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Research article

A nonconvex modeling approach to cost–volume–profit analysis with variable price in an oligopolistic market

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Abstract: Classical cost–volume–profit (CVP) analysis usually provides point-wise break-even or target-profit calculations under fixed-price assumptions. This paper developed a nonconvex feasible–profit framework for multi-producer CVP analysis in an oligopolistic market where the common price is endogenously determined by the joint production vector. The framework was applied to a three-country copper export market consisting of Mongolia, Chile, and Peru, which are interpreted as aggregate producers, using annual data for 2013–2025. Export volumes, China's copper import demand, and London Metal Exchange copper prices were obtained from Bloomberg, while cost parameters were calibrated benchmark values. For the three reduced-form price environments, the price coefficient was estimated by least squares, feasible–profit membership was tested, and a penalty method was used to compute local stationary points. The results showed that feasible–profit membership depends strongly on the assumed price mechanism. Under the linear aggregate-output and pairwise interaction specifications, the observed export vector is infeasible only in the early years and becomes feasible later, while under the inverse aggregate-output specification, the observed vector remains feasible throughout the sample but the computed stationary points are infeasible in 2024–2025. The findings suggest that multi–producer CVP analysis should account for endogenous price formation and export competition, and the proposed framework provides an interpretable tool for examining nonconvex feasible–profit regions.

Keywords: CVP analysis; oligopolistic market; feasible–profit set; nonconvex optimality
Mathematics Subject Classification: 90C30, 90C26, 90C20, 52C17

1. Introduction

Cost–volume–profit (CVP) analysis is a classical tool in managerial decision–making for examining the relationship among cost, sales volume, and profit [1–3]. In its standard form, CVP analysis assumes fixed selling prices, linear revenue and cost relationships, fixed input prices, fixed total fixed costs, and variable costs proportional to output [1]. Under these assumptions, CVP analysis is mainly used to determine the sales or production level required to cover total cost or attain a target profit. Break-even analysis is therefore a special case of CVP analysis [1, 2]. If sales fall below the break-even point, losses arise; hence, the break-even level and margin of safety remain central indicators for managerial planning [1, 3].

A large body of research has applied break-even and CVP analysis in managerial and production contexts, including multi-product contribution analysis, micro and small enterprises, and agricultural production [3–5]. Further applications include reusable product systems, milk production and pricing, and sector-specific production analysis [6–8]. More recent studies have extended these approaches to pricing, material-flow analysis, and related industrial decision-making contexts [9–11]. Related applications have also been discussed in production and sustainability-oriented settings [12]. These studies confirm the usefulness of CVP analysis but also reveal a structural limitation: most CVP models remain concentrated on single-product, single-firm, fixed-price, or deterministic target-profit settings. As a result, the classical framework usually identifies a single break-even point or target-profit point rather than a multidimensional set of profitable operating conditions.

Production planning and optimization studies have developed broader tools for complex decision environments. Mathematical programming, nonlinear programming, equilibrium models, game-theoretical approaches, and multiobjective optimization have been applied to production quantities, capacity allocation, pricing, scheduling, inventory control, and resource planning [13, 14]. Recent studies also combine optimization with machine learning, uncertainty modeling, and metaheuristic algorithms in project scheduling, operating-room assignment, and renewable-energy portfolio selection [15–17]. Related applications have also been developed for agricultural supply-chain design, biodiesel supply-chain design, and circular supply-chain network design [18–20]. These studies demonstrate the growing importance of advanced optimization in planning problems. However, their primary focus is usually scheduling, allocation, project selection, or supply-chain network design rather than the CVP-specific question of whether multiple producers or exporting countries can remain profitable simultaneously when the common price is endogenously generated by their joint output decisions.

A natural step beyond point-wise CVP analysis is the study of profitability regions. Enkhbat and Tungalag [21] introduced a sphere-packing approach to break-even and profitability analysis in which profitable ranges of prices, outputs, and variable costs are described through inscribed spheres. This approach extends CVP analysis from isolated break-even points to profitability regions. However, it is developed for reduced settings in which some parameter groups are fixed, so that the resulting sets remain convex and can be handled by linear programming. Once several economically relevant variables are allowed to vary simultaneously, especially when price depends on output, the convex structure that supports reduced formulations need not be preserved.

Relative to Enkhbat and Tungalag [21] and Barintag et al. [22], the novelty of the present paper lies in four aspects. First, it introduces endogenous common-price determination into a multi-producer CVP framework. Second, it compares three reduced-form price mechanisms to show how feasible–profit membership changes with the assumed price-output relationship. Third, it applies the framework to country-level aggregate producers in the copper export market. Fourth, it distinguishes between observed feasible–profit membership and the feasibility of penalty-method stationary points. Thus, the contribution is not only an application of a prior nonconvex CVP method, but an extension of CVP analysis toward endogenous-price feasible-set interpretation.

Recent work in operations research and industrial organization shows that endogenous-price oligopoly models often generate structurally difficult optimization problems, including multilevel, bi-level, and equilibrium formulations [23–25]. Related studies also examine multi-leader/multi-follower structures, further showing the complexity of optimization problems with strategic price and output interactions [26]. Recent advances in constrained and nonconvex optimization also emphasize the importance of penalty methods and related local computational techniques for difficult nonlinear constraint systems [27–29]. These strands of literature are closely related to the present study, but they have not yet been integrated into a framework directly suitable for multi-producer CVP analysis under endogenous common-price determination.

The central research gap addressed in this paper is the absence of a tractable nonconvex modeling framework for multi-producer CVP analysis in which joint profitability under an endogenous common price is treated as a feasible-point problem. Existing CVP studies usually focus on break-even points or target-profit calculations under fixed-price assumptions. Existing production planning and metaheuristic approaches provide useful tools for complex optimization problems, but they usually do not formulate joint CVP profitability as a feasible–profit region. Existing endogenous-price oligopoly models capture strategic price-output interactions, but they are not designed specifically to extend CVP analysis from point-wise break-even analysis to nonlinear feasible-set membership.

The proposed framework differs from general nonconvex production optimization and metaheuristic solution approaches in both objective and interpretation. It is not intended to serve as a general-purpose global optimizer or to dominate metaheuristic algorithms for large-scale production planning. Instead, it provides a transparent feasible–profit formulation in which the profitability inequalities, market-capacity restriction, and endogenous price mechanism are stated explicitly. A penalty method is then used as a local computational device to construct stationary points of the associated penalized problem and to evaluate whether those points satisfy the feasible–profit conditions. The practical advantage of this approach is interpretability: computed points can be assessed directly in terms of joint profitability and market feasibility.

Motivated by this perspective, the paper develops a nonconvex modeling framework for multi-producer CVP analysis in an oligopolistic market with variable price. The model asks whether there exists a joint production vector under which all producers remain profitable simultaneously while satisfying market-capacity and non-negativity restrictions. Since the common price depends on the joint production vector, the feasible–profit set may become nonlinear and nonconvex. The detailed system description, price-function interpretation, source of nonconvexity, and penalty-method construction are presented in the Methods section.

To illustrate the framework, the paper considers a three-country copper export market consisting of Mongolia, Chile, and Peru, interpreted as aggregate producers. Three reduced-form endogenous price environments are examined: a linear aggregate-output price, an inverse aggregate-output price, and a pairwise interaction price. For each environment, the price coefficient is estimated by least squares, the observed export vector is evaluated against the profitability and market-capacity conditions, and the penalty method is used to compute stationary points of the penalized problem. The numerical experiment is designed not to estimate a structural demand system or to produce globally exact solutions, but to examine feasible-set membership and local stationary-point behavior under alternative price mechanisms.

The contribution of the paper is threefold. First, it formulates variable-price multi-producer CVP analysis as a nonconvex feasible–profit problem with endogenous common-price determination, thereby extending CVP analysis from point-wise break-even calculations to nonlinear feasibility analysis. Second, it clarifies how endogenous price-output interaction creates nonconvex feasible–profit structures and shows how a penalty-based local method can be used to construct and evaluate stationary points. Third, it provides a numerical comparison of three reduced-form price environments using copper export data for Mongolia, Chile, and Peru, showing that feasible-set membership and stationary-point admissibility depend strongly on the assumed price mechanism.

Several limitations should be noted at the outset. The three price equations are treated as reduced-form endogenous price environments rather than fully structural demand systems. The annual cost parameters used in the empirical implementation are calibrated benchmark values rather than directly observed harmonized accounting data for all countries. In addition, the penalty method is local in nature and does not guarantee global solutions, nor does it necessarily preserve the observed export vector even when that vector is already feasible. These limitations do not undermine the modeling contribution of the paper, but they clarify the interpretation of the numerical results.

The remainder of the paper is organized as follows. Section 2 presents the system description, model formulation, endogenous price functions, source of nonconvexity, least-squares estimation of the price coefficient, and penalty-method algorithm. Section 3 reports the computational experiment for the three-country copper market application, including the input data, case-by-case results, sensitivity analysis, and comparative summary. Section 4 discusses the results in relation to the existing literature, practical implications, and methodological limitations. Section 5 concludes the paper and outlines directions for future research.

2. Model formulation

2.1. System description and modeling rationale

Classical cost–volume–profit (CVP) analysis is usually developed for a single firm under fixed-price and linear-cost assumptions. In that setting, price is treated as exogenous, variable cost is assumed to be proportional to output, and the main analytical objective is to determine a break-even output level or a target-profit output level. Although this framework is useful for managerial planning, it is limited when several producers supply the same market and when the market price is affected by their joint production decisions.

In an oligopolistic production or export market, producers do not operate independently. The output decision of one producer may affect the common market price faced by all producers. Therefore, profitability cannot be evaluated only through a single-firm break-even condition. Instead, it is necessary to examine whether there exists a joint production vector under which all producers remain profitable simultaneously while satisfying market-capacity and non-negativity restrictions. This motivates a feasible–profit approach rather than a conventional single-objective profit-maximization approach.

In the theoretical formulation, the decision-making units are described generally as producers. In the empirical application, the same mathematical structure is applied to exporting countries, which are interpreted as aggregate producers in the international copper export market. Thus, Mongolia, Chile, and Peru are not treated as individual firms, but as country-level aggregate suppliers whose export volumes represent aggregate output supplied to the common destination market. This interpretation aligns the theoretical model with the country-level data used in the computational experiment.

The proposed framework is related to production planning and nonconvex optimization problems, where mathematical programming, nonlinear programming, equilibrium models, and metaheuristic algorithms are often used to analyze production, pricing, scheduling, and allocation decisions. Modern metaheuristic methods, such as genetic algorithms, particle swarm optimization, simulated annealing, and other population-based search methods, can be useful for finding high-quality numerical solutions in complex production systems. However, the objective of the present paper is different. The aim is not to develop a general-purpose global solver for nonconvex production optimization. Rather, the aim is to formulate multi-producer CVP analysis as a transparent feasible–profit problem under endogenous common-price determination.

The practical advantage of the proposed penalty-based framework is that it keeps the economic structure of the CVP problem explicit. The model directly represents the profitability conditions, market-capacity restriction, and endogenous price mechanism. The penalty method is then used as a local computational device to construct stationary points of the associated penalized problem and to evaluate whether these points satisfy the feasible–profit conditions. Thus, the framework provides an interpretable method for studying feasible-set membership and local stationary-point behavior in a nonconvex CVP setting.

2.2. Model formulation of the feasible–profit problem

Consider n producers operating in an oligopolistic market. Let x_j denote the output of producer j, let c_j denote its variable cost per unit, and let F_j denote its fixed-cost benchmark. The producers face a common market price

$$p = p(x_1, x_2, \dots, x_n),$$

which depends on the joint production vector. The profit of producer j is therefore given by

$$\Pi_j(x) = p(x_1, x_2, \dots, x_n)x_j - c_j x_j - F_j, \qquad j = 1, 2, \dots, n. \tag{2.1}$$

In a general oligopolistic setting, one may consider a multiobjective profit-maximization problem of the form [23]

$$\max \Pi_j(x), \qquad j = 1, 2, \dots, n, \tag{2.2}$$

subject to

$$\sum_{j=1}^{n} x_j \le Q, \tag{2.3}$$

$$x_j^{\min} \le x_j \le x_j^{\max}, \qquad j = 1, 2, \dots, n, \tag{2.4}$$

where Q denotes the total market-capacity parameter.

A common way to handle a multiobjective problem is to use scalarization. For example, one may consider

$$\max_{x \in S} \sum_{j=1}^{n} \alpha_j \Pi_j(x), \tag{2.5}$$

where

$$\alpha_j \geq 0, \qquad j = 1, 2, \ldots, n,$$

are nonnegative weights and S denotes the admissible set determined by (2.3)–(2.4). However, the main objective of the present paper is not to select a Pareto-optimal production vector or to maximize a weighted aggregate profit function. Instead, the central question is whether the system admits at least one joint production vector under which all producers remain profitable at the same time.

For this purpose, we introduce the following profitability conditions [21, 22]:

$$\Pi_j(x) = p(x_1, x_2, \ldots, x_n)x_j - c_j x_j - F_j \geq 0, \qquad j = 1, 2, \ldots, n, \tag{2.6}$$

$$\sum_{j=1}^{n} x_j \leq Q, \tag{2.7}$$

$$x_j^{\min} \leq x_j \leq x_j^{\max}, \qquad j = 1, 2, \ldots, n. \tag{2.8}$$

The transition from individual profitability conditions to the feasible–profit region is economically important. Each inequality in (2.6) requires one producer to operate without loss under the common endogenous price. However, in a multi-producer market, the feasibility of one producer cannot be evaluated independently of the others because all producers are connected through the same price function. Therefore, the relevant object is the set of all joint production vectors that satisfy the profitability conditions for every producer simultaneously.

Denote this feasible–profit region by Δ:

$$\Delta = \left\{ x \in \mathbb{R}^n \;\middle|\; \Pi_j(x) \geq 0,\ j = 1, 2, \ldots, n,\ \sum_{j=1}^{n} x_j \leq Q,\ x_j^{\min} \leq x_j \leq x_j^{\max},\ j = 1, 2, \ldots, n \right\}. \tag{2.9}$$

Thus, the main problem considered in this paper is the following feasible–point problem:

$$\text{Find a point } x \in \Delta.$$

This formulation extends classical CVP analysis from a point-wise break-even calculation to a multidimensional feasible–profit problem. The model asks whether there exists a joint production configuration that secures nonnegative profit for all producers while respecting market-capacity and output restrictions.

2.3. Endogenous price function and the source of nonconvexity

The function $p(\cdot)$ denotes the common market price generated by the joint production vector. Unlike classical CVP analysis, where the selling price is fixed, the present framework assumes that the market price depends on the production decisions of all producers:

$$p = p(x_1, x_2, \ldots, x_n).$$

Economically, this means that the revenue of producer j depends not only on its own output x_j, but also on the output decisions of the other producers in the same market.

The interaction between price and output is the main source of nonconvexity. In the profit function (2.1), the revenue term is $p(x)x_j$. If $p(x)$ depends on aggregate output, inverse aggregate output, or pairwise interaction terms, then $p(x)x_j$ becomes nonlinear in the decision vector. As a result, the profitability condition

$$p(x)x_j - c_j x_j - F_j \geq 0$$

defines a nonlinear inequality. The intersection of several such inequalities with market-capacity and output restrictions may generate a nonconvex feasible–profit region. Therefore, the problem is more naturally interpreted as a nonconvex feasible–point problem than as a classical convex profit–maximization problem.

2.4. Reduced-form price environments

To examine how alternative endogenous price mechanisms affect feasible–profit membership, we consider three reduced-form price environments. These specifications are not intended to estimate a full structural demand system. Instead, they are used to study how the geometry and membership of the feasible–profit region change when the common price is linked to aggregate output or interaction effects among suppliers.

It is important to emphasize that the three specifications should not be interpreted as structural demand functions. In particular, the linear aggregate-output and pairwise interaction specifications are not intended to represent conventional downward-sloping commodity demand. They are included as reduced-form computational environments that generate different nonlinear price-output relationships and allow the feasible–profit region to be examined under alternative market assumptions. Among the three cases, the inverse aggregate-output specification is closest to the usual downward-sloping price intuition. By contrast, the linear aggregate-output and pairwise interaction specifications should be interpreted as mathematical benchmark cases that capture a positive association between market activity and the reference price index, rather than as demand curves.

In the three-country copper export application, let x_m, x_c, and x_p denote the export quantities of Mongolia, Chile, and Peru, respectively. The three countries are treated as aggregate suppliers facing the same common reference price:

$$p_m = p_c = p_p = p(x_m, x_c, x_p). \tag{2.10}$$

Case 1. Linear aggregate-output price.

$$p(x_m, x_c, x_p) = a(x_m + x_c + x_p). \tag{2.11}$$

This specification represents a price index that changes proportionally with the scale of total market activity. It is included as a simple aggregate-output benchmark and should not be interpreted as a downward-sloping demand function.

Case 2. Inverse aggregate-output price.

$$p(x_m, x_c, x_p) = \frac{a}{x_m + x_c + x_p}. \tag{2.12}$$

This specification reflects the standard economic intuition that, other things being equal, a larger aggregate supply is associated with a lower market price.

Case 3. Pairwise interaction price.

$$p(x_m, x_c, x_p) = a(x_m x_c + x_m x_p + x_c x_p). \tag{2.13}$$

This specification allows the common reference price to depend on pairwise interaction effects among the three aggregate suppliers. It is included to examine how nonlinear cross-country output interactions affect feasible–profit membership and should be interpreted as a benchmark nonlinear price-output environment rather than as a structural demand curve.

Together, the three cases provide alternative reduced-form price environments for evaluating how the feasible–profit region changes under different market assumptions. They also allow the numerical experiment to compare whether the observed export vector and the penalty-method stationary points belong to the feasible–profit region under different price mechanisms.

2.5. Estimation of the price coefficient

For each price specification, the unknown parameter a is estimated by least squares. Suppose that the common market price can be represented in the general form

$$p(x_1, \ldots, x_n) = a\, f(x_1, \ldots, x_n), \tag{2.14}$$

where $f(\cdot)$ represents the selected output-based price structure. In the three-country case, this becomes

$$p(x_m, x_c, x_p) = a\, f(x_m, x_c, x_p). \tag{2.15}$$

Let $\{(x_m^i, x_c^i, x_p^i, p^i)\}_{i=1}^k$ denote the observed sample. The least-squares objective function is

$$E(a) = \sum_{i=1}^{k} \left[a\, f(x_m^i, x_c^i, x_p^i) - p^i\right]^2. \tag{2.16}$$

The least-squares estimator of a is obtained by solving

$$\min_a E(a). \tag{2.17}$$

From the first-order optimality condition,

$$\frac{dE}{da} = 2\sum_{i=1}^{k} \left(a\, f(x_m^i, x_c^i, x_p^i) - p^i\right) f(x_m^i, x_c^i, x_p^i) = 0. \tag{2.18}$$

Therefore,

$$\hat{a} = \frac{\sum_{i=1}^{k} p^i f(x_m^i, x_c^i, x_p^i)}{\sum_{i=1}^{k} f^2(x_m^i, x_c^i, x_p^i)}. \tag{2.19}$$

Equation (2.19) is used to estimate the price coefficient under each of the three reduced-form price environments.

2.6. *Penalty method for stationary-point construction*

Because the feasible–profit set may be nonconvex, finding a feasible point is computationally difficult. This paper does not attempt to solve the problem globally. Instead, following Barintag et al. [22], we use a penalty method to construct local stationary points of an associated penalized problem. The purpose of the penalty method is not to guarantee global feasibility or global optimality, but to generate computable candidate points that can be evaluated against the profitability and market-capacity conditions.

Before introducing the penalized objective formally, it is useful to explain the intuition of the penalty approach. The original feasible-point problem requires the output vector to satisfy the profitability and market-capacity restrictions. Instead of enforcing these restrictions only as hard constraints, the penalty method incorporates violations of the restrictions into the objective function. A candidate output vector that violates a profitability condition or the market-capacity condition receives an additional penalty. As the penalty parameters increase, violations become more costly, and the computed solution is pushed toward the feasible–profit set. Therefore, the penalty terms can be interpreted as artificial costs assigned to infeasibility. In economic terms, the method searches for output configurations that are attractive in terms of the weighted profit expression while progressively discouraging violations of the nonnegative-profit and market-capacity requirements.

To construct stationary points associated with the profitability conditions, consider the auxiliary nonconvex optimization problem

$$\max \ F(x) = \sum_{j=1}^{n} \alpha_j \Pi_j(x), \qquad x \in \mathbb{R}^n, \qquad \alpha_j \geq 0, \quad j = 1, 2, \ldots, n, \tag{2.20}$$

subject to

$$p(x_1, x_2, \ldots, x_n)x_j - c_j x_j - F_j \geq 0, \qquad j = 1, 2, \ldots, n, \tag{2.21}$$

$$\sum_{j=1}^{n} x_j \leq Q, \qquad x_j \geq 0, \qquad j = 1, 2, \ldots, n. \tag{2.22}$$

For numerical implementation, the maximization problem is converted into a minimization problem by minimizing the negative weighted-profit function with penalty terms. The penalized objective function is defined as

$$\Phi(x, c^k) = -\sum_{j=1}^{n} \alpha_j \Pi_j(x) + c_1^k \sum_{j=1}^{n} (F_j + c_j x_j - p(x)x_j)^2 + c_2^k \left(\sum_{j=1}^{n} x_j - Q \right)^2, \tag{2.23}$$

where c_1^k and c_2^k are positive penalty parameters that increase with iteration k.

The first penalty term penalizes violations of the profitability conditions, since

$$F_j + c_j x_j - p(x)x_j = -\Pi_j(x).$$

The second penalty term penalizes deviations from the market-capacity condition. As the penalty parameters increase, the computed solutions are forced toward points that better satisfy the profitability and capacity restrictions, whenever such points can be found by the local method.

For the penalty–method implementation, we restrict attention to the nonnegative domain

$$D = \left\{ x \in \mathbb{R}^n \;\middle|\; x_j \geq 0, \ j = 1, \ldots, n \right\}.$$

Starting from an initial point $u^0 \in D$, the penalty parameters are increased iteratively until the generated sequence stabilizes. The procedure is summarized in Algorithm 1.

Algorithm 1 Penalty method for stationary-point construction

Choose an initial point $u^0 \in D$, penalty parameters $c_1^1 > 0$, $c_2^1 > 0$, update factor $m > 1$, and tolerance $\varepsilon > 0$. Set $k = 1$.
repeat
  Solve the penalized subproblem

$$\min_{x \in D} \Phi(x, c^k),$$

  and denote a computed solution by x^k.
  if $\|u^{k-1} - x^k\| \leq \varepsilon$ **then**
    Stop.
  else
    Set $u^k = x^k$.
    Update the penalty parameters by

$$c_1^{k+1} = mc_1^k, \qquad c_2^{k+1} = mc_2^k.$$

    Set $k \leftarrow k + 1$.
  end if
until a stopping criterion is satisfied.

Accordingly, in the computational experiment, the penalty method is used as a practical local procedure for identifying stationary points of the penalized problem. A computed point is then evaluated to determine whether it belongs to the feasible–profit region. If the computed point satisfies the profitability and market-capacity conditions, it provides evidence that the feasible–profit region is nonempty under the corresponding price environment. If it does not, the result is interpreted as a limitation of the local penalty-based computation under that particular nonconvex structure.

2.7. Data description and numerical implementation

The proposed framework is implemented for a three-country copper export market consisting of Mongolia, Chile, and Peru. The output variables x_m, x_c, and x_p represent the copper export volumes of Mongolia, Chile, and Peru to China, respectively. The variable Q denotes China's copper import demand, and p denotes the London Metal Exchange three-month forward copper price. The empirical sample covers annual observations for 2013–2025.

The export, demand, and price variables are obtained from Bloomberg, and the corresponding tickers are reported in Supplementary File 1. The cost parameters used in the numerical implementation are calibrated benchmark values. They are not interpreted as directly observed harmonized accounting data for the three countries. Instead, they are used to implement the profitability conditions on a comparable scale and to illustrate the behavior of the proposed nonconvex feasible–profit model.

For each of the three price environments, the computational procedure consists of four steps. First, the parameter a is estimated using the least-squares formula in (2.19). Second, the observed export vector is evaluated against the profitability and market-capacity conditions. Third, the penalty method is applied to compute a stationary point of the penalized problem. Fourth, the computed stationary point is evaluated to determine whether it belongs to the feasible–profit region. This procedure allows the numerical experiment to compare feasible region membership and stationary-point behavior across the three endogenous price environments.

3. Computational experiment

This section presents the numerical implementation of the proposed nonconvex feasible–profit framework for the three-country copper export market consisting of Mongolia, Chile, and Peru. The computational experiment has four objectives. First, it describes the data and calibrated benchmark parameters used in the numerical implementation. Second, it estimates the price coefficient under each endogenous price environment. Third, it evaluates whether the observed export vector and the computed stationary points belong to the feasible–profit region. Fourth, it conducts a sensitivity analysis based on 2025 market conditions to examine the robustness of feasible–profit membership under alternative parameter settings.

3.1. Input data

The computational experiment is based on annual data for a three–country copper export market consisting of Mongolia, Chile, and Peru. Let x_1, x_2, and x_3 denote the export volumes of Mongolia, Chile, and Peru, respectively, and let Q denote China's copper import demand. The variable p denotes the observed copper market price, while c_t and F_t denote the calibrated annual variable-cost and fixed-cost parameters used in the numerical implementation.

The quantities x_1, x_2, x_3, and Q are measured in million tonnes, whereas the price variable p is measured in U.S. dollars per metric ton. The variables x_1, x_2, x_3, Q, and p are taken from observable international market data. In particular, p is based on the international copper price quoted on the London Metal Exchange and is used here as the common reference price for the three-country market, while Q is represented by China's copper import demand.

By contrast, directly comparable annual cost data for the three-country market are not readily available in a uniform form. For this reason, the cost parameters are calibrated for empirical implementation. The variable-cost parameter is benchmarked from an industry average production-cost value in 2013 and then adjusted year by year to obtain the annual series c_t. The fixed-cost parameter F_t is subsequently derived from this calibrated benchmark, so that both cost parameters vary over time while remaining consistent with the observed price scale. Accordingly, c_t and F_t are not interpreted as directly observed harmonized accounting data, but as calibrated annual benchmark values used to evaluate the profitability conditions on a comparable basis.

The calibration is implemented as follows. The initial value c_{2013} is set equal to the benchmark production-cost value used for numerical implementation. Specifically, the baseline value $c_{2013} = 2000$ USD per metric ton is an author-calibrated industry-cost benchmark reported in Supplementary File 1 and is used as the common initial cost scale for the numerical experiment. For subsequent years, the annual variable-cost benchmark is updated proportionally with the observed international copper price:

$$c_t = c_{2013}\frac{p_t}{p_{2013}}, \qquad t = 2013, \ldots, 2025. \tag{3.1}$$

The fixed-cost benchmark is then derived as a proportional benchmark linked to the same annual cost scale:

$$F_t = \gamma c_t, \tag{3.2}$$

where γ is the fixed-cost calibration coefficient. In the numerical experiment, $\gamma = 0.38$.

These calibrated values are applied commonly to Mongolia, Chile, and Peru in order to isolate the effect of alternative endogenous price mechanisms and joint export quantities on feasible–profit membership. This choice improves comparability across the three-country system, but it is also a limitation of the empirical application because the values do not represent country-specific accounting costs. For this reason, the sensitivity analysis in Section 3.4 varies both c and F by −20%, −10%, −5%, 0%, +5%, +10%, and +20% to examine whether the feasible–profit classification is robust to alternative cost assumptions.

It should be emphasized that this data configuration is used for empirical illustration of the proposed nonconvex modeling framework. The copper market case provides a convenient application because international price data, export quantities, and demand proxies are observable and economically interpretable. However, the mathematical structure of the model is not specific to the copper industry. The same framework may be applied to other industries in which aggregate suppliers face common market prices, capacity restrictions, and profitability conditions under endogenous or jointly determined price mechanisms.

Table 1. Input data for the three-country copper market case.

Year	x_1	x_2	x_3	Q	p	F_t	c_t
2013	0.566293	2.788759	1.917270	10.080023	7332.74	760.00	2000.00
2014	1.352802	2.942108	2.038112	11.861833	6867.39	735.49	1935.50
2015	1.418173	3.745941	2.665904	13.318840	5525.17	659.71	1736.08
2016	1.498215	4.744769	4.607897	17.051826	4877.54	619.85	1631.17
2017	1.386865	4.592277	4.937263	17.333307	6181.98	697.82	1836.37
2018	1.399692	5.822399	5.380562	19.719957	6525.34	716.94	1886.68
2019	1.326698	7.729349	5.956183	22.023818	6012.37	688.18	1811.01
2020	1.300094	7.717852	4.809581	21.776728	6169.85	697.12	1834.57
2021	1.204075	8.896671	5.547811	23.427846	9315.01	856.59	2254.18
2022	1.317491	8.482143	6.268964	25.317628	8827.44	833.87	2194.39
2023	1.419753	8.446724	7.264334	27.588108	8491.84	817.87	2152.28
2024	1.548466	9.234782	6.993802	28.164599	9159.07	849.39	2235.23
2025	2.073402	9.492014	7.569675	30.365196	9938.07	884.78	2328.35

Notes: x_1, x_2, and x_3 denote the export volumes of Mongolia, Chile, and Peru, respectively, measured in million tonnes. Q denotes China's copper import demand, also measured in million tonnes. p is the observed international copper price quoted on the London Metal Exchange, measured in U.S. dollars per metric ton. The annual cost parameters c_t and F_t are calibrated benchmark values used for numerical implementation. The baseline value $c_{2013} = 2000$ USD per metric ton is an author-calibrated industry-cost benchmark reported in Supplementary File 1. The variable-cost benchmark is then updated proportionally with the observed international copper price according to equation (3.1). The fixed-cost benchmark is derived from the same calibration scale according to equation (3.2). In the computational experiment, these annual benchmark values are applied commonly across the three countries in each year. Accordingly, c_t and F_t are not directly observed accounting data, but calibration values used to examine the nonconvex model and its associated local stationary-point behavior.

Table 1 reports the input data used in the numerical study. The sample covers the period spanning 2013–2025 and provides the empirical basis for estimating the price coefficient under each endogenous price environment and for evaluating the model's feasibility and stationary-point behavior. Thus, the purpose of the data in this section is not to claim that the framework is specific only to copper markets, but to provide an empirical setting in which the proposed nonconvex model and its associated local penalty method can be implemented and interpreted numerically.

3.2. Computational procedure

The computational experiment is designed to evaluate the behavior of the proposed nonconvex feasible–profit framework under the three reduced-form endogenous price environments introduced in Section 2. The procedure is applied separately to each year in the sample period spanning 2013–2025 and to each price specification. The objective is not to compute a globally optimal production vector, but to examine feasible–profit membership and the behavior of the local penalty-based stationary-point method.

For each price environment, the computational procedure consists of four main steps. First, the price coefficient a is estimated using the least-squares estimator in (2.19). The functional form $f(x_m, x_c, x_p)$ differs across the three cases: aggregate output in Case 1, inverse aggregate output in Case 2, and pairwise output interaction in Case 3. This step produces one estimated price equation for each endogenous price environment.

Second, the observed export vector for each year,

$$x^{obs} = (x_m^{obs}, x_c^{obs}, x_p^{obs}),$$

is evaluated against the feasible–profit conditions. Specifically, the model checks whether the profitability inequality

$$\Pi_j(x^{obs}) \geq 0, \qquad j = m, c, p,$$

and the market-capacity condition

$$x_m^{obs} + x_c^{obs} + x_p^{obs} \leq Q$$

are satisfied. If all conditions hold, the observed export vector is classified as belonging to the feasible–profit region. Otherwise, it is classified as outside the feasible–profit region.

Third, the penalty method described in Algorithm 1 is applied to compute a stationary point of the penalized problem. Starting from an initial point in the nonnegative domain, the method solves a sequence of penalized minimization problems with increasing penalty parameters. The computed point is denoted by x^k. Since the model is nonconvex, the resulting point is interpreted as a local stationary point of the penalized problem rather than as a globally optimal solution.

Fourth, the computed stationary point x^k is evaluated against the same profitability and market-capacity conditions used for the observed export vector. This allows comparison between the feasibility status of the observed export vector and that of the computed stationary point. In addition, the squared Euclidean distance between the observed vector and the computed stationary point is calculated as

$$\|x^k - x^{obs}\|^2.$$

This distance is used only as a descriptive measure of how far the computed stationary point is from the observed export configuration. It is not interpreted as an objective minimized by the penalty method.

The numerical implementation is carried out in the R programming environment. For each case and year, the outputs reported in the following subsections include the modeled price, the total observed export volume, the total computed export volume, the feasibility status of the observed vector, the feasibility status of the computed stationary point, and the squared distance between the two vectors. This procedure allows the three endogenous price environments to be compared in terms of feasible-set membership, stationary-point admissibility, and sensitivity to the assumed price mechanism.

3.3. Computational results under three price environments

This subsection reports the numerical results for Mongolia, Chile, and Peru under the three reduced-form price environments introduced in Section 2.4. For each case, the price coefficient a is estimated using the least-squares formula in equation (2.19). The observed export vector is then evaluated against the profitability and market-capacity conditions. The penalty method described in Algorithm 1 is also applied to compute local stationary points of the associated penalized problem.

The numerical implementation is carried out in the R programming environment. For each price environment, the computation proceeds in three steps. First, the price coefficient is estimated from the observed data. Second, the observed export vector is tested for membership in the feasible–profit region. Third, the penalty-based stationary point is computed and evaluated against the same feasibility conditions. This procedure allows comparison of observed feasible-region membership and computed stationary-point admissibility across the three alternative price mechanisms.

The purpose of the numerical experiment is not to estimate a structural demand system, recover the observed export vector, or obtain globally optimal production quantities. Rather, it is to examine how the local penalty-based stationary-point method behaves under alternative reduced-form price specifications and whether the computed points satisfy the feasible–profit conditions. The results are reported case by case below.

3.3.1. Case 1: Linear aggregate-output price

In the first price environment, the common market price is specified by

$$p(x_1, x_2, x_3) = a(x_1 + x_2 + x_3).$$

Using the data in Table 1, the least-squares estimate is $a = 532.884$. Hence, the fitted price equation is

$$p(x_1, x_2, x_3) = 532.884(x_1 + x_2 + x_3).$$

Table 2 reports the observed and computed results under this specification.

Table 2. Observed and computed results under Case 1: Linear aggregate-output price.

Year	$p(x^{obs})$	$\sum x^{obs}$	$\sum x^k$	$p(x^k)$	Obs. in set	Stat. pt. in set	Sq. dist.
2013	2809.54	5.2723	4.6695	2488.28	No	Yes	2.6291
2014	3374.77	6.3330	4.5435	2421.14	Yes	Yes	2.3384
2015	4172.49	7.8300	4.1523	2212.71	Yes	Yes	7.2224
2016	5782.26	10.8509	3.9455	2102.48	Yes	Yes	22.6378
2017	5817.18	10.9164	4.3494	2317.70	Yes	Yes	22.0417
2018	6715.75	12.6027	4.4479	2370.24	Yes	Yes	34.0341
2019	7999.78	15.0122	4.2996	2291.19	Yes	Yes	60.1102
2020	7368.47	13.8275	4.3458	2315.81	Yes	Yes	50.6217
2021	8338.87	15.6486	5.1640	2751.81	Yes	Yes	66.3949
2022	8562.70	16.0686	5.0479	2689.96	Yes	Yes	67.4008
2023	9128.74	17.1308	4.9661	2646.36	Yes	Yes	77.6386
2024	9473.11	17.7771	5.1272	2732.22	Yes	Yes	84.5903
2025	10196.78	19.1351	5.3078	2828.44	Yes	Yes	93.3781

Notes: x^k denotes a stationary point computed by the penalty-method procedure. “Sq. dist.” denotes the squared Euclidean distance between x^k and the observed export vector.

Under Case 1, the observed export vector lies outside the feasible–profit set only in 2013 and belongs to that set from 2014 onward. The penalty method computes a feasible stationary point for 2013, indicating that the feasible–profit set is nonempty in that year. In later years, the observed vector is already feasible, but the computed stationary points differ substantially from the observed configuration. Thus, the numerical results should be interpreted as evidence on feasible-set membership and local stationary-point behavior, not as recovery of the observed export vector.

3.3.2. Case 2: Inverse aggregate-output price

In the second price environment, the common market price is specified by

$$p(x_1, x_2, x_3) = \frac{a}{x_1 + x_2 + x_3}.$$

Using the data in Table 1, the least-squares estimate is $a = 64684.22$. Hence, the fitted price equation is

$$p(x_1, x_2, x_3) = \frac{64684.22}{x_1 + x_2 + x_3}.$$

Table 3 reports the observed and computed results under this specification.

Table 3. Observed and computed results under Case 2: Inverse aggregate–output price.

Year	$p(x^{obs})$	$\sum x^{obs}$	$\sum x^k$	$p(x^k)$	Obs. in set	Stat. pt. in set	Sq. dist.
2013	12268.64	5.2723	10.0800	6417.07	Yes	Yes	10.2127
2014	10213.80	6.3330	11.8618	5453.14	Yes	Yes	11.4602
2015	8261.06	7.8300	13.3188	4856.60	Yes	Yes	12.7563
2016	5961.20	10.8509	17.0518	3793.39	Yes	Yes	19.5602
2017	5925.41	10.9164	17.3333	3731.79	Yes	Yes	21.3919
2018	5132.59	12.6027	19.7200	3280.14	Yes	Yes	28.7530
2019	4308.77	15.0122	22.0238	2937.01	Yes	Yes	38.2442
2020	4677.93	13.8275	21.7767	2970.34	Yes	Yes	41.7173
2021	4133.56	15.6486	23.4278	2761.00	Yes	Yes	49.9254
2022	4025.51	16.0686	25.3176	2554.91	Yes	Yes	55.4307
2023	3775.90	17.1308	27.5881	2344.64	Yes	Yes	64.7635
2024	3638.64	17.7771	27.7985	2326.90	Yes	No	64.7276
2025	3380.40	19.1351	26.6411	2427.98	Yes	No	48.4270

Notes: x^k denotes a stationary point computed by the penalty-method procedure. "Sq. dist." denotes the squared Euclidean distance between x^k and the observed export vector.

Under Case 2, the observed export vector belongs to the feasible–profit set throughout the sample period. However, the computed stationary points belong to the feasible–profit set only from 2013 through 2023 and not in 2024–2025 under the present computational setup. This case therefore shows that feasibility of the observed vector and feasibility of the computed stationary point are distinct outcomes. It also illustrates the sensitivity of local penalty-based computation to the assumed endogenous price mechanism.

3.3.3. Case 3: Pairwise interaction price

In the third price environment, the common market price is specified by

$$p(x_1, x_2, x_3) = a(x_1x_2 + x_1x_3 + x_2x_3).$$

Using the data in Table 1, the least-squares estimate is $a = 113.759$. Hence, the fitted price equation is

$$p(x_1, x_2, x_3) = 113.759(x_1x_2 + x_1x_3 + x_2x_3).$$

Table 4 reports the observed and computed results under this specification.

Table 4. Observed and computed results under Case 3: Pairwise interaction price.

Year	$p(x^{obs})$	$\sum x^{obs}$	$\sum x^k$	$p(x^k)$	Obs. in set	Stat. pt. in set	Sq. dist.
2013	911.41	5.2723	7.7766	2293.19	No	Yes	4.5984
2014	1448.56	6.3330	7.6577	2223.64	No	Yes	1.8559
2015	2170.45	7.8300	7.2771	2008.05	No	Yes	2.8159
2016	4081.17	10.8509	7.0679	1894.27	Yes	Yes	11.5133
2017	4082.74	10.9164	7.4711	2116.58	Yes	Yes	11.6230
2018	5347.62	12.6027	7.5665	2170.94	Yes	Yes	20.3221
2019	7302.62	15.0122	7.4226	2089.16	Yes	Yes	41.0578
2020	6075.45	13.8275	7.4677	2114.63	Yes	Yes	34.1366
2021	7593.31	15.6486	8.2270	2566.54	Yes	Yes	48.1127
2022	8259.87	16.0686	8.1235	2502.34	Yes	Yes	47.9575
2023	9517.69	17.1308	8.0497	2457.09	Yes	Yes	55.8008
2024	10205.93	17.7771	8.1944	2546.20	Yes	Yes	61.8604
2025	12198.02	19.1351	8.3536	2646.10	Yes	Yes	68.3939

Notes: x^k denotes a stationary point computed by the penalty-method procedure. "Sq. dist." denotes the squared Euclidean distance between x^k and the observed export vector.

Under Case 3, the observed export vector lies outside the feasible–profit set in 2013–2015 and belongs to that set from 2016 onward. The penalty method computes feasible stationary points for the early infeasible years, indicating that the feasible–profit set is nonempty in those years. As in Case 1, the computed stationary points in later years differ from the observed configuration, so the results should be interpreted as evidence on feasible-set membership and local stationary-point behavior rather than as restoration of the observed vector.

3.4. Sensitivity analysis based on 2025 market conditions

To evaluate the robustness of the proposed feasible–profit framework under alternative market conditions, we conduct a sensitivity analysis based on the 2025 observation. The year 2025 is selected because it represents the most recent market condition in the sample and allows a direct comparison of Mongolia, Chile, and Peru under the three endogenous price environments. In the baseline 2025 case, the observed export vector is

$$(x_m, x_c, x_p) = (2.0734,\ 9.4920,\ 7.5697),$$

and the total export volume is 19.1351 million metric tons. Since China's copper import demand is 30.3652 million metric tons, the market-capacity condition is satisfied in the baseline case.

The sensitivity analysis varies four key parameters: the price coefficient a, the calibrated variable-cost benchmark c, the calibrated fixed-cost benchmark F, and the market-capacity parameter Q. Each parameter is varied by -20%, -10%, -5%, 0%, $+5\%$, $+10\%$, and $+20\%$, while the remaining parameters are held fixed at their 2025 baseline values. For each perturbation, the observed 2025 export vector is tested for membership in the feasible–profit set under the three price environments. This design allows us to examine whether the feasible–profit classification remains stable when price responsiveness, cost benchmarks, and market-capacity conditions change within a moderate range.

Table 5. Sensitivity summary for 2025 feasible–profit membership.

parameter	case	feasible count	feasible share
F	1	7/7	1.000
F	2	7/7	1.000
F	3	7/7	1.000
Q	1	7/7	1.000
Q	2	7/7	1.000
Q	3	7/7	1.000
a	1	7/7	1.000
a	2	6/7	0.857
a	3	7/7	1.000
c	1	7/7	1.000
c	2	7/7	1.000
c	3	7/7	1.000

Notes: Each parameter is varied by −20%, −10%, −5%, 0%, +5%, +10%, and +20%, while the remaining parameters are held fixed at their 2025 baseline values. Feasible count reports the number of perturbation scenarios in which the observed 2025 export vector remains in the feasible–profit set.

Table 5 shows that the observed 2025 export vector is generally robust to the tested parameter perturbations. For the calibrated fixed-cost benchmark F, the market-capacity parameter Q, and the calibrated variable-cost benchmark c, the observed vector remains feasible at all seven perturbation levels under all three price environments. The same result holds for the price coefficient a under the linear aggregate-output and pairwise interaction specifications. The only exception occurs in Case 2, where the observed vector remains feasible in six of the seven perturbation scenarios. Thus, the inverse aggregate-output specification is the most sensitive price environment in the 2025 robustness test.

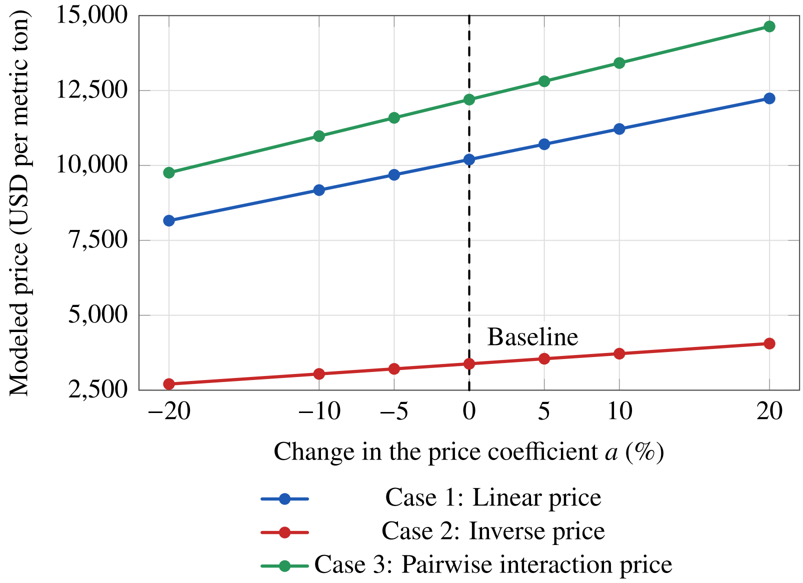

Figure 1. Sensitivity of modeled price to changes in the price coefficient a under the three reduced-form price environments.

Figure 1 shows that the modeled price responds monotonically to changes in the price coefficient a under all three price environments. However, the level of the modeled price differs substantially across the three specifications. In the 2025 sensitivity analysis, the inverse aggregate-output specification produces the lowest modeled price, whereas the pairwise interaction specification produces the highest modeled price. This difference is important because a lower modeled price reduces the profitability margin available for satisfying the nonnegative-profit conditions.

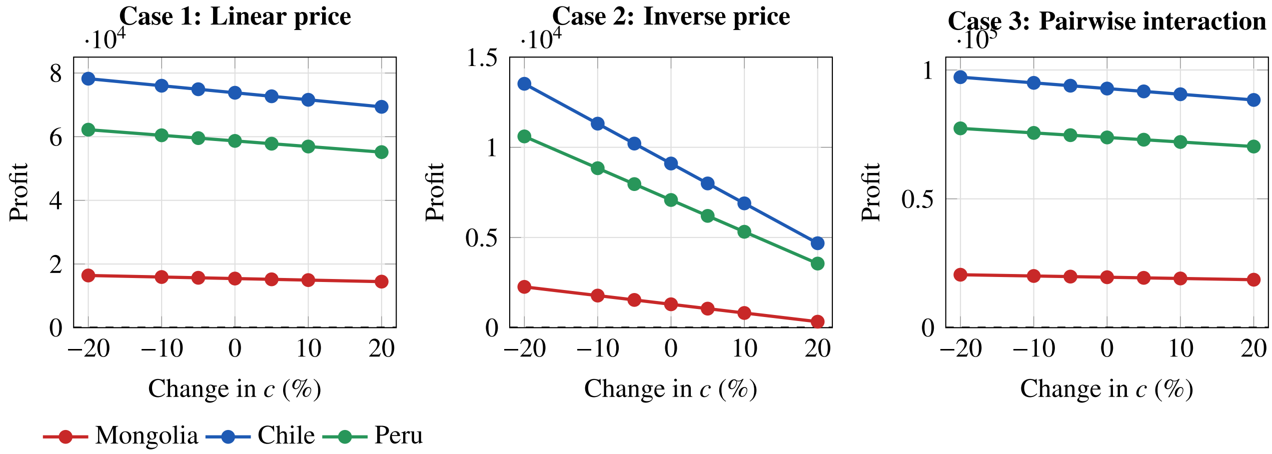

Figure 2. Sensitivity of country-level profit to changes in the calibrated variable-cost benchmark c under the three price environments. The dashed horizontal line indicates the zero-profit threshold.

Figure 2 shows that higher values of the calibrated variable-cost benchmark c reduce country-level profit in all three cases. Nevertheless, profit remains above the zero-profit threshold for all countries across the tested range. This indicates that the 2025 feasible–profit classification is not highly sensitive to moderate changes in the calibrated variable-cost benchmark. The figure also shows that Case 2 has the narrowest profitability margin, which is consistent with its lower modeled price.

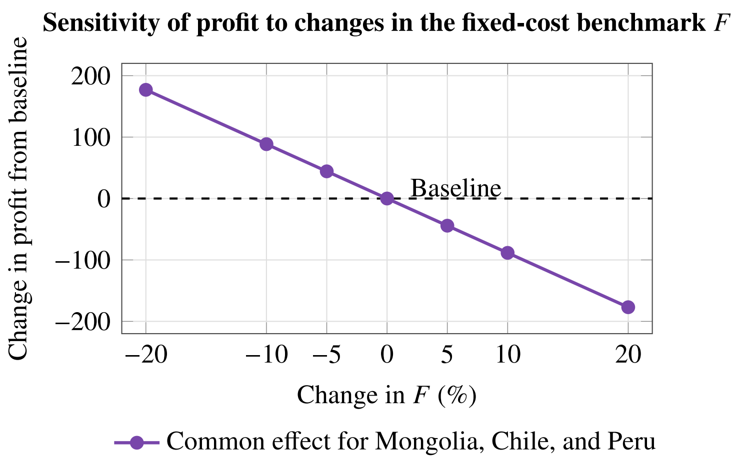

Figure 3. Sensitivity of profit to changes in the calibrated fixed-cost benchmark F. Since the same fixed-cost benchmark is applied across the three aggregate producers, the absolute effect of F perturbations is identical for Mongolia, Chile, and Peru.

Figure 3 reports the change in country-level profit relative to the 2025 baseline when the calibrated fixed-cost benchmark F is perturbed. Compared with the variable-cost benchmark c, the effect of F is smaller because F enters the profit condition as a fixed benchmark term, whereas c is multiplied by the export quantity. Because the same fixed-cost benchmark is applied across the three aggregate producers, the absolute effect of changes in F is identical for Mongolia, Chile, and Peru. Since the baseline profit levels are positive and the fixed-cost perturbations generate only limited absolute changes, all three countries remain profitable across the tested range. This confirms that the baseline 2025 feasible–profit classification is robust to moderate changes in the calibrated fixed-cost benchmark.

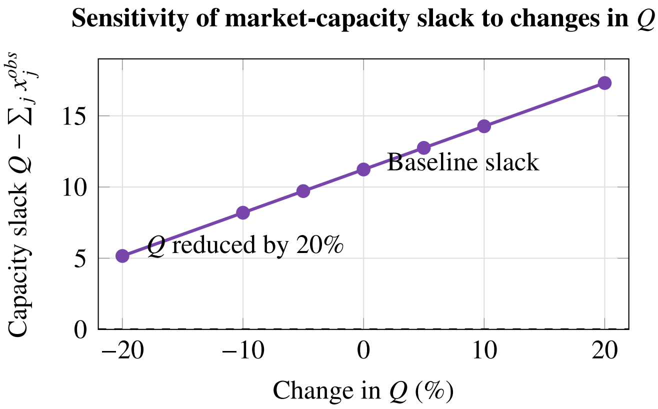

Figure 4. Sensitivity of market-capacity slack to changes in the market-capacity parameter Q. Positive values indicate that the observed 2025 export vector satisfies the capacity restriction. The result shows that capacity is not the binding source of infeasibility in the 2025 sensitivity analysis.

Figure 4 shows that the market-capacity condition remains satisfied under all tested perturbations of Q. Even when Q is reduced by 20 percent, the adjusted market-capacity parameter remains above the total 2025 export volume of the three countries. Thus, changes in Q within the tested range do not affect feasible–profit membership. This result indicates that, for the 2025 observation, potential infeasibility is not driven by the capacity restriction, but rather by the profitability condition under a low-price environment.

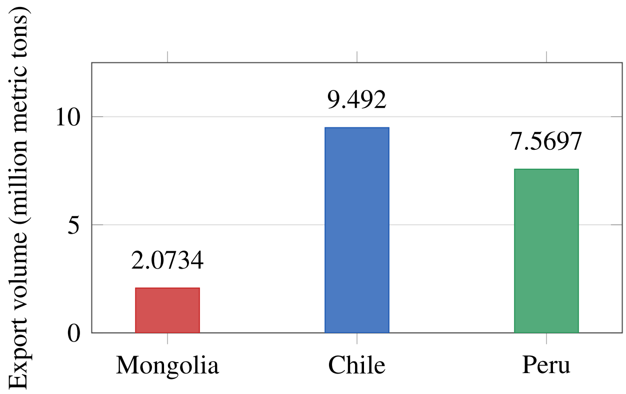

Figure 5. Copper export volumes of Mongolia, Chile, and Peru in 2025.

Figure 5 illustrates the asymmetric structure of the three-country export system in 2025. Chile has the largest export volume, at 9.4920 million metric tons, followed by Peru with 7.5697 million metric tons, while Mongolia records 2.0734 million metric tons. In terms of the total export volume of the three countries, Chile accounts for approximately 49.61 percent, Peru for 39.56 percent, and Mongolia for 10.84 percent. This asymmetry is important in an endogenous-price framework because the common market price is generated by the joint export vector, whereas the profitability condition is evaluated separately for each country. Larger exporters contribute more strongly to aggregate-output and interaction-based price terms, while the smallest exporter is more vulnerable when the common price declines.

The only infeasible scenario in the sensitivity analysis occurs under the inverse aggregate-output price specification when the price coefficient a is reduced by 20 percent. In this case, the modeled price falls from 3380.40 to 2704.32. As a result, Mongolia's profit becomes negative, at approximately -105.25, while Chile and Peru remain profitable. Therefore, the observed 2025 export vector no longer belongs to the feasible–profit set under Case 2 when a is reduced by 20 percent. This finding indicates that the inverse aggregate-output price environment is more sensitive to downward changes in the price coefficient than the linear aggregate-output and pairwise interaction specifications.

Economically, this result is intuitive. Under the inverse price function, the modeled price is already lower than under the other two price environments. A further reduction in the price coefficient lowers the common price below the level required for all three countries to satisfy the profitability conditions simultaneously. Mongolia is the first country to become infeasible because it has the smallest export volume in the 2025 three-country system. By contrast, Chile and Peru maintain positive profit values under the same perturbation because their larger export volumes generate higher revenue terms under the common price.

Overall, the sensitivity analysis confirms that the 2025 feasible–profit result is robust under most parameter perturbations. It also shows that the feasible–profit region is most vulnerable when the price mechanism produces a relatively low common price, as in the inverse aggregate-output specification. This supports the broader conclusion that feasible–profit membership depends not only on export quantities and cost benchmarks, but also on the assumed structure of endogenous price formation.

3.5. Comparative summary

The numerical results show that the proposed nonconvex feasible–profit model behaves differently across the three endogenous price environments. The comparison is based on whether the observed export vector and the penalty-method stationary point belong to the feasible–profit set.

Under the linear aggregate-output price specification (Case 1), the observed export vector is infeasible only in 2013 and feasible from 2014 onward. The penalty method computes feasible stationary points in all years, including 2013, indicating that the feasible–profit set is nonempty even when the observed export configuration is infeasible. However, the squared distance between the observed and computed vectors increases from 2.6291 in 2013 to 93.3781 in 2025, showing that the penalty method should not be interpreted as a procedure for recovering the observed vector.

Under the inverse aggregate-output price specification (Case 2), the observed export vector is feasible in all 13 sample years. However, the computed stationary points are feasible only from 2013 to 2023 and become infeasible in 2024–2025. This case shows most clearly that feasibility of the observed vector and feasibility of the computed stationary point are distinct outcomes.

Table 6. Summary of observed and computed feasible–set membership under the three price environments.

Year	Case	Obs. in set	Stat. pt. in set	Sq. dist.
2013	1	No	Yes	2.6291
2013	2	Yes	Yes	10.2127
2013	3	No	Yes	4.5984
2014	1	Yes	Yes	2.3384
2014	2	Yes	Yes	11.4602
2014	3	No	Yes	1.8559
2015	1	Yes	Yes	7.2224
2015	2	Yes	Yes	12.7563
2015	3	No	Yes	2.8159
2016	1	Yes	Yes	22.6378
2016	2	Yes	Yes	19.5602
2016	3	Yes	Yes	11.5133
2017	1	Yes	Yes	22.0417
2017	2	Yes	Yes	21.3919
2017	3	Yes	Yes	11.6230
2018	1	Yes	Yes	34.0341
2018	2	Yes	Yes	28.7530
2018	3	Yes	Yes	20.3221
2019	1	Yes	Yes	60.1102
2019	2	Yes	Yes	38.2442
2019	3	Yes	Yes	41.0578
2020	1	Yes	Yes	50.6217
2020	2	Yes	Yes	41.7173
2020	3	Yes	Yes	34.1366
2021	1	Yes	Yes	66.3949
2021	2	Yes	Yes	49.9254
2021	3	Yes	Yes	48.1127
2022	1	Yes	Yes	67.4008
2022	2	Yes	Yes	55.4307
2022	3	Yes	Yes	47.9575
2023	1	Yes	Yes	77.6386
2023	2	Yes	Yes	64.7635
2023	3	Yes	Yes	55.8008
2024	1	Yes	Yes	84.5903
2024	2	Yes	No	64.7276
2024	3	Yes	Yes	61.8604
2025	1	Yes	Yes	93.3781
2025	2	Yes	No	48.4270
2025	3	Yes	Yes	68.3939

Notes: "Obs. in set" indicates whether the observed export vector belongs to the feasible–profit set, "Stat. pt. in set" indicates whether the computed stationary point belongs to that set, and "Sq. dist." denotes the squared Euclidean distance between the computed stationary point and the observed export vector.

Under the pairwise interaction price specification (Case 3), the observed export vector is infeasible in 2013–2015 and feasible from 2016 onward. For the early infeasible years, the penalty method computes feasible stationary points, again suggesting that the feasible–profit set may be nonempty even when the observed export vector lies outside it. From 2016 onward, both the observed vector and the computed stationary point are feasible.

Overall, the same three-country export system produces different feasible-set and stationary-point behavior depending on the assumed price mechanism. Case 1 shows a transition from early non-membership to later membership under a linear aggregate-output price relation. Case 2 shows full feasibility of the observed vector but weaker feasibility of computed stationary points in the final years. Case 3 shows a similar transition to feasibility, driven by nonlinear pairwise interaction effects.

Table 6 summarizes the observed and computed feasible–set membership results under the three price environments. The table clarifies the role of the penalty method as a local stationary-point construction procedure rather than a global feasibility guarantee. Therefore, failure of a computed point to belong to the feasible–profit set should not be interpreted as proof that the set is empty. Economically, the comparison shows that feasible–profit membership depends not only on export quantities and cost benchmarks, but also on the assumed structure of endogenous price formation.

4. Discussion

The numerical results show that the proposed nonconvex feasible–profit framework extends classical CVP analysis from point-wise break-even calculation to feasible-region analysis. Classical CVP analysis usually identifies a break-even point or target-profit output under fixed-price assumptions. In contrast, the present framework examines whether a joint production vector belongs to a feasible–profit set when the common reference price is linked to the output vector. This distinction is important in oligopolistic production and export settings, where the profitability of one producer cannot be evaluated independently of the output decisions of other producers.

The results also clarify how the proposed framework differs from conventional production optimization and metaheuristic approaches. Many production-planning studies focus on finding high-quality or near-optimal solutions for scheduling, inventory, allocation, project selection, or supply-chain network design. The present study has a different purpose. It formulates the CVP question as a feasible-set membership problem and uses a penalty method as a local computational tool for constructing candidate stationary points. These points are then evaluated directly against the profitability and market-capacity conditions. The practical advantage of this approach is interpretability: the computed points can be assessed in terms of whether all aggregate producers remain profitable simultaneously under a shared price environment.

The comparison across the three price environments confirms that feasible–profit membership depends strongly on the assumed price-output mechanism. Under the linear aggregate-output specification, the observed export vector moves from early non-membership to later membership. Under the inverse aggregate-output specification, the observed vector remains feasible throughout the sample, but the computed stationary points are infeasible in 2024–2025. Under the pairwise interaction specification, the observed vector is infeasible in the early years but becomes feasible later as interaction effects increase. These differences show that feasible–profit membership is shaped not only by export quantities and calibrated cost benchmarks, but also by how the common price is represented.

These findings should be interpreted in light of the reduced-form nature of the price specifications. The inverse aggregate-output case is closest to the conventional downward-sloping price intuition, where a larger aggregate supply is associated with a lower price. By contrast, the linear aggregate-output and pairwise interaction specifications should not be interpreted as structural commodity demand functions. They are better understood as benchmark computational environments that capture alternative positive associations between market activity and the reference price index. Therefore, the purpose of comparing the three cases is not to estimate a structural demand system, but to examine how the geometry and membership of the feasible–profit region change under alternative price-output mechanisms.

The 2025 comparison of Mongolia, Chile, and Peru provides additional economic insight. In the empirical application, the three countries are interpreted as aggregate producers rather than individual firms. Chile and Peru have much larger export volumes than Mongolia, while Mongolia is the smallest exporter in the three-country system. This asymmetry matters because the common price is generated by the joint export vector, whereas profitability is evaluated separately for each aggregate producer. Larger exporters contribute more strongly to aggregate-output and interaction-based price terms, while the smaller exporter is more vulnerable when the common price decreases. This result is consistent with the sensitivity analysis, where Mongolia becomes infeasible under the inverse price specification when the price coefficient is reduced by 20 percent.

From a managerial and export–planning perspective, the results suggest that export profitability should not be assessed only through historical price-cost margins or single-country break-even calculations. In markets where several aggregate suppliers compete for the same demand source, profitability depends on the joint configuration of supply, demand capacity, cost conditions, and price responsiveness. Therefore, export planning should consider the behavior of competing suppliers and the price mechanism linking aggregate supply to market outcomes.

The main theoretical contribution of the paper is to connect set-based CVP analysis with endogenous common-price modeling and local nonconvex optimization. Relative to prior CVP studies, the contribution is not only the use of a penalty method, but the formulation of multi-producer CVP analysis as a feasible–profit membership problem under alternative price mechanisms. The framework shifts the object of analysis from a single break-even point to a feasible–profit region and provides a computational procedure for examining whether observed or computed production vectors satisfy joint profitability conditions.

These results should be interpreted with caution. The three price equations are reduced-form computational scenarios rather than full structural demand models. The cost parameters are calibrated benchmark values rather than directly observed country-specific accounting data. In addition, the penalty method is local in nature and does not guarantee global feasibility or global optimality. Therefore, failure to compute a feasible stationary point should not be interpreted as proof that the feasible–profit set is empty; rather, it may reflect the sensitivity of the local method to the geometry of the nonconvex feasible region.

5. Conclusion

This paper developed a nonconvex feasible–profit framework for multi-producer cost–volume–profit analysis under endogenous common-price determination. Unlike classical CVP analysis, which usually focuses on single-firm break-even or target-profit calculations under fixed-price assumptions, the proposed framework examines whether a joint production vector can satisfy profitability, market-capacity, and non-negativity restrictions simultaneously.

The framework was applied to a three-country copper export market consisting of Mongolia, Chile, and Peru, interpreted as aggregate producers. Three reduced-form price environments were considered: a linear aggregate-output price, an inverse aggregate-output price, and a pairwise interaction price. These specifications were not treated as structural demand functions, but as alternative computational environments for examining how feasible–profit membership changes when the common price is linked to joint output.

The results show that feasible–profit membership and penalty-method stationary-point behavior depend strongly on the assumed price mechanism. Under the linear aggregate-output and pairwise interaction specifications, the observed export vector is infeasible only in the early years and becomes feasible later. Under the inverse aggregate-output specification, the observed export vector remains feasible throughout the sample, but the computed stationary points are infeasible in 2024–2025. The 2025 sensitivity analysis further shows that the feasible–profit classification is generally robust to moderate changes in the price coefficient, calibrated cost benchmarks, and market-capacity parameter, except when the inverse price coefficient is reduced by 20 percent.

The scientific value of the paper lies in connecting set-based CVP analysis, endogenous common-price modeling, and local nonconvex optimization. Relative to prior CVP studies, the contribution is not only the use of a penalty method, but the formulation of multi-producer CVP analysis as a feasible–profit membership problem under alternative endogenous price mechanisms. The framework is useful because computed points can be interpreted directly in terms of whether all producers remain profitable under a shared market price.

Several limitations remain. The empirical application uses annual data for 2013–2025, which provides a small sample for estimating reduced-form price relationships. The cost parameters are calibrated benchmark values rather than directly observed country-specific accounting costs. The three price specifications are reduced-form computational scenarios rather than full structural demand models. In addition, the penalty method is local and does not guarantee global feasibility or global optimality; failure to compute a feasible stationary point should not be interpreted as proof that the feasible–profit set is empty.

Future research may extend the framework by estimating richer structural or dynamic price equations, using firm-level or country-specific cost data, comparing the local penalty method with global optimization and hybrid metaheuristic algorithms, and applying geometric inner-approximation methods such as sphere packing to study the size and robustness of feasible–profit regions. The framework may also be applied to other commodity markets and production systems where aggregate suppliers face common prices, capacity restrictions, and endogenous price-output interactions.

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Author contributions

Munkhbayar Byambadash: Conceptualization. Bayanjargal Darkhijav: Formal analysis, Methodology. Tumendelger Lkhagvasuren: Writing–original draft, Software. Enkhbat Rentsen: Supervision.

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Data availability

The dataset used in the computational experiment is provided as Supplementary File 1. It contains annual observations for 2013–2025 on copper export volumes from Mongolia, Chile, and Peru to China, China's copper import demand, the London Metal Exchange three-month forward copper price, and the calibrated cost benchmarks used in the numerical implementation. The export, demand, and price variables are obtained from Bloomberg using the tickers reported in Supplementary File 1. The cost parameters are calibrated benchmark values constructed for numerical analysis and are not directly observed harmonized accounting data. Supplementary File 1 also reports variable definitions, units of measurement, Bloomberg tickers, and explanatory notes to support transparency and reproducibility.

Conflict of interest

The authors declare no conflict of interest.

Use of Generative-AI tools declaration

During the preparation of this manuscript, the authors used ChatGPT for language editing and polishing. All AI-assisted content was critically reviewed and verified by the authors.

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