0}, (R_t^*)_{t \geq 0} \in \Pi$ that pays a dividend for any surplus exceeding the level b and injects capital if the surplus is between $[-\underline{x}, 0]$; otherwise, if $x < -\underline{x}$, we liquidate immediately. We then have

$$V(x) = J(x, \pi^*) = v(x).$$

Proof. Note that we only need to show the result for $x \in [0, b]$; otherwise, it holds obviously.

We first prove that $v(x) = J(x, \pi^*) \leq V(x)$. Let U^* be the controlled surplus process under the strategy π^* . Define $\Lambda_t^{L^*} = \{s \leq t : L_{s-}^* \neq L_s^*\}$ as the set of all jumps in the cumulative dividend process L^* up to time t , and let $\hat{L}_t^* = \sum_{s \in \Lambda_t^{L^*}} (L_s^* - L_{s-}^*)$ be the discontinuous part of L_t^* and $\tilde{L}_t^* = L_t^* - \hat{L}_t^*$ be the continuous

part of L_t^* . Similarly, $\Lambda_t^{R*} = \{s \leq t : R_{s-}^* \neq R_s^*\}$ denotes the set of jump time points of the cumulative capital injection process R up to time t , and $\widetilde{R}_t^*$ is the corresponding continuous part of R_t^* . Then, by Itô's formula, we have

$$\begin{aligned} e^{-r(t \wedge \tau^{\pi*})} v(U_{t \wedge \tau^{\pi*}}^*) &= v(x) + \int_0^{t \wedge \tau^{\pi*}} e^{-rs} \left(\frac{1}{2} \sigma^2 v''(U_{s-}^*) + cv'(U_{s-}^*) - rv(U_{s-}^*) \right) ds \\ &\quad + \int_0^{t \wedge \tau^{\pi*}} e^{-rs} \sigma v'(U_{s-}^*) dB_s - \int_0^{t \wedge \tau^{\pi*}} e^{-rs} v'(U_{s-}^*) (d\widetilde{L}_s^* - d\widetilde{R}_s^*) \\ &\quad + \sum_{s \in \Lambda_{t \wedge \tau^{\pi*}}^{L*} \cup \Lambda_{t \wedge \tau^{\pi*}}^{R*}} e^{-rs} (v(U_s^*) - v(U_{s-}^*)) \\ &\quad + \sum_{0 \leq T_i \leq t \wedge \tau^{\pi*}} e^{-rT_i} (v(U_{T_i}^* + Y_i) - v(U_{T_i}^*)), \end{aligned} \quad (3.7)$$

where $\{T_i\}_{i \geq 1}$ represents the sequence of the arrival time of the jumps, and $\{Y_i\}_{i \geq 1}$ represents the corresponding jump sizes. Note that we assume that the dividend and the capital injection are implemented after the jumps of the surplus. For the time s when dividends are paid, we have $v'(U_s^*) = \phi_1$, and for the time s when capital is injected, we have $v'(U_s^*) = \phi_2$, and then we obtain

$$\begin{aligned} v(U_s^*) - v(U_{s-}^*) &= -\phi_1 \Delta L_s^*, \quad \text{for } \forall s \in \Lambda_{t \wedge \tau^{\pi*}}^{L*}, \\ v(U_s^*) - v(U_{s-}^*) &= \phi_2 \Delta R_s^*, \quad \text{for } \forall s \in \Lambda_{t \wedge \tau^{\pi*}}^{R*}, \end{aligned}$$

where ΔL^* and ΔR^* denote the jump of the dividend and capital injection process.

Hence, by taking expectation on both sides of (3.7), and with the help of the compensate Poisson random measure, and the martingale property of Itô's integral, we have

$$\begin{aligned} \mathbb{E}_x \left[e^{-r(t \wedge \tau^{\pi*})} v(U_{t \wedge \tau^{\pi*}}^*) \right] &= v(x) + \mathbb{E}_x \left[\int_0^{t \wedge \tau^{\pi*}} e^{-rs} \left(\frac{1}{2} \sigma^2 v''(U_{s-}^*) + cv'(U_{s-}^*) - rv(U_{s-}^*) \right) ds \right] \\ &\quad + \int_0^{t \wedge \tau^{\pi*}} \int_{\mathbb{R} \setminus \{0\}} \lambda e^{-rs} (v(U_{s-}^* + y) - v(U_{s-}^*)) f(y) dy ds \\ &\quad - \mathbb{E}_x \left[\int_0^{t \wedge \tau^{\pi*}} e^{-rs} (\phi_1 dL_s^* - \phi_2 dR_s^*) \right]. \end{aligned}$$

With the fact that $\mathcal{L}v(x) = 0$ for any $x \in [0, b]$, we arrive at

$$v(x) = \mathbb{E}_x \left[\int_0^{t \wedge \tau^{\pi*}} e^{-rs} (\phi_1 dL_s^* - \phi_2 dR_s^*) + e^{-r(t \wedge \tau^{\pi*})} v(U_{t \wedge \tau^{\pi*}}^*) \right].$$

By letting $t \rightarrow \infty$ with the help of dominated and monotone convergence theorems, we arrive at

$$\begin{aligned} v(x) &= \lim_{t \rightarrow \infty} \mathbb{E}_x \left[\int_0^{t \wedge \tau^{\pi*}} e^{-rs} (\phi_1 dL_s^* - \phi_2 dR_s^*) + e^{-r(t \wedge \tau^{\pi*})} v(U_{t \wedge \tau^{\pi*}}^*) \right] \\ &= \mathbb{E}_x \left[\int_0^{\tau^{\pi*}} e^{-rs} (\phi_1 dL_s^* - \phi_2 dR_s^*) + e^{-r\tau^{\pi*}} v(U_{\tau^{\pi*}}^*) \right] \end{aligned}$$

$$= \mathbb{E}_x \left[\int_0^{\tau^{\pi^*}} e^{-rs} (\phi_1 dL_s^* - \phi_2 dR_s^*) + e^{-r\tau^{\pi^*}} P(U_{\tau^{\pi^*}}^*) \right] = J(x, \pi^*) \leq V(x).$$

For the other direction of the inequality, we consider any admissible strategy $\pi = ((L_t)_{t \geq 0}, (R_t)_{t \geq 0}) \in \Pi$, and the associated controlled surplus process U . With a similar analysis as above, one can obtain

$$\begin{aligned} \mathbb{E}_x \left[e^{-r(t \wedge \tau^\pi)} v(U_{t \wedge \tau^\pi}) \right] &= v(x) + \mathbb{E}_x \left[\int_0^{t \wedge \tau^\pi} e^{-rs} \mathcal{L}v(U_{s^-}) ds \right] \\ &\quad - \mathbb{E}_x \left[\int_0^{t \wedge \tau^\pi} e^{-rs} v'(U_{s^-}) (d\tilde{L}_s - d\tilde{R}_s) \right] \\ &\quad - \mathbb{E}_x \left[\sum_{s \in \Lambda_{t \wedge \tau^\pi}^L} e^{-rs} \int_0^{\Delta L_s} v'(U_{s^-} - u) du - \sum_{s \in \Lambda_{t \wedge \tau^\pi}^R} e^{-rs} \int_0^{\Delta R_s} v'(U_{s^-} + w) dw \right]. \end{aligned}$$

With some simple algebra, one obtains

$$\begin{aligned} \mathbb{E}_x \left[\int_0^{t \wedge \tau^\pi} e^{-rs} (\phi_1 dL_s - \phi_2 dR_s) + e^{-r(t \wedge \tau^\pi)} P(U_{t \wedge \tau^\pi}) \right] \\ &= v(x) + \mathbb{E}_x \left[\int_0^{t \wedge \tau^\pi} e^{-rs} \mathcal{L}v(U_{s^-}) ds \right] \\ &\quad + \mathbb{E}_x \left[\int_0^{t \wedge \tau^\pi} e^{-rs} (\phi_1 - v'(U_{s^-})) d\tilde{L}_s \right] + \mathbb{E}_x \left[\int_0^{t \wedge \tau^\pi} e^{-rs} (v'(U_{s^-}) - \phi_2) d\tilde{R}_s \right] \\ &\quad + \mathbb{E}_x \left[\sum_{s \in \Lambda_{t \wedge \tau^\pi}^L} e^{-rs} \int_0^{\Delta L_s} (\phi_1 - v'(U_{s^-} - u)) du + \sum_{s \in \Lambda_{t \wedge \tau^\pi}^R} e^{-rs} \int_0^{\Delta R_s} (v'(U_{s^-} + w) - \phi_2) dw \right] \\ &\quad + \mathbb{E}_x \left[e^{-r(t \wedge \tau^\pi)} (P(U_{t \wedge \tau^\pi}) - v(U_{t \wedge \tau^\pi})) \right]. \end{aligned}$$

Since v satisfies the HJB equation (3.3), along with the prescribed boundary conditions (2.11), all expectation terms on the right-hand side of the equation must be non-positive. This property directly implies

$$J(x, \pi) = \mathbb{E}_x \left[\int_0^{t \wedge \tau^\pi} e^{-rs} (\phi_1 dL_s - \phi_2 dR_s) + e^{-r(t \wedge \tau^\pi)} P(U_{t \wedge \tau^\pi}) \right] \leq v(x), \quad (3.8)$$

and by the arbitrariness of the strategy π , we have $V(x) \leq v(x)$. $\square$

The preceding analysis in (3.8) yields the following characterization of the value function.

Proposition 3.3. The value function V given in (2.10) is the smallest solution of the HJB equation (3.3) with the boundary conditions (2.11).

In subsequent sections, we numerically solve the HJB equation (3.3) subject to the boundary condition in (2.11), where the liquidation threshold (if this exists) is defined as

$$\underline{x} = \frac{V(0) - K}{\phi_2 - \varphi}. \quad (3.9)$$

According to Proposition 3.2, we implement an upper boundary condition $V'(B) = \phi_1$ at a sufficiently large threshold B . This formulation leads to the following system of equations for the numerical solution:

$$\begin{cases} \max \{ \mathcal{L}V(x), \phi_1 - V'(x), V'(x) - \phi_2 \} = 0 & x \in (0, B), \\ V'(0) = \phi_2, \quad V'(B) = \phi_1. \end{cases} \quad (3.10)$$

with the boundary conditions (2.11); that is, $V(x) = V(0) - \phi_2 x$ for $x \in [-\underline{x}, 0)$, and $V(x) = P(x)$ for $x < -\underline{x}$, where $\underline{x}$ is given by (3.9). The robustness of the choices of B will be examined numerically in Section 6 (see Example 6.4).

4. Numerical algorithm

In this section, we solve the control problem (2.10) and the associated HJB equation (3.10) numerically using the MCA method (see, e.g., [43]). The MCA methods with value/policy iteration algorithm have been widely applied to various stochastic control problems, including optimal dividend problems. For instance, Jin et al. [44] used this approach to study optimal reinsurance and dividend payout strategies under regime-switching diffusion models. Their work involved constructing a locally consistent discrete-time controlled Markov chain to approximate the underlying risk process. By discretizing the associated HJB equation via finite differences and applying the dynamic programming principle, they developed a value iteration algorithm to compute the value function and optimal strategy. Similar applications can be found in [45] and [46] for problems involving capital injection and investment strategies; see also [47] for an application in the portfolio selection problem, and [48] for an application in the mean-field game in the queueing problem. Some more recent extensions and applications include, among others, [49](consumption and investment), [50](optimal insurance), and [51](optimal control in economics).

Given the well-established theoretical foundations of the MCA method, including its proven convergence properties (see Section 5), we adapt this framework to solve our optimal dividend and capital injection problem with liquidation. Specifically, our numerical approach involves three key steps.

1. Surplus dynamics approximation

We approximate the controlled continuous time (two-sided) jump-diffusion model with a discrete-time controlled Markov chain, ensuring local consistency with the original dynamics (Eq (4.6) and Definition 4.1). This involves the following

- Discretizing the state space with a step size h ;
- Deriving the transition probabilities and interpolation intervals that match the drift and diffusion terms (Eqs (4.17)–(4.20));
- Incorporating two-sided Poisson jumps via properties of the Poisson process (Definition 4.1).

2. Dynamic programming equation

Using finite differences, we derive the discretized dynamic programming equation (Eq (4.12)) for the approximate value function (Eq (4.11)).

3. Numerical solution via value iteration

For a given step size h , we iteratively compute the value function V_n^h until convergence using the value iteration method.

The MCA method guarantees convergence of the discrete-time controlled Markov chain to the original jump-diffusion process by using locally consistent transition probabilities, as in (4.17)–(4.20). This convergence is theoretically validated through weak convergence theory (Propositions 5.1–5.3), where the interpolated process (5.12) replicates the dynamics of the original surplus model (2.3). Leveraging the Skorohod representation theorem, we further establish the convergence of the value function (Propositions 5.4–5.5). Finally, numerical experiments (Figure 1(b)) confirm the convergence of the value iteration algorithm used to compute the approximate value function under the MCA framework.

To formally construct the numerical approximation, we first revisit the surplus process given by (2.1) in the form of jump arrival times and jump sizes. Let $\nu_0 = 0$ and let $\{\nu_n, n \geq 1\}$ be the sequence of the arrival time of the jumps, and $\{Y_n, n \geq 1\}$ is the corresponding sequence of jump sizes. Let $\omega_n = \nu_n - \nu_{n-1}$ for $n \geq 1$ be the waiting time of the n th jump, and $\{\omega_n, n \geq 1\}$ is a sequence of mutually independent random variables following a common exponential distribution with mean $1/\lambda$, and is independent of $\{Y_n, n \geq 1\}$. Meanwhile, we assume that $\{\omega_k, Y_k, k \geq n\}$ is independent of X_s for all $s < \nu_n$, ω_k , and Y_k , for all $k < n$. Then the aggregate jump amounts denoted as S_t can be written as

$$S_t = \sum_{\nu_n \leq t} Y_n.$$

We focus on the local nature of jumps for the surplus process (2.1). Since the waiting times $\{\omega_k\}_{k \geq 1}$ are exponentially distributed, we see that

$$\mathbb{P}(\text{jump happens on } [t, t + \Delta t) \mid X_s, N_s, s \leq t) = \lambda \Delta t + o(\Delta t). \quad (4.1)$$

Therefore, according to the definition and independence of Y_n , for any $y \in \mathbb{R} \setminus \{0\}$, we have

$$\begin{aligned} \mathbb{P}(X_t \in (x + y, x + y + dy) \mid t = \nu_n \text{ for some } n; X_s, N_s, s < t; X_{t-} = x) \\ = (pf_+(y)\mathbf{1}_{\{y>0\}} + (1-p)f_-(-y)\mathbf{1}_{\{y<0\}})dy. \end{aligned} \quad (4.2)$$

In the following, we first consider the local behavior of the controlled surplus process (2.3) between two consecutive jumps, which is essentially a controlled diffusion process given as

$$dU_t = cdt + \sigma dB_t - dL_t + dR_t, \quad \nu_{n-1} \leq t < \nu_n, \quad n = 1, 2, \dots \quad (4.3)$$

We then construct a discrete-time controlled Markov chain that is locally consistent (we define local consistency later in this section) with (4.3). Let $h > 0$ be a discretization parameter representing one step-size of the surplus. Define $P_h = \{x : x = kh, k = 0, \pm 1, \pm 2, \dots\} \cap G$, where $G = (-\infty, B]$. Without loss of generality, we assume that B is divisible by h . We define $\{\xi_n^h, n < \infty\}$ as a controlled discrete-time Markov chain on P_h and denote its transition probability from a state x to another state y under the control π^h by $p_D^h(x, y | \pi^h)$, where we introduce π^h later. Our objective is to identify suitable values for $p_D^h(x, y | \pi^h)$ so that the constructed discrete-time Markov chain can approximate the local nature of the controlled diffusion process (4.3) arbitrarily closely when $h \rightarrow 0$. Since, at any discrete time n , we can

either pay dividends, inject capital, or continue with no action when $\xi_n^h \in (0, B)$, and inject capital or stop with liquidation when $\xi_n^h \leq 0$. We then denote the one-step increment of the discrete-time Markov chain ξ_n^h as $\Delta\xi_n^h = \xi_{n+1}^h - \xi_n^h$, with

$$\begin{aligned} \Delta\xi_n^h = & \Delta\xi_n^h \mathbf{1}_{\{\text{pay dividends at } n\}} + \Delta\xi_n^h \mathbf{1}_{\{\text{inject capital at } n\}} \\ & + \Delta\xi_n^h \mathbf{1}_{\{\text{continue at } n\}}, \text{ for } \xi_n^h \in (0, B), \end{aligned} \quad (4.4)$$

and

$$\Delta\xi_n^h = \Delta\xi_n^h \mathbf{1}_{\{\text{inject capital at } n\}} + \Delta\xi_n^h \mathbf{1}_{\{\text{stop at } n\}}, \quad \text{for } \xi_n^h \in (-\infty, 0]. \quad (4.5)$$

Note that the Markov chain and the control will be determined to ensure that the sum of the indicator functions above equals one both in (4.4) and (4.5); i.e., there is one and only one nonzero term in (4.4) and (4.5), respectively. Denote $\{\pi_n^h : n = 0, 1, \dots\}$ as a sequence of control actions, where $\pi_n^h = 0$ represents the case of continuation at time n (during the period from time n to $n + 1$), $\pi_n^h = 1$ means that we inject capital at time n , $\pi_n^h = 2$ is the case that we pay out dividends at time n , and $\pi_n^h = 3$ represents the scenario of stopping by liquidation at time n .

For the case when $\pi_n^h = 1$, let Δg_n^h denote the corresponding capital amount injected at time n ; that is $\Delta\xi_n^h = \Delta g_n^h$. On the other hand, when $\pi_n^h = 2$, or $\xi_n^h = B$, a certain amount of dividend is paid out, such that the surplus level is decreased by one step compared with the level before the dividend payment; in particular, when $\xi_n^h = B$, we require an immediate dividend payment with an amount h such that the Markov chain is reflected at Level B . Let us denote Δl_n^h as the amount of dividend paid out at time n when $\pi_n^h = 2$, and then $\Delta\xi_n^h = -\Delta l_n^h$. In addition, when $\xi_n^h \leq 0$ and $\pi_n^h = 3$, we stop immediately, and hence the surplus will stay at the same value of ξ_n^h . Finally, if $\pi_n^h = 0$ when $\xi_n^h > 0$, then we continue with no action. Let $\tilde{\Delta}^h(\cdot) > 0$ be the interpolation interval on P_h such that $\inf_x \tilde{\Delta}^h(x) > 0$ for each $h > 0$ and all $x \in P_h$, and $\lim_{h \rightarrow 0} \sup_{x \in P_h} \tilde{\Delta}^h(x) \rightarrow 0$.

Furthermore, let $\mathbb{E}_{x,n}^{h,0}$ and $\text{Var}_{x,n}^{h,0}$ denote the conditional expectation and conditional variance given $\{\xi_k^h, k \leq n; \pi_k^h, k \leq n-1\}$, $\xi_n^h = x$, and $\pi_n^h = 0$, respectively. Then, the sequence $\{\xi_n^h, n \geq 0\}$ is said to be *locally consistent* with respect to (4.3) if it satisfies

$$\begin{aligned} \mathbb{E}_{x,n}^{h,0}(\Delta\xi_n^h) &= c\tilde{\Delta}^h(x) + o(\tilde{\Delta}^h(x)), \\ \text{Var}_{x,n}^{h,0}(\Delta\xi_n^h) &= \sigma^2\tilde{\Delta}^h(x) + o(\tilde{\Delta}^h(x)), \\ \sup_{n,\omega \in \Omega} |\Delta\xi_n^h(\omega)| &\rightarrow 0 \quad \text{as } h \rightarrow 0. \end{aligned} \quad (4.6)$$

Note that, due to the nature of singular control, the capital injection and dividend payout can be treated as “instantaneous”. Hence, we can extend the interpolation interval above to the space with all control actions when $x > 0$, which becomes

$$\Delta t^h(x, i) = \tilde{\Delta}^h(x) \mathbf{1}_{\{i=0\}}, \text{ for any } (x, i) \in P_h \cap (0, B) \times \{0, 1, 2\}. \quad (4.7)$$

For $x \leq 0$, all control actions are “instantaneous”, and we do not need to define any interpolation interval there. Let $\pi^h := \{\pi_n^h; n \geq 0\}$ be a sequence of control strategies that belong to Π^h , which is a set of piecewise constant admissible controls. The sequence π^h is regarded as admissible control under the discretization step h if π_n^h is $\sigma\{\xi_0^h, \dots, \xi_n^h, \pi_0^h, \dots, \pi_{n-1}^h\}$ -adapted for all n , where $\sigma\{\xi_0^h, \dots, \xi_n^h, \pi_0^h, \dots, \pi_{n-1}^h\}$ is the σ -algebra generated by the sequence of random variables $\{\xi_i^h, i \leq n\}$.

and $\{\pi_i^h, i \leq n-1\}$. We then introduce the transition probability associated with the controlled Markov chain $\{\xi_n^h, n \geq 0\}$, namely that for any $E \in \mathcal{B}(P_h)$ (Borel set of P_h), we have

$$\mathbb{P}(\xi_{n+1}^h \in E \mid \sigma\{\xi_0^h, \dots, \xi_n^h, \pi_0^h, \dots, \pi_n^h\}) = p_D^h(\xi_n^h, E \mid \pi_n^h).$$

At the upper boundary, we have a reflection, that is

$$\mathbb{P}(\xi_{n+1}^h = B - h \mid \xi_n^h = B, \sigma\{\xi_0^h, \dots, \xi_n^h, \pi_0^h, \dots, \pi_n^h\}) = p_D^h(B, B - h \mid \pi_n^h) = 1.$$

Put

$$t_0^h := 0, \quad t_n^h := \sum_{k=0}^{n-1} \Delta t^h(\xi_k^h, \pi_k^h), \quad n^h(t) := \max\{n : t_n^h \leq t\}.$$

Then the piecewise constant interpolations of the controlled Markov chain and the two control processes, denoted, respectively, by ξ^h , g^h , and l^h , which are naturally defined as follows for any $t \in [t_n^h, t_{n+1}^h)$:

$$\xi_t^h = \xi_n^h, \quad g_t^h = \sum_{k \leq n^h(t)} \Delta g_k^h \mathbf{1}_{\{\pi_k^h=1\}}, \quad l_t^h = \sum_{k \leq n^h(t)} \Delta l_k^h \mathbf{1}_{\{\pi_k^h=2\}}. \quad (4.8)$$

Next, we extend the controlled Markov Chain to be one that is locally consistent with the controlled jump-diffusion model given in (2.3). That is, the value of the Markov chain ξ^h in time t_{n+1}^h can be expressed according to the following two scenarios.

- (i) No jumps during the time interval $[t_n^h, t_{n+1}^h)$ with the probability $(1 - \lambda \Delta t^h(x, 0) + o(\Delta t^h(x, 0)))$; one can determine ξ_{n+1}^h by the transition probability p_D^h of the controlled diffusion case given above.
- (ii) There is one jump during the time interval $[t_n^h, t_{n+1}^h)$ with the probability $\lambda \Delta t^h(x, 0) + o(\Delta t^h(x, 0))$; one can determine ξ_{n+1}^h by $\xi_{n+1}^h = \xi_n^h + y_h$, where $y_h \in P_h$ is the nearest value of the jump size y such that $\xi_{n+1}^h \in P_h$, where y has a density function $f(y)$. It is obvious that $|y_h - y| \rightarrow 0$ as $h \rightarrow 0$ uniformly in x .

Let H_n^h denotes the event that there is no Poisson jump in the period $[t_n^h, t_{n+1}^h)$ so that ξ_{n+1}^h is determined by Scenario (i) above. K_n^h denotes the event of the second case (ii), where there is one jump occurring in the period. We then have $\mathbf{1}_{H_n^h} + \mathbf{1}_{K_n^h} = 1$. Hence, one can redefine the local consistency of a controlled Markov chain as shown below.

Definition 4.1. A controlled Markov chain $\{\xi_n^h, n \geq 0\}$ is said to be locally consistent with (2.3), if one can find an interpolation interval $\Delta t^h(x, 0) \rightarrow 0$, uniformly in x , as $h \rightarrow 0$ such that

- (i) A transition probability $p_D^h(\cdot)$ that is locally consistent with (4.3) exists such that (4.6) holds true;
- (ii) There is a $\beta^h(x, 0) = o(\Delta t^h(x, 0))$ such that the one-step transition probability $p^h(x, y \mid \pi^h)$ for any $y \in P_h$ is given by

$$p^h(x, y \mid \pi^h) = (1 - \lambda \Delta t^h(x, 0) + \beta^h(x, 0)) p_D^h(x, y \mid \pi^h) + (\lambda \Delta t^h(x, 0) - \beta^h(x, 0)) \mathbb{P}_x\left(Y \in \left[x + y - \frac{h}{2}, x + y + \frac{h}{2}\right]\right).$$

Next, we consider the liquidation time associated with the controlled Markov chain. According to (3.9), we liquidate immediately whenever $\xi_n^h < -\underline{x}$ at some step n . Let $\widehat{n} := \inf\{n : \xi_n^h < -\underline{x}\}$. Hence, we define the liquidation time as

$$\tau^h := t_{\widehat{n}}^h = \sum_{k=0}^{\widehat{n}-1} \Delta t^h(\xi_k^h, \pi_k^h). \quad (4.9)$$

Let $\xi_0^h = x \in P_h$, and $\pi^h \in \Pi^h$ is an admissible control. The performance function under the discrete-time controlled Markov chain now can be written as

$$J^h(x; \pi^h) = \mathbb{E}_x \left[\sum_{k=0}^{\widehat{n}-1} e^{-rt_k^h} (\phi_1 \Delta l_k^h - \phi_2 \Delta g_k^h) + e^{-r\tau^h} P(\xi_{\widehat{n}}^h) \right]. \quad (4.10)$$

The corresponding value function is defined as

$$V^h(x) := \sup_{\pi^h \in \Pi^h} J^h(x; \pi^h). \quad (4.11)$$

According to the dynamic programming principle in (3.1), it is easy to show that $V^h(x)$ satisfies the following dynamic programming equation:

$$V^h(x) = \max \left\{ \sum_{y \in P_h} e^{-r\Delta t^h(x,0)} p^h(x, y | \pi^h) V^h(y), \sum_{y \in P_h} p^h(x, y | \pi^h) (V^h(y) + \phi_1(x - y)), \right. \\ \left. \sum_{y \in P_h} p^h(x, y | \pi^h) (V^h(y) - \phi_2(y - x)) \right\}, \quad \text{for } x \in P_h \cap (0, B), \quad (4.12)$$

with $V^h(x) = V^h(0) + \phi_2 x$ for $x \in P_h \cap [-\underline{x}, 0]$, and $V^h(x) = P(x)$ for $x \in P_h \cap (-\infty, -\underline{x})$. Note that the discounting with the interpolation interval does not appear in the second and third part in (4.12), since both the dividend payout and capital injection are “instantaneous”. Moreover, when x reaches the upper boundary B , the optimal control action is to pay dividends immediately; that is

$$V^h(B) = V^h(B - h) + \phi_1 h. \quad (4.13)$$

When it comes to numerical computation, we solve V^h with the help of iterations over the value space using the (accelerated) Gauss–Seidel method. To identify the transition probabilities $p_D^h(\cdot)$ and the value of the interpolation interval $\Delta t^h(\cdot, 0)$, we utilize the finite difference method to define the first and second derivatives of $V(x)$ with respect to x in (3.10) with a step size $h > 0$ as follows

$$V(x) \rightarrow V^h(x), \\ V'(x) \rightarrow \frac{V^h(x+h) - V^h(x)}{h}, \\ V''(x) \rightarrow \frac{V^h(x+h) - 2V^h(x) + V^h(x-h)}{h^2}.$$

Therefore, if we combine this with the operator $\mathcal{L}$ given in (3.2) without Poisson jumps, we can rewrite (3.10) as

$$\max \left\{ \frac{\sigma^2}{2} \frac{V^h(x+h) - 2V^h(x) + V^h(x-h)}{h^2} + c \frac{V^h(x+h) - V^h(x)}{h} - rV^h(x), \right. \\ \left. \phi_1 - \frac{V^h(x) - V^h(x-h)}{h}, \frac{V^h(x+h) - V^h(x)}{h} - \phi_2 \right\} = 0, \quad x \in P_h \cap (0, B), \quad (4.14)$$

where we use the backward difference method instead of the forward difference for the term related to the immediate dividend payout.

To solve the transition probabilities $p_D^h(\cdot)$ and the value of interpolation interval, we should simplify (4.14) and compare it with (4.12) when there are no Poisson jumps. For the case when the maximum in (4.14) is taken from the first term, we multiply both sides of the equation by h^2 and arrive at

$$\frac{\sigma^2}{2} (V^h(x+h) - 2V^h(x) + V^h(x-h)) + ch(V^h(x+h) - V^h(x)) - h^2 r V^h(x) = 0.$$

Reorganize the coefficients of $V^h(\cdot)$, we get

$$\left(\frac{\sigma^2}{2} + ch\right)V^h(x+h) - (ch + \sigma^2 + h^2 r)V^h(x) + \frac{\sigma^2}{2}V^h(x-h) = 0. \quad (4.15)$$

On the other hand, we rewrite the first part of (4.12) so that we only keep the terms related to $V^h(x-h)$, $V^h(x)$, $V^h(x+h)$, and we arrive at

$$e^{-r\Delta t^h(x,0)} \left[p_D^h(x, x+h|\pi^h)V^h(x+h) + p_D^h(x, x-h|\pi^h)V^h(x-h) \right] - V^h(x) = 0. \quad (4.16)$$

Note that (4.15) and (4.16) hold simultaneously for all $x \in P_h \cap (0, B)$. Therefore, by equaling the coefficient of $V^h(\cdot)$ (with a possible difference up to a constant proportion), one gets the following results:

$$\begin{aligned} p_D^h(x, x+h|\pi^h) &= \frac{\frac{\sigma^2}{2} + ch}{\sigma^2 + ch}, \\ p_D^h(x, x-h|\pi^h) &= \frac{\frac{\sigma^2}{2}}{\sigma^2 + ch}, \\ p_D^h(\cdot) &= 0, \text{ otherwise,} \end{aligned} \quad (4.17)$$

as well as

$$\Delta t^h(x, 0) = \frac{h^2}{\sigma^2 + ch}, \quad (4.18)$$

Note that we have applied the approximation $e^{-r\Delta t} \approx (1 + r\Delta t)^{-1}$ in the calculation above to simplify the expression of $\Delta t^h(x, 0)$. We also find the transition probability for the second and third parts of the right side of (4.12), which are

$$p_D^h(x, x-h|\pi^h) = 1, \quad (4.19)$$

for an immediate dividend payment, and

$$p_D^h(x, x+h|\pi^h) = 1, \quad (4.20)$$

for an immediate capital injection.

On the basis of the discussions above, one arrives at the following dynamic programming equations:

$$V^h(x) = \begin{cases} \max_{\pi^h \in \Pi^h} \left[\left(1 - \lambda \Delta t^h(x, 0) + \beta^h(x, 0)\right) e^{-r \Delta t^h(x, 0)} \sum_{y \in P_h} p_D^h(x, y | \pi^h) V^h(y) \right. \\ \quad + \left(\lambda \Delta t^h(x, 0) + \beta^h(x, 0) \right) e^{-r \Delta t^h(x, 0)} p \int_0^\infty V^h(x + y) f_+(y) dy \\ \quad + \left(\lambda \Delta t^h(x, 0) + \beta^h(x, 0) \right) e^{-r \Delta t^h(x, 0)} (1 - p) \int_0^\infty V^h(x - y) f_-(y) dy, \\ \left. V^h(x - h) + \phi_1 h, V^h(x + h) - \phi_2 h \right], \quad x \in P_h \cap (0, B), \\ V^h(h) - \phi_2 h, \quad x = 0, \\ V^h(B - h) + \phi_1 h, \quad x = B, \end{cases} \quad (4.21)$$

with $V^h(x) = V^h(0) + \phi_2 x$ for $x \in P_h \cap [-x, 0)$, and $V^h(x) = P(x)$ for $x \in P_h \cap (-\infty, -x)$.

Building on the formulation in (4.21), we solve the optimal control problem numerically using the Gauss–Seidel iteration method in the value space. While more advanced iterative methods (e.g., [43]) could further improve the computational efficiency, we use the classical Gauss–Seidel approach in our numerical studies for clarity and illustrative purposes.

5. Convergence analysis

In this section, we provide a convergence analysis for the above-mentioned MCA method. The main technique is the theory of weak convergence (see, for example, [43, 45]), where the key is to prove the tightness of a collection of processes indexed by the discretization parameter h such that the limit is exactly our controlled surplus process given by (2.3). In turn, we can show that the value function V^h given in (4.11) converges to V given in (2.10) on a sufficiently large interval when $h \rightarrow 0$.

For the approximating Markov chain constructed above, piecewise constant interpolation is obtained in (4.8) based on the appropriately chosen interpolation interval. For any fixed $h > 0$, let $\{v_j^h\}_{j \geq 1}$ denote the sequence of discrete time points of the jump arrivals, and let $\{y_j^h\}_{j \geq 1}$ denote the corresponding jump sizes, i.e., $y_j^h = \xi_{v_j^h}^h - \xi_{v_{j-1}^h}^h$. Let $\mathcal{F}_n^h$ be the smallest σ -algebra generated by $\{\xi_k^h, l_k^h, g_k^h, H_k^h$ for $k \leq n$; v_j^h, y_j^h for $v_j^h < n\}$ for any $n \geq 0$, and $\{\mathcal{F}_n^h\}_{n \geq 0}$ is increasing in n . With the help of the representations of singular control and the interpolations defined above, we then have

$$\begin{aligned} \xi_n^h &= x + \sum_{k=0}^{n-1} \left(\Delta \xi_k^h \mathbf{1}_{H_k^h} + \Delta \xi_k^h (1 - \mathbf{1}_{H_k^h}) \right) - \sum_{k=0}^{n-1} \Delta l_k^h + \sum_{k=0}^{n-1} \Delta g_k^h \\ &= x + \sum_{k=0}^{n-1} \mathbb{E}_{x,k}^{h,0} (\Delta \xi_k^h) \mathbf{1}_{H_k^h} + \sum_{k=0}^{n-1} \left(\Delta \xi_k^h - \mathbb{E}_{x,k}^{h,0} (\Delta \xi_k^h) \right) \mathbf{1}_{H_k^h} \\ &\quad + \sum_{k=0}^{n-1} \Delta \xi_k^h (1 - \mathbf{1}_{H_k^h}) - \sum_{k=0}^{n-1} \Delta l_k^h + \sum_{k=0}^{n-1} \Delta g_k^h. \end{aligned} \quad (5.1)$$

According to the local consistency given in (4.6), one has

$$\begin{aligned}\sum_{k=0}^{n-1} \mathbb{E}_{x,k}^{h,0}(\Delta \xi_k^h) \mathbf{1}_{H_k^h} &= \sum_{k=0}^{n-1} (c\Delta t_k^h + o(\Delta t_k^h)) \mathbf{1}_{H_k^h}, \\ &= \sum_{k=0}^{n-1} (c\Delta t_k^h + o(\Delta t_k^h)) - \sum_{k=0}^{n-1} (c\Delta t_k^h + o(\Delta t_k^h)) \mathbf{1}_{K_k^h},\end{aligned}$$

where to simplify the notation, we let Δt_k^h denotes the interpolation interval $\Delta t^h(\xi_k^h, \pi_k^h)$. In addition, it is easy to see that

$$\mathbb{E}\left(\sum_{k=0}^{n^h(t)-1} \mathbf{1}_{K_k^h}\right) = \mathbb{E}(\text{number of jumps until time } t) \rightarrow \lambda t, \quad \text{as } h \rightarrow 0. \quad (5.2)$$

We then see that $\sum_{k=0}^{n^h(t)-1} \mathbf{1}_{K_k^h}$ converges to λt in probability as $h \rightarrow 0$. Therefore

$$\sum_{k=0}^{n-1} (c\Delta t_k^h + o(\Delta t_k^h)) \mathbf{1}_{K_k^h} = \max_{k \leq n-1} (\Delta t_k^h) O\left(\sum_{k=0}^{n-1} \mathbf{1}_{K_k^h}\right) \rightarrow 0 \quad \text{in probability as } h \rightarrow 0. \quad (5.3)$$

In addition, let

$$M_n^h := \sum_{k=0}^{n-1} (\Delta \xi_k^h - \mathbb{E}_{x,k}^{h,0}(\Delta \xi_k^h)) \mathbf{1}_{H_k^h}, \quad (5.4)$$

which is a zero-mean martingale with respect to the filtration $\{\mathcal{F}_n^h\}$. We then define the process $(B_t^h)_{t \geq 0}$ as

$$B_t^h := \frac{1}{\sigma} M_t^h, \quad \text{with } M_t^h = M_n^h \quad \text{for } t \in [t_n^h, t_{n+1}^h), \quad (5.5)$$

where σ is the volatility in (2.1), and M_t^h is the continuous-time piecewise interpolation of the discrete martingale M_n^h given by (5.4). Further, let

$$S_n^h := \sum_{k=0}^{n-1} \Delta \xi_k^h (1 - \mathbf{1}_{H_k^h}) = \sum_{j: \nu_j^h \leq n} y_j^h. \quad (5.6)$$

Hence, we can rewrite (5.1) as the corresponding continuous-time interpolation

$$\xi_t^h = x + ct + \sigma B_t^h + S_t^h - l_t^h + g_t^h + \delta_t^h, \quad (5.7)$$

where

$$S_t^h = \sum_{j: \nu_j^h \leq n^h(t)} y_j^h, \quad (5.8)$$

δ_t^h is a negligible error satisfying $\lim_{h \rightarrow 0} \sup_{t \in [0, T]} \delta_t^h = 0$ for any $T \in (0, \infty)$. One can further define the continuous-time interpolation of the point process $N_t^h = \inf\{n : \nu_n^h \leq n^h(t)\}$, and hence (5.8) can be expressed as $S_t^h = \sum_{j=1}^{N_t^h} y_j^h$ as well. Equation (5.3) indicates that we may ignore the term related to $\mathbf{1}_{K_k^h}$ in (5.1) (as well as in (5.7)) when taking limits for $h \rightarrow 0$.

On the other hand, due to the instantaneous jumps from the processes $(l_t^h)_{t \geq 0}$ and $(g_t^h)_{t \geq 0}$, one may fail to show the tightness of the sequence $\{l_t^h, g_t^h\}$ and hence one may not be able to have convergence under the Skorokhod topology. A conventional method to overcome such a difficulty is to rescale the time such that the above-mentioned processes are tight under the rescaled version (see e.g., [43]). Let us define a new time increment denoted by $\Delta \tilde{t}_n^h$ as follows:

$$\Delta \tilde{t}_n^h = \begin{cases} \Delta t_n^h, & \text{no action on step } n, \\ |\Delta l_n^h|, & \text{pay dividend on step } n, \\ |\Delta g_n^h|, & \text{inject capital on step } n. \end{cases} \quad (5.9)$$

Let $\tilde{t}_n^h = \sum_{k=0}^{n-1} \Delta \tilde{t}_k^h$. We then define $\tilde{T}_t^h$ as

$$\tilde{T}_t^h = \sum_{k=0}^{n-1} \Delta \tilde{t}_k^h, \quad \text{for } t \in [\tilde{t}_n^h, \tilde{t}_{n+1}^h).$$

Hence, $\tilde{T}_t^h$ represents the original length of time under the new timescale, which increases with a slope of one during the “period” of no action and with a slope of zero in the “period” of dividend payment or capital injection. Therefore, we define the continuous-time piecewise interpolation process as well as the other processes in (5.7) (namely $B_t^h, N_t^h, S_t^h, l_t^h, g_t^h$) under the rescaled time as $\tilde{\xi}_t^h = \xi_{\tilde{T}_t^h}^h, \tilde{B}_t^h = B_{\tilde{T}_t^h}^h, \tilde{N}_t^h = N_{\tilde{T}_t^h}^h, \tilde{S}_t^h = S_{\tilde{T}_t^h}^h, \tilde{l}_t^h = l_{\tilde{T}_t^h}^h, \tilde{g}_t^h = g_{\tilde{T}_t^h}^h$. In consequence

$$\tilde{\xi}_t^h = x + c \int_0^t d\tilde{T}_s^h + \sigma \tilde{B}_t^h + \tilde{S}_t^h - \tilde{l}_t^h + \tilde{g}_t^h + \delta_t^h. \quad (5.10)$$

Proposition 5.1. Let $\{\xi_n^h, n < \infty\}$ and $(\xi_t^h)_{t \geq 0}$ be the approximating discrete-time Markov chain with the transition probabilities given in (4.17), (4.19), (4.20), and Definition 4.1 that is locally consistent with (2.3) and the corresponding continuous-time piecewise interpolation given in (4.8), respectively. If we let $\{\tilde{\xi}_t^h, \tilde{B}_t^h, \tilde{N}_t^h, \tilde{S}_t^h, \tilde{l}_t^h, \tilde{g}_t^h, \tilde{T}_t^h\}$ be the collection of the rescaled processes, then $\{\tilde{\xi}_t^h, \tilde{B}_t^h, \tilde{N}_t^h, \tilde{S}_t^h, \tilde{l}_t^h, \tilde{g}_t^h, \tilde{T}_t^h\}$ is tight.

Proof. First, we show that the collection of processes $\{B_t^h\}$ and $\{S_t^h\}$ given in (5.5) and (5.8), respectively, are tight in the view of the tightness definition given by [43, Theorem 9.2.1]. To simplify the notation, we assume that all the processes are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we consider any fixed time horizon $T < \infty$. Let ω_h be an $\mathcal{F}_t$ -liquidation time and $\omega_h \leq T$ with probability one, where $\mathcal{F}_t^h = \mathcal{F}_n^h$ for $t \in [t_n^h, t_{n+1}^h)$. Then for any $\varepsilon > 0$, we have

$$\begin{aligned} \mathbb{E}_{\omega_h}^h ((B_{\omega_h+\varepsilon}^h - B_{\omega_h}^h)^2) &= \mathbb{E}_{\omega_h}^h \left(\frac{1}{\sigma^2} (M_{\omega_h+\varepsilon}^h - M_{\omega_h}^h)^2 \right) \\ &= \frac{1}{\sigma^2} \text{Var}_{\omega_h}^h \left(\sum_{k: t_k^h \in [\omega_h, \omega_h+\varepsilon]} \Delta \xi_k^h \right) = \varepsilon + \delta_h, \end{aligned} \quad (5.11)$$

where $\limsup_{h \rightarrow 0} \delta_h = 0$ uniformly in the liquidation time ω_h , and $\mathbb{E}_{\omega_h}^h$ and $\text{Var}_{\omega_h}^h$ denote the conditional expectation and variance given $\mathcal{F}_{\omega_h}^h$. Hence, one has

$$\lim_{\varepsilon \rightarrow 0} \limsup_{h \rightarrow 0} \sup_{\omega_h} \mathbb{E}_{\omega_h}^h ((B_{\omega_h+\varepsilon}^h - B_{\omega_h}^h)^2) = 0.$$

Therefore, one arrives at the tightness of $\{B^h\}$. The tightness of $\{S^h\}$ is obvious by (5.8) and the fact that

$$\liminf_{\varepsilon \rightarrow 0} \mathbb{P}(v_{n+1}^h - v_n^h > \varepsilon \mid \mathcal{F}_{v_n}^h) = 1.$$

In addition, the tightness of $\{N^h\}$ can be obtained from the fact that for any bounded interval $[t, t + s]$, we have

$$\mathbb{E}(N_{t+s}^h - N_t^h) \leq \lambda s + \epsilon^h(s), \quad \text{for } \epsilon^h(s) \rightarrow 0 \text{ as } h \rightarrow 0.$$

The tightness of the corresponding rescaled processes $\{\tilde{B}^h\}$, $\{\tilde{N}^h\}$, and $\{\tilde{S}^h\}$ can be obtained easily by noting that $\mathbb{E}((\tilde{B}_{t+\varepsilon}^h - \tilde{B}_t^h)^2 \mid \mathcal{F}_t^h) \leq \mathbb{E}((B_{t+\varepsilon}^h - B_t^h)^2 \mid \mathcal{F}_t^h)$; this inequality holds true if we replace $\tilde{B}^h$ and B^h by $\tilde{N}^h$ and N^h , as well as $\tilde{S}^h$ and S^h , respectively. Furthermore, for the rescaled processes $\{\tilde{l}^h\}$ and $\{\tilde{g}^h\}$, one finds that

$$|\tilde{l}_{\omega_h+\varepsilon}^h - \tilde{l}_{\omega_h}^h| \leq |\varepsilon| + O(h),$$

and

$$|\tilde{g}_{\omega_h+\varepsilon}^h - \tilde{g}_{\omega_h}^h| \leq |\varepsilon| + O(h).$$

Hence, $\{\tilde{l}^h\}$ and $\{\tilde{g}^h\}$ are tight. Finally, with the boundedness of the constants c and σ in (5.7), one obtains the tightness of $\{\tilde{\xi}^h\}$ and therefore completes the proof. $\square$

According to Proposition 5.1, $\{\tilde{\xi}_t^h, \tilde{B}_t^h, \tilde{N}_t^h, \tilde{S}_t^h, \tilde{l}_t^h, \tilde{g}_t^h, \tilde{T}_t^h\}$ is tight; hence there is a subsequence, say $\{\tilde{\xi}_t^{h_i}, \tilde{B}_t^{h_i}, \tilde{N}_t^{h_i}, \tilde{S}_t^{h_i}, \tilde{l}_t^{h_i}, \tilde{g}_t^{h_i}, \tilde{T}_t^{h_i}, i \geq 1\}$, that weakly converges to a limit. If we let $\{\tilde{U}_t, \tilde{B}_t, \tilde{N}_t, \tilde{S}_t, \tilde{l}_t, \tilde{g}_t, \tilde{T}_t\}$ denotes such limit, then we have the following proposition.

Proposition 5.2. Let $\{\tilde{U}_t, \tilde{B}_t, \tilde{N}_t, \tilde{S}_t, \tilde{l}_t, \tilde{g}_t, \tilde{T}_t\}$ be the limit of the weakly convergent subsequence of $\{\tilde{\xi}_t^h, \tilde{B}_t^h, \tilde{N}_t^h, \tilde{S}_t^h, \tilde{l}_t^h, \tilde{g}_t^h, \tilde{T}_t^h\}$. If we let $\tilde{\mathcal{F}}_t$ be the filtration generated by $\{\tilde{U}_s, \tilde{B}_s, \tilde{N}_s, \tilde{S}_s, \tilde{l}_s, \tilde{g}_s, \tilde{T}_s\}$ for $s \leq t$, then the rescaled process $\tilde{B}_t = B_{\tilde{T}_t}$ is an $\tilde{\mathcal{F}}_t$ -martingale with the quadratic variation $\tilde{T}_t$, where B_t is a standard Brownian motion adapted to $\{\mathcal{F}_t\}$. In addition, the limiting process $\tilde{U}_t$ satisfies

$$\tilde{U}_t = x + c \int_0^t d\tilde{T}_t + \sigma \tilde{B}_t + \tilde{S}_t - \tilde{l}_t + \tilde{g}_t. \quad (5.12)$$

Proof. We first prove that $\tilde{B}_t$ is indeed an $\tilde{\mathcal{F}}_t$ -martingale, where we follow the similar arguments in the proof of Theorem 9.4.2 in [43]. Let us fix $t > 0, \delta > 0$ and a finite partition $\{t_k\}_{k \leq p}$ of the interval $[0, t]$ for some $p < \infty$; further let $\varphi(\cdot)$ be a real-valued and continuous function with compact support. According to (5.5), one has

$$\mathbb{E}(\varphi(\tilde{\xi}_{t_k}^h, \tilde{B}_{t_k}^h, \tilde{S}_{t_k}^h, \tilde{l}_{t_k}^h, \tilde{g}_{t_k}^h, k \leq p) [\tilde{B}_{t+\delta}^h - \tilde{B}_t^h]) = 0. \quad (5.13)$$

By using the Skorohod representation theorem [43, Theorem 9.1.7], there exists a probability measure, say $\mathbb{P}$, such that the weak convergence of $\{\tilde{\xi}_t^h, \tilde{B}_t^h, \tilde{S}_t^h, \tilde{l}_t^h, \tilde{g}_t^h, \tilde{T}_t^h\}$ to a limit $\{\tilde{U}_t, \tilde{B}_t, \tilde{S}_t, \tilde{l}_t, \tilde{g}_t, \tilde{T}_t\}$ is equivalent to $\mathbb{P}$ -almost surely convergence. Hence, with the dominant convergence theorem, by letting $h \rightarrow 0$ in (5.13), we have

$$\mathbb{E}(\varphi(\tilde{U}_{t_k}, \tilde{B}_{t_k}, \tilde{S}_{t_k}, \tilde{l}_{t_k}, \tilde{g}_{t_k}, k \leq p) [\tilde{B}_{t+\delta} - \tilde{B}_t]) = 0. \quad (5.14)$$

That is to say, for all $A \in \tilde{\mathcal{F}}_t$, we have

$$\mathbb{E}([\tilde{B}_{t+\delta} - \tilde{B}_t] \mathbf{1}_A) = 0,$$

which is equivalent to

$$\mathbb{E}(\widetilde{B}_{t+\delta} - \widetilde{B}_t | \mathcal{F}_t) = 0.$$

Hence, $\widetilde{B}_t$ is an $\widetilde{\mathcal{F}}_t$ -martingale. In addition, with a similar analysis, one can obtain

$$\mathbb{E}(\varphi(\widetilde{U}_{t_k}, \widetilde{B}_{t_k}, \widetilde{S}_{t_k}, \widetilde{l}_{t_k}, \widetilde{g}_{t_k}, k \leq p) [\widetilde{B}_{t+\delta} - \widetilde{B}_t]^2 - (\widetilde{T}_{t+\delta} - \widetilde{T}_t)) = 0.$$

Therefore, $\widetilde{B}_t$ is an $\widetilde{\mathcal{F}}_t$ -martingale with the quadratic variation $\widetilde{T}_t$ (i.e., an $\widetilde{\mathcal{F}}_t$ -Brownian motion).

Finally, by letting $h \rightarrow 0$ in (5.10) yields (5.12).

□

Proposition 5.3. Define the right-continuous inverse of the function $\widetilde{T}_t$ as $T_t := \inf\{s : \widetilde{T}_s > t\}$ for any $t < \infty$, then $\lim_{t \rightarrow \infty} T_t = \infty$ with probability one. For the limiting process $\widetilde{U}_t$ given in (5.12), define the inverses process $U_t = \widetilde{U}_{T_t}$, $B_t = \widetilde{B}_{T_t}$, and $N_t = \widetilde{N}_{T_t}$. Then U_t satisfies (2.3), B_t is an $\mathcal{F}_t$ -adapted standard Brownian motion, and N_t is a Poisson process with intensity λ .

Proof. Note that $\lim_{t \rightarrow \infty} \widetilde{T}_t = \infty$ with probability one; hence, T_t exists for all $t < \infty$ and $\lim_{t \rightarrow \infty} T_t = \infty$ with probability one. The proof that B_t is a standard Brownian motion adapted to $\mathcal{F}_t$ is straightforward. According to Proposition 5.2, $\widetilde{B}_t$ is an $\widetilde{\mathcal{F}}_t$ -martingale with the quadratic variation $\widetilde{T}_t$. By the definition of T_t and the optional sampling theorem, the time-changed process $B_t = \widetilde{B}_{T_t}$ is a martingale with respect to the filtration $\mathcal{F}_t = \widetilde{\mathcal{F}}_{T_t}$ with the quadratic variation $\widetilde{T}_{T_t} = t$. Thus, B_t is a standard Brownian motion. The proof that N_t is a Poisson process follows a similar argument. Let $M_t^{h,N} := N_t^h - \lambda t_{n^h(t)}^h - \epsilon_t^h$ be the continuous-time interpolation of the discrete-time martingale $M_n^{h,N} = N_n^h - \sum_{k=0}^{n-1} \mathbb{E}[\mathbf{1}_{K_k^h} | \mathcal{F}_k^h]$ (see (5.2)), where $\epsilon_t^h = \sum_{k=0}^{n^h(t)-1} o(\Delta t^h(\xi_k^h, 0))$ is an accumulated error term that vanishes as $h \rightarrow 0$. Under the rescaled time, we obtain the $\widetilde{\mathcal{F}}_t^{h,N}$ -martingale $\widetilde{M}_t^{h,N} = \widetilde{N}_t^h - \lambda \widetilde{T}_t^h - \widetilde{\epsilon}_t^h$. Therefore, for any t and $\delta > 0$ (see the proof of Proposition 5.2 above), we have

$$\mathbb{E}(\varphi(\widetilde{\xi}_{t_k}^h, \widetilde{B}_{t_k}^h, \widetilde{S}_{t_k}^h, \widetilde{l}_{t_k}^h, \widetilde{g}_{t_k}^h, k \leq p) (\widetilde{M}_{t+\delta}^{h,N} - \widetilde{M}_t^{h,N})) = 0.$$

Taking the limit as $h \rightarrow 0$, the error term $\widetilde{\epsilon}_t^h$ vanishes. By the dominated convergence theorem, we obtain

$$\mathbb{E}(\varphi(\widetilde{U}_{t_k}, \widetilde{B}_{t_k}, \widetilde{S}_{t_k}, \widetilde{l}_{t_k}, \widetilde{g}_{t_k}, k \leq p) ((\widetilde{N}_{t+\delta} - \lambda \widetilde{T}_{t+\delta}) - (\widetilde{N}_t - \lambda \widetilde{T}_t))) = 0.$$

This implies that $\widetilde{M}_t^N = \widetilde{N}_t - \lambda \widetilde{T}_t$ is an $\widetilde{\mathcal{F}}_t$ -martingale. Evaluating this rescaled martingale at the stopping time T_t via the optional sampling theorem yields $\widetilde{M}_{T_t}^N = \widetilde{N}_{T_t} - \lambda \widetilde{T}_{T_t}$, which is a martingale with respect to the filtration $\mathcal{F}_t$. Because $\widetilde{T}_{T_t} = t$ and $\widetilde{N}_{T_t} = N_t$, it follows that $N_t - \lambda t$ is an $\mathcal{F}_t$ -martingale. Since N_t is a counting process with a deterministic compensator λt , we find that N_t is a Poisson process with intensity λ .

Finally, by rescaling the time in (5.12), one obtains

$$U_t = x + ct + \sigma B_t + \sum_{i=1}^{N_t} Y_i - l_t + g_t.$$

This completes the proof.

□

To show the convergence in the performance function and value function, we need the following technical assumption.

- (A) Let H^o be an interior of a compact set on $\mathbb{R}$, and define the mapping from $D[0, \infty)$ (the space of right continuous with left limits functions) to $[0, \infty]$ (the one-point compactification of $[0, \infty)$) as

$$\Gamma(\varphi(\cdot)) := \inf\{t : \varphi(t) \notin H^o\},$$

such that $\Gamma(\varphi(\cdot)) = \infty$ if $\varphi(t) \in H^o$ for all $t < \infty$. Then the function $\Gamma(\cdot)$ is continuous with probability one with respect to the measure induced by $\widetilde{U}$.

Proposition 5.4. Let the weakly convergent subsequence of $\{\widetilde{\xi}_t^h, \widetilde{B}_t^h, \widetilde{N}_t^h, \widetilde{S}_t^h, \widetilde{l}_t^h, \widetilde{g}_t^h, \widetilde{T}_t^h\}$ indexed by h , with a limit $\{\widetilde{U}_t, \widetilde{B}_t, \widetilde{N}_t, \widetilde{S}_t, \widetilde{l}_t, \widetilde{g}_t, \widetilde{T}_t\}$ when $h \rightarrow 0$. Let $J^h(x; \pi^h)$ and $J(x; \pi)$ be the performance functions defined in (4.10) and (2.4), respectively. We then have

$$J^h(x; \pi^h) \rightarrow \mathbb{E}_x \left[\int_{[0, \tau^\pi)} e^{-rt} (\phi_1 dL_t - \phi_2 dR_t) + e^{-r\tau^\pi} P(U_{\tau^\pi}) \right], \quad \text{as } h \rightarrow 0. \quad (5.15)$$

Proof. We shall first discuss the convergence of the liquidation times. Let $\widetilde{\tau}^h = \widetilde{l}_n^h$ where $\widehat{n}$ is given by (4.9), and $\widetilde{\tau} = \inf\{t \geq 0 : \widetilde{U}_t < -x\}$. Let $[0, \infty]$ denote the one-point compactification of $[0, \infty)$; that is, the point $\{\infty\}$ is appended to the set $[0, \infty)$ as the limit point of any increasing and unbounded sequence in $[0, \infty)$. Note that $[0, \infty]$ is compact, and hence the sequence of liquidation times $\{\widetilde{\tau}^h, h > 0\}$, which takes values in $[0, \infty]$, is tight. Therefore, there is a convergent subsequence of $\{\widetilde{\tau}^h\}$ with the limit denoted by $\widehat{\tau}$. Thus, according to Assumption (A), we have $\widehat{\tau} = \widetilde{\tau}$ with probability one, since

$$\widehat{\tau} = \lim_{h \rightarrow 0} \widetilde{\tau}^h = \lim_{h \rightarrow 0} \Gamma(\widetilde{\xi}^h) = \Gamma(\widetilde{U}) = \widetilde{\tau}.$$

In addition, since $\Delta l_n^h = h$ for all n , and, due to the tightness and uniform integrability properties, for any t , the integral $\int_{[0, t)} d\widetilde{l}_t$ can be adequately approximated by a Riemann sum uniformly in h ; hence, by the rescale transformation, weak convergence, and the Skorohod representation theorem, one has

$$\mathbb{E}_x^{\pi^h} \left(\sum_{k=0}^{\widehat{n}-1} e^{-rt_k^h} \phi_1 \Delta l_k^h \right) \rightarrow \mathbb{E}_x^\pi \left(\int_{[0, \widetilde{\tau})} e^{-r\widetilde{T}_t} \phi_1 d\widetilde{l}_t \right), \quad \text{as } h \rightarrow 0.$$

Then, by an inverse transformation of the time, one arrives at

$$\mathbb{E}_x^\pi \left(\int_{[0, \widetilde{\tau})} e^{-r\widetilde{T}_t} \phi_1 d\widetilde{l}_t \right) = \mathbb{E}_x \left(\int_{[0, \tau^\pi)} e^{-rt} \phi_1 dL_t \right).$$

Similarly, for the liquidation payoff, with (2.12) and the dominated convergence theorem, one has

$$\mathbb{E}_x^{\pi^h} \left(e^{-r\tau^h} P(\xi_n^h) \right) \rightarrow \mathbb{E}_x^\pi \left(e^{-r\widetilde{\tau}} P(\widetilde{U}_{\widetilde{\tau}}) \right) = \mathbb{E}_x \left(e^{-r\tau^\pi} P(U_{\tau^\pi}) \right) \quad \text{as } h \rightarrow 0.$$

Finally, for capital injection, which is similar to the dividend part, we have

$$\mathbb{E}_x^{\pi^h} \left(\sum_{k=0}^{\widehat{n}-1} e^{-rt_k^h} \phi_2 \Delta g_k^h \right) \rightarrow \mathbb{E}_x \left(\int_{[0, \tau^\pi)} e^{-rt} \phi_2 dR_t \right) \quad \text{as } h \rightarrow 0.$$

Therefore, one arrives at

$$J^h(x; \pi^h) \rightarrow \mathbb{E}_x \left[\int_{[0, \tau^\pi)} e^{-rt} (\phi_1 dL_t - \phi_2 dR_t) + e^{-r\tau^\pi} P(U_{\tau^\pi}) \right], \quad \text{as } h \rightarrow 0.$$

□

Next, we prove the convergence of the value function in the following proposition.

Proposition 5.5. Let $V(x)$ and $V^h(x)$ be the value functions defined in (2.10) and (4.11), respectively. Then $V^h(x) \rightarrow V(x)$ as $h \rightarrow 0$.

Proof. We first show that

$$\limsup_{h \downarrow 0} V^h(x) \leq V(x).$$

If we consider any admissible control $\pi \in \Pi$, we have

$$J(x; \pi) := \mathbb{E}_x \left[\int_{[0, \tau^\pi)} e^{-rt} (\phi_1 dL_t - \phi_2 dR_t) + e^{-r\tau^\pi} P(U_{\tau^\pi}) \right] \leq V(x),$$

by the definition of the value function (see (2.10)). Let π^{h*} be an optimal control strategy for $(\xi_t^h)_{t \geq 0}$ given by (5.7) with the discretization parameter h , i.e. $\pi^{h*} = (l_t^{h*}, g_t^{h*})_{t \geq 0}$. We then have

$$V^h(x) = J^h(x; \pi^{h*}) = \sup_{\pi^h \in \Pi^h} J^h(x; \pi^h).$$

Let $\{\hat{h}\}$ be a subsequence of $\{h\}$ such that

$$\lim_{\hat{h} \downarrow 0} V^{\hat{h}}(x) = \limsup_{\hat{h} \downarrow 0} V^{\hat{h}}(x) = \lim_{\hat{h} \downarrow 0} J^{\hat{h}}(x; \pi^{\hat{h}}).$$

We assume that $\{\xi_t^{\hat{h}}, S_t^{\hat{h}}, N_t^{\hat{h}}, B_t^{\hat{h}}, l_t^{\hat{h}}, g_t^{\hat{h}}, \tau^{\hat{h}}\}$ converges weakly to $\{U, S, N, B, l, g, \tau^\pi\}$ as $\hat{h} \rightarrow 0$ (passing to an additional subsequence if needed), where $\pi \in \Pi$ is an admissible control. Then, with the weak convergence, Skorohod representation, and Proposition 5.4, one has

$$\lim_{\hat{h} \downarrow 0} J^{\hat{h}}(x; \pi^{\hat{h}}) = J(x; \pi).$$

Hence,

$$\limsup_{\hat{h} \downarrow 0} V^{\hat{h}}(x) = J(x; \pi) \leq V(x).$$

The last inequality holds true, since π is admissible. Next, we show that

$$\liminf_{h \downarrow 0} V^h(x) \geq V(x).$$

Assume that $\widehat{\pi}$ is an optimal control with the compound Poisson process $(S_t)_{t \geq 0}$ and the Brownian motion $(B_t)_{t \geq 0}$ such that $J(x; \widehat{\pi}) = V(x)$, and we denote the corresponding controlled trajectory of the surplus process as $(\widehat{U}_t)_{t \geq 0}$. According to Theorem 11.3.1 in [43] (see also [45]) and Assumption (A), for each $\epsilon > 0$, there exists a $\delta > 0$ and a probability space on which it is defined $(B^\epsilon, S^\epsilon, N^\epsilon, \tau^\epsilon)$,

an ordinary control $\widehat{\pi}^\epsilon$ which is piecewise constant (i.e., constant on the interval $[k\delta, (k+1)\delta)$ for $k \geq 0$) and takes only finitely many values, and the associated controlled dynamics U^ϵ , such that $|J(x; \widehat{\pi}^\epsilon) - J(x; \widehat{\pi})| \leq \epsilon$. In addition, for each $\epsilon > 0$ and the corresponding $\delta > 0$, there is a $\theta > 0$ and a $\rho > 0$ with jumps of the dynamics that only take values in the discrete set $k\rho$ for $k = 0, \pm 1, \pm 2, \dots$, such that the approximating ordinary control $\widehat{\pi}^\epsilon$ can be chosen so that its probability law at any time $n\delta$, conditioned on $\{B_s^\epsilon, S_s^\epsilon, N_s^\epsilon, s \leq n\delta, \widehat{\pi}^\epsilon(i\delta), i < n, \mathbf{1}_{\{\tau^\epsilon < n\delta\}}\}$, depends only on the initial condition x and the sample paths at those discrete time points $p\theta < n\delta$ for $p = 0, 1, \dots$ with the jumps taking values in the discrete set $\{k\rho, k = 0, \pm 1, \pm 2, \dots\}$. Therefore, we adapt it for use on the above-mentioned controlled Markov chain, and one has $|J^h(x; \widehat{\pi}^\epsilon) - J(x; \widehat{\pi}^\epsilon)| \leq \epsilon$. Hence,

$$\liminf_{h \downarrow 0} V^h(x) \geq J^h(x; \widehat{\pi}^\epsilon) \geq J(x; \widehat{\pi}) - 2\epsilon = V(x) - 2\epsilon.$$

Then, by letting $\epsilon \rightarrow 0$, one obtains $\liminf_{h \downarrow 0} V^h(x) \geq V(x)$, which completes the proof. $\square$

6. Numerical examples

In this section, we calculate the optimal value functions and identify the associated optimal strategies for various scenarios based on the MCA method given in Section 4.

According to the dynamic programming equations (4.21), we numerically solve the optimal control problem by performing the calculation with value iterations. Then for any $n = 0, 1, 2, \dots$, we define the vectors $V_n^h = (V_n^h(0), V_n^h(h), \dots, V_n^h(B))^\top$ and $V^h = (V^h(0), V^h(h), \dots, V^h(B))^\top$, which are the value function at the n th iteration and the corresponding converged value function in the algorithm, respectively. Furthermore, we let $\pi_n^h = (\pi_n^h(0), \pi_n^h(h), \dots, \pi_n^h(B))^\top$ denotes the vector of optimal control action for each surplus value during the n th iteration, where $\pi_n^h(x) = 0$ denotes the control of continuation with no action when the current surplus is at x , and $\pi_n^h(x) = 1$ denotes the control of immediate capital injection, and $\pi_n^h(x) = 2$ represents the control of immediate dividend payout.

The numerical algorithm based on the value iteration method is given in Algorithm 1 below. For all numerical examples in the subsequent sections, we adopt a step size of $h = 0.055$, set the convergence tolerance to 10^{-6} , and choose the upper bound $B = 30$. The stability of the value functions and optimal strategies when varying B is also examined.

Algorithm 1 MCA with value iteration

- 1: **Input:** State P_h , model parameters $(h, c, \sigma, r, \lambda, p, \phi_1, \phi_2, \alpha, \varphi)$, transaction probabilities $p_D^h(\cdot)$, interpolation interval $\Delta t(\cdot)$.
- 2: **Output:** Value function V^h , optimal strategy π^h .
- 3: **Initialize:** Set $n = 0$, set $V_0^h = \mathbf{0}$, and π_0^h to arbitrary initial strategy
- 4: $\underline{x} \leftarrow V_0^h(0)(1 - \alpha)/(\phi_2 - \varphi)$
- 5: **repeat**
- 6: For each $x \in P_h \cap [0, B]$

$$V_{n+1}^h(x) \leftarrow \begin{cases} \max_{\pi^h \in \Pi^h} \left[\left(1 - \lambda \Delta t^h(x, 0) + \beta^h(x, 0)\right) e^{-r \Delta t^h(x, 0)} \sum_{y \in P_h} p_D^h(x, y | \pi^h) V_n^h(y) \right. \\ \quad + \left(\lambda \Delta t^h(x, 0) + \beta^h(x, 0) \right) e^{-r \Delta t^h(x, 0)} p \int_0^\infty V_n^h(x + y) f_+(y) dy \\ \quad + \left(\lambda \Delta t^h(x, 0) + \beta^h(x, 0) \right) e^{-r \Delta t^h(x, 0)} (1 - p) \int_0^\infty V_n^h(x - y) f_-(y) dy, \\ \left. V_n^h(x - h) + \phi_1 h, V_n^h(x + h) - \phi_2 h \right], \quad x \in P_h \cap (0, B), \\ V_n^h(0) = V_n^h(h) - \phi_2 h, \\ V_n^h(B) = V_n^h(B - h) + \phi_1 h. \end{cases}$$

$$\pi_{n+1}^h(x) \leftarrow \begin{cases} \operatorname{argmax}_{\pi^h \in \Pi^h} \left[\left(1 - \lambda \Delta t^h(x, 0) + \beta^h(x, 0)\right) e^{-r \Delta t^h(x, 0)} \sum_{y \in P_h} p_D^h(x, y | \pi^h) V_n^h(y) \right. \\ \quad + \left(\lambda \Delta t^h(x, 0) + \beta^h(x, 0) \right) e^{-r \Delta t^h(x, 0)} p \int_0^\infty V_n^h(x + y) f_+(y) dy \\ \quad + \left(\lambda \Delta t^h(x, 0) + \beta^h(x, 0) \right) e^{-r \Delta t^h(x, 0)} (1 - p) \int_0^\infty V_n^h(x - y) f_-(y) dy, \\ \left. V_n^h(x - h) + \phi_1 h, V_n^h(x + h) - \phi_2 h \right], \quad x \in P_h \cap (0, B), \\ 1, \quad x = 0, \\ 2, \quad x = B. \end{cases}$$

- 7: $\underline{x} \leftarrow V_{n+1}^h(0)(1 - \alpha)/(\phi_2 - \varphi)$
- 8: $n \leftarrow n + 1$
- 9: **until** $\sup_{x \in P_h \cap [0, B]} |V_{n+1}^h(x) - V_n^h(x)| < \text{tolerance}$
- 10: **return:** V_n^h and π_{n+1}^h .

In the numerical studies, we focus on the scenario of Proposition 2.1(i), where the liquidation function $P(x) = \alpha V(0) + \varphi x$ is affine, with $\varphi < \phi_2$. In the first benchmark of Example 6.1, we compare two jump density distributions: (i) In the double exponential case, both positive and negative jumps follow exponential distributions; (ii) in the exponential-Pareto case, the positive jumps are exponential, while the negative jumps follow a Pareto distribution. Both distributions share the same mean, and all other model parameters remain identical.

Example 6.1. We adopt the following baseline parameters:

$$c = 6, \quad r = 0.1, \quad \sigma = 0.2, \quad \lambda = 1, \quad p = 0.2, \quad \phi_1 = 0.9, \quad \phi_2 = 1.1, \quad \varphi = 0.8, \quad \alpha = 0.8.$$

Case (i): Double exponential jumps. Assume that both the positive and negative jumps follow

exponential distributions. The jump densities are given by

$$f_+(y) = a_+ e^{-a_+ y}, \quad f_-(y) = a_- e^{a_- y}, \quad y > 0,$$

with $a_+ = 1$ and $a_- = 0.2$.

Case (ii): Exponential-Pareto jumps. Assume that the positive jumps follow an exponential distribution, while the negative jumps follow a Pareto distribution. The jump densities are given by

$$f_+(y) = a_+ e^{-a_+ y}, \quad f_-(y) = \frac{\rho \theta^\rho}{(y + \theta)^{\rho+1}}, \quad y > 0,$$

with $a_+ = 1$, $\rho = 2$, and $\theta = 5$, so that $\mathbb{E}[Y] = \theta/(\rho - 1) = 5$.

The resulting optimal liquidation thresholds $-x^*$ and dividend barriers b^* are given in Table 1, and the value functions are plotted in Figure 1(a). We also show the convergence of the value iteration algorithm in Figure 1(b) for both cases.

Table 1. Benchmark example: Optimal strategy.

	$-x^*$	b^*
Double exponential	-8.672	0.440
Exponential-Pareto	-9.825	0.220

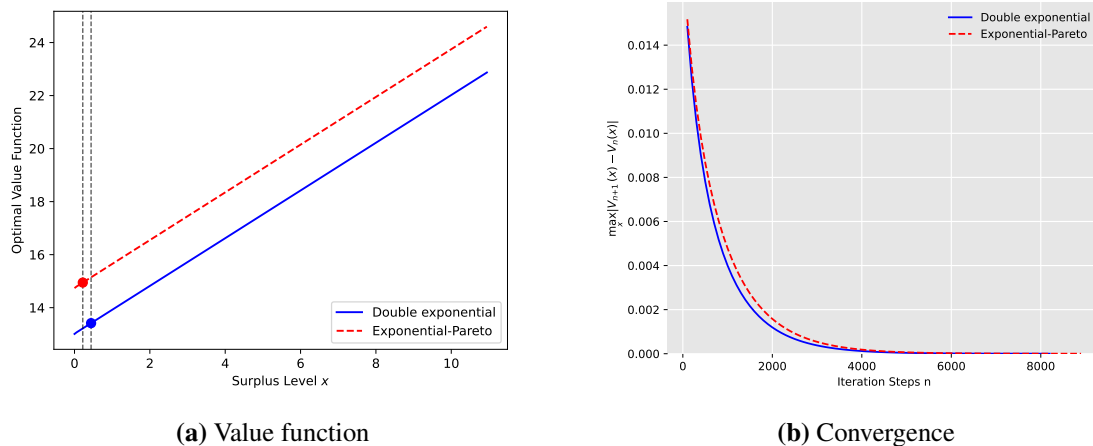


Figure 1. Benchmark example: Double exponential vs. exponential-Pareto.

Discussion of Example 6.1 Table 1 presents the optimal policy thresholds under the double exponential case and the exponential-Pareto case. Figure 1(a) illustrates the corresponding value functions over a truncated surplus interval (for illustration purposes) encompassing the optimal dividend barriers, and Figure 1(b) demonstrates the convergence of the value iteration algorithm.

In Figure 1(a), the value function exhibits a monotonic (concave) increase with respect to the initial surplus x , and the vertical dashed line marks the optimal dividend barrier b^* . In addition, for the same

surplus level, the value function under the exponential-Pareto case is higher than that under the double exponential case.

Table 1 reveals that the double exponential case produces a higher dividend barrier ($b^* = 0.440$) compared with the exponential-Pareto case ($b^* = 0.220$). This divergence stems from the distinct risk profiles, where light-tailed exponential downward jumps allow for greater patience in the dividend distribution, while heavy-tailed Pareto jumps necessitate earlier dividend extraction to mitigate potential profit losses from extreme downside risk. The liquidation threshold also differs markedly, as the exponential-Pareto case yields a lower liquidation level $-\underline{x}^* = -9.825$ than the double exponential case $-\underline{x}^* = -8.672$, indicating that heavy-tailed negative jumps lead the decision-maker to tolerate larger deficits before liquidating.

To validate the convergence of the value iteration scheme, we track the “error” value $\max_{x \in \mathcal{S}_h} |V_{n+1}(x) - V_n(x)|$ across iteration steps n (sampled every 100 iterations). Figure 1(b) shows that the error declines monotonically, dropping rapidly in the early iterations and approaching zero smoothly. The double exponential case converges slightly faster than the exponential-Pareto case, but, for both models, the value function converges eventually after around 6000 iterations.

Example 6.2. We continue the numerical study of Example 6.1 under the double exponential case by performing sensitivity analyses with respect to the affine liquidation cost parameters φ and α , and the proportional transaction costs for dividends ϕ_1 and capital injections ϕ_2 . Table 2 reports the resulting optimal liquidation thresholds $-\underline{x}^*$ and dividend barriers b^* .

Table 2. The influences of liquidation and transaction costs on the optimal policy.

	$-\underline{x}^*$	b^*		$-\underline{x}^*$	b^*
$\varphi = 0.6$	-6.438	0.220	$\alpha = 0.5$	-19.349	2.035
$\varphi = 0.8$	-8.672	0.440	$\alpha = 0.7$	-12.165	1.375
$\varphi = 0.9$	-11.938	1.560	$\alpha = 0.9$	-5.024	0.220
(a) Affine liquidation parameter φ			(b) Affine liquidation parameter α		
	$-\underline{x}^*$	b^*		$-\underline{x}^*$	b^*
$\phi_1 = 0.7$	-4.309	5.555	$\phi_2 = 1.1$	-8.672	0.440
$\phi_1 = 0.8$	-6.199	3.245	$\phi_2 = 1.5$	-3.079	2.915
$\phi_1 = 0.9$	-8.672	0.440	$\phi_2 = 2.0$	-1.689	3.575
(c) Dividend transaction cost ϕ_1			(d) Capital injection transaction cost ϕ_2		

Discussion of Example 6.2

- (1) Impact of the proportional liquidation cost φ (Table 2(a)). Table 2(a) illustrates the impact of the liquidation cost parameter φ on the optimal strategies. As φ increases, the optimal liquidation threshold $-\underline{x}^*$ decreases, meaning that the firm is willing to tolerate larger deficits before opting for liquidation. This is because a higher φ implies a higher proportional cost upon liquidation, making liquidation less attractive compared with continuing the business via capital injections. Simultaneously, the dividend barrier b^* increases with φ . The increased cost of liquidation

incentivizes the decision-maker to maintain a larger surplus buffer to avoid entering the liquidation region. Therefore, the firm becomes more cautious in paying dividends and only does so at higher surplus levels.

- (2) Impact of a fixed liquidation payoff parameter α (Table 2(b)). Note that for $\alpha \in [0, 1]$, the fixed liquidation payoff is $K = \alpha V(0) \leq V(0)$, which implies that immediate liquidation at zero surplus is never optimal. Intuitively, increasing α enhances the attractiveness of the liquidation option by raising the terminal payoff. Consequently, the decision-maker becomes less inclined to wait for uncertain future recovery with capital injection, leading to earlier liquidation decisions. Moreover, a higher liquidation payoff diminishes the value of retaining a surplus within the firm. This reduces the incentive to build up large surplus buffers before distributing dividends, resulting in a lower optimal dividend barrier and a more aggressive dividend payout policy.
- (3) Impact of the dividend proportional rate ϕ_1 (Table 2(c)). As ϕ_1 increases, the liquidation threshold $-\underline{x}^*$ becomes more negative and the dividend barrier b^* decreases. This indicates that the decision-maker becomes willing to tolerate larger deficits before liquidating while initiating dividend payments at lower surplus levels. A higher ϕ_1 increases the marginal value of any future surplus through dividend distributions, encouraging the firm to commence dividend payments earlier and thereby increasing the continuation value. Consequently, liquidation becomes relatively less attractive compared with continuation via capital injection, leading to a lower liquidation threshold.
- (4) Impact of the capital injection transaction cost ϕ_2 (Table 2(d)). When ϕ_2 increases, capital injections become more expensive, making the decision-maker less willing to rescue the company following significant losses from negative jumps and thereby raising the liquidation threshold $-\underline{x}^*$. Meanwhile, the dividend barrier b^* increases with ϕ_2 , as the decision-maker prefers to maintain a larger precautionary surplus buffer to avoid costly future injections, and hence dividend payments are initiated only at higher surplus levels.

Example 6.3. This example presents sensitivity analyses of the optimal policy thresholds with respect to key model parameters under the double exponential case. The parameters examined include the premium rate c , the jump size parameters a_+ and a_- , the Poisson intensity λ , the probability of positive jump p , the diffusion coefficient σ , and the discount rate r . The resulting optimal liquidation thresholds $-\underline{x}^*$ and dividend barriers b^* are reported in Table 3.

Discussion of Example 6.3

- (1) Impact of the premium rate c (Tables 3(a)). Table 3(a) demonstrates that higher premium rates c significantly reduce the optimal liquidation threshold $-\underline{x}^*$, reflecting the increased willingness of the decision-maker to tolerate larger deficits before liquidation. This occurs because a stronger and stable cash inflow improves the firm's long-term recovery prospects, making capital injections economically viable even for substantial shortfalls.

The optimal dividend barrier b^* exhibits a non-monotonic relationship with c . For low to moderate values of c , the decision-maker adopts a more conservative dividend policy, retaining larger capital buffers to hedge against liquidation risks even when c increases. However, as c grows beyond a certain level, the marginal benefit of holding excess capital decreases, and the opportunity cost

of delaying dividend payouts dominates. Consequently, b^* declines when c continues to grow, reflecting a shift toward more aggressive dividend distributions when profitability is sufficiently high.

- (2) Impact of the mean positive jump a_+ (Table 3(b)). Similar to the premium rate c , the mean positive jump size a_+ enhances profitability of the firm, causing the dividend barrier b^* first rise and then decline as a_+ decreases (i.e., as jumps grow larger) as shown in Table 3(b). However, unlike premium rate c , which uniformly increase deficit tolerance (decrease of $-\underline{x}^*$), we observe that, for small to moderate positive jumps, the liquidation threshold $-\underline{x}^*$ declines only marginally when a_+ decreases, indicating a consistent but limited profitability effects on $-\underline{x}^*$ compared with the deterministic profitability parameter c . For very large positive jumps, (e.g., $a_+ = 0.01$), the firm adopts an aggressive dividend policy ($b^* = 0.165$) and liquidates earlier ($-\underline{x}^* = -5.444$), likely due to the increased uncertainty associated with infrequent but extreme upside jumps.
- (3) Impact of the mean negative jump a_- (Table 3(d)). As the mean negative jump size decreases (i.e., a_- increases), both $-\underline{x}^*$ and b^* decline. Lower downside risk improves profitability, making the firm more reluctant to liquidate and allowing for earlier dividend payouts due to reduced precautionary needs.
- (4) Impact of the Poisson intensity λ and the probability of a positive jump p (Tables 3(c) and 3(e)). These parameters jointly characterize the jump frequency in the model. A higher jump frequency λ leads to a less negative liquidation threshold $-\underline{x}^*$, as increased downside exposure makes continuation less attractive. However, b^* initially increases with λ (e.g., from 0.165 to 0.275), but eventually declines (e.g., when $\lambda = 1.3$). It is reasonable that when λ increases, it is optimal for the firm to adopt a conservative dividend policy. Meanwhile, due to the present of positive jumps in our model, high-frequency jump arrivals (even if small in size) may provide “self-insurance”, reducing the need for a large retained surplus. Thus, we observe a decrease in optimal dividend barrier, when λ reaches some high values.

Similarly, a higher p (more positive jumps) lowers $-\underline{x}^*$ and b^* , as improved recovery prospects encourage deeper deficit tolerance and an earlier dividend payout.

- (5) Impact of the diffusion coefficient σ (Table 3(f)). Table 3(f) shows that the optimal liquidation threshold $-\underline{x}^*$ remains relatively stable with only marginal changes, suggesting that volatility has a limited effect on the liquidation decision in this setting. In contrast, the dividend barrier b^* increases substantially with σ . Higher volatility amplifies the uncertainty of surplus fluctuations, increasing the risk of large downward movements even when the current surplus is positive. Consequently, the decision-maker is willing to maintain a larger precautionary surplus buffer and initiates dividends only at higher surplus levels, resulting in a more conservative dividend policy.
- (6) Impact of discount rate r (Table 3(g)). Table 3(g) shows that as r increases, the optimal liquidation threshold $-\underline{x}^*$ moves upward, indicating that the decision-maker tolerates smaller deficits before liquidation. Since heavier discounting lowers the present value of future cash flow, making continuation via capital injection less attractive for a certain negative surplus region. Meanwhile, the dividend barrier b^* decreases with r , as retaining a surplus for future use becomes less valuable, leading to dividend payments at lower surplus levels.

Table 3. The influences of the model parameters on the optimal policy.

	$-x^*$	b^*		$-x^*$	b^*
$c = 6.0$	-8.672	0.440	$c = 9.0$	-24.487	1.320
$c = 7.0$	-13.592	1.320	$c = 10.0$	-30.085	1.045
$c = 8.0$	-18.955	1.485	$c = 12.0$	-41.366	0.440

(a) Premium rate c

	$-x^*$	b^*		$-x^*$	b^*
$a_+ = 0.01$	-5.444	0.165	$a_+ = 0.5$	-9.815	0.935
$a_+ = 0.05$	-12.233	0.165	$a_+ = 1.0$	-8.672	0.440
$a_+ = 0.10$	-14.907	0.220	$a_+ = 2.0$	-7.912	0.220

(b) Positive jump parameter a_+

	$-x^*$	b^*		$-x^*$	b^*
$\lambda = 0.1$	-33.114	0.165	$\lambda = 1.0$	-8.672	0.440
$\lambda = 0.5$	-21.415	0.220	$\lambda = 1.3$	-3.751	0.220
$\lambda = 0.8$	-13.298	0.275	$\lambda = 1.5$	-1.712	0.165

(c) Poisson intensity λ

	$-x^*$	b^*
$a_- = 0.2$	-8.672	0.440
$a_- = 0.4$	-21.075	0.220
$a_- = 1.0$	-28.813	0.165

(d) Negative jump parameter a_-

	$-x^*$	b^*
$p = 0.2$	-8.672	0.440
$p = 0.5$	-18.102	0.220
$p = 0.8$	-27.202	0.165

(e) Positive jump probability p

	$-x^*$	b^*
$\sigma = 0.02$	-8.669	0.440
$\sigma = 1.0$	-8.639	0.880
$\sigma = 2.0$	-8.400	1.980

(f) Diffusion coefficient σ

	$-x^*$	b^*
$r = 0.08$	-10.439	1.925
$r = 0.10$	-8.672	0.440
$r = 0.15$	-6.359	0.220

(g) Discount rate r

Example 6.4. To verify that the finite computational domain does not distort the optimal strategy, we conduct a grid truncation robustness analysis by varying the upper bound B while holding all the other parameters and the grid step size h fixed (consistent with Example 6.1 with the double exponential case). Specifically, we compare the baseline truncation level $B = 30$ with a larger value of $B \in \{40, 60, 80\}$. The corresponding results are listed in Table 4. Larger values of B introduce no changes to the optimal policy thresholds. The results affirm that the baseline truncation $B = 30$ is numerically adequate for our numerical examples, as further domain expansion does not materially alter the optimal strategy.

Table 4. Robustness analysis with respect to the truncation threshold B .

	$-\underline{x}^*$	b^*
$B = 30$	-8.672	0.440
$B = 40$	-8.672	0.440
$B = 60$	-8.672	0.440
$B = 80$	-8.672	0.440

Example 6.5. In this final example, we calculate the optimal dividend barrier when the liquidation function is an exponential function (a pure penalty structure); that is, $P(x) = -Ae^{-ax}$ for $A, a > 0$. In particular, we let $A = 5$ and $a = 0.3$, such that $Aa > \phi_2$, and consider only the scenario with double exponential jumps. The other model parameters are set to the same values as in Example 6.1. According to Proposition 2.1(iii), one notes that the optimal liquidation threshold is $-\underline{x}^* = -\infty$, i.e., no liquidation*. The optimal dividend barrier is $b^* = 2.2$. Without the choice of liquidation, the decision-maker is willing to maintain a larger precautionary surplus buffer and initiates dividends only at higher surplus levels to avoid costly capital injections, compared with the one ($b^* = 0.44$) in Example 6.1.

7. Conclusion

In this paper, we study the optimal dividend and capital injection control under a two-sided jump-diffusion model with strategic liquidation options. Our research addresses a gap in the literature by allowing decision-makers the flexibility to choose between rescuing the firm through capital injection or liquidating it with a payoff when the surplus becomes negative, rather than imposing rigid constraints that either force capital injection or force liquidation. We have derived the HJB equation and characterized the critical thresholds for optimal strategies through the dynamic programming principle and a verification theorem. Given the analytical challenges posed by two-sided jumps, which preclude the use of traditional scale function methods, we adopted the MCA approach to seek for numerical solutions. We constructed a discrete-time controlled Markov chain that is locally consistent with the original jump-diffusion process, and rigorously established the convergence of both the approximating chain and the corresponding value function using weak convergence theory.

Our numerical implementation, based on the value iteration algorithm, provides robust computational tools for solving this complex stochastic control problem. Extensive numerical studies were conducted for the double exponential and exponential-Pareto jump distributions under affine liquidation functions. The results reveal several important insights. (1) The optimal liquidation threshold and dividend barrier are significantly influenced by the model's parameters. The premium rates, the positive jump potentials, and the Poisson jump intensity exhibit non-monotonic effects on the optimal strategies. (2) The interplay among dividend payments, capital injections, and liquidation decisions demonstrates sophisticated risk management behavior, namely, the decision-maker may balance the costs of capital injection against the benefits of liquidation with optimal strategies reflecting the trade-off between preserving future growth opportunities and minimizing the downside risks. (3) For affine liquidation functions, the optimal strategies exhibit distinct liquidation regions, while exponential penalty functions typically result in the classical double-barrier strategy without liquidation.

*Note that Algorithm 1 can be slightly modified by removing the liquidation threshold calculations in the iteration and replacing the boundary condition by $V_n^h(x) = V_n^h(0) + \phi_2 x$ for all $x \in (-\infty, 0)$. This result is also applicable to the scenario of Proposition 2.1(ii).

The flexibility of our numerical method enables future research in several directions, including multi-dimensional surplus dynamics, incorporating a stochastic economic environment (regime-switching models), and more complex jump structures and dependencies in the model.

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Author contributions

Ningning Feng: Writing – original draft; writing – review & editing; formal analysis; visualization. Yixuan Liu: Writing – original draft. Ran Xu: Conceptualization; writing – original draft; writing – review & editing; supervision; funding acquisition.

Use of Generative–AI tools declaration

The authors declare that they did not utilize any artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

Ran Xu is a guest editor for *Journal of Industrial and Management Optimization* and was not involved in the editorial review or the decision to publish this article. The authors declare no competing interests.

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