



Research article

Climate uncertainty and asset pricing decisions: A machine learning analysis

Shuhua Zhang^{1,2}, Tong Wang¹, Ning Ye¹, Xinyu Wang¹ and Rui Wang^{3,*}

¹ Coordinated Innovation Center for Computable Modelling in Management Science Tianjin University of Finance and Economics, Tianjin 300222, China

² Zhujiang College, South China Agricultural University, Guangzhou 510900, China

³ School of Law, Tianjin University of Finance and Economics, Tianjin 300222, China

* **Correspondence:** Email: 15922218980@163.com.

Abstract: Climate change and climate-model uncertainty have important implications for macroeconomic allocation, carbon policy, and asset pricing. This paper develops a multi-sector Dynamic Stochastic General Equilibrium (DSGE) model with temperature dynamics, climate damages, carbon accumulation, productivity shocks, and ambiguity-averse preferences. Methodologically, we integrate climate-model uncertainty, ambiguity aversion, DSGE modeling, and the Deep Galerkin Method (DGM) into a unified computational framework. This approach enables us to solve a high-dimensional equilibrium system and examine resource allocation, carbon taxation, and asset-pricing responses under uncertainty. The framework extends conventional climate-economy and carbon-pricing studies by jointly analyzing macroeconomic and financial effects in the presence of climate-model uncertainty. The numerical results show that uncertainty accelerates the reallocation from brown inputs to green inputs, raises the model-implied carbon tax, and affects the equilibrium risk-free rate and price of risk. These findings provide a model-based benchmark for carbon-pricing design, sustainable investment, and climate-related risk management by policymakers, regulators, and financial market participants.

Keywords: climate change; model uncertainty; machine learning; production-based carbon tax

Mathematics Subject Classification: 91G80, 93E20, 60J75, 49L20

1. Introduction

Industrialization and modernization have significantly altered the ecological environment and economic activity through increased greenhouse gas emissions, particularly carbon dioxide, generated by human activities. Rising carbon emissions intensify global climate change, directly resulting in significant environmental pollution and potential catastrophic calamities. The indirect detrimental

impacts of climate change on economic development and industry activities must not be overlooked. The impact is far-reaching and challenging to precisely measure, yet it can provide substantial adverse impacts on economic structure and social development over time. We develop a stochastic dynamic general equilibrium model to examine the macroeconomic and asset-pricing implications of climate change and model uncertainty.

While current research in earth science reveals that all climate models and their economic impacts exhibit specific evolutionary patterns and characteristics, these models generally contain uncertainties. The repercussions of these uncertainties on the economy are frequently more severe than the apparent risks. There are substantial discrepancies in the Carbon-Climate Response (CCR) estimations produced by various models. For instance, from 11 coupled climate-carbon cycle models, the CCR values range from 1.0 to 2.1 °C/TtC [1, 2] (Figure 1). Although part of the discrepancy may reflect estimation error in the volatility parameters, the realized temperature path remains intrinsically uncertain even after parameter calibration. Furthermore, it is challenging to characterize and specify the effects of climate change on the economy, and this uncertainty is exacerbated by dramatic temperature fluctuations [3], geographical disparities, and high and low temperatures [4]. Using 15 Earth system models of varying complexity, the study assessed how alternative model architectures and assumptions affect carbon storage and derived the atmospheric CO₂ impulse response function (IRF_{CO_2}) [5]. Their analysis documented substantial cross-model variation in simulated atmospheric carbon stocks, highlighting the extent of uncertainty in carbon-cycle dynamics.

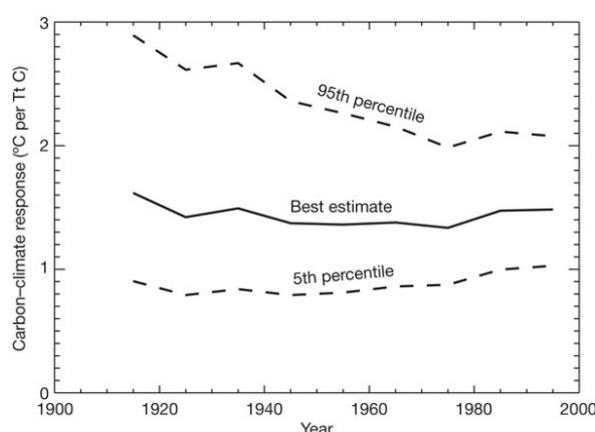


Figure 1. Observational estimation of the carbon-climate response.

The substantial uncertainty in climate and economic models raises the need for robust decision-making [6–8]. Such concerns arise from model misspecification, climate-damage uncertainty, carbon-stock dynamics, and economic projections. Additional ambiguity also stems from parameter differences across alternative climate scenarios. We introduce ambiguity-averse preferences to capture these concerns in a formal way. In the model, uncertainty is represented through penalty terms. The strength of ambiguity aversion is governed by penalty parameters. These parameters affect optimal allocation, carbon taxation, and asset-pricing outcomes.

In order to account for many uncertainties, such as shifts in atmospheric carbon stocks, variations in global temperatures, shifts in productivity, and economic harm from climate change, we build a multisectoral dynamic general equilibrium model. Our model specifically distinguishes three primary

sectors of social production: two sectors of production inputs and one sector of final output, where the goods directly meet societal consumption demands. The “brown sector” is one of the production input sectors that produces a lot of carbon emissions during the production process, which has an adverse effect on the environment. The “green sector”, on the other hand, uses non-polluting production methods, but using green inputs is more expensive. All socially productive activities require labor, which will be divided among these three sectors. The goal of this model is to maximize social welfare, or the amount of consumption, in order to determine the social planner’s best course of action.

We formulate the economic decision problem in continuous time and analyze it within a dynamic programming framework. This approach is closely related to the literature on stochastic differential equations and stochastic differential games, where value functions are characterized under regime-switching dynamics, jump-diffusion processes, and random horizons [9–11]. The central object of the analysis is the value function, which represents the optimal expected payoff attainable from the current state until the terminal time. Under appropriate regularity conditions, the value function is characterized by the associated Hamilton-Jacobi-Bellman (HJB) equation, together with the terminal or boundary conditions implied by the underlying economic model. Because the HJB equation generally takes the form of a nonlinear partial differential equation, numerical methods such as neural network approximation, finite difference schemes, and finite element methods are typically employed for its solution. In special cases where the value function admits a tractable functional form, an analytical solution may be derived. Such analytical solutions provide useful benchmarks for validating numerical results.

This study employs the Deep Galerkin Method (DGM) to solve the high-dimensional HJB equation generated by the model. DGM approximates the unknown value function with a neural network. The network is trained by minimizing a loss function that measures the residual of the HJB equation and the associated boundary conditions. Unlike grid-based methods, DGM uses random sampling over the relevant computational domain. This feature reduces the computational burden in high-dimensional settings. It is particularly useful in our framework, where the state variables include temperature, atmospheric carbon stocks, productivity, and climate damages. DGM therefore provides a tractable numerical approach for approximating the value function and recovering the corresponding optimal policies.

After calibrating the model parameters, we solve the model to examine the long-run effects of climate change and climate-model uncertainty on macroeconomic allocation and asset pricing. We use a social-planner framework and compare the benchmark economy with a no-uncertainty case. The numerical results show that climate-model uncertainty affects resource allocation, carbon taxation, and asset-pricing outcomes. In the calibrated economy, uncertainty is associated with a reallocation from brown inputs to green inputs and a higher model-implied carbon tax. It also affects the equilibrium risk-free rate and the price of risk. These patterns provide a model-based benchmark for carbon-pricing design, sustainable investment, and climate-related risk management. We further conduct sensitivity analyses to assess how changes in key parameters influence the numerical results.

This paper contributes to the literature in three ways. First, we develop a stochastic multisector dynamic general equilibrium model that incorporates climate-model uncertainty, model misspecification, and ambiguity-averse preferences. By distinguishing brown and green inputs, the model links climate uncertainty to production choices, resource allocation, and carbon-policy design. This extends conventional climate-economy and carbon-pricing studies, which often focus on climate

damages or optimal taxation in isolation. Second, we formulate the social planner's problem in continuous time and reduce it to a high-dimensional HJB equation. We then apply DGM to obtain numerical solutions. This provides a computational framework for analyzing high-dimensional climate-economy models under uncertainty. Third, the numerical analysis provides a model-based benchmark for understanding how climate-model uncertainty affects carbon taxation, sectoral reallocation, and asset prices. These results offer useful implications for policymakers, regulators, financial market participants, and firms. They are relevant to carbon-pricing design, sustainable investment, corporate technology choice, and climate-related risk management.

The remaining structure of this paper is as follows: Key research in related fields is introduced in Section 2. The baseline model and climate modeling uncertainties are presented in Section 3. The welfare maximization problem for social planners with recursive preferences is formulated in Section 4. Results pertaining to asset pricing and macroeconomics are presented in Section 5. Experimental results and numerical methods are described in detail in Section 6. Key parameters are subjected to sensitivity analysis in Section 7. The research findings are presented in Section 8.

2. Related literature

The climate-economy literature has substantially extended early integrated assessment models, such as Nordhaus's DICE framework. To account for multiple uncertainties in policy analysis, researchers have developed dynamic stochastic general equilibrium models [12, 13], incorporated recursive preferences to evaluate future climate risks [14, 15], and applied robust preference theory to characterize aversion to model uncertainty [14–16]. Related studies in stochastic control and differential games have further expanded the modeling of uncertainty to Markov regime-switching and jump-diffusion settings [17–19]. Against this background, our paper extends Barnett's multisectoral production integrated assessment framework [20] along the preference dimension by moving beyond the standard log-log utility specification to a general CRRA utility function. This is a key innovation of the paper. It enables the model to flexibly capture heterogeneous degrees of risk aversion and intertemporal substitution, thereby providing a richer welfare assessment of long-run climate risks. More importantly, by explicitly linking risk aversion to model uncertainty, the framework offers a more realistic basis for evaluating social planners' climate policy choices in complex information environments.

To evaluate carbon pricing and the social cost of carbon, it is crucial to characterize the multiple sources of uncertainty embedded in the climate-economy system. The literature has examined uncertainty related to productivity, climate change, and carbon stock [5, 21, 22], while broader surveys also underscore the importance of uncertainty in climate-economy analysis [23, 24]. Prior studies have incorporated growth and damage uncertainty into optimal carbon tax design [25], analyzed carbon pricing under alternative sources of risk [26], and shown that uncertainty can significantly increase the social cost of carbon [27, 28]. Related work further documents the macroeconomic effects of the low-carbon transition [29] and the role of uncertainty in asset pricing [30]. This paper extends the literature by centering on carbon stock uncertainty and its implications for optimal carbon pricing. Specifically, by explicitly modeling the dynamic link between carbon emissions and greenhouse gas concentrations, we show how carbon stock uncertainty affects temperature projections and the evolution of optimal carbon prices. Moreover, by solving for social welfare and optimal carbon pricing

endogenously and incorporating CRRA preferences, we identify how different forms of uncertainty interact with societal risk aversion in shaping climate policy.

The literature has developed a broad set of numerical tools to study forecasting and optimal carbon pricing under uncertainty, including Monte Carlo simulations [31, 32], stochastic dynamic optimization methods [33, 34], and other numerical techniques for stochastic optimal control [12, 27, 35]. This research has further expanded to stochastic hybrid systems with jumps and higher-order numerical schemes [7, 36, 37], alongside pseudo-analytical [28] and approximate closed-form solutions [20]. Despite these advances, conventional approaches become computationally burdensome in the presence of high-dimensional, multi-source uncertainty. Building on this literature, we solve the model with the Deep Galerkin Method (DGM), which uses neural networks to approximate the value function over the high-dimensional state space. The use of DGM is a key innovation of this paper, as it provides a tractable and scalable framework for robust optimal policy analysis and long-horizon carbon-pricing problems under complex uncertainty.

3. Model of climate risk and uncertainty

3.1. Climate change

According to previous climate modeling studies, climate warming and CO₂ emissions are strongly correlated, and global mean surface temperature exhibits an essentially linear characteristic for cumulative CO₂ emissions, as verified in simulations of historical and future climate scenarios [1, 38]. This linear relationship can be described by the transient climate response to cumulative CO₂ emissions (TCRE) [39], which, on the one hand, establishes a simple quantitative relationship between CO₂ emissions and the magnitude of warming and, on the other hand, covers the rate of warming. We set the temperature above the pre-industrial value as T_t , which gives the dynamics of temperature T_t based on the TCRE setting:

$$dT_t = \lambda F_t dt + \sigma'_T dW_t, \quad (3.1)$$

where F_t is the carbon emissions generated by the production process. The parameter λ measures climate sensitivity, representing the ratio between the change in global average temperature and cumulative emissions when CO₂ concentration is doubled (°C/TtC), the TCRE parameter. The constant fluctuation rate of temperature is denoted by σ_T , and W_t is the Brownian motion.

Uncertainty in temperature variation arises from numerous sources. Uncertainty in TCRE primarily stems from ambiguities in physical climate parameters, such as CO₂ radiative forcing, climate feedback coefficients, and ocean heat uptake efficiency. The uncertainties in these parameters influence the precision of TCRE estimates [40–42]. Moreover, terrestrial and oceanic processes are modeled and disseminated distinctly in Earth system models. The degree to which terrestrial and oceanic processes influence the response to climate change is also a factor to consider [43]. Consequently, there are constraints in simulating TCRE, and existing Earth system models may not comprehensively encompass all pertinent physical, chemical, and biological variables, thereby introducing uncertainty in the estimations.

The uncertainty in TCRE directly influences the evaluation of the carbon budget required to meet the climate objectives of the Paris Agreement. This ambiguity may pose challenges for policymakers

in establishing emission reduction targets, potentially impacting economic growth and investment choices [44]. This uncertainty can induce reluctance in investing in low-carbon technologies, thereby impacting industrial transformation and international trade dynamics. Moreover, there exists ambiguity regarding the extent and intensity of substantial and swift decreases in carbon dioxide emissions, which may induce volatility in energy markets, thereby impacting economic stability and corporate profitability. This requires managers to thoroughly consider these potential uncertainties when developing their strategies.

3.2. Production

The production system of society as a whole is divided into three main sectors. The first sector is responsible for the production of goods for social consumption C_t and is called the final goods sector Y_t . The remaining two input sectors are responsible for the production of intermediate goods, which are not used for consumption but are inputs into the final goods sector as factors of production. For the two sectors producing intermediate goods, one emits substantial carbon during production and is termed the “brown sector” or “polluting sector”, with output denoted as $B_t = F_t$. The other sector generates negligible carbon emissions during production and is termed the “green sector” or “clean sector”, with output denoted as G_t .

Competitive industries with declining scale technologies create the two distinct inputs required in the final goods sector. Labor serves as an input in production technology and is adjusted for deflation using productivity factors. Utilizing the Cobb-Douglas production function, the formulas for the final product sector Y_t and the two input sectors B_t and G_t are as follows:

$$Y_t = A_t L_{C,t}^{1-\alpha-\nu} \left[(B_t^\omega + G_t^\omega)^{\frac{1}{\omega}} \right]^\nu, \quad (3.2)$$

$$B_t = A_{B,t} (L_{B,t})^{\alpha_B}, \quad G_t = A_{G,t} (L_{G,t})^{\alpha_G}, \quad (3.3)$$

where t is time, A_t is the total factor productivity of the final goods sector, and $L_{i,t}$, $i \in \{C, B, G\}$ is the labor used in the production process of the final goods sector, the brown sector, and the green sector, respectively. The parameter $0 \leq \alpha < 1$ is the diminishing returns to scale. Specifically, α_i , $i \in \{B, G\}$ denotes the diminishing returns to scale parameter for labor used in input production in the brown and green sectors. ν is the share of input demand for non-labor inputs, and ω is the elasticity of substitution between green and brown product inputs. To simplify the problem, the productivity of the input sector is assumed to be a given constant A_B, A_G .

Further assume that A_t follows the geometric Brownian motion process:

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma'_A dW_t, \quad (3.4)$$

where μ_A is the drift of total factor productivity, σ_A is the constant volatility of productivity, and W_t is Brownian motion. Our framework abstracts from two additional emission channels. It does not explicitly model carbon emissions generated by the final goods sector during production, nor does it fully incorporate the relatively small emissions associated with green energy production and use. This modeling choice preserves tractability and allows the analysis to focus on the central mechanism of substitution between brown and green inputs. It should therefore be understood as a deliberate

abstraction rather than an assumption that these emissions are irrelevant. This abstraction may affect the quantitative magnitude of the simulation results. If these additional emission channels are incorporated, total emissions, atmospheric carbon accumulation, temperature changes, and climate damages may be higher than those reported in the baseline simulations. The net mitigation effect of replacing brown inputs with green inputs may also become less pronounced. Nevertheless, the main economic mechanism remains valid, as a shift toward green inputs still reduces exposure to carbon-intensive production. Therefore, the numerical results should be interpreted as benchmark outcomes under a simplified emissions structure, rather than as a full life-cycle accounting of all emission sources. Future research may extend the framework by incorporating these additional emission channels.

3.3. Carbon stocks

The average global temperature rises as atmospheric carbon concentrations rise. According to [25], a doubling of atmospheric carbon content might result in a world average temperature increase of 3°C. Diverse assumptions regarding carbon stock modeling can substantially influence energy consumption trajectories, which subsequently alter carbon emission levels and atmospheric carbon concentration trends, ultimately impacting global temperature variations. In a wholly unregulated market economy, disregarding the constraints on energy use imposed by carbon reserves could lead to the excessive exploitation of coal resources, assuming an inexhaustible supply. This would exacerbate temperature increases and inflict significant harm on the economy.

The utilization of fossil fuels in human industrial activities is a significant source of carbon emissions, which are discharged into the atmosphere and thereby contribute to the global carbon cycle. During this cycle, carbon is exchanged and sequestered across several reservoirs, including the atmosphere, terrestrial biosphere, and oceans. We are especially focused on the atmospheric CO₂ concentration, which is a crucial factor in assessing climate change. Consequently, a connection must be established to demonstrate the impact of carbon emissions on atmospheric CO₂ concentrations.

The total carbon emissions by humans since industrialization can be represented as $E = S - S_{PI}$, where S denotes the current atmospheric carbon stock, and S_{PI} signifies the pre-industrial carbon stock. According to the research conducted by [28], the dynamics of the carbon stock E_t progresses as follows:

$$dE_t = [\psi B_t - \phi E_t] dt + \sigma'_T dW_t, \quad (3.5)$$

where $0 < \psi < 1$ is the proportion of carbon emitted from fossil fuels into the atmosphere, and $\phi \geq 0$ indicates the rate of atmospheric carbon decay. σ_T is constant volatility, and W_t is Brownian motion.

3.4. Climate damages

The damage function $D(T_t)$, which finds widespread use in Integrated Assessment Models (IAMs), measures the effect of temperature fluctuations on social welfare, economic output, and policy decisions. A variety of probabilistic damage functions, including quadratic, exponential, and sixth-order forms, are summarized and fitted by [45]. However, there is a significant amount of structural uncertainty because these functions are fitted only using the data that is currently available.

Damage functions of level-affected and growth-rate-affected types are common. Level-impact functions, like the inverse quadratic function (L-N norm) in the DICE model, make the assumption that current temperatures have a direct impact on GDP levels. At moderate temperatures, these functions

behave similarly, but at higher temperatures, they diverge greatly. Growth-rate impact functions, on the other hand, represent damage as having a detrimental impact on GDP growth rates. This is accomplished by adjusting parameters to account for the negative effects of global warming on capital depreciation, which in turn have an impact on economic growth.

Following related literature [25, 46, 47], we adopt a composite functional form combining quadratic polynomials and exponentials. The correlation between the damage function and ultimate output, societal consumption, is represented by $C_t = Y_t/D(T_t)$ and diminishes as the temperature T_t rises. The first- and second-order derivatives of the damage function $D(T_t)$ are denoted as $[\nabla D](T_t)$ and $[\nabla^2](T_t)$, respectively, with the condition that $[\nabla D](T_t) > 0$. The functional representation of climatic damage is expressed as $\mathbb{D}(T_t) = \exp(D(T_t))$, where

$$D(T_t) = \gamma_1 T_t + \frac{\gamma_2}{2} T_t^2, \quad (3.6)$$

where γ_1 and γ_2 denote damage parameters that govern the sensitivity of economic output to temperature increases. The damage function is specified in logarithmic form as $Z_t = \log(\mathbb{D}(T_t)) = D(T_t)$, which evolves according to the following expression:

$$dZ_t = [\nabla D](T_t) \lambda B_t dt + \frac{|\sigma_T|^2}{2} [\nabla^2 D](T_t) dt + \sigma'_T [\nabla D](T_t) dW_t. \quad (3.7)$$

The function's form and parameterization mostly reflect the climatic damage function's uncertainty. Economic effects can vary depending on the type of damage function. For instance, [48] showed that the exponential form, when combined with a heavy-tailed probability distribution, significantly increases the willingness to pay to mitigate uncertainty. In contrast, the quadratic polynomial form in current IAMs may underestimate damage at higher temperatures. Furthermore, there are uncertainties in the damage function's key parameters and climate sensitivity parameters, which interact to influence social welfare. According to [47], efficiency-driven planners typically postpone abatement until the risks of climate change become apparent, while social planners who favor early resolution of uncertainty will choose immediate and relatively strict abatement policies to prevent a major future catastrophe.

3.5. Uncertainty

Numerous studies have investigated methods to reduce uncertainty in climate models, focusing on enhancing the description of physical processes and incorporating parameter uncertainty and sensitivity analysis. These efforts have led to valuable suggestions for improving the accuracy and reliability of climate models. In practice, various other disciplines and interdisciplinary research are involved in this source of climate model uncertainty. Numerous factors, including the urban heat island effect, topography, and geomorphology, influence atmospheric motions, which are inherently complex and exhibit random variability. Additionally, there are errors in both weather observations and ocean temperature measurements, and radiative forcing models can affect the prediction of global temperature. These uncertainties occasionally have a major impact on the outcomes of our analyses, but they cannot always be completely eliminated.

We call the temperature, productivity, carbon stock, and climate damage evolution outlined in the preceding section the baseline model for policymakers. We include a drift perturbation term (Λ_s) to represent the introduction of an alternative model because the baseline model might not be

the most accurate and ideal setup due to inaccurate settings or measurement errors, as well as the complex diversity and uncertainty of the real situation. By expressing the basic Brownian motion W_t as $W_t + \int_0^t \Lambda_s ds$, we derive a climate model that incorporates uncertainty:

$$\begin{aligned} dT_t &= [\lambda B_t + \sigma'_T \Lambda_{T,t}] dt + \sigma'_T dW_t, \\ dZ_t &= [\nabla D](T_t) [\lambda B_t + \sigma'_T \Lambda_{T,t}] dt + \frac{|\sigma_T|^2}{2} [\nabla^2 D](T_t) dt + \\ &\quad \sigma'_T [\nabla D](T_t) dW_t, \\ dE_t &= [\psi B_t - \phi E_t + \sigma'_T \Lambda_{T,t}] dt + \sigma'_T dW_t, \\ \frac{dA_t}{A_t} &= [\mu_A + \sigma'_A \Lambda_{A,t}] dt + \sigma'_A dW_t. \end{aligned}$$

Uncertainty is divided into two dimensions: one based on temperature change and uncertainty in the carbon stock and damage functions associated with the climate dimension, and the other considering uncertainty in productivity. The great advantage of using alternative models is that we do not need to give specific parameterized forms of the specific alternative climate models and perturbation terms, but instead use Λ_s to represent a very broad and multiple form of alternative choices.

4. Solution of the social planner's problem

4.1. Robust preference

We include robust preference terms to reflect uncertainty aversion in the model. In models of climate change, carbon stocks, climate damage, and economic productivity, robust preferences are linked to uncertainty and are constrained by the uncertainty aversion parameter ϱ . The uncertainty in both dimensions is described independently, with offset distortions Λ_A in the productivity model deflated by the penalty parameter ϱ_A , and offset distortions Λ_T in the climate change-related model deflated by the penalty parameter ϱ_T .

The utility function is augmented with quadratic expressions $\frac{1}{2}\varrho_T \Lambda'_T \Lambda_T$ and $\frac{1}{2}\varrho_A \Lambda'_A \Lambda_A$, both of which represent possible worst-case model offsets. To guard against potential misspecification in the baseline model—such as calibration errors or incorrect parameterization—that could distort analytical outcomes and policy decisions, the worst-case model is characterized as an adversarial constraint within the social planner's problem. The planner then maximizes social welfare subject to this constraint, yielding a robust optimal policy. Robust preference not only motivates the planner to be more cautious in decision making but also enhances the adaptability of the optimal decision under different model scenarios to better cope with the risks associated with uncertainty.

4.2. Social planner's problem

Consider the social planner's optimization problem and use the Constant Relative Risk Aversion (CRRA) utility function to carve out consumption. The marginal utility of consumption is assumed to have a constant elasticity η of the form:

$$U(C_t) = \frac{C_t^{1-\eta}}{1-\eta},$$

where η denotes the coefficient of relative risk aversion. As $\eta \rightarrow 0$, the level of risk aversion is minimal. A larger η indicates a greater degree of risk aversion, signifying that society prioritizes risk aversion in welfare distribution and economic activities, favoring stable and certain inputs. In the extreme case where $\eta = 1$, the utility function simplifies to the logarithmic form $U(C_t) = \ln(C_t)$. [49] interpreted this as a preference against intergenerational inequality, with the elasticity indicating the decreasing social value of consumption over generations.

Labor, as a shared input factor across all three production sectors, must be allocated efficiently among them. The equilibrium solution of the social planner is dictated by the optimal labor allocation $(L_{C,t}^*, L_{B,t}^*, L_{G,t}^*)$ for each production technology, as the selection of labor allocation directly influences consumption and production activities in equilibrium.

Our optimization problem is a max-min problem that initially minimizes the error distortion $(\Lambda_{T,t}, \Lambda_{A,t})$ of the model, constrained by the penalty parameter (ϱ_T, ϱ_A) of uncertainty. Subsequently, the optimal labor allocation is evaluated alongside the identified worst-case models $\Lambda_{T,t}^*$ and $\Lambda_{A,t}^*$ to determine the maximized social welfare for the economy. The value function and the optimal policy function depend on the state variables A_t, Z_t, T_t, E_t , indicating that the social planner problem possesses a recursive Markov equilibrium solution.

The problem of maximizing social welfare for social planners is as follows:

$$J(A_t, Z_t, T_t, E_t) = \max_{L_{C,t}, L_{B,t}, L_{G,t}} \min_{\Lambda_{A,t}, \Lambda_{T,t}} E_t \left[\int_t^\infty \exp(-\rho s) \left\{ \rho \frac{C_s^{1-\eta}}{1-\eta} + \varrho_A \frac{1}{2} \Lambda'_{A,s} \Lambda_{A,s} + \varrho_T \frac{1}{2} \Lambda'_{T,s} \Lambda_{T,s} \right\} ds \right].$$

Here, ρ represents the subjective discount rate. The planner's problem is subject to the following production constraints:

$$\begin{aligned} C_t &= \exp(-Z_t) Y_t = \exp(-Z_t) A_t (L_{C,t})^{1-\alpha-v} (B_t^\omega + G_t^\omega)^{v/\omega}, \\ B_t &= \bar{A}_B (L_{B,t})^{\alpha_B}, \\ G_t &= \bar{A}_G (L_{G,t})^{\alpha_G}, \\ L_t &= L_{C,t} + L_{B,t} + L_{G,t} = 1. \end{aligned}$$

Additionally, the model incorporates stochastic processes for temperature, climate damage, productivity, and carbon stocks:

$$\begin{aligned} dT_t &= [\lambda B_t + \sigma'_T \Lambda_{T,t}] dt + \sigma'_T dW_t, \\ dZ_t &= [\nabla D](T_t) [\lambda B_t + \sigma'_T \Lambda_{T,t}] dt + \frac{|\sigma_T|^2}{2} [\nabla^2 D](T_t) dt + \\ &\quad \sigma'_T [\nabla D](T_t) dW_t, \\ dE_t &= [\psi B_t - \phi E_t + \sigma'_T \Lambda_{T,t}] dt + \sigma'_T dW_t, \\ \frac{dA_t}{A_t} &= [\mu_A + \sigma'_A \Lambda_{A,t}] dt + \sigma'_A dW_t. \end{aligned}$$

By integrating dynamic optimization with optimal control theory, we can address the social planner problem and obtain a socially optimal solution to the model. The subsequent proposition will present the value function of the social planner, defined by the HJB equation. Resolving the first-order conditions (FOC) of the HJB equation produces the optimal decision for the control function, subsequently solved numerically for the value function. The solution of the value function enables an analysis of how climate change and the uncertainty inherent in climate models influence social welfare and optimal policy decisions.

Proposition 1 (Social planner's HJB equation). *The value function $J(A_t, Z_t, T_t, E_t)$ of the social planner's problem satisfies the following equation:*

$$\begin{aligned} \rho J = & \frac{\rho}{1-\eta} \exp(-Z_t(1-\eta)) A_t^{1-\eta} L_{C,t}^{(1-\alpha-\nu)(1-\eta)} (B_t^\omega + G_t^\omega)^{\frac{\nu(1-\eta)}{\omega}} - \frac{1}{2\varrho_A} \sigma_A^2 A_t^2 J_A^2 \\ & - \frac{1}{2\varrho_T} \sigma_T^2 [J_T + J_E + [\nabla D] J_Z]^2 + \mu_A A_t J_A + \frac{1}{2} \sigma_A^2 A_t^2 J_{AA} - \phi E_t J_E \\ & + [\lambda J_T + [\nabla D] \lambda J_Z + \psi J_E] B_t + \frac{1}{2} \sigma_T^2 (J_{TT} + J_{EE} + [\nabla D]^2 J_{ZZ} + [\nabla^2 D] J_Z). \end{aligned}$$

The final output labor force is represented as a function of B_t and G_t , with the worst-case model deviations initially provided as

$$\begin{aligned} \Lambda_{A,t}^* &= -\frac{1}{\varrho_A} \sigma_A A_t J_A, \\ \Lambda_{T,t}^* &= -\frac{1}{\varrho_T} \sigma_T [J_T + J_E + [\nabla D](T_t) J_Z]. \end{aligned}$$

The value function is a solution of the state variables A_t, Z_t, T_t, E_t , as proven in the Appendix A.1.

Optimal policy requires balancing short-term economic growth against long-term climate risks. Beyond the current output generated by the brown sector (B_t), social planners must account for future climate damages (Z_t) arising from temperature dynamics (T_t) and carbon accumulation (E_t). Policy design therefore needs to consider adverse climate realizations and tail risks. Appropriately pricing climate risk and reallocating resources toward the green sector (G_t) contribute to enhancing long-term social welfare. The following proposition gives the optimal control equations for B_t and G_t .

Proposition 2 (Optimal policies). *The optimal choices of B_t and G_t are given in the form of implicit equations:*

$$\begin{aligned} 0 = & \rho \exp(-Z_t(1-\eta)) A_t^{1-\eta} L_{C,t}^{(1-\alpha-\nu)(1-\eta)} (B_t^\omega + G_t^\omega)^{\frac{\nu(1-\eta)}{\omega}} \left\{ -\frac{(1-\alpha-\nu)}{\alpha_B A_B^{1/\alpha_B} L_{C,t}} B_t^{1/\alpha_B-1} + \frac{\nu}{(B_t^\omega + G_t^\omega)} B_t^{\omega-1} \right\} \\ & + \lambda J_T + \lambda [\nabla D] J_Z + \psi J_E, \\ 0 = & \rho \exp(-Z_t(1-\eta)) A_t^{1-\eta} L_{C,t}^{(1-\alpha-\nu)(1-\eta)} (B_t^\omega + G_t^\omega)^{\frac{\nu(1-\eta)}{\omega}} \left\{ -\frac{(1-\alpha-\nu)}{\alpha_G A_G^{1/\alpha_G} L_{C,t}} G_t^{1/\alpha_G-1} + \frac{\nu}{(B_t^\omega + G_t^\omega)} G_t^{\omega-1} \right\}. \end{aligned}$$

Similar to the value function, B_t and G_t are solutions pertaining to the state variables A_t, Z_t, T_t, E_t . In the next section, the HJB equation and the governing equations are solved simultaneously by numerical methods, and the proof procedure and code for the propositions can be seen in the Appendix A.1.

According to the previously mentioned optimal policy equation, determining a fair price for the environmental effects of the brown sector B_t is at the heart of management choices. The cost of brown inputs therefore incorporates both conventional production costs and the marginal climate damages associated with carbon emissions, which are specifically represented by the term $\lambda J_T + \lambda[\nabla D]J_Z + \psi J_E$. This formulation implies that efficient policy design requires mechanisms that internalize these external costs, such as carbon taxation or equivalent pricing instruments.

In contrast, the green sector G_t has an inherent competitive advantage because it is not obligated to pay for such climate damage costs. With the help of price signals, this mechanism ensures that resources will naturally move from the brown sector to the green sector in terms of allocation while pursuing economic output, ultimately aiding in the accomplishment of sustainable development goals.

4.3. Asset prices

A market economy must be introduced into a decentralized economy in order to arrive at the optimal asset pricing outcomes for the social planner's problem, including the carbon tax. By enforcing a carbon tax on the profits invested in production by the polluting sector, we establish a decentralized mechanism for the prices generated in the social planner's problem. This is a brief summary of the carbon tax's findings. Appendix A.2 contains the detailed extended model and derivation. Due to the externality of carbon emissions from the use of polluting inputs in the manufacture of final goods, the decentralization mechanism has been chosen. In other words, when the solution to the social planner's problem agrees with the decentralized market's solution, the optimal tax is achieved. This rate, under the ideal social planning scenario, is given by the following expression.

Proposition 3 (Optimal carbon tax). *The social planner's allocation can be decentralized through market-clearing prices by imposing a carbon tax τ_t on polluting input use in the brown sector. The optimal carbon tax rate satisfies:*

$$\tau_t = \frac{-[\lambda J_T + \lambda[\nabla D]J_Z + \psi J_E]}{\nu \rho e^{-Z_t(1-\eta)} A_t^{1-\eta} L_{C,t}^{(1-\alpha-\nu)(1-\eta)} (B_t^\omega + G_t^\omega)^{\frac{\nu(1-\eta)}{\omega}-1} B_t^{\omega-1}},$$

where each variable attains its socially optimal value, coinciding with the allocation chosen by the social planner.

The carbon tax rates are not set arbitrarily; rather, they dynamically reflect the balance between the economic contributions made by brown production activities and the full marginal damages that these activities impose on society, including harm to ecosystems, temperature, and carbon stocks. Policymakers have access to an essential management tool through this mechanism. Price signals can be used to direct market resources to naturally shift from the brown sector to the green sector by enacting this social cost-based carbon tax. In the end, this makes it possible to meet climate goals and maximize social welfare at the lowest possible cost.

After that, we go over the stochastic discount factor and calculate the price of risk and uncertainty as well as the risk-free rate. Initially, the stochastic discount factor (SDF) $\hat{\pi}_t$ for the CRRA utility function can be written as $\hat{\pi}_t = \exp(-\rho t)\rho C_t^{-\eta}$. [50] showed that in the robust preference setting, the SDF must be updated by multiplying it by an adjustment term for every model uncertainty channel. Accordingly, the stochastic discount factor takes the following modified form under model uncertainty:

$$\pi_t = \exp(-\rho t)\rho C_t^{-\eta}\zeta_{A,t}\zeta_{T,t},$$

where the Radon-Nikodym derivatives linked to the worst-case model results are exponentially harnessed to these adjustment terms $\zeta_{i,t}(i \in A, T)$:

$$\begin{aligned}\zeta_{i,t} &= \exp\left(-\frac{1}{2} \int_0^t (\Lambda_{i,\tau}^*)'(\Lambda_{i,\tau}^*)d\tau + \int_0^t (\Lambda_{i,\tau}^*)'dW_\tau\right), \\ d\zeta_{i,t} &= \zeta_{i,t}(\Lambda_{i,t}^*)'dW_t.\end{aligned}$$

The following proposition gives an expression for the risk-free interest rate based on the social planner model, and the full derivation of this result can be found in Appendix A.3.

Proposition 4 (Risk-free rate). *The stochastic discount factor's evolution formula is displayed below:*

$$\frac{d\pi_t}{\pi_t} = -r_{f,t}dt - \sigma'_{\pi,T}dW_t - \sigma'_{\pi,Z}dW_t - \sigma'_{\pi,E}dW_t - \sigma'_{\pi,A}dW_t,$$

where $\sigma_{\pi,T}, \sigma_{\pi,Z}, \sigma_{\pi,E}, \sigma_{\pi,A}$ is the price of risk, and $r_{f,t}$ is the risk-free rate. The risk-free rate can be expressed as follows:

$$\begin{aligned}r_{f,t} &= \rho + \left(\mu_A - \frac{1}{\varrho_A}J_A|\sigma_A|^2A_t\right) + \eta\left(\lambda B_t - \frac{1}{\varrho_T}[J_T + J_E + [\nabla D](T_t)J_Z]|\sigma_T|^2\right)\frac{C_T}{C} \\ &+ \frac{1}{2}\eta|\sigma_T|^2\left(\frac{C_{TT}}{C} - (\eta + 1)\left(\frac{C_T}{C}\right)^2\right) + [\nabla D](T_t)\left(\lambda B_t - \frac{1}{\varrho_T}[J_T + J_E + [\nabla D](T_t)J_Z]|\sigma_T|^2\right) \\ &+ \frac{1}{2}|\sigma_T|^2[\nabla^2 D](T_t) + \left(\psi B_t - \phi E_t - \frac{1}{\varrho_T}[J_T + J_E + [\nabla D](T_t)J_Z]|\sigma_T|^2\right) \\ &- \frac{1}{2}|\sigma_T|^2 - \frac{1}{2}[[\nabla D](T_t)]^2|\sigma_T|^2 - \frac{1}{2}|\sigma_A|^2(A_t)^2.\end{aligned}$$

The consumption C_t is shortened to C , where C_T and C_{TT} represent the first-and second-order derivatives of C_t with respect to temperature T_t , respectively.

Climate risks are actually implied by the “risk-free” interest rate. High temperatures and high carbon emissions will increase the level of economic uncertainty in the future, which will raise interest rates. As a result, the financing costs for the entire society will increase. In the short term, brown economic activities B_t that depend on fossil fuels can boost output, but in the long run, they will put more strain on themselves and the economy as a whole due to higher capital costs as a result of the climate crisis. Proposition 5 illustrates the market cost of risk and uncertainty.

Proposition 5 (Market price of risk, market price of uncertainty). *Equations for temperature, climate damage, carbon stocks, and productivity risk prices and uncertainty prices (i.e., the impact of model uncertainty on market risk prices) are as follows:*

$$\begin{aligned}\sigma_{\pi,T} &= \sigma_T \left[\eta \frac{C_T}{C} + \frac{1}{Q_T} J_T \right], \\ \sigma_{\pi,Z} &= \sigma_T [\nabla D](T_t) \left[\frac{1}{Q_T} J_Z + 1 \right], \\ \sigma_{\pi,E} &= \sigma_T \left[\frac{1}{Q_T} J_E + 1 \right], \\ \sigma_{\pi,A} &= \sigma_A A_t \left[\frac{1}{Q_A} J_A + 1 \right], \\ \Lambda_{A,t}^* &= -\frac{1}{Q_A} \sigma_A A_t J_A, \Lambda_{T,t}^* = -\frac{1}{Q_T} \sigma_T [J_T + J_E + [\nabla D](T_t) J_Z].\end{aligned}$$

Proposition 5 shows that climate risks affect economic and financial systems through several channels, including temperature dynamics, climate damage, accumulated carbon stocks, and changes in production efficiency. As temperatures rise, climate damage becomes more severe, and production conditions deteriorate. As a result, the market requires higher compensation for climate-related risks. Higher temperatures also increase risk exposure by directly reducing consumption, such as through lower agricultural output and higher energy costs. They further weaken expectations of future social welfare. Since carbon emissions accumulate over time, current emissions have persistent effects on future carbon stocks, climate conditions, and risk exposure.

Model uncertainty further amplifies these effects and strengthens the rationale for precautionary policy. The model therefore supports a two-dimensional policy response. First, adaptation investment should be increased to enhance the economy's resilience to climate risks. Second, mitigation policies should promote green industries, constrain carbon-intensive sectors, and accelerate the energy transition. Financial markets should also be encouraged to recognize climate risks and guide capital from high-carbon sectors toward low-carbon activities. In the subsequent numerical analysis, temperature is treated as the state variable determining the price of climate risk. The climate risk price, the risk-free rate, and the carbon tax are then computed to quantify how the climate model and model uncertainty affect asset-pricing outcomes.

5. Implementation details of the DGM algorithm

High-dimensional HJB equations are challenging to solve using conventional numerical methods because of the high dimensionality of the state space, nonlinear dynamics, and nonlocal terms. To address this challenge, this paper employs the DGM neural network to approximate the value function. DGM replaces the fixed basis functions used in classical Galerkin methods with a deep neural network, thereby reformulating the PDE system as a machine-learning-based optimization problem. The loss function is constructed from the residuals of the governing PDEs and the violations of the initial and boundary conditions, and the network parameters are updated by stochastic gradient descent over

randomly sampled points. Previous studies have shown that DGM can solve high-dimensional HJB equations with high accuracy [51]. In this model, DGM jointly approximates the value function and the associated first-order conditions (FOCs), providing a numerical basis for analyzing macroeconomic dynamics and the interaction between climate processes and economic decision-making.

In the numerical implementation, the input vector is $x = (A_t, Z_t, T_t, E_t)$, where A_t , Z_t , T_t , and E_t denote productivity, climate damage, temperature, and atmospheric carbon stocks, respectively. The network output is a scalar $J_\theta(x)$, which approximates the value function $J(A_t, Z_t, T_t, E_t)$. Since the social planner's problem is stationary and infinite-horizon, time is not included as an additional input.

We use a DGM-type fully connected neural network within the residual-minimization framework. The activation function is $\tanh$. This function is used because it is smooth and suitable for automatic differentiation. This is important for the present HJB equation, which contains first-order and second-order derivatives of the value function. The output layer is linear. All state variables are normalized to the unit interval before training. The normalized computational domain is therefore $[0, 1]^4$. When the normalized variables are used as network inputs, the derivatives entering the HJB equation are converted back to the original economic scale by the chain rule. At each iteration, 4096 interior collocation points and 1024 boundary points are randomly sampled. This random sampling strategy avoids the construction of a full tensor grid over the four-dimensional state space.

The derivatives of $J_\theta(x)$ are computed by automatic differentiation and substituted into the HJB equation in Proposition 1. Let $\mathcal{R}_{HJB}(x; \theta) = \rho J_\theta(x) - \mathcal{H}(x, J_\theta, \nabla J_\theta, \nabla^2 J_\theta)$ denote the HJB residual, where $\mathcal{H}(\cdot)$ is the right-hand side of the HJB equation. The training loss is defined as

$$\mathcal{L}(\theta) = \frac{1}{N_\Omega} \sum_{i=1}^{N_\Omega} |\mathcal{R}_{HJB}(x_i; \theta)|^2 + \omega_b \frac{1}{N_b} \sum_{j=1}^{N_b} |\mathcal{R}_{HJB}(x_j^b; \theta)|^2,$$

where $N_\Omega = 4096$, $N_b = 1024$, and $\omega_b = 1$. The first term measures the residual at interior points. The second term stabilizes the approximation near the boundary of the computational domain. The optimal brown and green inputs are then recovered from the first-order conditions in Proposition 2 by using the derivatives of the approximated value function.

The network is trained with the Adam optimizer. The learning rate is set to 10^{-4} , and the model is trained for 30,000 iterations. These parameters are selected based on preliminary numerical experiments and convergence diagnostics. A larger learning rate leads to less stable residual paths, while a smaller learning rate slows convergence. After 30,000 iterations, both the total loss and the PDE residual remain at low levels. Additional training produces only limited improvement. Table 1 summarizes the main implementation settings.

Table 1. DGM architecture and training settings.

Item	Setting	Description
Input variables	(A_t, Z_t, T_t, E_t)	State variables
Output	$J_\theta(A_t, Z_t, T_t, E_t)$	Value function
Hidden layers	4	Network depth
Neurons per layer	50	Network width
Activation function	tanh	Smooth activation
Output layer	Linear	Scalar approximation
Optimizer	Adam	Gradient-based training
Learning rate	10^{-4}	Training stability
Iterations	30,000	Loss convergence
Interior points	4096	Random collocation points
Boundary points	1024	Boundary stabilization
Derivatives	Automatic differentiation	HJB residual evaluation

This implementation provides a tractable numerical approximation to the high-dimensional HJB equation. The resulting value function is used to compute optimal input allocation, the carbon tax, the risk-free rate, and the market price of risk. The accuracy and computational feasibility of the DGM solution are assessed in the following validation subsection.

5.1. Validation and accuracy assessment of the DGM numerical framework

We compare the DGM method with two conventional numerical methods: the finite difference method and the finite element method. The comparison is conducted under the same calibrated parameter values as those used in the main numerical analysis. We consider two benchmark settings. The first is a reduced low-dimensional benchmark. It is used to examine numerical accuracy in a setting where grid-based methods are reliable and computationally tractable. The second is a higher-dimensional benchmark. It is used to evaluate the numerical stability and scalability of the DGM method when the dimension of the state space increases.

For the reduced benchmark, the reference solution is obtained from a high-resolution finite difference scheme. This reference solution is computed on a finer grid than the one used for the finite difference comparison reported in Table 2. For the higher-dimensional benchmark, an analytical solution is not available. We therefore construct the reference solution from the most refined numerical solution that remains computationally feasible under the same parameter values and boundary conditions. This reference solution is not obtained from the DGM training loss. It is used only as an external numerical benchmark for evaluating approximation accuracy.

Let $(J^m(x_i))$ denote the value function obtained by method (m) at evaluation point (x_i) . Let $(J^{ref}(x_i))$ denote the corresponding benchmark solution. The relative approximation error is defined as

$$\text{Error}_m = \frac{\left(\sum_{i=1}^N |J^m(x_i) - J^{ref}(x_i)|^2 \right)^{1/2}}{\left(\sum_{i=1}^N |J^{ref}(x_i)|^2 \right)^{1/2}} \times 100\%.$$

This error metric is a relative (L^2)-type measure over the evaluation domain. It captures the overall deviation of each numerical solution from the benchmark solution. It also provides a more direct assessment of numerical accuracy than the training residual alone.

The computation time is measured as the total wall-clock time required to obtain the numerical solution. For the DGM method, this includes the neural network training time. For the finite difference and finite element methods, this includes the discretization and numerical iteration time. Since these methods have different computational structures, the reported times should be interpreted as practical implementation costs. They are not intended to provide hardware-normalized complexity estimates.

Table 2 reports the validation results. In the low-dimensional benchmark, all three methods produce small relative approximation errors. The DGM method achieves an error of 1.2%. The finite difference and finite element methods achieve errors of 1.5% and 1.3%, respectively. These results indicate that the DGM approximation is comparable in accuracy to standard numerical methods in a setting where grid-based benchmarks are reliable.

Table 2. Validation of the DGM solution against numerical benchmarks.

Method	Dimension	Computation time (seconds)	Relative approximation error (%)
DGM	2	150	1.2
Finite difference method	2	500	1.5
Finite element method	2	480	1.3
DGM	4	800	1.8
Finite difference method	4	2500	2.0
Finite element method	4	2400	2.2

Note: Computation time refers to the total wall-clock time required to obtain the numerical solution. The relative approximation error is computed as the relative L^2 -type deviation from the benchmark solution over the evaluation domain. For the reduced benchmark, the benchmark solution is obtained from a high-resolution finite difference scheme. For the higher-dimensional benchmark, the benchmark solution is constructed from the most refined numerical solution that remains computationally feasible under the same parameter values and boundary conditions. The benchmark solution is computed independently of the DGM approximation. The approximation error is not computed from the DGM training loss.

In the higher-dimensional benchmark, the DGM method remains computationally feasible. Its computation time is 800 seconds, and its relative approximation error is 1.8%. The finite difference method requires 2500 seconds and produces an error of 2.0%. The finite element method requires 2400 seconds and produces an error of 2.2%. These results suggest that the computational burden of conventional grid-based methods increases more rapidly as the dimensionality rises. By contrast, the DGM method provides a stable and computationally feasible approximation for the high-dimensional HJB equation considered in this study.

Overall, the validation exercise supports the use of DGM in the present climate-economy model. The comparison should not be interpreted as showing that DGM is uniformly superior to all traditional numerical methods. Rather, it shows that DGM delivers accurate and feasible approximations in the benchmark settings considered here.

6. Baseline numerical results

6.1. Empirical analysis

The empirical data used in this study are taken from China as a case example. Annual CO₂ emissions from fossil-fuel combustion and industrial processes in China, covering the period from 1949 to 2024, are obtained from Our World in Data. Annual average temperature data for China over the same period are retrieved from the World Bank. These two datasets provide a long and consistent empirical basis for tracing the evolution of China's carbon output and the associated temperature changes. Their selection is motivated by both data availability and policy relevance. China is a major carbon-emitting economy and provides an important context for studying carbon-intensive growth, climate risk, and low-carbon transition. The interaction between emissions and temperature changes is also central to climate-policy design, carbon-reduction strategies, and sustainable development objectives.

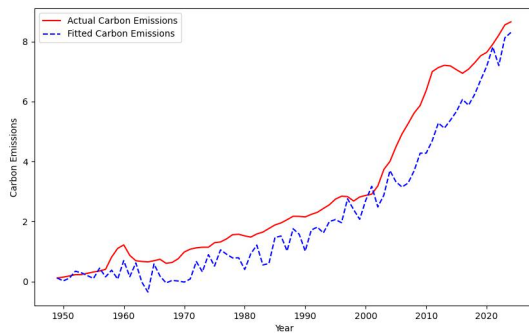
The empirical estimation in this subsection is intended to provide data-informed calibration for the climate block of the model. Since the theoretical framework is general, it requires a representative empirical setting to discipline the key parameters related to carbon emissions, temperature dynamics, and carbon-stock adjustment. The resulting estimates are used as benchmark values in the numerical solution of the HJB equation and in the simulation of carbon taxes, sectoral input shares, and asset-pricing variables. Accordingly, the numerical results should be interpreted as a quantitative implementation of the general climate-economy model using China-based data for empirical calibration. The purpose is not to develop a country-specific forecasting model. The qualitative mechanism is general: stronger climate-model uncertainty increases the incentive to internalize carbon externalities and to reallocate resources from brown inputs to green inputs. However, the quantitative magnitudes of the optimal carbon tax, the speed of sectoral transition, and the asset-pricing responses depend on the calibrated parameter values. Applications to other economies would require re-estimating or recalibrating the relevant parameters.

For the parameter fitting, we utilize maximum likelihood estimation (MLE), a widely used statistical method for estimating the parameters of a probabilistic model. MLE works by finding the parameter values that maximize the likelihood function, which measures the probability of the observed data given the model. This approach is particularly effective in stochastic process modeling, where data are inherently noisy and subject to random variation. The strength of MLE lies in its ability to produce parameter estimates that are asymptotically efficient and unbiased, making it well-suited for fitting models with random components. The fitted parameters and the corresponding estimated paths are presented in Table 3 and Figure 2.

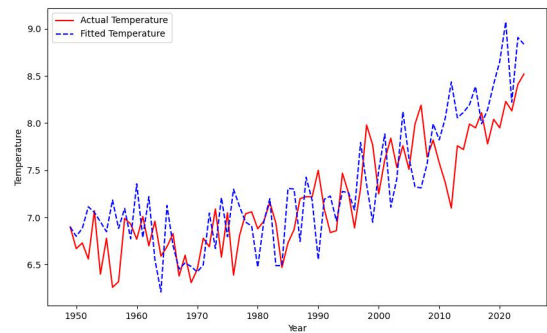
Table 3. Empirical parameter estimates.

Description	Parameter	Estimate	Std. Error
Temperature volatility	σ_T	0.33 ***	0.027
Climate sensitivity	λ	0.0082*	0.01
Fossil fuel carbon share	ψ	0.11 ***	0.0087
Carbon decay rate	ϕ	0.08**	0.033

Note: * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$.



(a)



(b)

Figure 2. Path fitting of predicted values and actual values.**Table 4.** Model parameters, calibration sources, and economic interpretation.

Parameter	Symbol	Value	Source or calibration basis	Economic interpretation
Discount factor	ρ	0.00375	Macro-finance literature [46, 49, 52]	Captures the social planner's intertemporal discounting of future welfare.
Final output DRS	α	0.05	Sectoral production estimates [53]	Measures decreasing returns to labor in the final-output sector.
Final output nonlabor exponent	ν	0.04	Non-labor input share estimates [54]	Determines the contribution of intermediate inputs to final output.
Input elasticity of substitution	ω	0.75	Elasticity range for clean and dirty inputs [55, 56]	Governs the substitutability between brown and green intermediate inputs.
Brown and green firm DRS	α_B, α_G	0.95	Sectoral production estimates [53]	Captures decreasing returns to labor in the brown and green input sectors.
Climate robustness parameter	ϱ_T	$\{\infty, 0.05\}$	Direct calibration; see Appendix B1	Measures the degree of aversion to climate-model misspecification. The case $\varrho_T = \infty$ represents no climate-model uncertainty.
Productivity robustness parameter	ϱ_A	0.05	Direct calibration; see Appendix B1	Measures the degree of aversion to productivity-model misspecification.
Aggregate productivity drift	μ_A	0.005	Prior macroeconomic studies and World Bank data [12, 46]	Determines the long-run growth trend of aggregate productivity.
Aggregate productivity volatility	σ_A	0.01	Direct calibration; see Appendix B2	Captures stochastic fluctuations in aggregate productivity.
Brown and green productivity	A_B, A_G	$\{100.43, 53.40\}$	Direct calibration; see Appendix B1	Determines the relative productivity of carbon-intensive and low-carbon inputs.
Climate damage curvature	γ_1, γ_2	$\{0.000177, 0.0044\}$	Climate-damage calibration following IAM studies [12]	Maps temperature increases into economic damages through the damage function.
CRRA parameter	η	1.45	Macro-finance literature [46, 49, 52]	Measures relative risk aversion in the social planner's utility function.
Initial final output productivity	A_0	19.46	Empirical calibration from baseline output productivity	Sets the initial level of aggregate productivity in the simulation.
Initial temperature	T_0	0°C	Normalized relative to the pre-industrial baseline	Defines the initial temperature anomaly in the climate module.
Initial carbon stocks	E_0	296 GtC	Carbon-stock calibration from climate data	Sets the initial atmospheric carbon stock used in the simulation.

Note: DRS denotes decreasing returns to scale. IAM denotes integrated assessment model. The parameters σ_T , λ , ψ , and ϕ are reported in Table 3 because they are estimated by maximum likelihood estimation. Table 3 reports the remaining parameters used in the numerical analysis, together with their calibration sources and economic interpretation.

The remaining parameters used in the numerical experiments are summarized in Table 4. To improve the transparency and reproducibility of the calibration procedure, we extend the original parameter table by reporting not only the benchmark values but also the source or calibration basis and the economic interpretation of each parameter. The parameters are classified according to their calibration sources. Some parameters are adopted from established studies in macroeconomics, environmental

economics, and climate-economy modeling. Other parameters are calibrated using external data sources or model-specific restrictions. The robustness parameters and sectoral productivity parameters are treated separately because they directly affect the strength of model-misspecification concerns and the relative attractiveness of brown and green inputs.

This expanded presentation also clarifies the role of calibration assumptions in the model results. Parameters related to robustness preferences, sectoral productivity, and climate damages are especially relevant for the optimal carbon tax and the transition from brown to green inputs. We therefore examine the sensitivity of the main conclusions to alternative parameter values in Section 7. This allows us to evaluate whether the key results are driven by specific benchmark choices or remain stable within reasonable parameter ranges.

6.2. Numerical results of the model with DGM algorithm

Before presenting the main numerical results, we examine the convergence behavior of the DGM training process. The DGM architecture and training settings have been described in Section 5. The training loss is used here as a diagnostic indicator. It shows whether the neural network approximation satisfies the HJB residual during optimization. However, a small training residual does not by itself guarantee accuracy over the whole state domain. For this reason, the DGM approximation is also compared with external numerical benchmarks in Section 5.1. The loss curves should therefore be interpreted as complementary evidence on the convergence of the neural network training process. In this experiment, the learning rate is set to 10^{-4} , and the model is trained for 30,000 iterations. The average loss and its logarithmic value are recorded every 50 iterations. Figure 3 reports the convergence pattern of the total loss and the PDE operator loss.

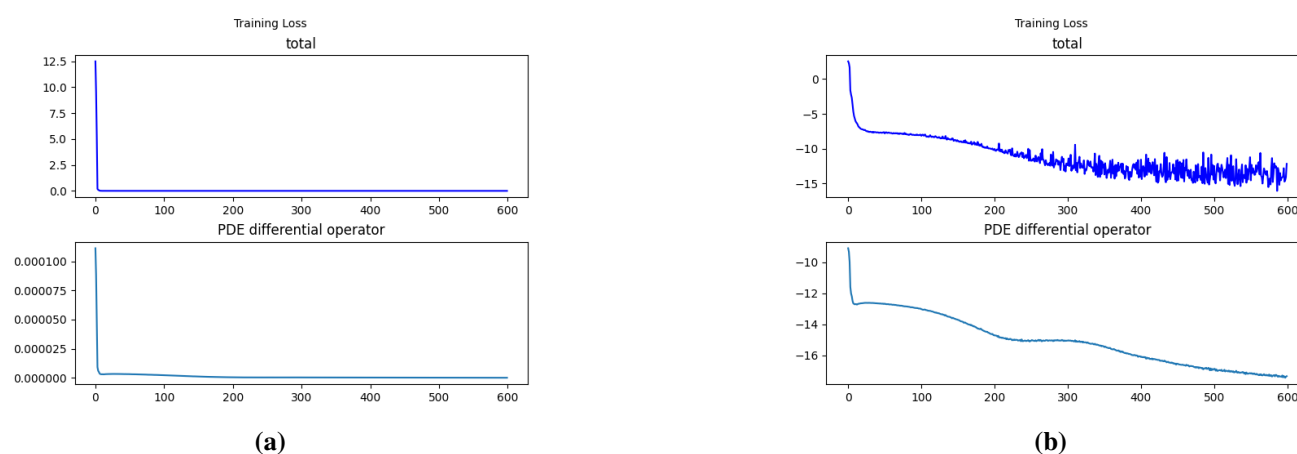


Figure 3. Convergence of total and PDE losses under DGM training.

Figure 3(a) shows that both the total training loss and the PDE operator loss decrease sharply during the early stage of training. The total loss falls from approximately 12.5 to a value close to zero after only a few iterations. The PDE operator loss follows a similar pattern. In later iterations, both loss values remain at low levels, suggesting that the model reaches a stable convergence state. Figure 3(b) further confirms this pattern on a logarithmic scale. The logarithmic total loss decreases from about 0 to around -15, while the logarithmic PDE operator loss declines from nearly -10 to about -16. Although

short-term fluctuations are observed, the overall downward trend indicates effective convergence of the DGM training process.

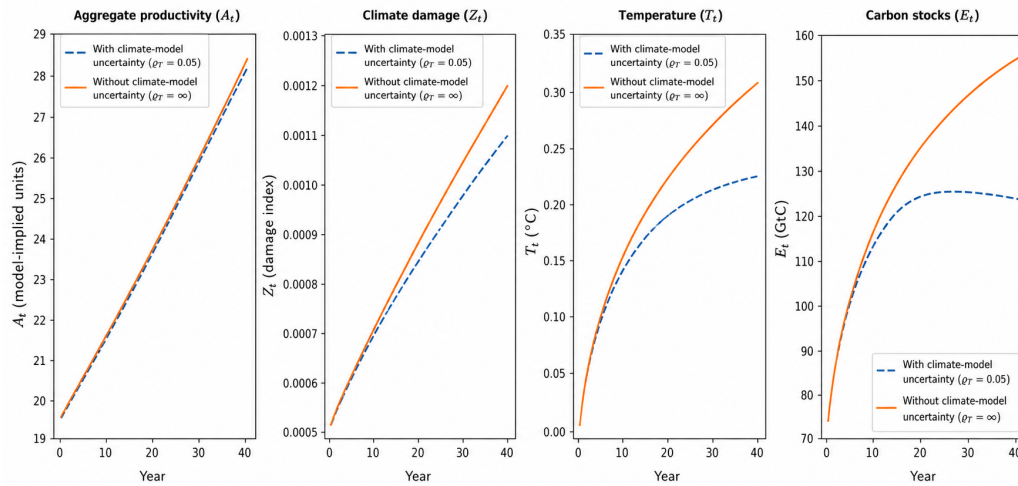


Figure 4. Simulation results based on numerical solutions.

Note: From left to right, the panels depict the productivity of the final output sector, the climate-induced economic damage function, temperature dynamics, and atmospheric carbon reserves.

Figures 4 and 5 summarize the dynamic evolution of the state variables and policy outcomes. We compare two scenarios: one with climate model uncertainty $\varrho_T = 0.05$ and one without it, $\varrho_T = \infty$. Economic model uncertainty is held constant at $\varrho_A = 0.05$. The dashed line represents aversion to climate model uncertainty, while the solid line represents the benchmark case without such aversion. As shown in Figure 4, productivity A_t increases almost linearly over time. The two curves follow similar paths, indicating that climate model uncertainty has only a limited effect on productivity. By contrast, climate damage Z_t rises over time. Its growth is slightly slower when uncertainty is considered, but the gap between the two scenarios widens in the long run.

Temperature T_t also increases over time. Without climate model uncertainty, the rate of temperature increase remains relatively stable. With uncertainty, temperature rises more rapidly in the early stage and then levels off in later periods. This pattern reflects the precautionary response of the social planner in the calibrated simulation. It suggests that climate model uncertainty may moderate long-run temperature changes. The atmospheric carbon reservoir E_t increases gradually in both scenarios. Without uncertainty, it follows a more stable and faster growth path. When uncertainty is considered, E_t rises rapidly at first but grows more slowly later. In some simulated paths, it may even follow a declining trajectory.

Overall, the effect of climate-model uncertainty is relatively limited in the early phase of the simulation. In the model economy, output expansion and energy use are initially associated with higher climate damages, stronger warming, and larger carbon emissions. Atmospheric carbon stocks also increase during this stage. As climate risk becomes more relevant in the planner's allocation, more resources are gradually directed toward emission reduction and green technologies. This adjustment reduces carbon emissions and mitigates pollution damages in the simulated economy. As a result, the increase in temperature and climate damages slows in later periods. Atmospheric carbon stocks may also decline in the later stage of the simulation.

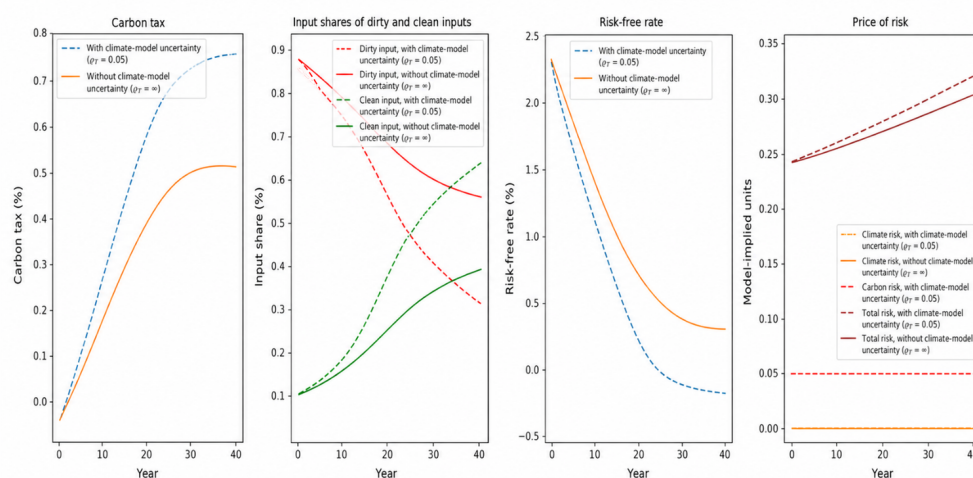


Figure 5. Macroeconomic results based on numerical solutions.

Note: Panel (1) reports the optimal carbon tax. Panel (2) presents the input shares of brown (carbon-intensive) and green (low-carbon) sectors. Panel (3) shows the equilibrium risk-free rate. Panel (4) displays the market prices of climate risk, carbon stock risk, and total risk.

Figure 5 reports the model-implied optimal carbon tax path for the brown sector. After climate-model uncertainty is introduced, the carbon tax follows an upward path. Its growth rate gradually declines in later periods, and the tax path converges to a stable level. Compared with the no-uncertainty case, the simulated carbon tax is higher. This result suggests that climate-model uncertainty strengthens the incentive to internalize carbon externalities within the model. The second panel illustrates the evolution of the input shares of brown and green sectors in final output production. Over time, the share of brown inputs declines, while the share of green inputs increases. This indicates a gradual reallocation of productive resources from brown inputs to green inputs within the simulated allocation. The dashed line exhibits a larger change than the solid line. This suggests that climate-model uncertainty is associated with a faster transition in the calibrated model.

The third panel shows that the risk-free rate declines as model uncertainty increases. The blue dashed line exhibits a steeper downward trend than the yellow solid line, indicating that climate model uncertainty lowers the risk-free rate. This pattern suggests a more cautious market assessment of future economic conditions. It also implies lower expected risk-free returns and weaker growth stability in the long run. The fourth panel reports the simulated prices of climate risk, carbon-stock risk, and total risk. The uncertainty scenario generates a higher total price of risk, while the price of pure climate risk remains close to zero. By contrast, the carbon-stock risk price is more persistent. This result suggests that carbon-stock-related risks are less easily predicted and hedged within the model. They therefore require greater risk compensation. Overall, the simulation indicates that uncertainty affects asset pricing mainly through the carbon-stock risk channel.

The numerical results suggest that climate-model uncertainty affects several macroeconomic and environmental outcomes in the calibrated economy. In the long run, it changes the simulated paths of temperature, atmospheric carbon stocks, and climate damages. Its effect on productivity is relatively limited. The simulations also generate a higher model-implied carbon tax and a faster reallocation of productive resources from brown inputs to green inputs. These patterns suggest that climate-model

uncertainty strengthens the incentive to internalize carbon externalities within the model. They also indicate a possible transition toward lower-carbon production in the simulated allocation. However, these findings are conditional on the benchmark calibration and the modeling assumptions. They should be interpreted as model-based implications rather than as universal empirical estimates.

From a policy perspective, the model suggests that incorporating climate-model uncertainty may improve welfare within the calibrated framework. This welfare gain arises because the carbon tax helps align decentralized decisions with the socially optimal allocation in the model. The asset-pricing results further suggest that climate-model uncertainty may influence capital allocation through the risk-free rate and the market price of risk. Nevertheless, these implications should be interpreted with caution. Their practical relevance depends on implementation costs, regulatory design, market structure, institutional capacity, sectoral heterogeneity, and country-specific economic conditions. Therefore, the model provides conditional policy insights rather than direct policy prescriptions.

7. Parametric sensitivity analysis

In this section, we provide an in-depth analysis of some of the key parameters, focusing on how the results of the study differ when these parameters are taken to different values. Similar sensitivity analyses have been conducted in [27]. The analysis considers variations in the elasticity of substitution (ω), temperature volatility (σ_T), instantaneous climate response parameter (λ), constant relative risk aversion coefficient (η), and carbon stock parameter (ψ). The corresponding changes in the figures are reported in Appendix C.

Table 5 reports the parameter values used in the sensitivity analysis. For each parameter, alternative calibrations above and below the baseline value are considered, with all other parameters held fixed. The analysis is conducted under model uncertainty, with ρ_T fixed at 0.05. To examine short-, medium-, and long-run effects, we compare equilibrium outcomes at $t = 0$ and at $t = 10, 20, 30$, and 40.

Table 5. Model sensitivity analysis parameters.

Parameter	Symbol	Alternative values
Input elasticity of substitution	ω	{0.35, 0.85}
Temperature volatility	σ_T	{0.045, 0.06}
TCRE parameter	λ	{0.0015, 0.002}
CRRA parameter	η	{1.35, 1.55}
Carbon stock parameter	ψ	{0.55, 0.75}

7.1. Elasticity of input substitution (ω)

The elasticity of input substitution ω determines the flexibility with which firms reallocate production between green inputs G_t and brown inputs B_t . A higher value of ω implies greater substitutability, whereas a lower value restricts adjustment. To examine the quantitative effects of alternative substitution regimes, the model is simulated under $\omega = 0.35$ and $\omega = 0.85$. As reported in Table 6, a lower substitution elasticity leads to greater variation in the damage function, temperature, and carbon stock. In particular, the carbon stock projections under the two scenarios differ by approximately 40 GtC by year 40. This result indicates that limited input substitutability amplifies the environmental consequences

of production. When firms face greater difficulty in replacing brown inputs with green inputs, their reliance on brown inputs persists. This reinforces carbon accumulation and intensifies climate-related damages.

Table 6 also shows that higher substitution elasticity accelerates the transition from brown to green inputs. As ω increases, the brown input share declines more sharply, while the green input share rises more strongly. By year 40, the green input share reaches 53.1%, exceeding the brown input share of 46.9%. This suggests that greater input flexibility strengthens firms' response to the carbon tax and supports emissions reduction. As input restructuring stabilizes over time, the carbon tax also becomes more stable. The macro-financial results show a gradual decline in the risk-free rate and an upward trend in the total risk price. However, under higher substitution elasticity, the risk-free rate decreases more slowly, and the total risk price rises at a more moderate pace. This is because firms can adjust their input structure more effectively and reduce their exposure to external shocks. The numerical illustration is provided in Figure 6 of Appendix C.

Table 6. Parameter sensitivity to elasticity of input substitution (ω).

Year	ω	Z_t	T_t	E_t	Tax	Rate of B_t (%)	Rate of G_t (%)	Risk-free rate (%)	Total risk price
0	0.35	5.00×10^{-4}	0	74	0.026	0.877	0.123	2.330	0.242
	0.85	5.00×10^{-4}	0	74	0.187	0.877	0.123	2.330	0.243
10	0.35	6.63×10^{-4}	0.120	112.466	0.030	0.836	0.164	1.152	0.263
	0.85	6.56×10^{-4}	0.110	108.974	0.260	0.797	0.203	1.554	0.261
20	0.35	8.58×10^{-4}	0.214	138.842	0.206	0.795	0.205	0.305	0.288
	0.85	8.13×10^{-4}	0.169	122.844	0.603	0.637	0.363	1.070	0.282
30	0.35	1.06×10^{-3}	0.288	156.548	0.361	0.739	0.261	0.054	0.316
	0.85	9.55×10^{-4}	0.202	126.291	0.703	0.522	0.478	0.659	0.307
40	0.35	1.26×10^{-3}	0.346	166.846	0.489	0.665	0.335	1.64×10^{-3}	0.347
	0.85	1.09×10^{-3}	0.228	127.324	0.715	0.469	0.531	0.333	0.334

Note: The symbol “%” denotes percentage points.

7.2. Temperature volatility (σ_T)

Temperature volatility σ_T captures uncertainty in atmospheric temperature dynamics. It is a key parameter in the baseline climate process and directly affects the optimized uncertainty contribution Λ_T . To examine its quantitative implications, the model is simulated under two calibrations, $\sigma_T = 0.045$ and $\sigma_T = 0.06$. As shown in Table 7, higher temperature volatility amplifies climate-related fluctuations and adverse climate effects. Under $\sigma_T = 0.06$, the carbon tax response becomes stronger. Brown input shares decline, while green input shares increase. These results indicate that higher climate uncertainty induces a more pronounced adjustment from pollution-intensive inputs toward low-carbon inputs.

The macro-financial outcomes are also affected by temperature volatility. Compared with $\sigma_T = 0.045$, the risk-free rate under $\sigma_T = 0.06$ is generally lower and declines more sharply over time. This

pattern suggests that long-term climate uncertainty depresses risk-free returns and increases pressure for green investment. Under lower volatility, both structural adjustment and climate dynamics remain more moderate. Overall, higher temperature volatility intensifies climate risk, strengthens carbon-tax adjustment, and promotes a faster transition toward low-carbon production. The numerical illustration is provided in Figure 7 of Appendix C.

Table 7. Parameter sensitivity to temperature volatility (σ_T).

Year	σ_T	Z_t	T_t	E_t	Tax	Rate of B_t (%)	Rate of G_t (%)	Risk-free rate (%)	Total risk price
0	0.045	5.00×10^{-4}	0	74	0.033	0.877	0.123	2.330	0.228
	0.06	5.00×10^{-4}	0	74	0.032	0.877	0.123	2.330	0.239
10	0.045	6.00×10^{-4}	0.110	108.790	0.219	0.819	0.181	1.471	0.249
	0.06	6.41×10^{-4}	0.118	111.789	0.338	0.797	0.203	1.174	0.259
20	0.045	7.02×10^{-4}	0.171	123.522	0.411	0.740	0.260	0.824	0.273
	0.06	8.04×10^{-4}	0.203	134.871	0.626	0.637	0.363	0.311	0.282
30	0.045	7.82×10^{-4}	0.197	124.530	0.540	0.645	0.355	0.394	0.300
	0.06	9.64×10^{-4}	0.262	147.115	0.807	0.356	0.644	-0.192	0.308
40	0.045	8.45×10^{-4}	0.206	119.287	0.594	0.570	0.430	0.186	0.330
	0.06	1.12×10^{-3}	0.306	153.201	0.879	0.144	0.856	-0.356	0.336

Note: The symbol “%” denotes percentage points.

7.3. $TCRE$ (λ)

The transient climate response parameter λ captures the relationship between carbon emissions and temperature. A higher value of λ implies a stronger temperature response to a given level of emissions, and therefore greater climate sensitivity. To examine its quantitative implications, the model is simulated under two calibrations, $\lambda = 0.0015$ and $\lambda = 0.002$. As shown in Table 8, the results are broadly consistent with those for temperature volatility σ_T . Under $\lambda = 0.002$, temperature increases more strongly, which leads to larger changes in the damage function and greater carbon accumulation. These results indicate that a higher rate of climate response amplifies the impact of emissions on the climate system. In response to more severe climate risks, carbon taxes increase to encourage emissions reduction. Economic agents also shift more actively from brown inputs to green inputs. In the asset-pricing results, the curve associated with $\lambda = 0.002$ remains higher, while the risk-free rate continues to decline. This pattern suggests that stronger climate sensitivity increases demand for risk-free assets as a hedge against climate risk. The numerical illustration is provided in Figure 8 of Appendix C.

Table 8. Parameter sensitivity to transient climate response parameter (λ).

Year	λ	Z_t	T_t	E_t	Tax	Rate of B_t (%)	Rate of G_t (%)	Risk-free rate (%)	Total risk price $r_{f,t}$
0	0.0015	5.00×10^{-4}	0	74	0.032	0.877	0.123	2.330	0.243
	0.002	5.00×10^{-4}	0	74	0.032	0.877	0.123	2.330	0.243
10	0.0015	6.46×10^{-4}	0.094	108.020	0.170	0.823	0.177	1.050	0.264
	0.002	6.67×10^{-4}	0.126	112.740	0.388	0.787	0.213	1.582	0.263
20	0.0015	7.83×10^{-4}	0.134	117.631	0.321	0.756	0.244	-0.013	0.287
	0.002	8.69×10^{-4}	0.223	139.125	0.759	0.496	0.504	1.023	0.286
30	0.0015	9.02×10^{-4}	0.145	114.121	0.458	0.663	0.337	-0.297	0.312
	0.002	1.07×10^{-3}	0.296	155.040	0.872	0.212	0.788	0.543	0.311
40	0.0015	1.02×10^{-3}	0.150	108.453	0.534	0.571	0.429	-0.322	0.340
	0.002	1.27×10^{-3}	0.349	163.240	0.882	0.142	0.858	0.254	0.339

Note: The symbol “%” denotes percentage points.

7.4. CRRA (η)

Table 9. Parameter sensitivity to CRRA coefficient (η).

Year	η	Z_t	T_t	E_t	Tax	Rate of B_t (%)	Rate of G_t (%)	Risk-free rate (%)	Total risk price $r_{f,t}$
0	1.35	5.00×10^{-4}	0	74	0.032	0.877	0.123	2.329	0.243
	1.55	5.00×10^{-4}	0	74	0.032	0.877	0.123	2.330	0.241
10	1.35	6.53×10^{-4}	0.105	107.149	0.224	0.770	0.230	1.035	0.264
	1.55	6.62×10^{-4}	0.118	111.858	0.383	0.831	0.169	1.484	0.263
20	1.35	8.02×10^{-4}	0.157	118.538	0.415	0.540	0.460	0.135	0.287
	1.55	8.48×10^{-4}	0.205	135.616	0.687	0.774	0.226	0.866	0.289
30	1.35	9.28×10^{-4}	0.175	116.964	0.572	0.240	0.760	-0.259	0.314
	1.55	1.03×10^{-3}	0.265	148.144	0.837	0.690	0.310	0.378	0.317
40	1.35	1.04×10^{-3}	0.181	111.416	0.642	0.123	0.877	-0.335	0.342
	1.55	1.20×10^{-3}	0.307	153.658	0.873	0.623	0.377	0.148	0.347

Note: The symbol “%” denotes percentage points.

The constant relative risk aversion coefficient η captures investors’ preferences under risk. A higher value of η implies stronger risk aversion and a greater preference for risk-free or low-risk assets. To

examine the role of risk preferences, the model is evaluated under two calibrations, $\eta = 1.35$ and $\eta = 1.55$. As shown in Table 9, a higher η strengthens precautionary behavior and changes equilibrium allocations. It reduces investment in environmentally beneficial but high-variance projects. As a result, the adjustment of climate-related state variables, including the damage function, temperature, and carbon stock, becomes more gradual.

Higher risk aversion also slows the transition from brown inputs to green inputs. Under $\eta = 1.55$, investors place greater weight on current stability and security. By contrast, lower risk aversion is associated with a stronger willingness to support long-term green investment. The asset-pricing results show that expected returns on risk-free assets decline over time. In the early transition stage, low- η investors require a higher risk price. In later stages, however, the risk price required by high- η investors becomes higher as physical risks accumulate. This shift raises the aggregate risk premium and leads to a gradual increase in the total price of risk. The numerical illustration is provided in Figure 9 of Appendix C.

7.5. Carbon stock parameter (ψ)

The carbon stock parameter ψ measures the proportion of fossil-fuel emissions entering the atmosphere. For a given level of fossil-fuel use, a higher value of ψ implies greater atmospheric carbon accumulation. To examine its quantitative implications, the model is evaluated under two calibrations, $\psi = 0.55$ and $\psi = 0.75$. As shown in Table 10, a higher ψ leads to a larger atmospheric carbon stock. This strengthens the greenhouse effect, raises temperature, and increases the damage function.

Table 10. Parameter sensitivity to carbon stock parameter (ψ).

Year	ψ	Z_t	T_t	E_t	Tax	Rate of B_t (%)	Rate of G_t (%)	Risk-free rate (%)	Total risk price $r_{f,t}$
0	0.55	5.00×10^{-4}	0	74	0.033	0.877	0.123	1.930	0.242
	0.75	5.00×10^{-4}	0	74	0.033	0.877	0.123	2.730	0.243
10	0.55	6.57×10^{-4}	0.111	102.975	0.138	0.818	0.182	0.985	0.263
	0.75	6.64×10^{-4}	0.122	120.007	0.334	0.813	0.187	1.932	0.264
20	0.55	8.18×10^{-4}	0.175	115.504	0.228	0.765	0.235	0.267	0.287
	0.75	8.68×10^{-4}	0.223	153.997	0.620	0.681	0.319	1.449	0.287
30	0.55	9.57×10^{-4}	0.204	116.629	0.314	0.706	0.294	-0.146	0.314
	0.75	1.09×10^{-3}	0.308	179.189	0.793	0.454	0.546	1.025	0.313
40	0.55	1.08×10^{-3}	0.217	112.767	0.393	0.638	0.362	-0.271	0.343
	0.75	1.32×10^{-3}	0.378	196.407	0.850	0.295	0.705	0.659	0.341

Note: The symbol “%” denotes percentage points.

The policy response is also more pronounced under a higher ψ . When $\psi = 0.55$, the inflow of carbon into the atmosphere is lower, and the increase in the carbon tax is limited. By contrast, the carbon tax in the high- ψ scenario is about twice as high as that in the low- ψ scenario. A higher carbon tax

reduces brown input shares and strengthens the shift toward eco-friendly inputs. Although a higher ψ worsens environmental conditions in the short run, it also increases incentives for long-term mitigation, technological progress, and industrial restructuring. The numerical illustration is provided in Figure 10 of Appendix C.

8. Conclusion

In this paper, we develop a theoretical framework that links climate dynamics and economic activity. The framework is used to examine the effects of climate change and climate-model uncertainty on economic production. We incorporate temperature dynamics, climate damages, carbon accumulation, and productivity shocks into the model. We also allow each channel to contain stochastic components. The resulting high-dimensional nonlinear Hamilton-Jacobi-Bellman (HJB) equation is solved using the Deep Galerkin Method (DGM). This method jointly approximates the value function and the optimal control policies. It also relaxes the smoothness requirements imposed by classical numerical schemes. The framework provides a numerical approximation to the equilibrium system and generates model-implied asset-pricing implications under climate and model uncertainty.

The numerical results provide a model-based benchmark for understanding how climate-model uncertainty affects economic allocation and asset pricing. In the calibrated economy, productive resources are gradually reallocated from brown inputs to green inputs, and this transition becomes faster when climate-model uncertainty is incorporated. The model-implied carbon tax also increases and then converges toward a stable level, suggesting that uncertainty strengthens the incentive to internalize carbon externalities. In addition, climate-model uncertainty is associated with a lower equilibrium risk-free rate and a higher price of risk, which is consistent with a precautionary valuation channel. These findings indicate that climate-related uncertainty may influence financing conditions, capital allocation, and risk compensation through equilibrium asset prices. Although the quantitative magnitudes depend on the benchmark calibration and modeling assumptions, the underlying mechanism provides useful implications for carbon-pricing design, corporate investment planning, technology choice, and climate-related risk management.

8.1. Extensions and future research

Several limitations of the present framework suggest important avenues for future research. First, the regulatory dimension of the model could be extended to incorporate alternative policy instruments, including carbon taxation schemes, emissions trading systems, and direct abatement regulations. Incorporating an explicit social cost of carbon would allow a more policy-relevant quantitative evaluation of climate risk.

Second, future work may consider alternative forms of model uncertainty, such as ambiguity in preferences, structured parametric misspecification, or entropy-based robustness frameworks. The present analysis adopts a conditional relative entropy approach, which imposes strict constraints on model distortions and may understate the adjustment costs associated with reducing polluting inputs. Exploring uncertainty specifications with varying degrees of robustness would therefore provide a broader perspective on climate-related macroeconomic risk.

Finally, the model could be enriched by incorporating additional state variables, including climate hazards, sector-specific technological progress in green and brown industries, and endogenous capital

accumulation. An important theoretical extension would be to characterize analytical solutions, when tractable, for the social planner's problem under climate uncertainty. Such developments would deepen our understanding of the macroeconomic and asset-pricing implications of climate risk and strengthen the theoretical foundations of the numerical results.

Author contributions

Shuhua Zhang: Identifying issues, questions or problems that warrant research.

Tong Wang: Integrating and aggregating data in diverse formats and from diverse sources.

Ning Ye: Drafting substantial original text within a section or across sections in an article.

Xinyu Wang: Reviewing, copy-editing, refining language and providing comments and suggestions.

Rui Wang: Testing computational models or simulations against known outcomes for accuracy.

Use of Generative-AI tools declaration

The authors declare that no Artificial Intelligence (AI) tools were used in the creation of this article.

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Conflict of interest

We have no conflicts of interest to disclose.

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