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Research article

A Stackelberg reinsurance-investment game on asset-liability management under CEV model

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Abstract: This paper investigates the optimal reinsurance-investment problem within the framework of a Stackelberg differential game. Under the constant elasticity of variance (CEV) model, a Stackelberg game model for investment-reinsurance asset-liability management between an insurer and a reinsurer is innovatively constructed. The two players are the insurer (follower) and the reinsurer (leader): The insurer aims to determine the optimal reinsurance and investment strategies, and the reinsurer seeks to formulate the optimal premium pricing and investment strategies. Both parties can allocate a risk-free asset and a risky asset whose price processes follow the CEV model, and both are exposed to uncontrollable external liabilities. To address the time–inconsistency issue in the game, the mean-variance criterion is adopted, formulating the optimization problems of both players as a nested game and deriving the corresponding extended Hamilton–Jacobi–Bellman equations. Based on this analytical framework, a verification theorem is formulated, which explicitly derives equilibrium reinsurance-investment strategies as well as the corresponding value functions for both the insurer and the reinsurer. Finally, numerical examples are provided to illustrate the rightness of our conclusions.

Keywords: reinsurance; asset-liability; Stackelberg game; HJB equation; investment-reinsurance strategy
Mathematics Subject Classification: 91A23, 93E20

1. Introduction

In the field of reinsurance research, insurers and reinsurers typically focus on two aspects to maximize their profits. On the one hand, they optimize their investment portfolios to boost returns; on the other hand, they purchase reinsurance to mitigate claim risks and minimize losses. In recent years, the insurance industry has witnessed vigorous development with continuous expansion of its market scale, and the total operating income of insurance companies has also maintained steady growth. The problem of optimal investment and reinsurance has become a hot research topic in the field of finance and insurance. At present, many scholars have theoretically studied the optimal investment and reinsurance strategies for insurance companies and reinsurance companies from various perspectives. Wang and Liu [1] investigated the optimal reinsurance strategy under the expected premium principle. They also explored the optimization problems pertaining to three types of reinsurance: proportional reinsurance, stop-loss reinsurance, and their hybrid form. Chen et al. [2] studied the robust optimal premium problem for insurers under the Black–Scholes model and the loss-dependent premium principle. Zhang et al. [3] discussed the insurer's optimal investment-reinsurance strategy based on hyperbolic absolute risk aversion (HARA) utility and the Ornstein–Uhlenbeck (O-U) process. Luo et al. [4] examined the optimal reinsurance and investment problem for an insurer whose surplus is governed by a linear diffusion process. Wang et al. [5] studied the stochastic differential investment and reinsurance game problem with delay and default risk between insurance companies and reinsurance companies. Zheng et al. [6] explored the decision-making behavior of policyholders when evaluating insurance under a narrow framework. Bernard et al. [7] investigated the optimal investment and insurance choices of insurance companies under interdependent insurance policies.

However, in most existing literature, scholars have confined their research primarily to the optimal asset investment and reinsurance strategies of insurers and reinsurers, while frequently overlooking the critical issue of asset-liability management. In the actual financial market, insurers encounter asset-liability management problems during their operations. Therefore, when considering the optimal asset investment and reinsurance strategies of insurers, their liability status should be taken into account. At present, the following scholars have conducted the following research on the optimal investment and reinsurance problem in the context of asset-liability issues. Sharpe and Tint [8] first studied the asset-liability problem using portfolio theory under the mean-variance framework. Chiu and Li [9] conducted an in-depth analysis of the asset-liability problem. They described the price process of risky assets and the liability process by means of geometric Brownian motion, and successfully derived the analytical equations corresponding to the optimal investment strategy and the efficient frontier. Chen et al. [10] linked the asset-liability management problem with Markov regime-switching. Wang et al. [11] studied the optimal dividend problem of insurance companies based on the diffusion risk model under stochastic liabilities, and formulated corresponding optimal strategies for practical scenarios according to the range of the liability ratio. Peng et al. [12] investigated the asset-liability management problem under the mean-variance criterion with insider information. Chen [13] described the robust continuous-time asset-liability management problem under Markovian regime switching. Wang et al. [14] examined the non-zero-sum stochastic differential game involving n competing constant absolute risk aversion (CARA) asset-liability managers, with a specific focus on potential model uncertainty and an explicit aim to derive robust investment strategies.

The wealth of insurers is far lower than that of reinsurers. Due to the principle of comparative psychology, insurers pay more attention to their own terminal wealth and the gap between themselves and reinsurers. Insurers aim to find a reinsurance and investment strategy that maximizes their relative performance under the defined mean-variance cost function. This also reflects the competition between insurers, as well as between insurers and reinsurers. Liang et al. [15] studied an insurer simultaneously negotiating proportional reinsurance contracts with two reinsurers, with pricing based on the variance premium principle. The two reinsurers resolved the competition through a noncooperative Nash game, where their competitive goal was to compete for the insurance company's business and optimize the so-called relative performance rather than their own residual profits. Yuan [16] designed a competitive framework for two insurance companies with ambiguity aversion under the utility framework and studied the resulting stochastic reinsurance game problem. Yang et al. [17] proposed a unified competition and cooperation framework applicable to n insurers and investigated the resulting reinsurance game problem. Their numerical results revealed the commonalities and differences between competitive and cooperative models, as well as the specific impacts of different competition and cooperation patterns. Yang [18] simultaneously considered the robust optimal reinsurance strategy incorporating ambiguity aversion, dependent claims, and the common interests between insurers and reinsurers, where the common interests are reflected in their competitive relationship. Lin et al. [19] used stochastic game theory to study the optimal reinsurance strategy between an insurer and two reinsurers in the market, and they adopted a Nash game model to simulate the price competition between the two reinsurers with different premium principles (variance premium principle and expected value premium principle). Both the insurer and the reinsurers aimed to maximize their respective mean-variance cost functions, thereby forming a time-inconsistent control problem. Wang and Chen [20] studied the design of reinsurance contracts that balance the common interests of multiple insurers and reinsurers involving both competition and cooperation. These contracts included optimal reinsurance strategies and premium prices, and the authors derived the insurers' optimal time-consistent reinsurance-investment strategies, as well as the reinsurers' optimal pricing and investment strategies.

In addition, most existing literature studies the problem under the mean-variance criterion combined with Stackelberg game theory. For example, Chen et al. [21] first used the Stackelberg differential game to investigate the optimal reinsurance problem from the perspective of the interaction between insurers and reinsurers. Under the utility maximization criterion, they obtained the optimal premium pricing strategy and optimal reinsurance strategy. Zhang et al. [22] considered the reinsurance and investment problem under the Stackelberg differential game framework, where the claim business between the insurer and the reinsurer is correlated through common shock dependence, and both are allowed to invest in a common risky asset whose price follows the O-U process. With the help of stochastic control theory, explicit expressions for the optimal control strategies and value functions of the insurer and the reinsurer were derived. Li et al. [23] studied the reinsurance problem of the mean-variance Stackelberg game with a random time horizon. They analyzed two specific types of reinsurance, namely excess-of-loss reinsurance and constant proportional reinsurance. Finally, examples illustrated that when the claim severity is light-tailed, and the hazard rate function is decreasing, the optimal form of reinsurance is excess-of-loss reinsurance; these examples also showed that when the claim severity is heavy-tailed, and the hazard rate function is decreasing, the equilibrium reinsurance exhibits a nontrivial coinsurance phenomenon. Zhou et al. [24] investigated the robust optimal reinsurance and investment problem between an insurer and a reinsurer with bounded memory under the Stackelberg differential game framework. Eventually, using stochastic control theory and backward induction, the Hamilton-Jacobi- Bellman (HJB) equation was established and solved, and the robust equilibrium strategy and the corresponding value function were given. Hu et al. [25] studied robust reinsurance contracts under Stackelberg game and market equilibrium. By using the stochastic dynamic programming method and backward induction, they derived the analytical expressions for the optimal risk allocation ratio and reinsurance price of the two types of reinsurance contracts. Bai et al. [26] studied a Stackelberg stochastic differential reinsurance-investment game problem, considering a wealth process with time delay to characterize bounded memory. By solving the corresponding Hamilton– Jacobi– Bellman (HJB) equation, the equilibrium investment strategy of the game was obtained. The equilibrium investment strategy showed that the impact of the time delay weight on the equilibrium strategy is related to the length of the time delay.

In the reinsurance market, a small number of large international reinsurance companies occupy a dominant position, boasting abundant capital, advanced underwriting technologies, and market pricing power, and they act as price leaders in transactions. In contrast, small and medium-sized insurance companies have weak bargaining power, and act as price takers, and they thus naturally become followers in the game. This reflects the leader–follower structure in the Stackelberg game of the reinsurance market. Hao [27] discussed the Stackelberg game-based reinsurance problem under the premise of credit risk in Chapter 3 of his doctoral dissertation, where the reinsurer, as the leader of the Stackelberg game, first determines the reinsurance price and investment strategy, and the insurer, as the follower of the game, can set the reinsurance ratio and its own investment strategy based on the reinsurance premium price. Cao et al. [28] investigated the reinsurance chain problem consisting of $n+1$ companies, with the first being the primary insurer and the rest reinsurers. Each reinsurance contract is constructed as a Stackelberg game, where the assuming reinsurer serves as the leader and the ceding company as the follower. Han et al. [29] studied the reinsurance problem within the theoretical framework of the Stackelberg game, in which the reinsurer acts as the game leader to determine the optimal reinsurance rate, and the insurer acts as the game follower to select the optimal loss-based reinsurance purchase. Yang [30] examined the problem of formulating optimal reinsurance contracts under competition between n cooperative insurance companies and one reinsurer. Within the Stackelberg game framework, the explicit solutions for the optimal claim risk-sharing strategy and reinsurance price were derived. Yuan et al. [31] established a robust reinsurance contract from the perspective of the common interests of insurers and reinsurers under the framework of the Stackelberg differential game. Specifically, the reinsurer acts as the game leader to determine the optimal reinsurance rate, and the insurer acts as the game follower to choose the optimal proportional reinsurance purchase.

Although prior research has applied Stackelberg games or asset-liability management (ALM) frameworks separately in finance; the integration of both for solving the reinsurance-investment problem remains unexplored. In this study, aiming at insurers and reinsurers that consider relative performance, we innovatively construct a Stackelberg game-based reinsurance model under asset-liability management. This research mainly refers to the achievements of Zhou [32], Luo [33], Sulem et al. [34], and Zou and Cadenillas [35]. By integrating their findings, we establish a robust Stackelberg game model for optimal investment and reinsurance strategy. The model aims to maximize the expected value of relative performance between insurers and reinsurers (rather than absolute performance) while minimizing the variance of relative wealth at the terminal time. In addition, we define net assets associated with liabilities. It is assumed that insurers can invest their wealth in a financial market consisting of risk-free assets and risky assets. By selecting and optimizing investment- reinsurance strategies under the mean-variance framework, we obtain the optimal investment and reinsurance strategies for both insurers and reinsurers. Therefore, this paper constructs an ALM- integrated Stackelberg stochastic differential game model, which is more consistent with real insurance and financial markets and renders the research results more scientifically reasonable. Finally, we analyze the results through numerical simulations and explore the impact of changes in model parameters on the equilibrium strategy.

Compared with the existing literature, the main innovations of this paper are summarized as follows: (i) Unlike Zhou et al. [32], who explored the optimal investment and equilibrium reinsurance strategies for insurers and reinsurers within the Stackelberg differential game framework, this paper pioneers the full embedding of the ALM structure within the Stackelberg differential game under the constant elasticity of variance (CEV) model, making the framework more consistent with how insurers and reinsurers actually operate in real-world ALM scenarios. Meanwhile, under the time-consistent mean-variance framework, we derive the optimal investment and reinsurance strategies for insurers, as well as the optimal premium pricing and investment strategies for reinsurers simultaneously. (ii) We extend the model framework by introducing the Stackelberg stochastic differential game, whereas the research of Luo et al. [33] does not involve game theory analysis. (iii) For modeling the surplus process, we adopt the classic Cramér–Lundberg model, which enhances the persuasiveness of the research conclusions. (iv) The investments in risky assets by both insurers and reinsurers follow the CEV model and are correlated with uncontrollable exogenous liabilities. Under the CEV model and the bilateral ALM structure, we obtain closed-form equilibrium strategies and value functions, which generalize the results of Bai et al. [26] and Zhou et al. [32]. A special numerical analysis is carried out for liability-related parameters, and new insights into the management of insurance and reinsurance strategies are derived therefrom.

The structure of this work is as follows: Section 2 presents the Stackelberg differential game for optimal investment between insurers and reinsurers under the mean-variance criterion within the Cramér–Lundberg model framework, and establishes the corresponding system of equations. Section 3 solves the system of HJB equations using backward induction. We derive the optimal investment strategies and equilibrium reinsurance strategies for both insurers and reinsurers. In Section 4, we conduct numerical analysis on the equilibrium solutions obtained in the previous section. The research and prospects for future research directions in this field are summarized in Section 5.

2. Mathematical model

In this paper, we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{F_t\}_{t\in[0,T]})$ that satisfies the given conditions: where $\mathbb{P}$ denotes the true probability measure and is complete, and the filtration $F := \{F_t\}_{t\in[0,T]}$ is right-continuous . Let $[0, T]$ be a finite investment time horizon for the insurer and reinsurer, during which all trading and risk management activities occur.

2.1. Financial market

We assume that both the insurer and reinsurer invest in the financial market to generate returns, where the financial market follows the classical Merton model [36]. Specifically, the financial market consists of a risk-free asset and a risky asset. The risk-free asset is described by the following differential equation:

$$\begin{cases} dS_0(t) = rS_0(t)dt, \quad t \in [0, T], \\ S_0(0) = 1, \end{cases}$$

where $r > 0$ is a constant, representing the risk-free asset interest rate.

Referring to Luo et al. [33] and Gu et al. [37], the price process of the risky asset follows a CEV model:

$$\begin{cases} dS(t) = S(t)[\alpha dt + \sigma S^{\beta}(t)dW_1(t)], \quad t \in [0, T], \\ S(0) = s_0, \end{cases}$$

where $\alpha (> r)$ denotes the expected rate of return of the risky asset, $\sigma > 0$ is the volatility coefficient of the risky asset, and $W_1(t)$ is a one-dimensional standard Brownian motion adapted to the filtration

$\{F_t\}_{t\in[0,T]}$. β is the elasticity parameter, and $\sigma S^{\beta}(t)$ denotes the instantaneous volatility. When $\beta = 0$, the CEV model simplifies to a geometric Brownian motion (GBM) model. When $\beta > 0$, $\sigma S^{\beta}(t)$ increases as the risky asset price increases. For $\beta < 0$, $\sigma S^{\beta}(t)$ increases as the stock price declines, it potentially results in a distribution with a heavier left tail. In order to keep the risky asset price process positive, we let $\beta \geq 0$.

2.2. *Dynamics of the surplus processes*

We consider an insurer and a reinsurer operating in the insurance market under a competitive framework. Assuming no reinsurance coverage is present, the surplus process of the insurer is modeled by the classical Cramér–Lundberg process:

$$R(t) = x_0 + ct - \sum_{i=1}^{N(t)} M_i,$$

where x_0 represents the initial wealth determined by the insurer, c denotes the premium rate charged by the insurer to policyholders, M_i denotes the size of the i-th claim, and they are independent and identically distributed random variables with distribution function $F_M(t)$. This sequence has finite mean and variance, and we may set $\mu_L = E(M_i), \mu_L' = E(M_i^2)$. Let $Q(t) := \sum_{i=1}^{N(t)} M_i$ be a compound Poisson process representing the aggregate claims amount payable by the insurer over $[0, t]$. Here, $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{F_t\}_{t\in[0,T]})$ counting the number of claims up to time t, and the claim sizes $M_i, i = 1, 2...$ are independent of the claim arrival process $N(t)$. Under the classical Cramér–Lundberg model, it is difficult to obtain explicit analytical solutions. According to Sulem et al. [34], a Lévy process can be decomposed into three components: a linear drift part, a Brownian motion part, and a pure jump part. Based on this decomposition, Zou and Cadenillas [35] proposed that the claim compound Poisson process of the insurer can be approximated by a Brownian motion. That is to say, we assume that $Q(t) := \sum_{i=1}^{N(t)} M_i$ is approximated by the following diffusion process:

$$Q(t) = at + bB(t) + hH(t),$$

where $B(t)$ is a one-dimensional standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{F_t\}_{t\in[0,T]})$. This Brownian motion captures the inherent randomness of the claims process. $H(t)$ represents a pure jump Poisson process with intensity parameter λ', and the processes $B(t), W(t), H(t), N(t)$ are mutually independent. We assume that the parameters a, b, h are all positive constants that satisfy the following condition:

$$a + \lambda' h = \lambda\mu_L, b^2 + \lambda' h^2 = \lambda\mu_L'.$$

Additionally, we assume that the insurer's premium rate is determined in accordance with the expected value principle.

$$c' = (1 + \theta)(a + \lambda' h),$$

where $\theta > 0$ represents the safety loading coefficient of the insurer. Therefore, the surplus process of the insurer is updated as.

$$R(t) = x_0 + \lambda' ht + (a + \lambda' h)\theta t - bB(t) - hH(t).$$

To mitigate risk, the insurer purchases proportional reinsurance, whereby the reinsurer assumes a portion of the risk according to a specified proportion. Let $q_1(t)$ denote the retention ratio of the insurer at time t after purchasing reinsurance. When a claim occurs, for each claim amount M_i, the insurer covers $q_1(t)M_i$, and the reinsurer pays the remaining $(1 - q_1(t))M_i$. The reinsurance premium rate is also calculated according to the expected value principle.

$$c'' = (1 - q_1(t))(1 + \eta)(a + \lambda' h),$$

where $\eta > \theta$ represents the safety loading coefficient of the reinsurance company. After purchasing reinsurance, the premium rate of the insurer becomes.

$$\begin{aligned} c = c' - c'' &= (1 + \theta)(a + \lambda' h) - (1 - q_1(t))(1 + \eta)(\alpha + \lambda' h) \\ &= \left[(1 + \eta)q_1(t) + \theta - \eta\right](a + \lambda' h). \end{aligned}$$

Thus, the insurer's surplus process with reinsurance strategy $q_1(t)$ becomes.

$$dR(t) = [(\eta q_1(t) + \theta - \eta)(a + \lambda' h) + \lambda' h q_1(t)]dt - q_1(t)(bdB(t) + hdH(t)).$$

Let $q_2(t)$ denote the reinsurance strategy chosen by the reinsurer at time t. In practice, if $q_2(t) \geq q_1(t)$, the reinsurer will accept the insurer's optimal reinsurance proposal; conversely, if $q_2(t) \leq q_1(t)$, the reinsurer will reject it. Thus, the surplus process of the reinsurer under reinsurance strategy $q_2(t)$ can be expressed as.

$$d\hat{R}(t) = (1 - q_2(t))[(1 + \eta)\lambda' h + a\eta]dt - (1 - q_2(t))(bdB(t) + hdH(t)).$$

2.3. Wealth processes

In order to optimize asset utilization, we assume that both the insurer and the reinsurer invest their surplus in a financial market consisting of risk-free and risky assets to generate additional investment income. Let $X_1(t)$ denote the total wealth of the insurer at time t; and $Y_1(t)$ signify the corresponding wealth of the reinsurer at time t. We define the investment-reinsurance strategies adopted by the two companies as $u_k(t) = (\pi_k(t), q_k(t)), k = 1, 2$, where $\pi_1(t), \pi_2(t)$ represent the amounts invested in the risky asset by the insurer and the reinsurer at time t, respectively, and $q_1(t), q_2(t)$ denote the reinsurance strategies selected by the insurer and the reinsurer at time t, respectively. When liabilities are not considered, $\forall t \in [0, T]$, and let $X_1^{u_1}(t)$ and $Y_1^{u_2}(t)$ denote the wealth under the investment- reinsurance strategies $u_1(t)$ and $u_2(t)$, respectively. Here, $X_1^{u_1}(t) - \pi_1(t)$ and $Y_1^{u_2}(t) - \pi_2(t)$ represent the amounts invested in the risk-free asset by the insurer and the reinsurer. Under the investment-reinsurance strategy $u_1(t)$, after straightforward calculations, the wealth process $X_1^{u_1}(t)$ of the insurer is described by the following stochastic differential equation (SDE):

$$\begin{aligned} dX_1^{u_1}(t) &= \frac{X_1^{u_1}(t) - \pi_1(t)}{S_0(t)} dS_0(t) + \frac{\pi_1(t)}{S(t)} dS(t) + dR(t) \\ &= \left[rX_1^{u_1}(t) + (\alpha - r)\pi_1(t) + (\eta q_1(t) + \theta - \eta)(a + \lambda' h) \right. \\ &\quad \left. + \lambda' h q_1(t)\right]dt + \pi_1(t)\sigma S^{\beta} dW_1(t) - q_1(t)(bdB(t) + hdH(t)). \end{aligned}$$

Under the investment and reinsurance strategy $u_2(t)$, the wealth process $Y_1^{u_2}(t)$ of the insurer is described by the following SDE:

$$\begin{aligned} dY_1^{u_2}(t) &= \frac{Y_1^{u_2}(t) - \pi_2(t)}{S_0(t)} dS_0(t) + \frac{\pi_2(t)}{S(t)} dS(t) + d\hat{R}(t) \\ &= [rY_1^{u_2}(t) + (\alpha - r)\pi_2(t) + (1 - q_2(t)((1+\eta)\lambda' h + a\eta)]dt \\ &\quad + \pi_2(t)\sigma S^\beta dW_1(t) - (1 - q_2(t))(bdB(t) + hdH(t)). \end{aligned}$$

To make our model discussion more aligned with reality, we consider the liability issues of both the insurer and the reinsurer. Referring to Li et al. [38], assume that the insurer and the reinsurer face the uncontrollable exogenous liability $L_1(t)$,$L_2(t)$, whose evolutions are described by the following differential equations:

$$dL_1(t) = \alpha_1(t)dt + \sigma_{11} S^\beta(t) dW_1(t) + \sigma_{12} dW_2(t), L_1(0) = l_{10} > 0;$$

$$dL_2(t) = \alpha_2(t)dt + \sigma_{21} S^\beta(t) dW_1(t) + \sigma_{22} dW_2(t), L_2(0) = l_{20} > 0.$$

Here, $\alpha_1 > 0$ and $\alpha_2 > 0$ represent debt ratio, and σ_{11} and σ_{21} represent volatility of uncontrollable exogenous liabilities, respectively. σ_{12} and σ_{22} represent liability volatility. $W_2(t)$ is one-dimensional standard Brownian motion, which is independent of both $W_1(t)$ and $W_0(t)$. In practice, the insurer's liabilities (e.g. claims, reserves) and the reinsurer's liabilities are often affected by macroeconomic factors that also drive financial market returns, such as interest rates, inflation, and economic cycles. The correlation between liability and financial market risks reflects the systematic risk shared by the insurance business and the financial market. This setup captures the realistic balance-sheet comovement and ensures the model is consistent with ALM practice.

After accounting for the liability, we define the net assets of the insurer and the reinsurer as.

$$\tilde{X}_1^{u_1}(t) = X_1^{u_1}(t) - L_1(t);$$

$$\tilde{Y}_1^{u_2}(t) = Y_1^{u_2}(t) - L_2(t).$$

The net wealth processes of the insurer and the reinsurer satisfy the following SDEs:

$$\begin{aligned} d\tilde{X}_1^{u_1}(t) &= dX_1^{u_1}(t) - dL_1(t) \\ &= \big[r(\tilde{X}_1^{u_1}(t) + L_1(t)) - \alpha_1 + (\eta q_1(t) + \theta - \eta)(a + \lambda' h) \\ &\quad + (\alpha - r)\pi_1(t) + \lambda' h q_1(t)\big]dt - q_1(t)(bdB(t) + hdH(t)) \\ &\quad + (\pi_1(t)\sigma - \sigma_{11})S^\beta(t) dW_1(t) - \sigma_{12} dW_2(t). \end{aligned} \tag{2.1}$$

$$\begin{aligned} d\tilde{Y}_1^{u_2}(t) &= dY_1^{u_2}(t) - dL_2(t) \\ &= \big[r(\tilde{Y}_1^{u_2}(t) + L_2(t)) - \alpha_2 + (1 - q_2(t))((1+\eta)\lambda' h + a\eta) \\ &\quad + (\alpha - r)\pi_2(t)\big]dt - (1 - q_2(t))(bdB(t) + hdH(t)) \\ &\quad + (\pi_2(t)\sigma - \sigma_{21})S^\beta(t) dW_1(t) - \sigma_{22} dW_2(t). \end{aligned} \tag{2.2}$$

2.4. Stackelberg different game

Currently, there is an imbalance in the status between insurers and reinsurers within the insurance market. This is because the entire insurance market involves competition among thousands of insurers, whereas the reinsurance sector is dominated by a few giants, such as Munich Re, China Re, and Swiss Re. Therefore, there exists a leader in the insurance market. Subsequently, we regard the reinsurer and the insurer as the leader and the follower in this game. To solve the Stackelberg game, we generally summarize these three steps:

Step 1: The reinsurer (as the leader), first declares its admissible strategy $(q_2(.), \pi_2(.))$.

Step 2: After observing the reinsurer's strategy, the insurer (as the follower) solves its own optimization problem and obtains its optimal strategy in response to the reinsurer's strategy $q_1{}^* = f^*(., q_2(.), \pi_2(.), \pi_1^*(.))$.

Step 3: Knowing that the insurer will execute the strategy $f^*(., q_2(.), \pi_2(.), \pi_1^*(.))$, the reinsurer selects its optimal strategy $(q_2^*(.), \pi_2^*(.))$ from the set of admissible strategies.

Remark 1: Both players in the game, the insurer and the reinsurer can clearly observe each other's strategies in the insurance market. However, they are more concerned with their own actual strategies. As a result, we derive the reinsurance strategy $(q_2(.), \pi_2(.))$ through the insurance company's HJB equation, and similarly, we solve for the insurer's strategy $(q_1(.), \pi_1(.))$ from the reinsurer's HJB equation. This ultimately allows us to resolve the optimization problem.

Definition 2.1. (Admissible strategy) For any $t \in [0, T]$, the strategy $u(t) := u_1 \times u_2$ is said to be admissible if it satisfies the following conditions:

1. $u_1(t)$ and $u_2(t)$ are $\{\mathcal{F}_t\}_{t\in[0,T]}$ progressively measurable, where $q_1(.) \in [0, 1]$ and $q_2(.) \in [0, 1]$;
2. $\forall s \in [t, T]$; there are $E[\int_t^T (|q_1(s)|^2 + |\pi_1(s)|^2)ds] < +\infty$ and $E[\int_t^T (|q_2(s)|^2 + |\pi_2(s)|^2)ds] < +\infty$;
3. Equations (2.1) and (2.2) associated with $(q_1(.), \pi_1(.), q_2(.), \pi_2(.))$ have unique strong solutions $\{\tilde{X}_1^{u_1}(t)\}_{t\in[0,T]}$ and $\{\tilde{Y}_1^{u_2}(t)\}_{t\in[0,T]}$ that satisfy $E[\sup_{t\in[0,T]} |\tilde{X}_1^{u_1}(t)|^2] < +\infty$ and $E[\sup_{t\in[0,T]} |\tilde{Y}_1^{u_2}(t)(t)|^2] < +\infty$.

Here, $U := U_1 \times U_2$ denotes the set of all admissible strategies, and the admissible strategy sets for the insurer and the reinsurer are denoted by U_1 and U_2, respectively. As the follower in the Stackelberg game, the insurer's wealth is significantly lower than that of the reinsurer. Due to the principle of comparison psychology, the insurer pays more attention to both its own terminal wealth and the wealth gap between itself and the reinsurer.

The insurer aims to find a reinsurance-investment strategy that maximizes the mean-variance cost function defined below under its relative performance. According to Espinosa et al. [39], the relative wealth of the insurer and the reinsurer is defined as follows:

$$\begin{aligned}\hat{X}_1^{u_1}(t) &= (1 - k_1)\tilde{X}_1^{u_1}(t) + k_1(\tilde{X}_1^{u_1}(t) - \tilde{Y}_1^{u_2}(t)) \\ &= \tilde{X}_1^{u_1}(t) - k_1\tilde{Y}_1^{u_2}(t);\end{aligned}$$

$$\begin{aligned}\hat{Y}_1^{u_2}(t) &= (1 - k_2)\tilde{Y}_1^{u_2}(t) + k_2(\tilde{Y}_1^{u_2}(t) - \tilde{X}_1^{u_1}(t)) \\ &= \tilde{Y}_1^{u_2}(t) - k_2\tilde{X}_1^{u_1}(t).\end{aligned}$$

Combining the above equations, the wealth processes of the insurer and the reinsurer satisfy the following dynamic equations:

$$d\hat{X}_1^{u_1}(t) = d\tilde{X}_1^{u_1}(t) - k_1 d\tilde{Y}_1^{u_2}(t)$$

$$
\begin{aligned}
&= [r(\hat{X}_1^{u_1}(t) + L_1(t) - k_1L_2(t)) + (\eta q_1(t) + \theta - \eta)(a + \lambda' h) + \lambda' h q_1(t) \\
&\quad - k_1(1 - q_2(t))((1 + \eta)\lambda' h + a\eta) - \alpha_1 + k_1\alpha_2 + (\alpha - r)(\pi_1(t) \\
&\quad - k_1\pi_2(t))]dt - (q_1(t) - k_1(1 - q_2(t)))(bdB(t) + hdH(t)) + (\sigma\pi_1(t) \\
&\quad - k_1\pi_2(t)\sigma - \sigma_{11} + k_1\sigma_{21})S^{\beta}(t)dW_1(t) - (\sigma_{12} - k_1\sigma_{22})dW_2(t); \qquad (2.3)
\end{aligned}
$$

$$
\begin{aligned}
d\hat{Y}_1^{u_2}(t) &= d\tilde{Y}_1^{u_2}(t) - k_2 d\tilde{X}_1^{u_1}(t) \\
&= [r(\hat{Y}_1^{u_2}(t) + L_2(t) - k_2L_1(t)) + (1 - q_2(t))((1 + \eta)\lambda' h + a\eta) \\
&\quad - k_2\lambda' h q_1(t) - k_2(\eta q_1(t) + \theta - \eta)(a + \lambda' h) - \alpha_2 + k_2\alpha_1 \\
&\quad + (\alpha - r)(\pi_2(t) - k_2\pi_1(t))]dt - (-k_2q_1(t) + 1 - q_2(t))(bdB(t) \\
&\quad + hdH(t)) + (\sigma\pi_2(t) - k_2\pi_1(t)\sigma - \sigma_{21} + k_2\sigma_{11})S^{\beta}(t)dW_1(t) \\
&\quad - (\sigma_{22} - k_2\sigma_{12})dW_2(t). \qquad (2.4)
\end{aligned}
$$

Assume that both the insurer and the reinsurer have their own mean-variance preferences. They aim to maximize the mean-variance utility of their terminal wealth. Under the mean-variance framework, the Stackelberg differential game between the insurer and the reinsurer is formulated as follows:

Denote by $\mathbb{E}_{t,y,l,s}[.] = \mathbb{E}[.|Y(t) = y, L(t) = l, S(t) = s]$ and $Var_{t,y,l,s}[.] = Var[.|Y(t) = y, L(t) = l, S(t) = s]$ the expectation and variance operators, with $Y(t) = y$ implying the initial condition $\hat{Y}(t) = \hat{y}$.

Definition 2.2. The optimization problem of the insurer as follows:

$$
\begin{cases}
\sup_{u_1\in\mathcal{U}_1} J_1(t, \hat{x}_1, l_1, s; u_1(\cdot), u_2(\cdot)) := \sup_{u_1\in\mathcal{U}_1}\{\mathbb{E}_{t,\hat{x}_1,l_1,s}[\hat{X}_1^{u_1}(T)] \\
-\frac{m_1}{2}\,\mathrm{Var}_{t,\hat{x}_1,l_1,s}[\hat{X}_1^{u_1}(T)]\} \text{ subject to } (\hat{X}_1, L_1, S; u_1(\cdot), u_2(\cdot)) \text{ satisfies (2.3)}, \\
\text{for any } u_2(\cdot) \in \mathcal{U}_2.
\end{cases} \qquad (2.5)
$$

The optimization problem of the reinsurer as follows:

$$
\begin{cases}
\sup_{u_2\in\mathcal{U}_2} J_2(t, \hat{y}_1, l_2, s; u_1^*(\cdot, u_2), u_2(\cdot)) := \sup_{u_2\in\mathcal{U}_2}\{\mathbb{E}_{t,\hat{y}_1,l_2,s}[\hat{Y}_1^{u_2}(T)] \\
-\frac{m_2}{2}\,\mathrm{Var}_{t,\hat{y}_1,l_2,s}[\hat{Y}_1^{u_2}(T)]\} \text{ subject to } (\hat{Y}_1, L_2, S; u_1^*(\cdot, u_2(\cdot)), u_2(\cdot)) \text{ satisfies} \\
\text{(2.4), and } u_1^*(\cdot, u_2(\cdot)) \text{ is an optimal strategy of the problem (2.5).}
\end{cases} \qquad (2.6)
$$

Here, $m_1 > 0$ and $m_2 > 0$ are the risk aversion coefficients of the insurer and the reinsurer, respectively. To determine the equilibrium strategies and equilibrium value functions, we introduce the following definition:

Definition 2.3. Consider an admissible strategy $u_1^*(., u_2(.)) \in U_1$ associated with any given reinsurer's strategy $u_2(.) \in U_2$. Choose arbitrarily modified strategy $\hat{u}_1 \in U_1$, a real positive number ε, and initial value $(t, \hat{x}_1, l_1, s) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$. We define a strategy u_1^{ε} that satisfies the following conditions:

$$
u_1^{\varepsilon}(s, \hat{x}_1, l_1, u_2(\cdot)) := \begin{cases}
\hat{u}_1(s, \hat{x}_1, l_1), & (s, \hat{x}_1, l_1) \in [t, t + \varepsilon) \times \mathbb{R} \times \mathbb{R}^+, \\
u_1^*(s, \hat{x}_1, l_1, u_2(\cdot)), & (s, \hat{x}_1, l_1) \in [t + \varepsilon, T) \times \mathbb{R} \times \mathbb{R}^+.
\end{cases}
$$

If we have

$$
\liminf_{\varepsilon\to 0} \frac{J_1(t, \hat{x}_1, l_1, s; u_1^*(., u_2(.)), u_2(.)) - J_1(t, \hat{x}_1, l_1, s; u_1^{\varepsilon}(., u_2(.)), u_2(\cdot))}{\varepsilon} \geq 0,
$$

then the equilibrium strategy of the insurer is $u_1^*(\cdot, u_2(\cdot))$, and the equilibrium value function is $V_1(t, \hat{x}_1, l_1, s)$.

$$V_1(t, \hat{x}_1, l_1, s; u_1(.)) = J_1(t, \hat{x}_1, l_1, s; u_1^*(., u_2(.)), u_2(.)).$$

Let $u_2^* \in U_2$ be the strategy of the reinsurer and $u_1^*(\cdot, u_2(\cdot)) \in U_1$ be the optimal strategy of the insurer. For any modified strategy $\hat{u}_2 \in U_2$ and positive number ε, with initial value $(t, \hat{y}_1, l_2, s) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$, we define the strategy u_2^ε that satisfies the following conditions:

$$u_2^\varepsilon(s, \hat{y}_1, l_2) := \begin{cases} \hat{u}_2(s, \hat{y}_1, l_2), & (s, \hat{y}_1, l_2) \in [t, t+\varepsilon) \times \mathbb{R} \times \mathbb{R}^+, \\ u_2^*(s, \hat{y}_1, l_2), & (s, \hat{y}_1, l_2), \in [t+\varepsilon, T) \times \mathbb{R} \times \mathbb{R}^+. \end{cases}$$

If we have

$$\liminf_{\varepsilon \to 0} \frac{J_2(t, \hat{y}_1, l_2, s; u_1^*(., u_2^*(.)), u_2^*(.)) - J_2(t, \hat{y}_1, l_2, s; u_1^*(., u_2^\varepsilon(.)), u_2^\varepsilon(\cdot))}{\varepsilon} \geq 0,$$

then we say that $u_2^*(.)$ is the equilibrium strategy of the reinsurer, and $V_2(t, \hat{y}_1, l_2, s)$ is its corresponding equilibrium value function.

$$V_2(t, \hat{y}_1, l_2, s) = J_2(t, \hat{y}_1, l_2, s; u_1^*(., u_2^*(.)), u_2^*(.)).$$

According to the above definition, the equilibrium strategy $u_1^* \times u_2^*$ is time-consistent and continuous. Our aim is to find the equilibrium strategy and the corresponding equilibrium value function.

Proposition 1. Under the mean-variance framework, the value functions of Problem (2.5) and Problem (2.6) are independent of their respective state variables $\hat{x}_1, \hat{y}_1$. That is, neither the insurer nor the reinsurer depends on their state variables.

By virtue of Proposition 1, the value functions of both the insurer and the reinsurer are denoted as $V_1(t, \hat{x}_1, l_1, s)$ and $V_2(t, \hat{y}_1, l_2, s)$, respectively, thereby facilitating the derivation of the extended HJB equations. We define two variational operators. For any function $\psi(t, \hat{x}_1, l_1, s)$ belonging to the continuous space $C^{1,2,2,2}([0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$, its partial derivatives $\psi_t(t, \hat{x}_1, l_1, s)$, $\psi_{\hat{x}_1}(t, \hat{x}_1, l_1, s)$, $\psi_s(t, \hat{x}_1, l_1, s)$,$\psi_{ss}(t, \hat{x}_1, l_1, s)$,$\psi_{\hat{x}_1\hat{x}_1}(t, \hat{x}_1, l_1, s)$, $\psi_{l_1}(t, \hat{x}_1, l_1, s)$, $\psi_{l_1 l_1}(t, \hat{x}_1, l_1, s)$, $\psi_{\hat{x}_1 l_1}(t, \hat{x}_1, l_1, s)$, $\psi_{l_1 s}(t, \hat{x}_1, l_1, s)$, $\psi_{\hat{x}_1 s}(t, \hat{x}_1, l_1, s)$ also are continuous on $C^{1,2,2,2}([0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$, $\forall \psi(t, \hat{x}_1, l_1, s) \in C^{1,2,2,2}([0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$. The variational operator is defined as follows:

$$\begin{aligned} \mathcal{L}_1^{u_1,u_2}\psi(t, \hat{x}_1, l_1, s) = {} & \frac{\partial \psi}{\partial t} + [r(\hat{x}_1(t) + l_1(t) - k_1 l_2(t)) + (\eta q_1(t) + \theta - \eta)(a \\ & + \lambda' h) + \lambda' h q_1(t) - k_1(1 - q_2(t))((1+\eta)\lambda' h + a\eta) \\ & - \alpha_1 + k_1\alpha_2 + (\alpha - r)(\pi_1(t) - k_1\pi_2(t))]\frac{\partial \psi}{\partial \hat{x}_1} + \alpha_1 \frac{\partial \psi}{\partial l_1} \\ & + \alpha s \frac{\partial \psi}{\partial s} + \frac{1}{2}[(\sigma\pi_1(t) - \sigma_{11} - k_1\pi_2(t)\sigma + k_1\sigma_{21})^2 s^{2\beta} \\ & + b^2(q_1(t) - k_1(1 - q_2(t)))^2 + (\sigma_{12} - k_1\sigma_{22})^2]\frac{\partial^2 \psi}{\partial \hat{x}_1^2} \\ & + \frac{1}{2}(\sigma_{11}^2 s^{2\beta} + \sigma_{12}^2)\frac{\partial^2 \psi}{\partial l_1^2} + [\sigma_{11}(\sigma\pi_1(t) - \sigma_{11} - k_1\sigma\pi_2(t) \end{aligned}$$

$$+ k_1\sigma_{21})s^{2\beta} - \sigma_{12}(\sigma_{12} - k_1\sigma_{22})]\frac{\partial^2\psi}{\partial\hat{x}_1\partial l_1} + \frac{1}{2}\sigma^2 s^{2\beta+2}\frac{\partial^2\psi}{\partial s^2}$$
$$+ \sigma\sigma_{11}s^{2\beta+1}\frac{\partial^2\psi}{\partial l_1\partial s} + \sigma(\sigma\pi_1(t) - \sigma_{11} - k_1\pi_2(t)\sigma$$
$$+ k_1\sigma_{21})s^{2\beta+1}\frac{\partial^2\psi}{\partial\hat{x}_1\partial s}. \tag{2.7}$$

$\forall\phi(t, \hat{y}_1, l_2, s) \in C^{1,2,2,2}([0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$. The variational operator is defined as follows:

$$\mathcal{L}_2^{u_1,u_2}\phi(t, \hat{y}_1, l_2, s) = \frac{\partial\phi}{\partial t} + [r(\hat{y}_1(t) + l_2(t) - k_2 l_1(t)) - \alpha_2 + k_2\alpha_1 + (1 - q_2(t))((1$$
$$+ \eta)\lambda' h + a\eta) - k_2(\eta q_1(t) + \theta - \eta)(a + \lambda' h) + (\alpha - r)(\pi_2(t)$$
$$- k_2\pi_1(t)) - k_2\lambda' h q_1(t)]\frac{\partial\phi}{\partial\hat{y}_1} + \alpha_2\frac{\partial\phi}{\partial l_2} + \alpha s\frac{\partial\phi}{\partial s} + \frac{1}{2}[(\sigma\pi_2(t)$$
$$- \sigma_{21} - k_2\pi_1(t)\sigma + k_2\sigma_{11})^2 s^{2\beta} + b^2(-k_2 q_1(t) + 1 - q_2(t))^2$$
$$+ (\sigma_{22} - k_2\sigma_{12})^2]\frac{\partial^2\phi}{\partial\hat{y}_1^2} + \frac{1}{2}(\sigma_{21}^2 s^{2\beta} + \sigma_{22}^2)\frac{\partial^2\phi}{\partial l_2^2} + \frac{1}{2}\sigma^2 s^{2\beta+2}\frac{\partial^2\phi}{\partial s^2}$$
$$- [\sigma_{21}(\sigma\pi_2(t) - \sigma_{21} - k_2\pi_1(t)\sigma + k_2\sigma_{11})s^{2\beta} + \sigma_{22}(\sigma_{22}$$
$$- k_2\sigma_{12})]\frac{\partial^2\phi}{\partial\hat{y}_1\partial l_2} + \sigma\sigma_{21}s^{2\beta+1}\frac{\partial^2\phi}{\partial l_2\partial s} + \sigma(\pi_2(t)\sigma - \sigma_{21}$$
$$- k_2\pi_1(t)\sigma + k_2\sigma_{11})s^{2\beta+1}\frac{\partial^2\phi}{\partial\hat{y}_1\partial s}. \tag{2.8}$$

Remark 2 (Regularity & integrability conditions for verification theorem) Assume there exist functions $D_1, f_1, D_2, f_2 \in C^{1,2,2,2}$ satisfying the following:

(i) $D_1, f_1, D_2, f_2 \in C^{1,2,2,2}([0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$, and there exists a constant $C > 0$ such that for all $(t, \hat{x}_1, l_1, s)$,
$$|D_1(t, \hat{x}_1, l_1, s)| + |f_1(t, \hat{x}_1, l_1, s)| \leq C(1 + |\hat{x}_1|^2 + |l_1|^2 + |s|^2),$$
and analogously for D_2, f_2 with respect to $(\hat{y}_1, l_2, s)$.

(ii) For any admissible strategy (u_1, u_2) satisfying Definition 2.1, the following integrability conditions hold:
$$\mathbb{E}\int_t^T \left(\left|\partial_{\hat{x}_1}D_1 \cdot \sigma_1\right|^2 + \left|\partial_{\hat{x}_1}f_1 \cdot \sigma_1\right|^2\right)ds < \infty,$$
and similarly for the derivatives with respect to l_1, s, as well as for D_2, f_2. These conditions ensure that the Itô integrals appearing in the verification argument are true martingales (not merely local martingales).

(iii) For each $(t, \hat{x}_1, l_1, s)$ and for any given u_2, the supremum
$$\sup_{u_1\in U_1}\left\{\mathcal{L}_1^{u_1,u_2}D_1 - \mathcal{L}_1^{u_1,u_2}\left(\tfrac{m_1}{2}f_1^2\right) + m_1 f_1\,\mathcal{L}_1^{u_1,u_2}f_1\right\}$$
is attained by some $u_1^* \in U_1$; similarly, the supremum in (2.11) is attained by some $u_2^* \in U_2$. This holds, for instance, if U_1, U_2 are compact, and the expression inside the supremum is continuous in u_i, or if it is strictly concave in u_i.

Under Remark 2, the verification theorem (Theorem 2.4) holds, that is, the functions D_1, f_1 (resp. D_2, f_2) coincide with the equilibrium value function and the conditional expectation of the terminal wealth, and u_1^*, u_2^* are the time-consistent equilibrium strategies.

The following theorem presents the extended HJB equations under the mean-variance problem.

Theorem 2.4. (Verification theorem) To solve the optimization Problem (2.5) and Problem (2.6), we suppose that there exist four real-valued functions: $D_1(t,\hat{x}_1,l_1,s), f_1(t,\hat{x}_1,l_1,s), D_2(t,\hat{y}_1,l_2,s)$, $f_2(t,\hat{y}_1,l_2,s) \in C^{1,2,2,2}([0,T]\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R})$ satisfy the following system of extended HJB equations:

$$\begin{aligned}\sup_{u_1\in\mathcal{U}_1}\Bigg\{&\mathcal{L}_1^{u_1,u_2}D_1(t,\hat{x}_1,l_1,s)-\mathcal{L}_1^{u_1,u_2}\left\{\frac{m_1}{2}f_1^2(t,\hat{x}_1,l_1,s)\right\}\\&+m_1f_1(t,\hat{x}_1,l_1,s)\times\mathcal{L}_1^{u_1,u_2}f_1(t,\hat{x}_1,l_1,s)\Bigg\}=0.\end{aligned}\tag{2.9}$$

$$\mathcal{L}_1^{u_1^*(u_2),u_2}f_1(T,\hat{x}_1,l_1,s)=0,\quad D_1(T,\hat{x}_1,l_1,s)=\hat{x}_1,\quad f_1(T,\hat{x}_1,l_1,s)=\hat{x}_1,\tag{2.10}$$

where

$$\begin{aligned}u_1^*(t,\hat{x}_1,l_1,s)=\arg\sup_{u_1\in\mathcal{U}_1}\Big\{&\mathcal{L}_1^{u_1,u_2}D_1(t,\hat{x}_1,l_1,s)-\mathcal{L}_1^{u_1,u_2}\Big(\frac{m_1}{2}f_1^2(t,\hat{x}_1,l_1,s)\Big)\\&+m_1f_1(t,\hat{x}_1,l_1,s)\times\mathcal{L}_1^{u_1,u_2}f_1(t,\hat{x}_1,l_1,s)\Big\}.\end{aligned}$$

$D_1(t,\hat{x}_1,l_1,s)=V_1(t,\hat{x}_1,l_1,s)$, $E_{t,\hat{x}_1,l_1,s}[\hat{X}_1^{u_1^*,u_2^*}(T)]=f_1(t,\hat{x}_1,l_1,s)$, u_1^* represents the Nash equilibrium strategy for the insurer.

$$\begin{aligned}\sup_{u_2\in\mathcal{U}_2}\Bigg\{&\mathcal{L}_2^{u_1,u_2}D_2(t,\hat{y}_1,l_2,s)-\mathcal{L}_2^{u_1,u_2}\left(\frac{m_2}{2}f_2^2(t,\hat{y}_1,l_2,s)\right)\\&+m_2f_2(t,\hat{y}_1,l_2,s)\times\mathcal{L}_2^{u_1,u_2}f_2(t,\hat{y}_1,l_2,s)\Bigg\}=0.\end{aligned}\tag{2.11}$$

$$\mathcal{L}_2^{u_1^*(u_2^*),u_2^*}f_2(T,\hat{y}_1,l_2,s)=0,\quad D_2(T,\hat{y}_1,l_2,s)=\hat{y}_1,\quad f_2(T,\hat{y}_1,l_2,s)=\hat{y}_1,\tag{2.12}$$

where

$$\begin{aligned}u_2^*(t,\hat{y}_1,l_2,s)=\arg\sup_{u_2\in\mathcal{U}_2}\Big\{&\mathcal{L}_2^{u_1^*,u_2}D_2(t,\hat{y}_1,l_2,s)-\mathcal{L}_2^{u_1^*,u_2}\Big(\frac{m_2}{2}f_2^2(t,\hat{y}_1,l_2,s)\Big)\\&+m_2f_2(t,\hat{y}_1,l_2,s)\times\mathcal{L}_2^{u_1^*,u_2}f_2(t,\hat{y}_1,l_2,s)\Big\}.\end{aligned}$$

$D_2(t,\hat{y}_1,l_2,s)=V_2(t,\hat{y}_1,l_2,s)$, $E_{t,\hat{y}_1,l_2,s}[\hat{Y}_1^{u_1^*,u_2^*}(T)]=f_2(t,\hat{y}_1,l_2,s)$, u_2^* represents the Nash equilibrium strategy for the reinsurer.

Proof. See Appendix A.

3. Main results

3.1. The solution to the Stackelberg game

This section derives the corresponding Nash equilibrium strategies and their associated value functions. The solutions are obtained using the backward induction approach outlined previously.

Theorem 3.1. *Based on the variational operator defined in Section 2 and Theorem 2.4, the corresponding HJB equations can be derived. After solving the HJB equations via the dynamic programming principle and backward induction, the following conclusions are obtained:*

(1) The optimal investment strategies for the insurer and the reinsurer are

$$\begin{cases}\pi_1^* = \frac{\sigma_{11}}{\sigma} + \frac{(\alpha-r)(m_2+m_1k_1)+m_1m_2\sigma s^{2\beta}(k_1\sigma_{12}-\sigma_{11})(e^{r(T-t)}-1)+m_1k_1\frac{(\alpha-r)^2}{r}(e^{2\beta r(t-T)}-1)}{(1-k_1k_2)m_1m_2\sigma^2 s^{2\beta}e^{r(T-t)}} \\ -\frac{2m_2\beta\sigma^2e^{2\alpha\beta t}(1-e^{2\alpha\beta(T-t)})}{(1-k_1k_2)m_1m_2\sigma^2 s^{2\beta}e^{r(T-t)}}, \\ \pi_2^* = \frac{\sigma_{21}}{\sigma} + \frac{(\alpha-r)(m_1+m_2k_2)+m_1m_2\sigma s^{2\beta}(\sigma_{12}-k_2\sigma_{11})(e^{r(T-t)}-1)+m_1\frac{(\alpha-r)^2}{r}(e^{2\beta r(t-T)}-1)}{(1-k_1k_2)m_1m_2\sigma^2 s^{2\beta}e^{r(T-t)}} \\ +\frac{2m_2k_2\beta\sigma^2e^{2\alpha\beta t}(1-e^{2\alpha\beta(T-t)})}{(1-k_1k_2)m_1m_2\sigma^2 s^{2\beta}e^{r(T-t)}}.\end{cases}$$

(2) The optimal reinsurance strategies for the insurer and the reinsurer are

$$\begin{cases}q_1^* = \frac{(k_1+m_2-k_1k_2m_2)[(1+\eta)\lambda' h+a\eta]}{(1-k_1k_2)m_1m_2b^2e^{r(T-t)}}, \\ q_2^* = 1 - \frac{[(1+\eta)\lambda' h+a\eta](m_1+m_2k_2)}{(1-k_1k_2)m_1m_2b^2e^{r(T-t)}}.\end{cases}$$

Specifically, if we assume no cooperation or competition, which corresponds to $k_1 = k_2 = 0$, then

$$\begin{cases}\pi_1^* = \frac{\sigma_{11}}{\sigma} + \frac{(\alpha-r)m_2-m_1m_2\sigma s^{2\beta}\sigma_{11}(e^{r(T-t)}-1)-2m_2\beta\sigma^2e^{2\alpha\beta t}(1-e^{2\alpha\beta(T-t)})}{m_1m_2\sigma^2 s^{2\beta}e^{r(T-t)}}, \\ \pi_2^* = \frac{\sigma_{21}}{\sigma} + \frac{(\alpha-r)m_1+m_1m_2\sigma s^{2\beta}\sigma_{12}(e^{r(T-t)}-1)+m_1\frac{(\alpha-r)^2}{r}(e^{2\beta r(t-T)}-1)}{m_1m_2\sigma^2 s^{2\beta}e^{r(T-t)}}.\end{cases}$$

$$\begin{cases}q_1^* = \frac{(1+\eta)\lambda' h+a\eta}{m_1b^2e^{r(T-t)}}, \\ q_2^* = 1 - \frac{(1+\eta)\lambda' h+a\eta}{m_2b^2e^{r(T-t)}}.\end{cases}$$

Proof: See Appendix B.

3.2. Reinsurance pricing

Reinsurance premiums impact not only the interests of reinsurance companies but also those of primary insurers. For reinsurers, appropriate pricing is the foundation of profitability. Excessively high pricing may lead to loss of insurance business in the market, whereas excessively low pricing may fail to cover claim payments, resulting in financial losses. For primary insurers, reinsurance pricing directly affects their costs and risk management strategies. Appropriate reinsurance pricing enables primary insurers to effectively transfer risks, optimize their risk-bearing structure, and ensure stable business development.

Based on the discussion in the previous section, the insurance contract between the insurer and the reinsurer can only be successfully established when $q_1^*(t) = q_2^*(t)$, at which point, the transaction can be finalized. The safety loading coefficient of the reinsurer affects the pricing of reinsurance products. According to the theorem above, the optimal reinsurance strategies adopted by the insurer and the reinsurer vary under different scenarios, leading to differences in reinsurance pricing across cases. Therefore, we will analyze the following scenarios separately:

(i) When $0 < \widetilde{q}_1^*(t) \le 1$, and $0 < \widetilde{q}_2^*(t) \le 1$, the optimal reinsurance strategy is

$$(q_1^*(t), q_2^*(t)) = (\widetilde{q}_1^*(t), \widetilde{q}_2^*(t)).$$

Under these circumstances, the reinsurance products of the insurer and reinsurer reach a state of supply–demand balance, resulting in the condition $\widetilde{q}_1^*(t) = \widetilde{q}_2^*(t)$. Therefore, the safety loading coefficient of the reinsurer can be derived through straightforward calculation:

$$\eta^*(t) = \frac{(1-k_1k_2)m_1m_2b^2e^r(T-t)}{(a+\lambda'h)(k_1+m_2-k_1k_2m_2+m_1+m_2k_2)} - \frac{\lambda'h}{a+\lambda'h}.$$

Therefore, the reinsurance contract is executed at a price of $\eta^*(t)$.

(ii) When $\widetilde{q}_1^*(t) > 1$, and $0 < \widetilde{q}_2^*(t) \leq 1$, the optimal reinsurance strategy is

$$(q_1^*(t), q_2^*(t)) = (1, 1 - \frac{[(1+\eta)\lambda'h + ah](m_1+m_2k_2)}{\lambda'h(a+\lambda'h)}).$$

Thus,

$$\eta^*(t) = \frac{(1-k_1k_2)m_1m_2b^2e^r(T-t)}{(a+\lambda'h)(k_1+m_2-k_1k_2m_2)} - \frac{\lambda'h}{a+\lambda'h}.$$

(iii) When $0 < \widetilde{q}_1^*(t) \leq 1$, and $\widetilde{q}_2^*(t) > 1$, the optimal reinsurance strategy is

$$(q_1^*(t), q_2^*(t)) = (\frac{(k_1+m_2-k_1k_2m_2)[(1+\eta)\lambda'h+ah]}{(1-k_1k_2)m_1m_2b^2e^r(T-t)}, 1).$$

In this case, the reinsurer has the right to refuse to sign the reinsurance contract, the reinsurance market is oversupplied, and the fierce competition makes the reinsurance contract price too low, so it is difficult to sign the contract.

(iv) When $\widetilde{q}_1^*(t) > 1$, and $\widetilde{q}_2^*(t) > 1$, the optimal reinsurance strategy is

$$(q_1^*(t), q_2^*(t)) = (1, 1).$$

Under these conditions, the reinsurance market achieves supply- demand equilibrium, with reinsurance supply and demand reaching a state of relative stability. At this juncture, reinsurance contract prices stabilize at a relatively constant level. When competitive factors are disregarded, that is, $k_1 = k_2 = 0$, the safety loading coefficient (reinsurance contract price) of the reinsurer can be derived as follows:

$$\eta^*(t) = \frac{m_1b^2e^r(T-t) - \lambda'h}{(a+\lambda'h)} - \frac{\lambda'h}{a+\lambda'h}.$$

4. Numerical simulation

In this section, we conduct a numerical analysis of the results obtained in the previous section to elucidate the impact of key parameters on the optimal investment and reinsurance strategies for both the insurer and reinsurer under the typical scenario of exponential utility (CARA) preferences. Throughout the numerical analysis, unless otherwise specified, the baseline parameter values are set as shown in Table 1. The values of some parameters are adopted from the studies of Bensoussan et al. [40] and Luo et al. [33] to ensure the rationality and feasibility of our analysis.

Table 1. Common parameter values for both insurer and reinsurer.

r	α	σ	a	b	h	λ'	η	m_1	m_2	k_1	k_2	β	σ_{11}	σ_{12}	σ_{21}
0.05	0.1	0.4	0.1	0.04	0.01	1	0.25	0.3	0.1	0.2	0.4	1	3	0.14	1

We first analyze the impact of various parameters on the optimal reinsurance strategies for both the insurer and reinsurer, accompanied by economic interpretations.

Figure 1 shows a positive relationship between the intensity of the insurance λ' claim process and the insurer's optimal reinsurance strategy, but a negative relationship with the reinsurer's. This is because higher claim intensity leads the insurer to increase proportional reinsurance to reduce its own risk. The reinsurer then increases insurance claims to reduce its own profits. As a result, the reinsurer is therefore unwilling to assume excessive claim risk and instead chooses to reduce the supply of reinsurance products.

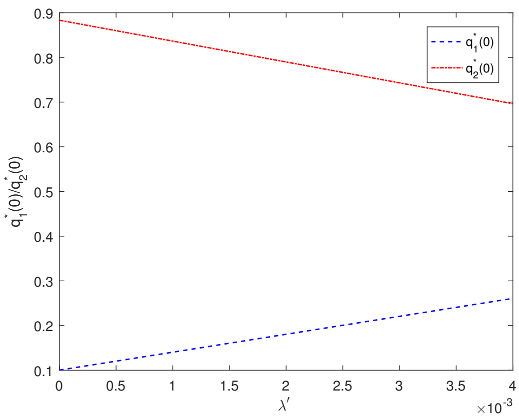

Figure 1. The effects of λ' on $q_k^*(t)$.

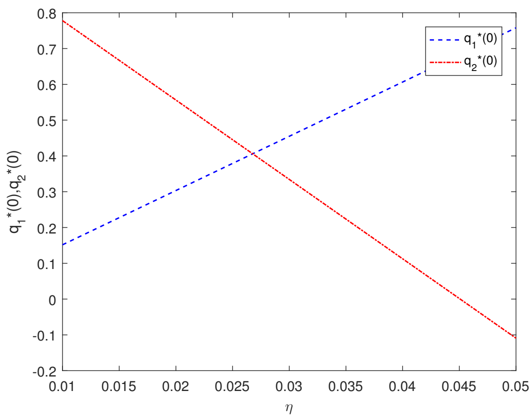

Figure 2. The effects of η on $q_k^*(t)$.

Figure 2 illustrates the impact of the reinsurer's safety loading factor η on the equilibrium reinsurance strategy. As observed from the graph, as η increases, the insurer's optimal reinsurance strategy $q_1^*(t)$ increases, whereas the reinsurer's optimal reinsurance strategy $q_2^*(t)$ decreases. The reasons are as follows: An increase in η implies a corresponding rise in reinsurance premium rates. Consequently, the cost of purchasing reinsurance for the insurer increases, meaning that the insurer must pay higher premiums for the same level of coverage and terms. For the reinsurer, however, higher premiums generate increased cash flow, which may encourage the reinsurer to offer relatively more reinsurance coverage to capture profits, although this also entails assuming greater risk. Nevertheless, a higher η is not always better. Excessively high safety loading factors lead to significantly increased premiums, which may deter insurers from purchasing reinsurance. In a competitive reinsurance market, this could put the reinsurer at a disadvantage and result in the loss of certain business opportunities. Conversely, if the safety loading factor is too low, the reinsurer collects relatively lower premiums, which may leave it inadequately prepared to handle large excess claims when they occur.

Figure 3 illustrates the influence of the risk aversion coefficient m_1 and the competition sensitivity coefficient k_1 on the insurer's optimal reinsurance strategy. As shown in the graph, $q_1^*(t)$ is negatively correlated with the risk aversion coefficient m_1. An increase in m_1 indicates that the insurer becomes more risk-averse and is more inclined to purchase reinsurance to transfer risk. The figure also reveals that the competition sensitivity coefficient k_1 increases, the insurer's optimal reinsurance strategy $q_1^*(t)$ also increases. That is, a higher competition sensitivity coefficient k_1 implies that the insurer is more sensitive to its relative wealth compared to reinsurers. In response, the insurer will reduce its reinsurance purchases and increase its retention level, thereby gaining a competitive edge in a highly competitive market environment. Although this approach exposes the insurer to greater risk, it can also lead to more substantial terminal wealth returns and enhance its competitiveness in the market.

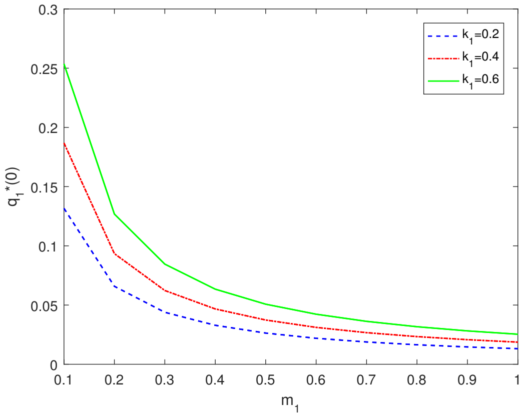

Figure 3. The effects of m_1, k_1 on $q_1^*(t)$.

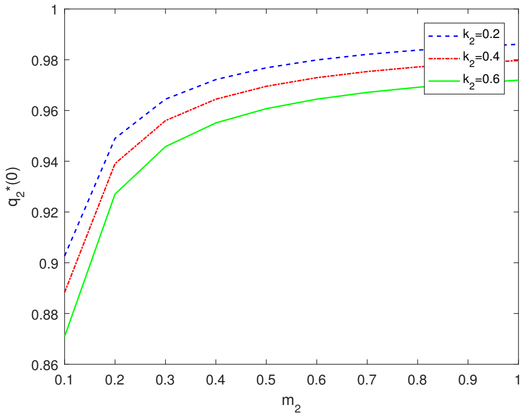

Figure 4. The effects of m_2, k_2 on $q_2^*(t)$.

Figure 4 depicts the influence of the risk aversion coefficient m_2 and the competition sensitivity coefficient k_2 on the reinsurer's optimal reinsurance strategy $q_2^*(t)$. As shown in the graph, $q_2^*(t)$ is positively correlated with the risk aversion coefficient m_2. An increase in m_2 indicates that the reinsurer becomes more risk-averse, leading to a reduced willingness to accept reinsurance contracts. This decreases potential risk payouts and raises the retention level $q_2^*(t)$. The figure also shows that as the sensitivity coefficient k_2 increases, the reinsurer's optimal strategy $q_2^*(t)$ decreases. This suggests that the reinsurer places greater emphasis on competitive sensitivity, lowering its retention level $q_2^*(t)$ and allocating more funds toward investments to generate higher returns. Such a strategy helps ensure a stronger position in a highly competitive environment.

Figure 5 illustrates the influence of the risk aversion coefficient m_1 and the competition sensitivity coefficient k_1 on the insurer's equilibrium investment strategy $\pi_1^*(t)$. As shown in the graph, $\pi_1^*(t)$ decreases as the risk aversion coefficient m_1 increases. A higher m_1 indicates that the insurer is more risk-averse and thus tends to allocate more funds to low-risk or risk-free assets to mitigate exposure, thereby reducing investment in risky assets. At the same time, it can be observed that the competition sensitivity coefficient k_1 is positively correlated with the insurer's investment strategy $\pi_1^*(t)$. An increase in k_1 implies that the insurer is more sensitive to competition with reinsurers. To achieve higher returns, the insurer will opt for investments with higher expected yields, which typically entail greater risks.

Figure 6 depicts the influence of the risk aversion coefficient m_2 and the competition sensitivity coefficient k_2 on the reinsurer's equilibrium investment strategy $\pi_2^*(t)$. As can be seen from the graph, m_2 is negatively correlated with the reinsurer's equilibrium investment strategy $\pi_2^*(t)$, while the competition sensitivity coefficient k_2 is positively correlated with $\pi_2^*(t)$. A higher value of m_2 indicates a greater degree of risk aversion on the part of the reinsurer. In such cases, the reinsurer will reduce its allocation to risky assets. Conversely, as k_2 increases, competition between the insurer and the reinsurer intensifies. The reinsurer thus seeks higher returns by allocating more funds to risky assets, that is, by raising the proportion of funds invested in such assets.

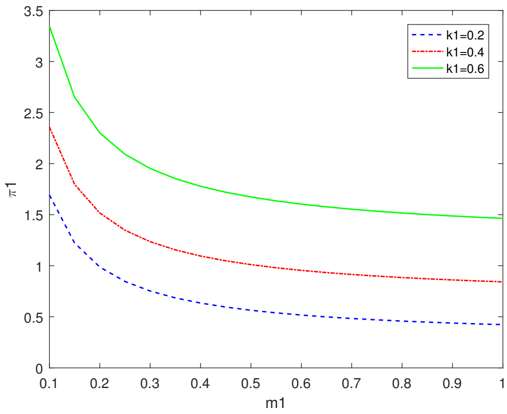

Figure 5. The effects of m_1, k_1 on $\pi_1^*(t)$.

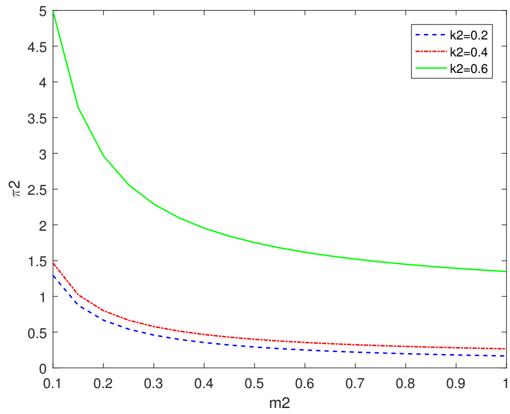

Figure 6. The effects of m_2, k_2 on $\pi_2^*(t)$.

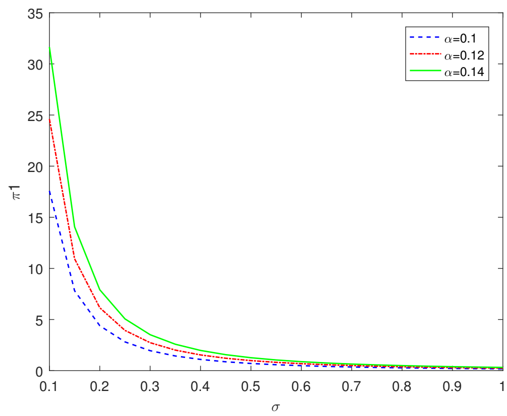

Figure 7. The effects of α, σ on $\pi_1^*(t)$.

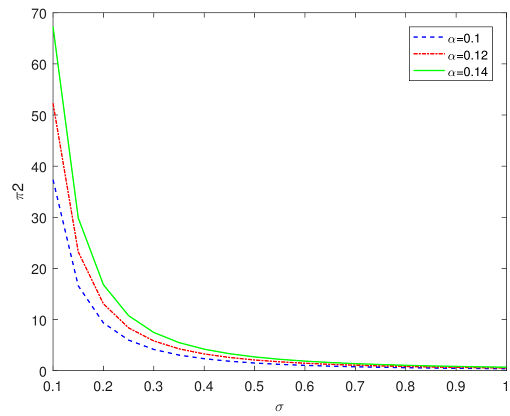

Figure 8. The effects of α, σ on $\pi_2^*(t)$.

Figures 7 and 8 illustrate the impact of the expected return rate α and the expected volatility σ of risky assets on the equilibrium investment strategies $\pi_k^*(t)$, $k = 1, 2$ for both the insurer and the reinsurer. As shown in the two figures, a higher expected return rate α of the risky asset leads to greater investment in the risky asset by both the insurer and the reinsurer. Hence, $\pi_k^*(t)$, $k = 1, 2$ increase as the expected return rate α rises. A higher α implies that the risky asset yields higher returns, which incentivizes both parties to invest more in such assets to pursue greater profits. Furthermore, as the expected volatility σ increases, the likelihood of adverse movements in the risky asset also rises. In response, both the insurer and the reinsurer tend to reduce their investments in high-risk assets and shift toward lower-risk investment alternatives.

Figures 9 and 10 respectively illustrate the impacts of σ_{11} and σ_{21}—the volatilities of uncontrollable exogenous liabilities—on the investment strategies of the insurer and the reinsurer. As can be seen from Figure 9, with other parameters remaining unchanged, the value of parameter σ_{11} determines the magnitude of the liability volatility of the insurer. A larger σ_{11} will lead to an increase in liabilities. Therefore, the insurer tends to increase their investment in risky assets to reduce liabilities. Similarly, it can be observed from Figure 10 that the reinsurer exhibits a similar trend.

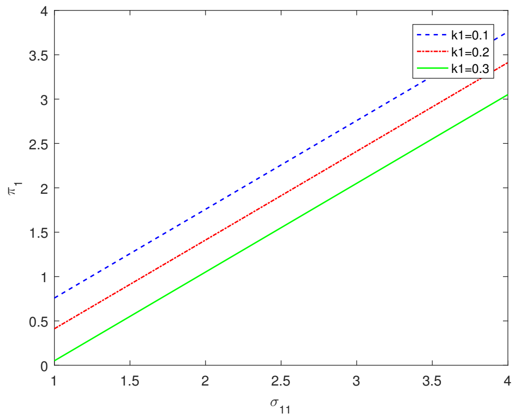

Figure 9. The effects of σ_{11} on $\pi_1^*(t)$.

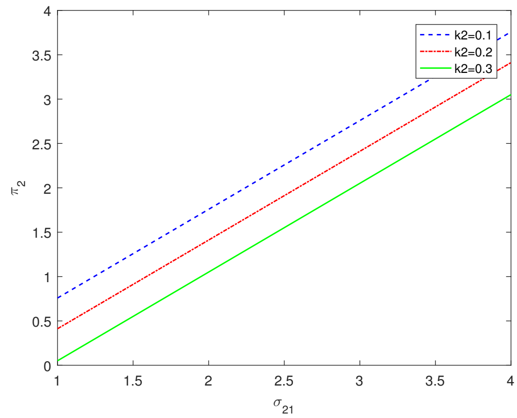

Figure 10. The effects of σ_{21} on $\pi_2^*(t)$.

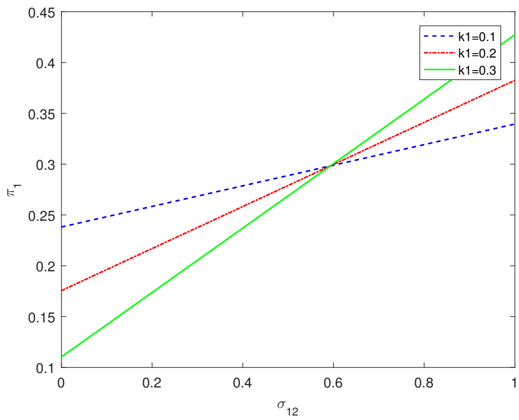

Figure 11. The effects of σ_{12} on $\pi_1^*(t)$.

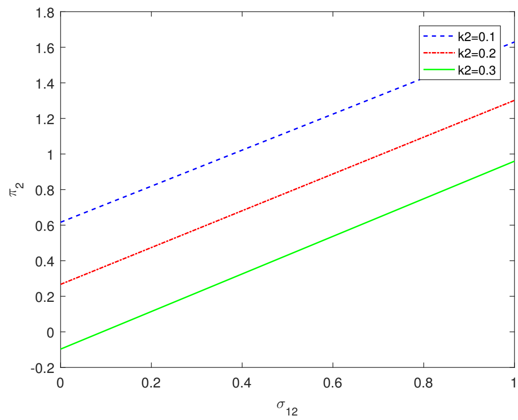

Figure 12. The effects of σ_{12} on $\pi_2^*(t)$.

Figures 11 and 12 graphically illustrate the impact of liability volatility σ_{12} on the equilibrium investment strategies $\pi_i, (i = 1, 2)$ of the insurer and the reinsurer. As shown in Figure 11, with all other parameters remaining constant, a larger value of this parameter σ_{12} leads to a higher debt level. Consequently, the insurer tends to increase their investment in risky assets to reduce the debt level. By the same token, the reinsurer exhibits a similar trend.

As illustrated in Figures 13 and 14, we analyze the impacts of the expected return on risky assets α and the elasticity parameter β on the equilibrium investment strategy $\pi_i, (i = 1, 2)$. Notably, π_1 increases with the rise of α and decreases with the fall of β; that is, a larger value of α indicates a higher return on risky assets. The insurer can achieve higher earnings through such investments and thus increase their allocation proportion of risky assets. In contrast, π_2 decreases with the rise of α and also declines as β increases. A higher return on risky assets is accompanied by greater risks, and the reinsurer will thus avoid risks by cutting their investments in risky assets. With the increase of β, the value of s^β rises, which leads to a larger magnitude of the expected decline in volatility. Meanwhile, the probability of significant adverse fluctuations in the prices of risky assets increases, which indicates that higher investment risks will diminish the attractiveness of risky assets. Consequently, investors will reduce their allocation of risky assets to lower their risk exposure.

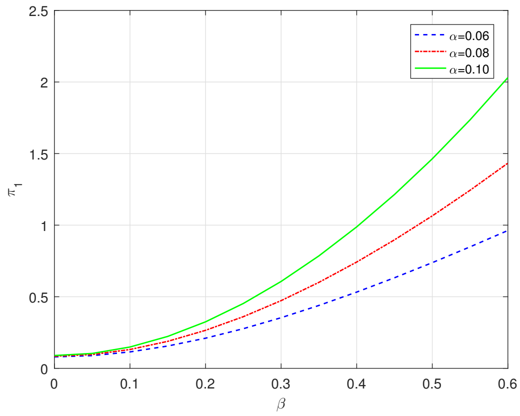

Figure 13. The effects of β on $\pi_1^*(t)$.

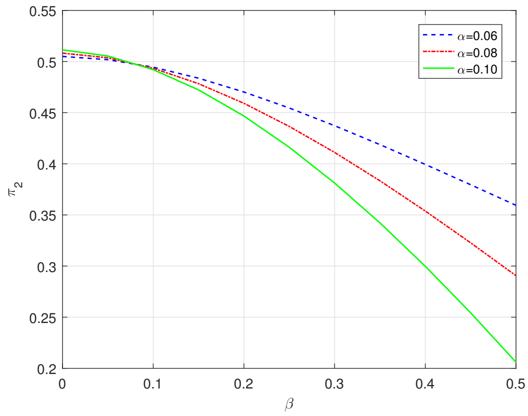

Figure 14. The effects of β on $\pi_2^*(t)$.

5. Conclusion

In this paper, we investigate the optimal reinsurance problem within the Stackelberg game framework, where both the insurer and the reinsurer are assumed to be subject to uncontrollable exogenous liabilities. To begin, we posit that the financial market comprises a risk-free asset and a risky asset. The price dynamics of the risky asset are governed by a simple Brownian motion model, whereas the surplus processes of both the insurer and the reinsurer follow the classic Cramér–Lundberg model. We define the relative wealth of the two parties to construct a stochastic differential game model that characterizes the investment and reinsurance decisions with embedded liability constraints between the insurer and the reinsurer. Subsequently, we apply stochastic dynamic programming theory to solve the Hamilton-Jacobi- Bellman (HJB) equations corresponding to each party and ultimately derive the equilibrium value function as well as the optimal equilibrium investment and reinsurance strategies. We then address the pricing issue in the reinsurance market by analyzing the optimal reinsurance strategy derived above. Finally, we perform a numerical analysis to explore the impact of key parameters on the equilibrium optimal strategies, and we visualize the relationships between these parameters and strategies via graphical illustrations. Based on the graphical results, we further interpret how the key parameters exert their influence on the equilibrium investment and reinsurance decisions.

The main conclusions are drawn as follows: (i) When the insurer prioritizes terminal wealth maximization, it transfers risks by purchasing proportional reinsurance and allocates capital to risky assets for return generation. (ii) As the risk aversion coefficient increases, both the insurer and the reinsurer will scale back their investments in risky assets. (iii) The higher the liability volatility, the more both the insurer and the reinsurer will increase their investments in high-risk assets to mitigate liability risks; in contrast, liability volatility exerts no impact on reinsurance strategies.

Based on the research foundation and limitations of this paper, in the future, we can expand the research scope and deepen the research content from the following aspects to further improve the application research of the Stackelberg differential game in insurance and financial markets: First, this paper adopts the most commonly used expected premium principle. We can extend the research to variance premiums, exponential premiums, the generalized Denneberg absolute deviation principle [41], and other pricing principles, and we can analyze the impacts of different pricing rules on equilibrium strategies as well as the robustness of the resulting outcomes. Second, this paper mainly conducts its analysis based on the form of proportional reinsurance. However, a variety of reinsurance forms exist in the actual insurance market, and significant differences are observed in the risk diversification effects and pricing mechanisms across different reinsurance forms. For future research, we can consider incorporating multiple reinsurance forms such as excess-of-loss reinsurance, stop-loss reinsurance, and composite reinsurance, and we can compare the optimal strategies and profit levels of both parties under different reinsurance forms. Meanwhile, we can explore hybrid reinsurance strategies that integrate various reinsurance forms, and analyze their advantages in terms of risk diversification and profit enhancement, and thereby providing a more comprehensive reference for the reinsurance decision-making of market participants. Third, in future research, we intend to incorporate more complex market environments and constraint conditions. Specifically, we will further relax the hypothetical premises adopted in this paper and take into account more reality-aligned market factors, such as information asymmetry (i.e., the two parties hold divergent information regarding the returns of risky assets and the risks of insurance claims), regulatory constraints (e.g., solvency supervision and capital adequacy requirements), macroeconomic fluctuations, default risks, and model uncertainty. We will then analyze the impacts of these factors on the equilibrium of the Stackelberg differential game, thereby enhancing the applicability and robustness of the research conclusions.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

Liangli Chen: formal analysis, methodology, conceptualization, software, validation, visualization, writing–review and editing, writing–original draft; Zhijian Qiu: data curation, supervision, resources; Lu Li: writing–review and editing, investigation, formal analysis. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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