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Research article

Controlled anti-trace Hamilton dynamics and related conservation laws

Savin Treanță[1,2,3,*] **and Marius-Alin Dragu**[4]

[1] Department of Applied Mathematics, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania

[2] Academy of Romanian Scientists, 54 Splaiul Independentei, 050094 Bucharest, Romania

[3] Fundamental Sciences Applied in Engineering - Research Center, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania

[4] Department of Bioengineering and Biotechnology, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania

* **Correspondence:** Email: savin.treanta@upb.ro.

Abstract: Based on the Euler–Lagrange partial differential equations (PDEs) of anti-trace type, interpreted as an extension of the standard Euler–Lagrange PDEs and generated by multi-dimensional autonomous controlled Lagrangians of second order, we introduce the associated anti-trace Hamilton PDEs. In addition, new conservation laws are established as a direct application of the mathematical concepts presented in this paper. In addition, by using block matrices built from Kronecker products, we formulate the compact (matrix) form of (generalized) Hamilton PDEs of the anti-trace type. The theory is further extended to Lagrangian and Hamiltonian tensor fields. To illustrate the applicability of the theoretical framework, we provide an illustrative application.

Keywords: optimal control; controlled Lagrangians; anti-trace Hamilton PDEs; conservation laws; Lagrangian and Hamiltonian tensors

Mathematics Subject Classification: 26B25, 90C17, 49K21

1. Introduction

1.1. Background and motivation

Over time, Hamilton-type partial differential equations (PDEs) have been identified as a robust framework for optimal control problems involving functionals generated by simple, multiple, or curvilinear integrals (see, for instance, Dragu and Treanţă [1] and Treanţă [2]). The significant advantages of the Hamiltonian approach are related to the idea that the Hamilton function admits more symmetries in

its coordinates, in contrast to the Lagrangian type. Hamiltonian dynamics are developed around the fundamental concepts of calculus as first-order ordinary differential equations (ODEs) or PDEs which are constructed either from second-order ODEs or PDEs of the Euler–Lagrange type, either as Euler–Lagrange equations associated with some particular functionals that contain Hamiltonian functions (see, for instance, Udrişte and Matei [3]). McLachlan [4] investigated a class of Hamiltonian PDEs via classical symplectic schemes and reported the stability, accuracy, and dispersion of various explicit splitting techniques. Marsden et al. [5] made a representative contribution to mechanics (continuous and discrete approach) and field theories, introducing a geometric-variational method. Later, Bridges and Reich [6] suggested a new theoretical mathematical pipeline for generalizing the symplectic numerical integrators for ODEs to Hamiltonian PDEs in $\mathbb{R}^2$. In Horwood et al. [7], the problem of the invariant classification of the orthogonal coordinate webs defined in Euclidean space is solved within the framework of Felix Klein's Erlangen program. Moore et al. [8], established conformal conservation laws for a class of multi-symplectic equations with added dissipation. Gong et al. [9] presented new structure-preserving algorithms for general multi-symplectic formulations of Hamiltonian PDEs. Brenier [10] considered the Euler equations of incompressible fluids and solved the initial value problem with the help of a concave maximization problem. Recently, Bambusi et al. [11] formulated and proved a result of almost global existence for some abstract nonlinear PDEs on flat tori and applied it to some concrete equations. Fagerholm et al. [12] studied the task of rendering neuronal state equations compatible with the principle of stationary action.

In the past several years, numerous researchers have thoroughly analyzed the multi-dimensional Hamilton or Hamilton–Jacobi theories. In this way, Wang [13] studied the dynamical vector fields of a regular controlled Hamiltonian subject to geometric constraints, investigating the canonical symplectic form and regular reduced symplectic forms. De Leon et al. [14] constructed the Hamilton–Jacobi theory for contact Hamiltonian systems of the time-dependent and independent type. Moreover, multi-time Hamilton-Jacobi of non-smooth type systems were deeply explored by Motta and Rampazzo [15], while commuting Hamiltonians and multi-time equations of Hamilton-Jacobi type represented the object of study for Cardin and Viterbo [16]. Treanţă [17] made important contributions to this field. The author inspected the Hamiltonian dynamics by applying a non-classical Legendrian duality approach and obtained a Hamilton–Jacobi type divergence PDE governed by Lagrangians of higher order. In recent times, Treanţă et al. [18] characterized the Lagrangians governed by higher-order derivatives of the state variables by higher-order Hamilton ODEs and Hamilton–Jacobi PDEs. Chen et al. [19] provided a new theoretical link between particular machine learning optimization issues and the multi-time Hopf formula, which is seen as a solution to certain multi-time Hamilton–Jacobi PDEs. Regarding the effective applications, bioengineering, dynamic rheology (see [3]), machine learning, and electromagnetic fields represent some particular areas with representative works based on the Hamilton approach. Yoo-Kong et al. investigated the Lagrangian formalism of the rational Calogero–Moser system, in both discrete and continuous time [20]. On the other hand, Boulaaras studied the existence and multiplicity of weak solutions for a family of nonlocal fourth-order Leray–Lions–Kirchhoff problems indexed by variable exponent growth, weighted Hardy–Rellich singular potentials with logarithmic corrections, and nonlinear Robin boundary conditions [21].

Moreover, Azevedo et al. extended dynamic programming techniques to stochastic optimal control problems whose state dynamics are governed by Markov-switching jump-diffusion stochastic differential equations [22]. In the work of Bakan et al., numerical approximation of stochastic optimal control was

also addressed through Runge–Kutta schemes for stochastic differential equations [23]. Runge–Kutta methods for the optimal control of stochastic differential equations have also been discussed in the works of Yılmaz et al. [24,25].

Notably, Baltas et al. applied robust control and dynamic programming techniques to the optimal management of defined contribution pension funds in the distribution phase, where inflation, member mortality, and model uncertainty were taken into account. In their proposed framework, closed-form optimal strategies are obtained for exponential utility, and the analysis shows how robustness and inflation shape the optimal investment decisions [26]. Savku et al. proposed stochastic hybrid systems with jumps as a flexible framework for modelling abrupt regime changes in financial markets (and, more broadly, sudden economic, cultural or societal shifts) [27]. The authors used an extension of Bellman's optimality principle and the corresponding Hamilton–Jacobi-Bellman equation to derive consumption/investment strategies in a Markov–switching jump-diffusion market. In the work of Temocin and Weber, stochastic hybrid systems with regime changes and impulsive behavior were also been treated through discrete approximation [28].

In terms of applications in bioengineering, due to its tremendous features, which are unique and more difficult to be generated artificially, reflecting an internal and personal process that cannot be experienced by others, electroencephalography signals became a topic of interest for researchers to analyze and mathematically model. Thus, computational neuroscience applications can be related by equations of motion, including information about membrane potentials, firing rates, or neuronal activity. Path-integral representations of stochastic dynamics have been created using appropriate variational techniques and neural networks (see [12]).

Second-order dynamical models and their qualitative behaviour have also been investigated from complementary viewpoints. For instance, Mahdi et al. [29] proposed a complete classification of Hyers–Ulam stability on the whole real line for the second-order constant-coefficient equation $m\ddot{x}+b\dot{x}+kx=0$, showing that stability holds precisely when the characteristic polynomial has no roots on the imaginary axis. In a different direction, Boulaaras [30] studied a $(2+1)$-dimensional variable-coefficient generalized Kadomtsev–Petviashvili model with time-dependent dispersion management, constructing exact rational lump solutions and quantifying how the modulated coefficients control the localization and propagation of the waves. These works are complementary to the variational and Hamiltonian approach developed in this paper, which is based on second-order controlled Lagrangians.

Encouraged and motivated by current trends and recent developments in this field (see, also, Treanţă and Ahmad [31]), the main goal of this paper is to develop a new point of view regarding Hamilton-type PDEs related to variational controlled models. Concretely, based on the Euler–Lagrange PDEs of the anti-trace type (obtained by multi-dimensional controlled Lagrangians of first- and second-order), seen as an extension of the standard Euler–Lagrange PDEs, we introduce the associated anti-trace Hamilton PDEs. Moreover, new conservation laws are established as a direct application of the concepts presented in this paper. We also formulate the compact (matrix) form of (generalized) anti-trace Hamilton PDEs.

1.2. Novelty and contributions

The first main contribution of the paper consists of associating a Hamiltonian-type system of PDEs with the anti-trace Euler–Lagrange dynamics, written in terms of the state, control, generalized multi-momenta, and Hamiltonian variables.

A second contribution concerns the compact matrix representation of the anti-trace Hamilton PDEs. By introducing suitable block matrices and applying the vectorization and Kronecker product operators, the PDEs system has been expressed in a compact form. Consequently, the proposed formulation provides a convenient algebraic representation of the generalized Hamiltonian dynamics that can be further considered in analytical and computational developments.

The corresponding generalized anti-trace Hamilton PDEs are also formulated in tensorial form, leading to a broader description of the controlled dynamics. Notably, the analysis shows that the energy-momentum tensor is conserved both for first-order and second-order autonomous controlled Lagrangians.

1.3. Organization of the paper

The paper is structured as follows. In Section 2, we deal with Euler–Lagrange and Hamilton PDEs of the anti-trace type based on multi-dimensional autonomous controlled Lagrangians of the first- and second-order type. In addition, a compact (matrix) form of anti-trace Hamilton PDEs is formulated. Moreover, new conservation laws of the Hamiltonian are established and analyzed. In the third section, we study similar issues in the case of Lagrangian and Hamiltonian tensors. Section 4 proposes an illustrative example that supports the theoretical findings. The last section is devoted to the concluding remarks.

2. Anti-trace Euler–Lagrange and Hamilton PDEs

To introduce anti-trace Hamilton PDEs, we start from the associated Euler–Lagrange PDEs of the anti-trace type. In this regard, let us consider the C^3-class multi-dimensional autonomous controlled Lagrangian of second order (contextually, it has partial derivatives of the second order)

$$L\Big(x(t), x_\alpha(t), x_{\pi\zeta}(t), \Psi(t), \Psi_\alpha(t), \theta(t)\Big).$$

Notably, the Lagrangian L embeds the following mathematical tools.

- $\mathbb{R}^m$, $\mathbb{R}^n$, and $\mathbb{R}^\epsilon$ are the finite-dimensional Euclidean space with dimension m, n, ϵ.
- $t = (t^\alpha) = (t^1, \dots, t^m) \in \Omega_{0,t_0} \subset \mathbb{R}^m_+$ is the multi dimensional evolution variable, $\alpha \in \{1, \dots, m\}$.
- $\Omega_{0,t_0} := [0, t_0]$ denotes a multi-dimensional interval in $\mathbb{R}^m_+$ regarding partial order product.
- $x = (x^i) = \left(x^1, \dots, x^n\right) : \Omega_{0,t_0} \to \mathbb{R}^n$ is the state variable, $i \in \{1, \dots, n\}$.
- $x_\alpha(t) := \dfrac{\partial x}{\partial t^\alpha}(t)$, $x_{\pi\zeta}(t) := \dfrac{\partial^2 x}{\partial t^\pi \partial t^\zeta}(t) = \dfrac{\partial^2 x}{\partial t^\zeta \partial t^\pi}(t)$ are the first- and second-order partial derivatives of the state function x, which appear as arguments of the Lagrangian L, $\alpha, \pi, \zeta \in \{1, \dots, m\}$.
- $\mathbb{X}$ represents the space of state functions of C^4-class equipped with the uniform norm $\| x \| = \| x \|_\infty + \sum_{\alpha=1}^m \| x_\alpha \|_\infty + \frac{1}{\mathcal{Z}(\pi,\zeta)} \sum_{\pi=1}^m \sum_{\zeta=1}^m \| x_{\pi\zeta} \|_\infty$, where $\| \cdot \|_\infty$ is the maximum norm ($\mathcal{Z}(\pi, \zeta)$ defined below).
- $\Psi = (\Psi^\kappa) = \left(\Psi^1, \dots, \Psi^\epsilon\right) : \Omega_{0,t_0} \to \mathbb{R}^\epsilon$ represents the control variable, $\kappa \in \{1, \dots, \epsilon\}$.
- $\Psi_\alpha(t) := \dfrac{\partial \Psi}{\partial t^\alpha}(t)$ denotes the first-order partial derivative of the control function, incorporated in the Lagrange function.
- $\mathbb{P}$ is the space of control functions of C^2-class equipped with the uniform norm $\| \Psi \| = \| \Psi \|_\infty + \sum_{\alpha=1}^m \| \Psi_\alpha \|_\infty$.
- $\theta = (\theta_i^{\pi\zeta}) : \Omega_{0,t_0} \to \mathbb{R}^{nm^2}$ is interpreted as a co-state variable.
- δ^α_β denotes the Kronecker symbol: $\delta^\alpha_\beta = 1$ if $\alpha = \beta$ and $\delta^\alpha_\beta = 0$ otherwise.

The partial derivatives of L with respect to the state and control variables are given by

$$L_x := \underbrace{\left[\frac{\partial L}{\partial x^1} \quad \frac{\partial L}{\partial x^2} \quad \cdots \quad \frac{\partial L}{\partial x^n}\right]^T}_{(n\times 1)}, \quad L_\Psi := \underbrace{\left[\frac{\partial L}{\partial \Psi^1} \quad \frac{\partial L}{\partial \Psi^2} \quad \cdots \quad \frac{\partial L}{\partial \Psi^\epsilon}\right]^T}_{(\epsilon\times 1)},$$

$$L_{x_\alpha} := \underbrace{\begin{bmatrix} \frac{\partial L}{\partial x_1^1} & \frac{\partial L}{\partial x_2^1} & \cdots & \frac{\partial L}{\partial x_m^1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial L}{\partial x_1^n} & \frac{\partial L}{\partial x_2^n} & \cdots & \frac{\partial L}{\partial x_m^n} \end{bmatrix}}_{(n\times m)}, \quad L_{\Psi_\alpha} := \underbrace{\begin{bmatrix} \frac{\partial L}{\partial \Psi_1^1} & \frac{\partial L}{\partial \Psi_2^1} & \cdots & \frac{\partial L}{\partial \Psi_m^1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial L}{\partial \Psi_1^\epsilon} & \frac{\partial L}{\partial \Psi_2^\epsilon} & \cdots & \frac{\partial L}{\partial \Psi_m^\epsilon} \end{bmatrix}}_{(\epsilon\times m)},$$

$$L_{x_{\pi\zeta}} := \underbrace{\begin{bmatrix} \frac{\partial L}{\partial x_{11}^1} & \frac{\partial L}{\partial x_{12}^1} & \cdots & \frac{\partial L}{\partial x_{1m}^1} & \cdots & \frac{\partial L}{\partial x_{mm}^1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial L}{\partial x_{11}^n} & \frac{\partial L}{\partial x_{12}^n} & \cdots & \frac{\partial L}{\partial x_{1m}^n} & \cdots & \frac{\partial L}{\partial x_{mm}^n} \end{bmatrix}}_{(n\times m^2)}, \quad \text{with } x_{\pi\zeta} = x_{\zeta\pi}.$$

To handle the symmetries related to the second-order partial derivative of x, we introduce Saunders's multi-index notation [32,33]. A multi-index represents an m-tuple $\mathcal{T}$ of natural numbers. $\mathcal{T}$ has the components denoted by $\mathcal{T}(\pi)$, with π being an ordinary index $1 \le \pi \le m$. In this context, we define the multi-index 1_π: $1_\pi(\pi) = 1$ and $1_\pi(\alpha) = 0$, for $\pi \neq \alpha$. The multi-indices are equipped with componentwise addition: $(\mathcal{T} + \mathcal{Y})(\pi) = \mathcal{T}(\pi) + \mathcal{Y}(\pi)$. The length of a multi-index is given by

$$|\mathcal{T}| = \sum_{\pi=1}^{m} \mathcal{T}(\pi).$$

Its factorial is defined as follows:

$$\mathcal{T}! = \prod_{\pi=1}^{m} (\mathcal{T}(\pi))!.$$

The number of distinct orderings of the indices $\pi_1, \pi_2, \ldots, \pi_r, \pi_w \in \{1, 2, \ldots, m\}, w \in \{1, 2, \ldots, r\}$ is computed as follows:

$$\mathcal{Z}(\pi_1, \pi_2, \ldots, \pi_r) = \frac{|1_{\pi_1} + 1_{\pi_2} + \cdots + 1_{\pi_r}|!}{(1_{\pi_1} + 1_{\pi_2} + \cdots + 1_{\pi_r})!}.$$

Next, we discuss three particular cases, namely $r \in \{1, 2, 3\}$.

- *Case 1: $r = 1$.* $|1_{\pi_1}| = 1, 1_{\pi_1}! = 1$, $\mathcal{Z}(\pi_1) = 1$.
- *Case 2.1: $r = 2, \pi_1 \neq \pi_2$.* $|1_{\pi_1} + 1_{\pi_2}| = 2, (1_{\pi_1} + 1_{\pi_2})! = 1$, $\mathcal{Z}(\pi_1, \pi_2) = 2$.
- Case 2.2: $r = 2, \pi_1 = \pi_2$. $|1_{\pi_1} + 1_{\pi_2}| = 2, (1_{\pi_1} + 1_{\pi_2})! = 2$, $\mathcal{Z}(\pi_1, \pi_2) = 1$.
- *Case 3.1: Pairwise distinct.* $|1_{\pi_1} + 1_{\pi_2} + 1_{\pi_3}| = 3$, $(1_{\pi_1} + 1_{\pi_2} + 1_{\pi_3})! = 1$, $\mathcal{Z}(\pi_1, \pi_2, \pi_3) = 6$.
- Case 3.2: $r = 3, \pi_1 = \pi_2 \neq \pi_3$. $|1_{\pi_1} + 1_{\pi_2} + 1_{\pi_3}| = 3$, $(1_{\pi_1} + 1_{\pi_2} + 1_{\pi_3})! = 2$, $\mathcal{Z}(\pi_1, \pi_2, \pi_3) = 3$.
- *Case 3.3: $r = 3, \pi_1 = \pi_2 = \pi_3$.* $|1_{\pi_1} + 1_{\pi_2} + 1_{\pi_3}| = 3$, $(1_{\pi_1} + 1_{\pi_2} + 1_{\pi_3})! = 6$, $\mathcal{Z}(\pi_1, \pi_2, \pi_3) = 1$.

Definition 2.1 The total derivative operators of first (D_ζ) and second order (D^2_{ab}) are given below.

$$D_\zeta := \frac{\partial}{\partial t^\zeta} + \frac{\partial}{\partial x^i}\left(\frac{\partial x^i}{\partial t^\zeta}\right) + \frac{\partial}{\partial x^i_\eta}\left(\frac{\partial x^i_\eta}{\partial t^\zeta}\right) + \frac{1}{\mathcal{Z}(a,b)}\frac{\partial}{\partial x^i_{ab}}\left(\frac{\partial x^i_{ab}}{\partial t^\zeta}\right) + \frac{\partial}{\partial \Psi^\kappa}\left(\frac{\partial \Psi^\kappa}{\partial t^\zeta}\right) + \frac{\partial}{\partial \Psi^\kappa_\eta}\left(\frac{\partial \Psi^\kappa_\eta}{\partial t^\zeta}\right) + \frac{1}{\mathcal{Z}(\pi,\zeta)}\frac{\partial}{\partial \theta_i^{\pi\zeta}}\left(\frac{\partial \theta_i^{\pi\zeta}}{\partial t^\zeta}\right)$$

(applicable to multi-dimensional non-autonomous Lagrangians of second-order),

$$D^2_{ab} := D_a(D_b), \quad \zeta, a, b = \overline{1, m}.$$

Recalling that the Lagrangian L does not depend explicitly on the temporal evolution variable t, the total derivative operator D_ζ is reduced to

$$D_\zeta := \frac{\partial}{\partial x^i}\Big(\frac{\partial x^i}{\partial t^\zeta}\Big) + \frac{\partial}{\partial x^i_\eta}\Big(\frac{\partial x^i_\eta}{\partial t^\zeta}\Big) + \frac{1}{\mathcal{Z}(a,b)}\frac{\partial}{\partial x^i_{ab}}\Big(\frac{\partial x^i_{ab}}{\partial t^\zeta}\Big) + \frac{\partial}{\partial \Psi^\kappa}\Big(\frac{\partial \Psi^\kappa}{\partial t^\zeta}\Big) + \frac{\partial}{\partial \Psi^\kappa_\eta}\Big(\frac{\partial \Psi^\kappa_\eta}{\partial t^\zeta}\Big) + \frac{1}{\mathcal{Z}(\pi,\zeta)}\frac{\partial}{\partial \theta^{\pi\zeta}_i}\Big(\frac{\partial \theta^{\pi\zeta}_i}{\partial t^\zeta}\Big).$$

With the previous mathematical objects, we formulate the controlled variational model $(\mathcal{P})$. Note that the Lagrangian L is seen as the auxiliary Lagrangian, with the help of which the constrained model $(\mathcal{P})$ is written as an unconstrained one.

Multi-dimensional controlled variational model. *Find the pair (x^*, Ψ^*) that extremizes the controlled functional given by the following multiple integral:*

$$(\mathcal{P}) \quad \mathscr{E} : \mathbb{X} \times \mathbb{P}, \quad \mathscr{E}(x, \Psi) = \int_{\Omega_{0,t_0}} W\left(x(t), x_\alpha(t), \Psi(t), \Psi_\alpha(t)\right) dt^1 \cdots dt^m$$

between all functions x subject to the state equations

$$X^i_{\pi\zeta}\Big(x(t), x_\alpha(t), x_{\pi\zeta}(t), \Psi(t), \Psi_\alpha(t)\Big) = 0, \qquad i \in \{1, \dots, n\}, \quad \pi, \zeta \in \{1, \dots, m\}.$$

In addition, the functions x fulfill

$$x(t)|_{\partial\Omega_{0,t_0}} = \text{given}, \;\; x_\alpha(t)|_{\partial\Omega_{0,t_0}} = \text{given}.$$

(the case in which $X^i_{\pi\zeta} = X^i_{\zeta\pi}$ are discussed in Remark 2.2) by considering the anti-trace procedure. We remark that the number of imposed conditions may vary according to the expression of the Lagrangian W. It is chosen so that the boundary value problem is not overdetermined, but it cannot exceed the number of conditions listed above.

With the problem $(\mathcal{P})$, the auxiliary Lagrangian L is computed as follows:

$$L\Big(x(t), x_\alpha(t), x_{\pi\zeta}(t), \Psi(t), \Psi_\alpha(t), \theta(t)\Big) = W\left(x(t), x_\alpha(t), \Psi(t), \Psi_\alpha(t)\right)$$

$$+\theta^{\pi\zeta}_i X^i_{\pi\zeta}\Big(x(t), x_\alpha(t), x_{\pi\zeta}(t), \Psi(t), \Psi_\alpha(t)\Big).$$

Definition 2.2 ([1]) Denote by $\mathcal{G}_x$ the class of functions $\chi = (\chi^i) : \Omega_{0,t_0} \to \mathbb{R}^n$ satisfying the admissibility conditions

$$\chi|_{\partial\Omega_{0,t_0}} = 0, \tag{1}$$

$$\chi_\beta|_{\partial\Omega_{0,t_0}} = 0, \; \beta \in \{1, \dots, m\}, \tag{2}$$

$$\chi_{\pi\zeta}|_{\partial\Omega_{0,t_0}} = 0, \; \pi, \zeta \in \{1, \dots, m\}. \tag{3}$$

The functions $\chi_\beta := (\chi_\beta^i) : \Omega_{0,t_0} \to \mathbb{R}^{nm}$ of C^2-class are called *admissible variations of gradient-type type* associated with the *state* variable x.

Definition 2.3 ([1]) Denote by $\mathcal{G}_\Psi$ the class of functions $\aleph = (\aleph^\kappa) : \Omega_{0,t_0} \to \mathbb{R}^\epsilon$ satisfying the admissibility conditions

$$\aleph|_{\partial\Omega_{0,t_0}} = 0, \tag{4}$$

$$\aleph_\beta|_{\partial\Omega_{0,t_0}} = 0,\ \beta \in \{1,\dots,m\}. \tag{5}$$

The functions $\aleph_\beta := (\aleph_\beta^\kappa) : \Omega_{0,t_0} \to \mathbb{R}^{\epsilon m}$ of C^1-class are called *admissible variations of gradient-type type* associated with the *control* variable Ψ.

Theorem 2.1 (Euler–Lagrange PDEs of anti-trace type) *If* $(x^*(\cdot), \Psi^*(\cdot), \theta^*(\cdot))$ *extremize* $(\mathcal{P})$*, where the anti-trace procedure and gradient-type variations are assumed, then* $(x^*(\cdot), \Psi^*(\cdot), \theta^*(\cdot))$ *is a solution of the following system of anti-trace Euler–Lagrange PDEs*

$$\frac{\delta_\beta^\eta}{m}\frac{\partial L}{\partial x^i} - D_\beta\frac{\partial L}{\partial x_\eta^i} + \frac{\delta_\beta^\eta}{m\cdot\mathcal{Z}(a,b)}D_{ab}^2\frac{\partial L}{\partial x_{ab}^i} = 0, \quad i \in \{1,\dots,n\}, \quad \beta,\eta \in \{1,\dots,m\}, \tag{6}$$

$$\frac{\delta_\beta^\eta}{m}\frac{\partial L}{\partial \Psi^\kappa} - D_\beta\frac{\partial L}{\partial \Psi_\eta^\kappa} = 0, \quad \kappa \in \{1,\dots,\epsilon\}, \tag{7}$$

$$\frac{\partial L}{\partial \theta_i^{\pi\zeta}} = 0. \tag{8}$$

Proof. We begin by introducing the gradient-type variation functions associated with the state variable, $\chi_\beta := (\chi_\beta^i) : \Omega_{0,t_0} \to \mathbb{R}^{nm}$, $\chi_\beta|_{\partial\Omega_{0,t_0}} = 0$, with the primitive $\chi : \Omega_{0,t_0} \to \mathbb{R}^n$, $\chi|_{\partial\Omega_{0,t_0}} = 0$ (see Definition 2.2).We also assume that χ_β^i satisfies the complete integrability conditions

$$\frac{\chi_\beta^i}{\partial t^q} = \frac{\chi_q^i}{\partial t^\beta},\ \beta \in \{1,\dots,m\}. \tag{9}$$

Now, we consider the small parameters $\Gamma := (\Gamma^\beta) = \left(\Gamma^1,\dots,\Gamma^m\right) \in \mathbb{R}^m$, with the help of which we design the perturbed functions $x^* + \Gamma^\beta\chi_\beta$ (see the Einstein summation over β).

Next, we consider the gradient-type variations corresponding to the control function: $\aleph_\beta := (\aleph_\beta^\kappa) : \Omega_{0,t_0} \to \mathbb{R}^{\epsilon m}$, $\aleph_\beta|_{\partial\Omega_{0,t_0}} = 0$, together with the primitive $\aleph : \Omega_{0,t_0} \to \mathbb{R}^\epsilon$, $\aleph|_{\partial\Omega_{0,t_0}} = 0$ (see Definition 2.3), and complete integrability conditions

$$\frac{\aleph_\beta^\kappa}{\partial t^q} = \frac{\aleph_q^\kappa}{\partial t^\beta}. \tag{10}$$

In this setting, the small parameters $\Upsilon := (\Upsilon^\beta) = \left(\Upsilon^1,\dots,\Upsilon^m\right) \in \mathbb{R}^m$ generate the perturbed functions $\Psi^* + \Upsilon^\beta\aleph_\beta$ (with summation over repeated indices β). The co-state functions are characterized by classical variations, namely $w : \Omega_{0,t_0} \to \mathbb{R}^{nm^2}$, $w|_{\partial\Omega_{0,t_0}} = 0$. Here, the small parameter is $\mathcal{B} \in \mathbb{R}$, involving $\theta^* + \mathcal{B}w$ as perturbed function.

By equipping the Lagrange function L with the above-mentioned perturbed functions, we obtain an integral with the parameters $(\Gamma, \Upsilon, \mathcal{B})$ (see Jayswal et al. [34] and Udrişte and Matei [3]), that is

$$\mathscr{E}(\Gamma,\Upsilon,\mathcal{B}) = \int_{\Omega_{0,t_0}} L\Big(x^*(t) + \Gamma^\beta\chi_\beta(t), \Psi^*(t) + \Upsilon^\beta\aleph_\beta(t), \theta^*(t) + \mathcal{B}w(t)\Big)\,dt^1\cdots dt^m$$

$$= \int_{\Omega_{0,t_0}} L\Big(x^*(t) + \Gamma^\beta \chi_\beta(t), \frac{\partial}{\partial t^\alpha}\left(x^*(t) + \Gamma^\beta \chi_\beta(t)\right), \frac{\partial^2}{\partial t^\pi \partial t^\zeta}\left(x^*(t) + \Gamma^\beta \chi_\beta(t)\right),$$

$$\Psi^*(t) + \Upsilon^\beta \aleph_\beta(t), \frac{\partial}{\partial t^\alpha}\left(\Psi^*(t) + \Upsilon^\beta \aleph_\beta(t)\right), \theta^*(t) + \mathcal{B}w(t)\Big). \tag{11}$$

We recall that in agreement with the hypothesis, $(x^*(\cdot), \Psi^*(\cdot), \theta^*(\cdot))$ can extremize the problem $(\mathcal{P})$. We notice that

$$\chi_\beta|_{\partial\Omega_{0,t_0}} = 0 \implies \left(x^*(t) + \Gamma^\beta \chi_\beta(t)\right)|_{\partial\Omega_{0,t_0}} = x^*(t)|_{\partial\Omega_{0,t_0}} + \Gamma^\beta \cdot 0 = x^*(t)|_{\partial\Omega_{0,t_0}},$$

$$\aleph_\beta|_{\partial\Omega_{0,t_0}} = 0 \implies \left(\Psi^*(t) + \Upsilon^\beta \aleph_\beta(t)\right)|_{\partial\Omega_{0,t_0}} = \Psi^*(t)|_{\partial\Omega_{0,t_0}} + \Upsilon^\beta \cdot 0 = \Psi^*(t)|_{\partial\Omega_{0,t_0}},$$

$$w|_{\partial\Omega_{0,t_0}} = 0 \implies (\theta^*(t) + \mathcal{B}w(t))|_{\partial\Omega_{0,t_0}} = \theta^*(t)|_{\partial\Omega_{0,t_0}} + \mathcal{B} \cdot 0 = \theta^*(t)|_{\partial\Omega_{0,t_0}},$$

meaning that the perturbed functions $x^*(t) + \Gamma^\beta \chi_\beta(t)$, $\Psi^*(t) + \Upsilon^\beta \aleph_\beta(t)$, and $\theta^*(t) + \mathcal{B}w(t)$ take the same values as $x^*(\cdot)$, $\Psi^*(\cdot)$, and $\theta^*(\cdot)$ on $\partial\Omega_{0,t_0}$. Consequently, we have

$$\mathscr{E}(\Gamma, \Upsilon, \mathcal{B}) \geq \mathscr{E}(x^*(\cdot), \Psi^*(\cdot), \theta^*(\cdot)) = \mathscr{E}(0,0,0).$$

Accordingly, $\mathscr{E}(\Gamma, \Upsilon, \mathcal{B})$ has a minimum at $(0,0,0)$, meaning that

$$(0,0,0) = \nabla \mathscr{E}(\Gamma, \Upsilon, \mathcal{B})|_{(\Gamma,\Upsilon,\mathcal{B})=(0,0,0)}$$

(regard ∇ as the gradient operator). Equivalently, we can write

$$(0,0,0) = \left(\frac{\partial \mathscr{E}}{\partial \Gamma}(0,0,0), \frac{\partial \mathscr{E}}{\partial \Upsilon}(0,0,0), \frac{\partial \mathscr{E}}{\partial \mathcal{B}}(0,0,0)\right). \tag{12}$$

In the next stage of the proof, we process the equality (12), taking the statement (11) into account. For convenience, we omit * ,writing $(x(\cdot),\ \Psi(\cdot),\ \theta(\cdot))$ instead of $(x^*(\cdot),\ \Psi^*(\cdot),\ \theta^*(\cdot))$. Specifically,

$$0 = \frac{\partial \mathscr{E}}{\partial \Gamma}(0,0,0) \implies \int_{\Omega_{0,t_0}} \left[\frac{\partial L}{\partial x^i} \cdot \frac{\partial}{\partial \Gamma}\left(x^i + \Gamma^\beta \chi_\beta^i\right) + \frac{\partial L}{\partial x_\alpha^i} \cdot \frac{\partial}{\partial \Gamma}\left(\frac{\partial}{\partial t^\alpha}\left(x^i(t) + \Gamma^\beta \chi_\beta^i\right)\right)\right.$$

$$\left. + \frac{1}{\mathcal{Z}(\pi,\zeta)} \frac{\partial L}{\partial x_{\pi\zeta}^i} \cdot \frac{\partial}{\partial \Gamma}\left(\frac{\partial^2}{\partial t^\pi \partial t^\zeta}\left(x^i + \Gamma^\beta \chi_\beta^i\right)\right)\right] dt^1 \cdots dt^m$$

$$= \int_{\Omega_{0,t_0}} \left[\frac{\partial L}{\partial x^i}\chi_\beta^i + \frac{\partial L}{\partial x_\alpha^i}\frac{\partial \chi_\beta^i}{\partial t^\alpha} + \frac{\partial L}{\partial x_{\pi\zeta}^i}\frac{\partial^2 \chi_\beta^i}{\partial t^\pi \partial t^\zeta}\right] dt^1 \cdots dt^m = 0, \tag{13}$$

$$0 = \frac{\partial \mathscr{E}}{\partial \Upsilon}(0,0,0) \implies \int_{\Omega_{0,t_0}} \left[\frac{\partial L}{\partial \Psi^\kappa} \cdot \frac{\partial}{\partial \Upsilon}\left(\Psi^\kappa + \Upsilon^\beta \aleph_\beta^\kappa\right) + \frac{\partial L}{\partial \Psi_\alpha^\kappa} \cdot \frac{\partial}{\partial \Upsilon}\left(\frac{\partial}{\partial t^\alpha}\left(\Psi^\kappa + \Upsilon^\beta \aleph_\beta^\kappa\right)\right)\right] dt^1 \cdots dt^m$$

$$= \int_{\Omega_{0,t_0}} \left[\frac{\partial L}{\partial \Psi^\kappa}\aleph_\beta^\kappa + \frac{\partial L}{\partial \Psi_\alpha^\kappa}\frac{\partial \aleph_\beta^\kappa}{\partial t^\alpha}\right] dt^1 \cdots dt^m = 0, \tag{14}$$

$$0 = \frac{\partial \mathscr{E}}{\partial \mathcal{B}}(0,0,0) \implies \int_{\Omega_{0,t_0}} \frac{\partial L}{\partial \theta_i^{\pi\zeta}} \cdot \frac{\partial}{\partial \mathcal{B}}\left(\theta_i^{\pi\zeta} + \mathcal{B}w_{\pi\zeta}^i\right) dt^1 \cdots dt^m$$

$$= \int_{\Omega_{0,t_0}} \frac{\partial L}{\partial \theta_i^{\pi\zeta}} w_{\pi\zeta}^i dt^1 \cdots dt^m = 0. \tag{15}$$

Under the complete integrability conditions in (9) and (10) associated with gradient-type variations, the relations in (13) and (14) become

$$\int_{\Omega_{0,t_0}} \left[\frac{\partial L}{\partial x^i} \chi_\beta^i + \frac{\partial L}{\partial x_\alpha^i} \frac{\partial \chi_\alpha^i}{\partial t^\beta} + \frac{1}{\mathcal{Z}(\pi,\zeta)} \frac{\partial L}{\partial x_{\pi\zeta}^i} \frac{\partial \chi_{\pi\zeta}^i}{\partial t^\beta} \right] dt^1 \cdots dt^m = 0, \tag{16}$$

$$\int_{\Omega_{0,t_0}} \left[\frac{\partial L}{\partial \Psi^\kappa} \aleph_\beta^\kappa + \frac{\partial L}{\partial \Psi_\alpha^\kappa} \frac{\partial \aleph_\alpha^\kappa}{\partial t^\beta} \right] dt^1 \cdots dt^m = 0. \tag{17}$$

In what follows, we apply integration by parts:

$$\frac{\partial L}{\partial x_\alpha^i} \frac{\partial \chi_\alpha^i}{\partial t^\beta} = -\chi_\alpha^i D_\beta \frac{\partial L}{\partial x_\alpha^i} + D_\beta \left(\chi_\alpha^i \frac{\partial L}{\partial x_\alpha^i} \right),$$

$$\frac{1}{\mathcal{Z}(\pi,\zeta)} \frac{\partial L}{\partial x_{\pi\zeta}^i} \frac{\partial \chi_{\pi\zeta}^i}{\partial t^\beta} = \frac{1}{\mathcal{Z}(\pi,\zeta)} \chi_\beta^i D_{\pi\zeta}^2 \frac{\partial L}{\partial x_{\pi\zeta}^i} - \frac{1}{\mathcal{Z}(\pi,\zeta)} D_\pi \left(\chi_\beta^i D_\zeta \frac{\partial L}{\partial x_{\pi\zeta}^i} \right) + \frac{1}{\mathcal{Z}(\pi,\zeta)} D_\zeta \left(\frac{\partial L}{\partial x_{\pi\zeta}^i} \frac{\partial \chi_\beta^i}{\partial t^\pi} \right)$$

$$\frac{\partial L}{\partial \Psi_\alpha^\kappa} \frac{\partial \aleph_\alpha^\kappa}{\partial t^\beta} = -\aleph_\alpha^\kappa D_\beta \frac{\partial L}{\partial \Psi_\alpha^\kappa} + D_\beta \left(\aleph_\alpha^\kappa \frac{\partial L}{\partial \Psi_\alpha^\kappa} \right),$$

obtaining

$$\int_{\Omega_{0,t_0}} \left[\frac{\partial L}{\partial x^i} \chi_\beta^i - D_\beta \frac{\partial L}{\partial x_\alpha^i} \chi_\alpha^i + \frac{1}{\mathcal{Z}(\pi,\zeta)} D_{\pi\zeta}^2 \frac{\partial L}{\partial x_{\pi\zeta}^i} \chi_\beta^i \right] dt^1 \cdots dt^m + \int_{\Omega_{0,t_0}} D_\beta \left(\chi_\alpha^i \frac{\partial L}{\partial x_\alpha^i} \right) dt^1 \cdots dt^m$$

$$- \int_{\Omega_{0,t_0}} \frac{1}{\mathcal{Z}(\pi,\zeta)} D_\pi \left(\chi_\beta^i D_\zeta \frac{\partial L}{\partial x_{\pi\zeta}^i} \right) dt^1 \cdots dt^m + \int_{\Omega_{0,t_0}} \frac{1}{\mathcal{Z}(\pi,\zeta)} D_\zeta \left(\frac{\partial L}{\partial x_{\pi\zeta}^i} \frac{\partial \chi_\beta^i}{\partial t^\pi} \right) dt^1 \cdots dt^m = 0,$$

$$\int_{\Omega_{0,t_0}} \left[\frac{\partial L}{\partial \Psi^\kappa} \aleph_\beta^\kappa - D_\beta \frac{\partial L}{\partial \Psi_\alpha^\kappa} \aleph_\alpha^\kappa \right] dt^1 \cdots dt^m + \int_{\Omega_{0,t_0}} D_\beta \left(\aleph_\alpha^\kappa \frac{\partial L}{\partial \Psi_\alpha^\kappa} \right) dt^1 \cdots dt^m = 0.$$

The Gauss–Ostrogradski formula leads us to

$$\int_{\Omega_{0,t_0}} D_\beta \left(\chi_\alpha^i \frac{\partial L}{\partial x_\alpha^i} \right) dt^1 \cdots dt^m = \int_{\partial\Omega_{0,t_0}} \delta_{z\iota} \chi_\alpha^i \frac{\partial L}{\partial x_z^i} \mathbf{n}^\iota dt^1 \cdots dt^m,$$

$$\int_{\Omega_{0,t_0}} \frac{1}{\mathcal{Z}(\pi,\zeta)} D_\pi \left(\chi_\beta^i D_\zeta \frac{\partial L}{\partial x_{\pi\zeta}^i} \right) dt^1 \cdots dt^m = \int_{\partial\Omega_{0,t_0}} \frac{1}{\mathcal{Z}(\pi,\zeta)} \delta_{z\iota} \chi_\beta^i D_\zeta \frac{\partial L}{\partial x_{z\zeta}^i} \mathbf{n}^\iota dt^1 \cdots dt^m,$$

$$\int_{\Omega_{0,t_0}} \frac{1}{\mathcal{Z}(\pi,\zeta)} D_\zeta \left(\frac{\partial L}{\partial x_{\pi\zeta}^i} \frac{\partial \chi_\beta^i}{\partial t^\pi} \right) dt^1 \cdots dt^m = \int_{\partial\Omega_{0,t_0}} \frac{1}{\mathcal{Z}(\pi,\zeta)} \delta_{z\iota} \frac{\partial L}{\partial x_{\pi z}^i} \frac{\partial \chi_\beta^i}{\partial t^\pi} \mathbf{n}^\iota dt^1 \cdots dt^m,$$

$$\int_{\Omega_{0,t_0}} D_\beta \left(\aleph_\alpha^\kappa \frac{\partial L}{\partial \Psi_\alpha^\kappa} \right) dt^1 \cdots dt^m = \int_{\partial\Omega_{0,t_0}} \delta_{z\iota} \aleph_\alpha^\kappa \frac{\partial L}{\partial \Psi_z^\kappa} \mathbf{n}^\iota dt^1 \cdots dt^m,$$

where $\delta_{z\iota}$ denotes the Kronecker symbol, and $\mathbf{n}^{\iota}(\alpha)$ is the unit normal vector on $\partial\Omega_{0,t_0}$. Thus, in agreement with the boundary conditions (2), (3), and (5) describing the gradient-type variations (see Definitions 2.2 and 2.3), we show that these terms (boundary terms) vanish and we have

$$\int_{\Omega_{0,t_0}}\left[\frac{\partial L}{\partial x^i}\chi^i_\beta - D_\beta\frac{\partial L}{\partial x^i_\alpha}\chi^i_\alpha + \frac{1}{\mathcal{Z}(\pi,\zeta)}D^2_{\pi\zeta}\frac{\partial L}{\partial x^i_{\pi\zeta}}\chi^i_\beta\right]dt^1\cdots dt^m = 0, \tag{18}$$

$$\int_{\Omega_{0,t_0}}\left[\frac{\partial L}{\partial \Psi^\kappa}\aleph^\kappa_\beta - D_\beta\frac{\partial L}{\partial \Psi^\kappa_\alpha}\aleph^\kappa_\alpha\right]dt^1\cdots dt^m = 0. \tag{19}$$

At this point of the proof, we rewrite relations (18) and (19) as follows:

$$0 = \int_{\Omega_{0,t_0}}\left[\frac{\partial L}{\partial x^i}\chi^i_\beta - D_\beta\frac{\partial L}{\partial x^i_\alpha}\chi^i_\alpha + \frac{1}{\mathcal{Z}(\pi,\zeta)}D^2_{\pi\zeta}\frac{\partial L}{\partial x^i_{\pi\zeta}}\chi^i_\beta\right]dt^1\cdots dt^m$$

$$= \int_{\Omega_{0,t_0}}\left[\frac{\partial L}{\partial x^i}\delta^\alpha_\beta - D_\beta\frac{\partial L}{\partial x^i_\alpha} + \frac{1}{\mathcal{Z}(\pi,\zeta)}D^2_{\pi\zeta}\frac{\partial L}{\partial x^i_{\pi\zeta}}\delta^\alpha_\beta\right]\chi^i_\alpha dt^1\cdots dt^m$$

$$= \int_{\Omega_{0,t_0}} D_\alpha\left[\left(\frac{\partial L}{\partial x^i}\delta^\alpha_\beta - \frac{\partial L}{\partial x^i_\alpha} + \frac{1}{\mathcal{Z}(\pi,\zeta)}D_\beta D^2_{\pi\zeta}\frac{\partial L}{\partial x^i_{\pi\zeta}}\delta^\alpha_\beta\right)\chi^i\right]dt^1\cdots dt^m$$

$$- \int_{\Omega_{0,t_0}} D_\alpha\left(\frac{\partial L}{\partial x^i}\delta^\alpha_\beta - D_\beta\frac{\partial L}{\partial x^i_\alpha} + \frac{1}{\mathcal{Z}(\pi,\zeta)}D^2_{\pi\zeta}\frac{\partial L}{\partial x^i_{\pi\zeta}}\delta^\alpha_\beta\right)\chi^i dt^1\cdots dt^m,$$

$$0 = \int_{\Omega_{0,t_0}}\left[\frac{\partial L}{\partial \Psi^\kappa}\aleph^\kappa_\beta - D_\beta\frac{\partial L}{\partial \Psi^\kappa_\alpha}\aleph^\kappa_\alpha\right]dt^1\cdots dt^m = \int_{\Omega_{0,t_0}}\left[\frac{\partial L}{\partial \Psi^\kappa}\delta^\alpha_\beta - D_\beta\frac{\partial L}{\partial \Psi^\kappa_\alpha}\right]\aleph^\kappa_\alpha dt^1\cdots dt^m$$

$$= \int_{\Omega_{0,t_0}} D_\alpha\left[\left(\frac{\partial L}{\partial \Psi^\kappa}\delta^\alpha_\beta - D_\beta\frac{\partial L}{\partial \Psi^\kappa_\alpha}\right)\aleph^\kappa\right]dt^1\cdots dt^m - \int_{\Omega_{0,t_0}} D_\alpha\left(\frac{\partial L}{\partial \Psi^\kappa}\delta^\alpha_\beta - D_\beta\frac{\partial L}{\partial \Psi^\kappa_\alpha}\right)\aleph^\kappa_\alpha dt^1\cdots dt^m.$$

Applying the Gauss–Ostrogradski formula again, we have

$$\int_{\Omega_{0,t_0}} D_\alpha\left[\left(\frac{\partial L}{\partial x^i}\delta^\alpha_\beta - D_\beta\frac{\partial L}{\partial x^i_\alpha} + \frac{1}{\mathcal{Z}(\pi,\zeta)}D^2_{\pi\zeta}\frac{\partial L}{\partial x^i_{\pi\zeta}}\delta^\alpha_\beta\right)\chi^i\right]dt^1\cdots dt^m$$

$$= \int_{\partial\Omega_{0,t_0}}\delta_{z\iota}\left(\frac{\partial L}{\partial x^i}\delta^z_\beta - D_\beta\frac{\partial L}{\partial x^i_z} + \frac{1}{\mathcal{Z}(\pi,\zeta)}D^2_{\pi\zeta}\frac{\partial L}{\partial x^i_{\pi\zeta}}\delta^z_\beta\right)\chi^i\mathbf{n}^\iota dt^1\cdots dt^m,$$

$$\int_{\Omega_{0,t_0}} D_\alpha\left[\left(\frac{\partial L}{\partial \Psi^\kappa}\delta^z_\beta - D_\beta\frac{\partial L}{\partial \Psi^\kappa_\alpha}\right)\aleph^\kappa\right]dt^1\cdots dt^m = \int_{\Omega_{0,t_0}}\delta_{z\iota}\left(\frac{\partial L}{\partial \Psi^\kappa}\delta^z_\beta - D_\beta\frac{\partial L}{\partial \Psi^\kappa_z}\right)\aleph^\kappa\mathbf{n}^\iota dt^1\cdots dt^m$$

and recalling that $\chi|_{\partial\Omega_{0,t_0}} = \aleph|_{\partial\Omega_{0,t_0}} = 0$ (see the boundary conditions given in (1) and (4), we get

$$\int_{\Omega_{0,t_0}} D_\alpha\left[\left(\frac{\partial L}{\partial x^i}\delta^\alpha_\beta - D_\beta\frac{\partial L}{\partial x^i_\alpha} + \frac{1}{\mathcal{Z}(\pi,\zeta)}D^2_{\pi\zeta}\frac{\partial L}{\partial x^i_{\pi\zeta}}\delta^\alpha_\beta\right)\chi^i\right]dt^1\cdots dt^m = 0,$$

$$\int_{\Omega_{0,t_0}} D_\alpha\left[\left(\frac{\partial L}{\partial \Psi^\kappa}\delta^\alpha_\beta - D_\beta\frac{\partial L}{\partial \Psi^\kappa_\alpha}\right)\aleph^\kappa\right]dt^1\cdots dt^m = 0,$$

and consequently

$$\int_{\Omega_{0,t_0}} D_\alpha\left(\frac{\partial L}{\partial x^i}\delta^\alpha_\beta - D_\beta\frac{\partial L}{\partial x^i_\alpha} + \frac{1}{\mathcal{Z}(\pi,\zeta)}D^2_{\pi\zeta}\frac{\partial L}{\partial x^i_{\pi\zeta}}\delta^\alpha_\beta\right)\chi^i dt^1\cdots dt^m = 0, \tag{20}$$

$$\int_{\Omega_{0,t_0}} D_\alpha\left(\frac{\partial L}{\partial \Psi^\kappa}\delta^\alpha_\beta - D_\beta\frac{\partial L}{\partial \Psi^\kappa_\alpha}\right)\aleph^\kappa dt^1\cdots dt^m = 0. \tag{21}$$

Considering that the equalities in (15), (20), and (21) are satisfied for all differentiable variations χ^i, $\aleph^\kappa$ with $\chi^i|_{\partial\Omega_{0,t_0}} = \aleph^\kappa|_{\partial\Omega_{0,t_0}} = w^{\pi\zeta}_i|_{\partial\Omega_{0,t_0}} = 0$ and normalize, we obtain

$$\varsigma_i\frac{\partial L}{\partial x^i}\delta^\alpha_\beta - D_\beta\frac{\partial L}{\partial x^i_\alpha} + \frac{\varsigma_i}{\mathcal{Z}(\pi,\zeta)}D^2_{\pi\zeta}\frac{\partial L}{\partial x^i_{\pi\zeta}}\delta^\alpha_\beta = c_i = \text{const.}, \tag{22}$$

$$\vartheta_\kappa\frac{\partial L}{\partial \Psi^\kappa}\delta^\alpha_\beta - D_\beta\frac{\partial L}{\partial \Psi^\kappa_\alpha} = c_\kappa = \text{const.}, \tag{23}$$

$$\frac{\partial L}{\partial \theta^{\pi\zeta}_i} = 0.$$

where ς_i and ϑ_κ are dimension-dependent factors. In the next step, we focus on determining these factors. By tracing equations (22) and (23), we get

$$m\varsigma_i\frac{\partial L}{\partial x^i}\delta^\alpha_\beta - D_\beta\frac{\partial L}{\partial x^i_\alpha} + \frac{m\varsigma_i}{\mathcal{Z}(\pi,\zeta)}D^2_{\pi\zeta}\frac{\partial L}{\partial x^i_{\pi\zeta}}\delta^\alpha_\beta = mc_i,$$

$$m\vartheta_\kappa\frac{\partial L}{\partial \Psi^\kappa}\delta^\alpha_\beta - D_\beta\frac{\partial L}{\partial \Psi^\kappa_\alpha} = mc_\kappa.$$

On the other hand, by tracing equations (20) and (21), we should recover classical Euler–Lagrange partial differential equations, namely

$$\frac{\partial L}{\partial x^i} - D_\alpha\frac{\partial L}{\partial x^i_\alpha} + \frac{1}{\mathcal{Z}(\pi,\zeta)}D^2_{\pi\zeta}\frac{\partial L}{\partial x^i_{\pi\zeta}} = 0,$$

$$\frac{\partial L}{\partial \Psi^\kappa} - D_\alpha\frac{\partial L}{\partial \Psi^\kappa_\alpha} = 0.$$

Finally, we find $m\varsigma_i = 1$, $m\vartheta_\kappa = 1$, $mc_i = 0$, $mc_\kappa = 0$, $\forall i \in \{1,\ldots,n\}$, and $\kappa \in \{1,\ldots,\epsilon\}$. It involves $\varsigma_i = \vartheta_\kappa = \dfrac{1}{m}$, $c_i = c_\kappa = 0$ $\forall i \in \{1,\ldots,n\}$, and $\kappa \in \{1,\ldots,\epsilon\}$, identifying thus the anti-trace Euler-Lagrange PDEs stated in Theorem 2.1. Note that the statements encompassed in Theorem 2.1 involve simple index changes that do not alter the previously proven result. This completes the proof.

Remark 2.1 The Euler–Lagrange PDEs of the anti-trace type given in (6)–(8) are derived by subjecting the state and control variables to gradient-type variations. This new class of admissible variations was first introduced and discussed by Treanţă and Dragu [35].

Remark 2.2 If symmetries related to the second-order partial derivative of x involve symmetries in the state equations (i.e., $X_{\pi\zeta} = X_{\zeta\pi}$), then we can handle only state equations with $\pi \leq \zeta$. In this setting,

we have $\theta = (\theta_i^{\pi\zeta}) : \Omega_{0,t_0} \to \mathbb{R}^{nm(m+1)/2}$ involving $nm(m+1)/2$ different co-state functions. Moreover, equations (6) and (8) can be written as

$$\frac{\delta_\beta^\eta}{m}\frac{\partial L}{\partial x^i} - D_\beta \frac{\partial L}{\partial x_\eta^i} + \frac{\delta_\beta^\eta}{m} D_{ab}^2 \frac{\partial L}{\partial x_{ab}^i} = 0, \quad a \le b,$$

$$\frac{\partial L}{\partial \theta_i^{\pi\zeta}} = 0, \quad \pi \le \zeta.$$

Remark 2.3 The anti-trace Euler–Lagrange system of PDEs (6)–(8) can be rewritten as

$$\frac{\delta_\beta^\eta}{m}\frac{\partial L}{\partial x^i} - D_\beta \frac{\partial L}{\partial x_\eta^i} + \frac{\delta_\beta^\eta}{m \cdot \mathcal{Z}(a,b)} D_{ab}^2 \frac{\partial L}{\partial x_{ab}^i} = 0,$$

$$\frac{\delta_\beta^\eta}{m}\frac{\partial L}{\partial \Psi^\kappa} - D_\beta \frac{\partial L}{\partial \Psi_\eta^\kappa} = 0,$$

$$X_{\pi\zeta}^i \Big(x(t), x_\alpha(t), x_{\pi\zeta}(t), \Psi(t), \Psi_\alpha(t) \Big) = 0.$$

Let us introduce the vector variable y, equipped with nm x_η^i functions, $nm(m+1)/2$ $x_{ab}^i, a \le b$ functions (here we take each symmetric pair (a,b) and (b,a) into account only once, since $x_{ab}^i = x_{ba}^i$), and ϵm Ψ_η^κ functions. Equivalently, we write $y := (y^h), h \in \{1, \dots, nm + nm(m+1)/2 + \epsilon m\}$. The anti-trace Euler–Lagrange PDEs can be formulated using canonical variables (multi-momenta) if

$$\det\left(\frac{\partial^2 L}{\partial y^{\tilde{h}} \partial y^{\hat{h}}} \right) \neq 0, \quad \tilde{h}, \hat{h} \in \{1, \dots, nm + nm(m+1)/2 + \epsilon m\}.$$

For an arbitrary fixed $x(\cdot)$ and $\Psi(\cdot)$, we introduce the generalized canonical variables (multi-momenta) $u = \left(u_i^\eta\right)$, $v = \left(v_i^{ab}\right)$, $s = (s_\kappa^\eta)$, $i \in \{1, 2, \dots, n\}$, $\kappa \in \{1, 2, \dots, \epsilon\}$, and $\eta, a, b \in \{1, 2, \dots, m\}$, computed as follows:

$$u_i^\eta := \frac{\partial L}{\partial x_\eta^i} \Big(x(t), x_\alpha(t), x_{\pi\zeta}(t), \Psi(t), \Psi_\alpha(t), \theta(t) \Big),$$

$$v_i^{ab} := \frac{\partial L}{\partial x_{ab}^i} \Big(x(t), x_\alpha(t), x_{\pi\zeta}(t), \Psi(t), \Psi_\alpha(t), \theta(t) \Big), \ v_i^{ab} = v_i^{ba},$$

$$s_\kappa^\eta := \frac{\partial L}{\partial \Psi_\eta^\kappa} \Big(x(t), x_\alpha(t), x_{\pi\zeta}(t), \Psi(t), \Psi_\alpha(t), \theta(t) \Big).$$

For $(x, \Psi, \theta) \in \mathbb{R}^n \times \mathbb{R}^\epsilon \times \mathbb{R}^{nm^2}$, $\theta_i^{\pi\zeta} = \theta_i^{\zeta\pi}$, let us consider the Legendre map

$$\mathcal{L} : \mathbb{R}^{nm} \times \mathbb{R}^{nm^2} \times \mathbb{R}^{\epsilon m} \to \mathbb{R}^{nm} \times \mathbb{R}^{nm^2} \times \mathbb{R}^{\epsilon m}, \quad (x_\eta, x_{ab}, \Psi_\eta) \longmapsto (u, v, s),$$

with $x_{ab} = x_{ba}$ and $v^{ab} = v^{ba}$. We assume that $\mathcal{L}$ is a C^2-local diffeomorphism (i.e., $\det\left(\frac{\partial^2 L}{\partial y^{\tilde{h}} \partial y^{\hat{h}}} \right) \neq 0$) so that the system determines x_η^i, x_{ab}^i and Ψ_η^κ as functions of $(x, u, v, \Psi, s, \theta)$. This is called the Legendre

transformation (local invertibility), and L is called *super-regular*. The Legendre transformation of the Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^{nm} \times \mathbb{R}^{nm^2} \times \mathbb{R}^{\epsilon} \times \mathbb{R}^{\epsilon m} \times \mathbb{R}^{nm^2} \to \mathbb{R}$ is the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^{nm} \times \mathbb{R}^{nm^2} \times \mathbb{R}^{\epsilon} \times \mathbb{R}^{\epsilon m} \times \mathbb{R}^{nm^2} \to \mathbb{R}$, defined as

$$H(x, u, v, \Psi, s, \theta) = x^i_\eta u^\eta_i + \frac{1}{\mathcal{Z}(a,b)} x^i_{ab} v^{ab}_i + \Psi^\kappa_\eta s^\eta_\kappa$$

$$-L(x, x_\eta(x, u, v, \Psi, s, \theta), x_{ab}(x, u, v, \Psi, s, \theta), \Psi, \Psi_\eta(x, u, v, \Psi, s, \theta), \theta).$$

Moreover, we have all mathematical objects to introduce the *energy-momentum tensor*, with the components described by

$$\Theta^\alpha_\beta(x, u, v, \Psi, s, \theta) = \left[u^\alpha_i - \frac{1}{\mathcal{Z}(a,\alpha)} \frac{\partial v^{a\alpha}_i}{\partial t^a}\right] x^i_\beta + s^\alpha_\kappa \Psi^\kappa_\beta + \frac{1}{\mathcal{Z}(a,\alpha)} x^i_{\beta a} v^{a\alpha}_i$$

$$-L(x, x_\eta(x, u, v, \Psi, s, \theta), x_{ab}(x, u, v, \Psi, s, \theta), \Psi, \Psi_\eta(x, u, v, \Psi, s, \theta), \theta)\delta^\alpha_\beta.$$

Theorem 2.2 (Hamilton PDEs of the anti-trace type) *If the Lagrangian L is super-regular and the pair $(x(\cdot), \Psi(\cdot), \theta(\cdot))$ is a solution for the above-mentioned Euler–Lagrange PDEs of the anti-trace type expressed in* (6)–(8)*, then the point $(x(\cdot), u(\cdot), v(\cdot), \Psi(\cdot), s(\cdot), \theta(\cdot))$ represents a solution of the following Hamiltonian-type differential system*:

$$\frac{\partial x^i}{\partial t^\beta}(t) = \frac{\partial H}{\partial u^\beta_i}(x, u, v, \Psi, s, \theta), \tag{24}$$

$$\frac{1}{\mathcal{Z}(a,b)} \frac{\partial^2 x^i}{\partial t^a \partial t^b}(t) = \frac{\partial H}{\partial v^{ab}_i}(x, u, v, \Psi, s, \theta), \tag{25}$$

$$D_\beta u^\eta_i(t) - \frac{\delta^\eta_\beta}{m \cdot \mathcal{Z}(a,b)} D^2_{ab} v^{ab}_i(t) = -\frac{\delta^\eta_\beta}{m} \frac{\partial H}{\partial x^i}(x, u, v, \Psi, s, \theta), \tag{26}$$

$$\frac{\partial \Psi^\kappa}{\partial t^\beta}(t) = \frac{\partial H}{\partial s^\beta_\kappa}(x, u, v, \Psi, s, \theta), \tag{27}$$

$$D_\beta s^\eta_\kappa(t) = -\frac{\delta^\eta_\beta}{m} \frac{\partial H}{\partial \Psi^\kappa}(x, u, v, \Psi, s, \theta), \tag{28}$$

$$\frac{\partial H}{\partial \theta^{\pi\zeta}_i}(x, u, v, \Psi, s, \theta) = 0. \tag{29}$$

Summation over repeated indices is assumed.

Proof. By direct computation, we find

$$\frac{\partial H}{\partial x^j} = u^\eta_i \frac{\partial x^i_\eta}{\partial x^j} + \frac{1}{\mathcal{Z}(a,b)} v^{ab}_i \frac{\partial x^i_{ab}}{\partial x^j} + s^\eta_\kappa \frac{\partial \Psi^\kappa_\eta}{\partial x^j} - \frac{\partial L}{\partial x^j} - \frac{\partial L}{\partial x^i_\eta} \frac{\partial x^i_\eta}{\partial x^j} - \frac{\partial L}{\partial x^i_{ab}} \frac{\partial x^i_{ab}}{\partial x^j}$$

$$-\frac{\partial L}{\partial \Psi^{\kappa}}\frac{\partial \Psi^{\kappa}}{\partial x^{j}}-\frac{\partial L}{\partial \Psi^{\kappa}_{\eta}}\frac{\partial \Psi^{\kappa}_{\eta}}{\partial x^{j}}-\frac{\partial L}{\partial \theta^{\pi\zeta}_{i}}\frac{\partial \theta^{\pi\zeta}_{i}}{\partial x^{j}}=-\frac{\partial L}{\partial x^{j}},\ j=\{1,\ldots,n\}$$

and

$$\frac{\partial H}{\partial \Psi^{r}}=u^{\eta}_{i}\frac{\partial x^{i}_{\eta}}{\partial \Psi^{r}}+\frac{1}{\mathcal{Z}(a,b)}v^{ab}_{i}\frac{\partial x^{i}_{ab}}{\partial \Psi^{r}}+s^{\eta}_{\kappa}\frac{\partial \Psi^{\kappa}_{\eta}}{\partial \Psi^{r}}-\frac{\partial L}{\partial x^{i}}\frac{\partial x^{i}}{\partial \Psi^{r}}-\frac{\partial L}{\partial x^{i}_{\eta}}\frac{\partial x^{i}_{\eta}}{\partial \Psi^{r}}-\frac{\partial L}{\partial x^{i}_{ab}}\frac{\partial x^{i}_{ab}}{\partial \Psi^{r}}$$

$$-\frac{\partial L}{\partial \Psi^{r}}-\frac{\partial L}{\partial \Psi^{\kappa}_{\eta}}\frac{\partial \Psi^{\kappa}_{\eta}}{\partial \Psi^{r}}-\frac{\partial L}{\partial \theta^{\pi\zeta}_{i}}\frac{\partial \theta^{\pi\zeta}_{i}}{\partial \Psi^{r}}=-\frac{\partial L}{\partial \Psi^{r}},\ r=\{1,\ldots,\epsilon\}.$$

Moreover, by processing the associated anti-trace Euler Lagrange PDEs in (6)–(7), we obtain

$$\frac{\delta^{\eta}_{\beta}}{m}\frac{\partial L}{\partial x^{i}}=D_{\beta}\frac{\partial L}{\partial x^{i}_{\eta}}-\frac{\delta^{\eta}_{\beta}}{m\cdot\mathcal{Z}(a,b)}D^{2}_{ab}\frac{\partial L}{\partial x^{i}_{ab}}\delta^{\eta}_{\beta}=D_{\beta}u^{\eta}_{i}-\frac{\delta^{\eta}_{\beta}}{m\cdot\mathcal{Z}(a,b)}D^{2}_{ab}v^{ab}_{i},$$

together with

$$\frac{\delta^{\eta}_{\beta}}{m}\frac{\partial L}{\partial \Psi^{\kappa}}\delta^{\eta}_{\beta}=D_{\beta}\frac{\partial L}{\partial \Psi^{\kappa}_{\eta}}=D_{\beta}s^{\eta}_{\kappa},$$

involving (see $\frac{\partial H}{\partial x^{j}}=-\frac{\partial L}{\partial x^{j}}$, $\frac{\partial H}{\partial \Psi^{r}}=-\frac{\partial L}{\partial \Psi^{r}}$) and

$$D_{\beta}u^{\eta}_{i}-\frac{\delta^{\eta}_{\beta}}{m\cdot\mathcal{Z}(a,b)}D^{2}_{ab}v^{ab}_{i}=-\frac{\delta^{\eta}_{\beta}}{m}\frac{\partial H}{\partial x^{i}},$$

$$D_{\beta}s^{\eta}_{\kappa}=-\frac{\delta^{\eta}_{\beta}}{m}\frac{\partial H}{\partial \Psi^{\kappa}}.$$

Moreover, we have

$$\frac{\partial H}{\partial u^{\beta}_{i}}=\frac{\partial x^{i}}{\partial t^{\beta}},\ \frac{\partial H}{\partial v^{ab}_{i}}=\frac{1}{\mathcal{Z}(a,b)}\frac{\partial^{2}x^{i}}{\partial t^{a}\partial t^{b}},\quad \frac{\partial H}{\partial s^{\beta}_{\kappa}}=\frac{\partial \Psi^{\kappa}}{\partial t^{\beta}},\quad \frac{\partial H}{\partial \theta^{\pi\zeta}_{i}}=0.$$

Thus, the proof is concluded.

Remark 2.4 The Hamilton PDEs of the anti-trace type given in (24)–(28) that were previously formulated can be rewritten as follows:

$$-\delta^{\alpha}_{\beta}\delta^{i}_{j}\frac{\partial x^{j}}{\partial t^{\alpha}}(t)+\frac{\partial H}{\partial u^{\beta}_{i}}(x,u,v,\Psi,s,\theta)=0,$$

$$-\delta^{i}_{j}\frac{\partial^{2}x^{j}}{\partial t^{a}\partial t^{b}}(t)+\frac{\partial H}{\partial v^{ab}_{i}}(x,u,v,\Psi,s,\theta)=0,$$

$$\delta^{i}_{j}\frac{\partial u^{\eta}_{j}}{\partial t^{\beta}}(t)-\frac{\delta^{\eta}_{\beta}\delta^{i}_{j}}{m\cdot\mathcal{Z}(a,b)}\frac{\partial^{2}v^{ab}_{j}}{\partial t^{a}\partial t^{b}}(t)+\frac{\delta^{\eta}_{\beta}}{m}\frac{\partial H}{\partial x^{i}}(x,u,v,\Psi,s,\theta)=0,$$

$$-\delta^{\alpha}_{\beta}\delta^{\kappa}_{r}\frac{\partial \Psi^{r}}{\partial t^{\alpha}}(t)+\frac{\partial H}{\partial s^{\beta}_{\kappa}}(x,u,v,\Psi,s,\theta)=0,$$

$$\delta_r^\kappa \frac{\partial s_r^\eta}{\partial t^\beta}(t) + \frac{\delta_\beta^\eta}{m}\frac{\partial H}{\partial \Psi^\kappa}(x,u,v,\Psi,s,\theta) = 0.$$

Now we introduce the vectorization operator vec : $\mathbb{R}^{m\times m} \to \mathbb{R}^{m^2}$, $\mathrm{vec}(\delta_\beta^\eta) := \mathrm{vec}(I_m)$, with the help of which, a square matrix is written as a column vector. In general, we denote I_m as the $m \times m$ identity matrix and $O_{n,m}$ as the zero matrix with n rows and m columns. We use the following block matrices:

$${}^{x}J = \begin{pmatrix} -\delta_j^i \otimes \delta_\beta^\alpha & O_{nm\times nm^2} & O_{nm\times nm^2} & O_{nm\times n} & O_{nm\times \epsilon m} & O_{nm\times \epsilon m^2} \\ O_{nm^2\times nm} & -\delta_j^i \otimes I_{m^2} & O_{nm^2\times nm^2} & O_{nm^2\times n} & O_{nm^2\times \epsilon m} & O_{nm^2\times \epsilon m^2} \\ O_{nm^2\times nm} & O_{nm^2\times nm^2} & \delta_j^i \otimes I_{m^2} & -\frac{1}{m}(\delta_j^i \otimes \mathrm{vec}(\delta_\beta^\eta)) & O_{nm^2\times \epsilon m} & O_{nm^2\times \epsilon m^2} \\ O_{\epsilon m\times nm} & O_{\epsilon m\times nm^2} & O_{\epsilon m\times nm^2} & O_{\epsilon m\times n} & O_{\epsilon m\times \epsilon m} & O_{\epsilon m\times \epsilon m^2} \\ O_{\epsilon m^2\times nm} & O_{\epsilon m^2\times nm^2} & O_{\epsilon m^2\times nm^2} & O_{\epsilon m^2\times n} & O_{\epsilon m^2\times \epsilon m} & O_{\epsilon m^2\times \epsilon m^2} \end{pmatrix},$$

$${}^{\Psi}J = \begin{pmatrix} O_{nm\times nm} & O_{nm\times nm^2} & O_{nm\times nm^2} & O_{nm\times n} & O_{nm\times \epsilon m} & O_{nm\times \epsilon m^2} \\ O_{nm^2\times nm} & O_{nm^2\times nm^2} & O_{nm^2\times nm^2} & O_{nm^2\times n} & O_{nm^2\times \epsilon m} & O_{nm^2\times \epsilon m^2} \\ O_{nm^2\times nm} & O_{nm^2\times nm^2} & O_{nm^2\times nm^2} & O_{nm^2\times n} & O_{nm^2\times \epsilon m} & O_{nm^2\times \epsilon m^2} \\ O_{\epsilon m\times nm} & O_{\epsilon m\times nm^2} & O_{\epsilon m\times nm^2} & O_{\epsilon m\times n} & -\delta_r^\kappa \otimes \delta_\beta^\alpha & O_{\epsilon m\times \epsilon m^2} \\ O_{\epsilon m^2\times nm} & O_{\epsilon m^2\times nm^2} & O_{\epsilon m^2\times nm^2} & O_{\epsilon m^2\times n} & O_{\epsilon m^2\times \epsilon m} & \delta_r^\kappa \otimes I_{m^2} \end{pmatrix},$$

where $\otimes$ denotes the Kronecker product and

$$\delta_j^i = I_n, \quad \delta_\kappa^r = I_\epsilon, \quad \delta_\beta^\alpha = I_m,$$

We establish the following *compact (matrix) form of anti-trace Hamilton PDEs*:

$$\left({}^{x}J + {}^{\Psi}J\right)\begin{pmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial^2 x}{\partial t^2} \\ \frac{\partial u}{\partial t} \\ \breve{v} \\ \frac{\partial \Psi}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} = -\begin{pmatrix} \frac{\partial H}{\partial u} \\ \frac{\partial H}{\partial v} \\ \frac{1}{m}(\frac{\partial H}{\partial x} \otimes \mathrm{vec}(\delta_\beta^\eta)) \\ \frac{\partial H}{\partial s} \\ \frac{1}{m}(\frac{\partial H}{\partial \Psi} \otimes \mathrm{vec}(\delta_\beta^\eta)) \end{pmatrix},$$

Here, we introduce

$$\breve{v} := (\breve{v}_i) = \left(\frac{1}{\mathcal{Z}_(a,b)}\frac{\partial^2 v_i^{ab}}{\partial t^a \partial t^b}\right)$$

(with summation over repeated indices a, b). We refer the reader to Table 1, which comprises observations related to the components of ${}^{x}J$ and ${}^{\Psi}J$. Table 2 collects the dimensions resulting from the various Kronecker products embedded within the matrix structures ${}^{x}J$ and ${}^{\Psi}J$. Table 3 details the column-vector dimensions that constitute the left- and right-hand sides of the compact system.

Table 1. Description of block matrix structures incorporated in ${}^{x}J$ and ${}^{\Psi}J$.

Component	Interpretation	Dimension
δ^i_j	Identity matrix (Kronecker delta)	(n, n)
δ^α_β	Identity matrix (Kronecker delta)	(m, m)
I_{m^2}	Identity matrix	(m^2, m^2)
$\mathrm{vec}(\delta^\eta_\beta)$	Column vector, vectorization of I_m	$(m^2, 1)$
δ^r_κ	Identity matrix (Kronecker delta)	(ϵ, ϵ)

Table 2. Dimensions of the Kronecker products embedded in the previous block matrices.

Kronecker product	Dimension
$\delta^i_j \otimes \delta^\alpha_\beta$	(nm, nm)
$\delta^i_j \otimes I_{m^2}$	(nm^2, nm^2)
$\delta^i_j \otimes \mathrm{vec}(\delta^\eta_\beta)$	(nm^2, n)
$\delta^r_\kappa \otimes \delta^\alpha_\beta$	$(\epsilon m, \epsilon m)$
$\delta^r_\kappa \otimes I_{m^2}$	$(\epsilon m^2, \epsilon m^2)$

Table 3. Description of the components appearing in the left- and right-hand sides of the compact form of anti-trace Hamilton PDEs.

Block matrix structure	Dimension
$\dfrac{\partial x}{\partial t}$	$(nm, 1)$
$\dfrac{\partial^2 x}{\partial t^2}$	$(nm^2, 1)$
$\dfrac{\partial u}{\partial t}$	$(nm^2, 1)$
$\check{v}$	$(n, 1)$
$\dfrac{\partial \Psi}{\partial t}$	$(\epsilon m, 1)$
$\dfrac{\partial s}{\partial t}$	$(\epsilon m^2, 1)$
$\dfrac{\partial H}{\partial u}$	$(nm, 1)$
$\dfrac{\partial H}{\partial v}$	$(nm^2, 1)$
$\dfrac{\partial H}{\partial x} \otimes \mathrm{vec}(\delta^\eta_\beta)$	$(nm^2, 1)$
$\dfrac{\partial H}{\partial s}$	$(\epsilon m, 1)$
$\dfrac{\partial H}{\partial \Psi} \otimes \mathrm{vec}(\delta^\eta_\beta)$	$(\epsilon m^2, 1)$

In this context, we derive the following result regarding the dimensions of the matrices involved in the compact (matrix) form of anti-trace Hamilton PDEs:

- Dimension of $\left({}^{x}J + {}^{\Psi}J\right)$ is $(nm + 2nm^2 + \epsilon m + \epsilon m^2, nm + 2nm^2 + n + \epsilon m + \epsilon m^2)$;
- Dimension of $\begin{pmatrix} \dfrac{\partial x}{\partial t} \\ \dfrac{\partial^2 x}{\partial t^2} \\ \dfrac{\partial u}{\partial t} \\ \check{v} \\ \dfrac{\partial \Psi}{\partial t} \\ \dfrac{\partial s}{\partial t} \end{pmatrix}$ is $(nm + 2nm^2 + n + \epsilon m + \epsilon m^2, 1)$;
- Dimension of $\begin{pmatrix} \dfrac{\partial H}{\partial u} \\ \dfrac{\partial H}{\partial v} \\ \dfrac{\partial H}{\partial x} \otimes \mathrm{vec}(\delta^{\eta}_{\beta}) \\ \dfrac{\partial H}{\partial s} \\ \dfrac{\partial H}{\partial \Psi} \otimes \mathrm{vec}(\delta^{\eta}_{\beta}) \end{pmatrix}$ is $(nm + 2nm^2 + \epsilon m + \epsilon m^2, 1)$. Notice that these dimensions are compatible with the matrix product. Indeed, we have

$$(nm + 2nm^2 + \epsilon m + \epsilon m^2, nm + 2nm^2 + n + \epsilon m + \epsilon m^2) \cdot (nm + 2nm^2 + n + \epsilon m + \epsilon m^2, 1)$$
$$= (nm + 2nm^2 + \epsilon m + \epsilon m^2, 1).$$

However, following the approach described in Remark 2.2, the block matrices are built explicitly as follows:

$${}^{x}J = \begin{pmatrix} -\delta^i_j \otimes \delta^{\alpha}_{\beta} & O_{nm\times nm(m+1)/2} & O_{nm\times nm^2} & O_{nm\times n} & O_{nm\times \epsilon m} & O_{nm\times \epsilon m^2} \\ O_{nm(m+1)/2\times nm} & -\delta^i_j \otimes I_{m(m+1)/2} & O_{nm(m+1)/2\times nm^2} & O_{nm(m+1)/2\times n} & O_{nm(m+1)/2\times \epsilon m} & O_{nm(m+1)/2\times \epsilon m^2} \\ O_{nm^2\times nm} & O_{nm^2\times nm(m+1)/2} & \delta^i_j \otimes I_{m^2} & -\dfrac{1}{m}(\delta^i_j \otimes \mathrm{vec}(\delta^{\eta}_{\beta})) & O_{nm^2\times \epsilon m} & O_{nm^2\times \epsilon m^2} \\ O_{\epsilon m\times nm} & O_{\epsilon m\times nm(m+1)/2} & O_{\epsilon m\times nm^2} & O_{\epsilon m\times n} & O_{\epsilon m\times \epsilon m} & O_{\epsilon m\times \epsilon m^2} \\ O_{\epsilon m^2\times nm} & O_{\epsilon m^2\times nm(m+1)/2} & O_{\epsilon m^2\times nm^2} & O_{\epsilon m^2\times n} & O_{\epsilon m^2\times \epsilon m} & O_{\epsilon m^2\times \epsilon m^2} \end{pmatrix}$$

and, respectively, we have

$${}^{\Psi}J = \begin{pmatrix} O_{nm\times nm} & O_{nm\times nm(m+1)/2} & O_{nm\times nm^2} & O_{nm\times n} & O_{nm\times \epsilon m} & O_{nm\times \epsilon m^2} \\ O_{nm(m+1)/2\times nm} & O_{nm(m+1)/2\times nm(m+1)/2} & O_{nm(m+1)/2\times nm^2} & O_{nm(m+1)/2\times n} & O_{nm(m+1)/2\times \epsilon m} & O_{nm(m+1)/2\times \epsilon m^2} \\ O_{nm^2\times nm} & O_{nm^2\times nm(m+1)/2} & O_{nm^2\times nm^2} & O_{nm^2\times n} & O_{nm^2\times \epsilon m} & O_{nm^2\times \epsilon m^2} \\ O_{\epsilon m\times nm} & O_{\epsilon m\times nm(m+1)/2} & O_{\epsilon m\times nm^2} & O_{\epsilon m\times n} & -\delta^{\kappa}_{r} \otimes \delta^{\alpha}_{\beta} & O_{\epsilon m\times \epsilon m^2} \\ O_{\epsilon m^2\times nm} & O_{\epsilon m^2\times nm(m+1)/2} & O_{\epsilon m^2\times nm^2} & O_{\epsilon m^2\times n} & O_{\epsilon m^2\times \epsilon m} & \delta^{\kappa}_{r} \otimes I_{m^2} \end{pmatrix}.$$

Note that in this case, the dimension of $\left({}^{x}J + {}^{\Psi}J\right)$ is $(nm + nm(m+1)/2 + nm^2 + \epsilon m + \epsilon m^2, nm + nm(m+1)/2 + nm^2 + n + \epsilon m + \epsilon m^2)$.

Remark 2.5 For multi-time evolutions determined by the Euler–Lagrange PDEs of the anti-trace type expressed in (6)–(8), the Hamiltonian is not conserved in general (see (i), and (ii), below). However, the energy-momentum tensor is conserved for both first-order and second-order autonomous Lagrangians.

(i) *Euler–Lagrange PDEs of the anti-trace type generated by Lagrangians of first-order*
Taking the C^2-class multi-dimensional autonomous Lagrangian of first-order into account, we have

$$L\left(x(t), x_\alpha(t), \Psi(t), \Psi_\alpha(t), \theta(t)\right),$$

associated with the controlled variational problem

$$\int_{\Omega_{0,t_0}} W\left(x(t), \Psi(t), \Psi_\alpha(t)\right) dt^1 \cdots dt^m \to \text{extremize},$$

$$X^i_\alpha\left(x(t), x_\alpha(t), \Psi(t), \Psi_\alpha(t)\right) = 0,\ i = \overline{1,n},\ \alpha = \overline{1,m},$$

$$x(t)|_{\partial\Omega_{0,t_0}} = \text{given}.$$

Here, $\theta = (\theta^\alpha_i) : \Omega_{0,t_0} \to \mathbb{R}^{nm}$ and the auxiliary Lagrangian L is computed as follows:

$$L\left(x(t), x_\alpha(t), \Psi(t), \theta(t)\right) = W\left(x(t), \Psi(t), \Psi_\alpha(t)\right) + \theta^\alpha_i X^i_\alpha\left(x(t), \Psi(t), x_\alpha(t), \Psi_\alpha(t)\right).$$

The associated Hamiltonian function is

$$H\left(x, u, \Psi, s, \theta\right) = x^i_\eta u^\eta_i + \Psi^\kappa_\eta s^\eta_\kappa - L,$$

and the system of Hamilton PDEs (24)–(29) is reduced to

$$\frac{\partial x^i}{\partial t^\beta}(t) = \frac{\partial H}{\partial u^\beta_i}\left(x, u, \Psi, s, \theta\right),$$

$$D_\beta u^\eta_i(t) = -\frac{\delta^\eta_\beta}{m}\frac{\partial H}{\partial x^i}\left(x, u, \Psi, s, \theta\right),$$

$$\frac{\partial \Psi^\kappa}{\partial t^\beta}(t) = \frac{\partial H}{\partial s^\beta_\kappa}\left(x, u, \Psi, s, \theta\right),$$

$$D_\beta s^\eta_\kappa(t) = -\frac{\delta^\eta_\beta}{m}\frac{\partial H}{\partial \Psi^\kappa}\left(x, u, \Psi, s, \theta\right),$$

$$\frac{\partial H}{\partial \theta^{\pi\zeta}_i}\left(x, u, \Psi, s, \theta\right) = 0.$$

In this way, we have

$$D_\rho H = \frac{\partial H}{\partial x^i}\frac{\partial x^i}{\partial t^\rho} + \frac{\partial H}{\partial u_i^\eta}\frac{\partial u_i^\eta}{\partial t^\rho} + \frac{\partial H}{\partial \Psi^\kappa}\frac{\partial \Psi^\kappa}{\partial t^\rho} + \frac{\partial H}{\partial s_\kappa^\eta}\frac{\partial s_\kappa^\eta}{\partial t^\rho} + \frac{\partial H}{\partial \theta_i^\alpha}\frac{\partial \theta_i^\alpha}{\partial t^\eta},$$

and by using the Hamilton-type PDEs, it becomes

$$D_\rho H = \left[\frac{\partial H}{\partial x^i}\frac{\partial x^i}{\partial t^\eta} - \frac{1}{m}\frac{\partial x^i}{\partial t^\eta}\frac{\partial H}{\partial x^i}\right] + \left[\frac{\partial H}{\partial \Psi^\kappa}\frac{\partial \Psi^\kappa}{\partial t^\eta} - \frac{1}{m}\frac{\partial \Psi^\kappa}{\partial t^\eta}\frac{\partial H}{\partial \Psi^\kappa}\right].$$

In particular, the systems modelled by one evolution parameter ($m = 1$), have $\dfrac{\partial}{\partial t}H = 0$ for all functions x, Ψ, θ that satisfy the classical Euler–Lagrange PDEs generated by Lagrangian of first-order

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0,$$

$$\frac{\partial L}{\partial \Psi^\kappa}\delta_\beta^\eta - \frac{d}{dt}\frac{\partial L}{\partial \dot{\Psi}^\kappa} = 0,$$

$$\frac{\partial L}{\partial \theta_i^\alpha} = 0.$$

Next, we investigate if the energy-momentum tensor Θ_β^α is conserved. Recalling that this point addresses first-order autonomous Lagrangians, we have

$$\Theta_\beta^\alpha = u_i^\alpha x_\beta^i + s_\kappa^\alpha \Psi_\beta^\kappa - L\delta_\beta^\alpha.$$

Accordingly,

$$D_\ell \Theta_\beta^\alpha = \frac{\partial u_i^\alpha}{\partial t^\ell}x_\beta^i + u_i^\alpha \frac{\partial x_\beta^i}{\partial t^\ell} + \frac{\partial s_\kappa^\alpha}{\partial t^\ell}\Psi_\beta^\kappa + s_\kappa^\alpha \frac{\partial \Psi_\beta^\kappa}{\partial t^\ell}$$

$$-\frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\ell}\delta_\beta^\alpha - \frac{\partial L}{\partial x_\xi^i}\frac{\partial x_\xi^i}{\partial t^\ell}\delta_\beta^\alpha - \frac{\partial L}{\partial \Psi^\kappa}\frac{\partial \Psi^\kappa}{\partial t^\ell}\delta_\beta^\alpha - \frac{\partial L}{\partial \Psi_\xi^\kappa}\frac{\partial \Psi_\xi^\kappa}{\partial t^\ell}\delta_\beta^\alpha - \frac{\partial L}{\partial \theta_i^{\pi\zeta}}\frac{\partial \theta_i^{\pi\zeta}}{\partial t^\ell}\delta_\beta^\alpha$$

$$= -\frac{\delta_\ell^\alpha}{m}\frac{\partial H}{\partial x^i}x_\beta^i + u_i^\alpha \frac{\partial x_\beta^i}{\partial t^\ell} - \frac{\delta_\ell^\alpha}{m}\frac{\partial H}{\partial \Psi^\kappa}\Psi_\beta^\kappa + s_\kappa^\alpha \frac{\partial \Psi_\beta^\kappa}{\partial t^\ell} - \frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\ell}\delta_\beta^\alpha - u_i^\xi \frac{\partial x_\xi^i}{\partial t^\ell}\delta_\beta^\alpha - \frac{\partial L}{\partial \Psi^\kappa}\frac{\partial \Psi^\kappa}{\partial t^\ell}\delta_\beta^\alpha - s_\kappa^\xi \frac{\partial \Psi_\xi^\kappa}{\partial t^\ell}\delta_\beta^\alpha,$$

with

$$\frac{\partial H}{\partial x^i} = -\frac{\partial L}{\partial x^i}, \quad \frac{\partial H}{\partial \Psi^\kappa} = -\frac{\partial L}{\partial \Psi^\kappa}.$$

Subsequently, we contract with δ_ℓ^α in the right-hand side expression: we set $\eta = \ell$, followed by summing over η, finding

$$\sum_\alpha D_\alpha \Theta_\beta^\alpha = -\frac{\partial H}{\partial x^i}x_\beta^i + u_i^\alpha \frac{\partial x_\beta^i}{\partial t^\alpha} - \frac{\partial H}{\partial \Psi^\kappa}\Psi_\beta^\kappa + s_\kappa^\alpha \frac{\partial \Psi_\gamma^\kappa}{\partial t^\alpha} - \frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\beta} - u_i^\alpha \frac{\partial x_\xi^i}{\partial t^\alpha} - \frac{\partial L_\beta^\alpha}{\partial \Psi^\kappa}\frac{\partial \Psi^\kappa}{\partial t^\beta} - s_\kappa^\alpha \frac{\partial \Psi_\xi^\kappa}{\partial t^\alpha}.$$

In this way, we show that

$$\sum_\alpha D_\alpha \Theta_\beta^\alpha = 0,$$

which is interpreted as a conservation law involving the energy-momentum tensor associated with multi-dimensional first-order autonomous Lagrangians and anti-trace Euler–Lagrange PDEs.

(ii) *Euler–Lagrange PDEs of the anti-trace type generated by second-order Lagrangians*
In this case, the divergence of H is computed, taking into account that $H = H(x, u, v, \Psi, s, \theta)$, and processed as follows:

$$D_\rho H = \frac{\partial H}{\partial x^i}\frac{\partial x^i}{\partial t^\rho} + \frac{\partial H}{\partial u_i^\eta}\frac{\partial u_i^\eta}{\partial t^\rho} + \frac{1}{\mathcal{Z}(a,b)}\frac{\partial H}{\partial v_i^{ab}}\frac{\partial v_i^{ab}}{\partial t^\rho} + \frac{\partial H}{\partial \Psi^\kappa}\frac{\partial \Psi^\kappa}{\partial t^\rho} + \frac{\partial H}{\partial s_\kappa^\eta}\frac{\partial s_\kappa^\eta}{\partial t^\rho} + \frac{\partial H}{\partial \theta_i^{\pi\zeta}}\frac{\partial \theta_i^{\pi\zeta}}{\partial t^\rho}$$

$$= \left[\frac{\partial H}{\partial x^i}\frac{\partial x^i}{\partial t^\eta} + \frac{\partial x^i}{\partial t^\eta}\left(\frac{1}{m\cdot\mathcal{Z}(a,b)}D^2_{ab}v_i^{ab} - \frac{1}{m}\frac{\partial H}{\partial x^i}\right) + \frac{1}{\mathcal{Z}(a,b)}\frac{\partial^2 x^i}{\partial t^a \partial t^b}\frac{\partial v_i^{ab}}{\partial t^\rho}\right] + \left[\frac{\partial H}{\partial \Psi^\kappa}\frac{\partial \Psi^\kappa}{\partial t^\eta} - \frac{1}{m}\frac{\partial \Psi^\kappa}{\partial t^\eta}\frac{\partial H}{\partial \Psi^\kappa}\right].$$

One can notice that the Hamiltonian is conserved, for instance, when the dynamics are described by only one evolution parameter (i.e., $m = 1$) and the Lagrangian L is affine with respect to the second-order derivative of x (i.e., $\ddot{x}$). However, the sufficient conditions constitute another direction for further investigation of conservation, which will be addressed in a future research paper.

Now, let us explore whether the energy-momentum tensor Θ_β^α is conserved. Here

$$\Theta_\beta^\alpha(x,u,v,\Psi,s,\theta) = \left[u_i^\alpha - \frac{1}{\mathcal{Z}(a,\alpha)}\frac{\partial v_i^{a\alpha}}{\partial t^a}\right]x_\beta^i + s_\kappa^\alpha \Psi_\beta^\kappa + \frac{1}{\mathcal{Z}(a,\alpha)}x_{\beta a}^i v_i^{a\alpha} - L\delta_\beta^\alpha$$

$$\Longrightarrow D_\ell \Theta_\beta^\alpha = \frac{\partial u_i^\alpha}{\partial t^\ell}x_\beta^i + u_i^\alpha\frac{\partial x_\beta^i}{\partial t^\ell} + \frac{\partial s_\kappa^\alpha}{\partial t^\ell}\Psi_\beta^\kappa + s_\kappa^\alpha\frac{\partial \Psi_\beta^\kappa}{\partial t^\ell}$$
$$-\frac{1}{\mathcal{Z}(a,\alpha)}\frac{\partial}{\partial t^\ell}\left(\frac{\partial v_i^{a\alpha}}{\partial t^a}\right)x_\beta^i - \frac{1}{\mathcal{Z}(a,\alpha)}\frac{\partial v_i^{a\alpha}}{\partial t^a}\frac{\partial x_\beta^i}{\partial t^\ell} + \frac{1}{\mathcal{Z}(a,\alpha)}\frac{\partial x_{\beta a}^i}{\partial t^\ell}v_i^{a\alpha} + \frac{1}{\mathcal{Z}(a,\alpha)}x_{\beta a}^i\frac{\partial v_i^{a\alpha}}{\partial t^\ell}$$
$$-\frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\ell}\delta_\beta^\alpha - \frac{\partial L}{\partial x_\xi^i}\frac{\partial x_\xi^i}{\partial t^\ell}\delta_\beta^\alpha - \frac{\partial L}{\partial \Psi^\kappa}\frac{\partial \Psi^\kappa}{\partial t^\ell}\delta_\beta^\alpha - \frac{\partial L}{\partial \Psi_\xi^\kappa}\frac{\partial \Psi_\xi^\kappa}{\partial t^\ell}\delta_\beta^\alpha - \frac{1}{\mathcal{Z}(\pi,\zeta)}\frac{\partial L}{\partial x_{\pi\zeta}^i}\frac{\partial x_{\pi\zeta}^i}{\partial t^\ell}\delta_\beta^\alpha - \frac{\partial L}{\partial \theta_i^{\pi\zeta}}\frac{\partial \theta_i^{\pi\zeta}}{\partial t^\ell}\delta_\beta^\alpha$$
$$= \delta_\ell^\alpha\left[\frac{1}{m\cdot\mathcal{Z}(a,b)}D^2_{ab}v_i^{ab} - \frac{1}{m}\frac{\partial H}{\partial x^i}\right]x_\beta^i + u_i^\alpha\frac{\partial x_\beta^i}{\partial t^\ell} - \frac{\delta_\ell^\alpha}{m}\frac{\partial H}{\partial \Psi^\kappa}\Psi_\beta^\kappa + s_\kappa^\alpha\frac{\partial \Psi_\beta^\kappa}{\partial t^\ell}$$
$$-\frac{1}{\mathcal{Z}(a,\alpha)}\frac{\partial}{\partial t^\ell}\left(\frac{\partial v_i^{a\alpha}}{\partial t^a}\right)x_\beta^i - \frac{1}{\mathcal{Z}(a,\alpha)}\frac{\partial v_i^{a\alpha}}{\partial t^a}\frac{\partial x_\beta^i}{\partial t^\ell} + \frac{1}{\mathcal{Z}(a,\alpha)}\frac{\partial x_{\beta a}^i}{\partial t^\ell}v_i^{a\alpha} + \frac{1}{\mathcal{Z}(a,\alpha)}x_{\beta a}^i\frac{\partial v_i^{a\alpha}}{\partial t^\ell}$$
$$-\frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\ell}\delta_\beta^\alpha - \frac{\partial L}{\partial x_\xi^i}\frac{\partial x_\xi^i}{\partial t^\ell}\delta_\beta^\alpha - \frac{\partial L}{\partial \Psi^\kappa}\frac{\partial \Psi^\kappa}{\partial t^\ell}\delta_\beta^\alpha - \frac{\partial L}{\partial \Psi_\xi^\kappa}\frac{\partial \Psi_\xi^\kappa}{\partial t^\ell}\delta_\beta^\alpha - \frac{1}{\mathcal{Z}(\pi,\zeta)}v_i^{\pi\zeta}\frac{\partial x_{\pi\zeta}^i}{\partial t^\ell}\delta_\beta^\alpha.$$

In the end, adopting the same procedure as in the case of the -order Lagrangian, we deduce

$$\sum_\alpha D_\alpha \Theta_\beta^\alpha = 0.$$

The conservation related to the tensor Θ_β^α holds even for multi-dimensional second-order autonomous Lagrangians.

Remark 2.6 It is well known that, under suitable invexity or pseudoinvexity assumptions, variational inequalities ensure the existence of solutions to optimization problems. However, this topic lies beyond the scope of the present paper and will be addressed in future research. Moreover, the derivation of sufficient optimality conditions for the problems considered here constitutes another promising direction for further investigation.

3. Lagrangian and Hamiltonian tensors

In this section, we extend the C^3-class multi-dimensional Lagrangian of second-order $L\Big(x(t), x_{\xi}(t), x_{\pi\zeta}(t), \Psi(t), \Psi_{\xi}(t), \theta(t), t\Big)$ to a *Lagrangian tensor field*,

$$L^{\alpha}_{\beta}\Big(x(t), x_{\xi}(t), x_{\pi\zeta}(t), \Psi(t), \Psi_{\xi}(t), \theta(t), t\Big), \quad \alpha, \beta, \xi, \pi, \zeta \in \{1, \dots, m\},$$

together with the state equations (see Remark 3.1 formulated below)

$$X^{\alpha\alpha i}_{\pi\zeta}\Big(x(t), x_{\alpha}(t), x_{\pi\zeta}(t), \Psi(t), \Psi_{\alpha}(t), t\Big) = 0$$

and the Lagrange multipliers

$$\theta = (\theta^{\pi\zeta}_{\alpha\alpha i}) : \Omega_{0,t_0} \to \mathbb{R}^{nm^3}.$$

We assume that the trace of the Lagrangian tensor field L^{α}_{β} is mL, as follows:

$$\sum_{\alpha} L^{\alpha}_{\alpha} = mL.$$

In addition, the functions L^{α}_{β} determines anti-trace Euler–Lagrange PDEs (30)–(32) given below,

$$\frac{\delta^{\eta}_{\lambda}}{m}\frac{\partial L^{\alpha}_{\beta}}{\partial x^{i}} - D_{\lambda}\frac{\partial L^{\alpha}_{\beta}}{\partial x^{i}_{\eta}} + \frac{\delta^{\eta}_{\lambda}}{m \cdot \mathcal{Z}(a,b)} D^{2}_{ab}\frac{\partial L^{\alpha}_{\beta}}{\partial x^{i}_{ab}} = 0, \quad i \in \{1, \dots, n\} \quad \eta, \lambda, a, b \in \{1, \dots, m\}, \tag{30}$$

$$\frac{\delta^{\eta}_{\lambda}}{m}\frac{\partial L^{\alpha}_{\beta}}{\partial \Psi^{\kappa}} - D_{\lambda}\frac{\partial L^{\alpha}_{\beta}}{\partial \Psi^{\kappa}_{\eta}} = 0, \quad \kappa \in \{1, \dots, \epsilon\}, \tag{31}$$

$$\frac{\partial L^{\alpha}_{\alpha}}{\partial \theta^{\pi\zeta}_{\alpha\alpha i}} = 0, \quad \pi, \zeta \in \{1, \dots, m\} \text{ (see Remark 3.1)}. \tag{32}$$

Using the component L^{α}_{β}, we extend the generalized multi-momenta as follows:

$$u^{\alpha\eta}_{\beta i}(t) = \frac{\partial}{\partial x^{i}_{\eta}} L^{\alpha}_{\beta}\Big(x(t), x_{\xi}(t), x_{\pi\zeta}(t), \Psi(t), \Psi_{\xi}(t), \theta(t), t\Big),$$

$$v^{ab}_{i}(t)\delta^{\alpha}_{\beta} = \frac{\partial}{\partial x^{i}_{ab}} L^{\alpha}_{\beta}\Big(x(t), x_{\xi}(t), x_{\pi\zeta}(t), \Psi(t), \Psi_{\xi}(t), \theta(t), t\Big),$$

$$s^{\alpha\eta}_{\beta\kappa}(t) = \frac{\partial}{\partial \Psi^{\kappa}_{\eta}} L^{\alpha}_{\beta}\Big(x(t), x_{\xi}(t), x_{\pi\zeta}(t), \Psi(t), \Psi_{\xi}(t), \theta(t), t\Big).$$

Remark 3.1 According to the definition of second-order momenta $\Big(v^{ab}_{i}\delta^{\alpha}_{\beta} = \frac{\partial}{\partial x^{i}_{ab}} L^{\alpha}_{\beta}\Big)$, $\frac{\partial}{\partial x^{i}_{ab}} L^{\alpha}_{\beta} = 0$ for $\alpha \neq \beta$. Consequently, L^{α}_{β} depends on x^{i}_{ab} if and only if $\alpha = \beta$. Equivalently, the state equations are formulated only for the diagonal tensor components L^{α}_{α}, namely

$$X^{\alpha\alpha i}_{\pi\zeta}\Big(x(t), x_{\alpha}(t), x_{\pi\zeta}(t), \Psi(t), \Psi_{\alpha}(t), t\Big) = 0.$$

No state equation (hence no multiplier) is associated with the components $\alpha \neq \beta$. For instance, a choice compatible with $\sum_\alpha L^\alpha_\alpha = mL$ is

$$L^\alpha_\beta\left(x(t), x_\xi(t), x_{\pi\zeta}(t), \Psi(t), \Psi_\xi(t), \theta(t), t\right) = W^\alpha_\beta\left(x(t), x_\xi(t), \Psi(t), \Psi_\xi(t), t\right)$$

$$+\delta^\alpha_\beta\, \theta^{\pi\zeta}_i X^i_{\pi\zeta}\left(x(t), x_\xi(t), x_{\pi\zeta}(t), \Psi(t), \Psi_\xi(t), t\right),$$

with the same multipliers $\theta^{\pi\zeta}_i$ as in Section 2, for every $\alpha = \beta$. In this case, $\theta \in \mathbb{R}^{nm^2}$ instead of $\mathbb{R}^{nm^3}$, and equation (32) is understood for the diagonal components.

For $m > 1$, the correspondence between $\left(x, x_\xi, x_{\pi\zeta}, \Psi, \Psi_\xi, \theta, t\right)$ and $(x, u, v, \Psi, s, \theta, t)$ is no longer bijective. However, we propose a framework in which the *Hamiltonian tensor field* depends on all variables $\left(x, x_\xi, u, x_{\pi\zeta}, v, \Psi, \Psi_\xi, s, \theta, t\right)$, and duality is not assumed, that is

$$H^\alpha_\beta\left(x, x_\xi, u, x_{\pi\zeta}, v, \Psi, \Psi_\xi, s, \theta, t\right) = u^{\alpha\eta}_{\beta i}(t)x^i_\eta(t) + \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,b)}v^{ab}_i(t)x^i_{ab}(t) + s^{\alpha\eta}_{\beta\kappa}(t)\Psi^\kappa_\eta(t)$$

$$-L^\alpha_\beta\left(x(t), x_\xi(t), x_{\pi\zeta}(t), \Psi(t), \Psi_\xi(t), \theta(t), t\right).$$

We introduce a *componentwise energy-momentum tensor*, defined here by

$$\Theta^{\alpha\eta}_{\beta\gamma}\left(x, x_\xi, u, x_{\pi\zeta}, v, \Psi, \Psi_\xi, s, \theta, t\right) = \left[u^{\alpha\eta}_{\beta i}(t) - \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\frac{\partial v^{a\eta}_i}{\partial t^a}(t)\right]x^i_\gamma(t)$$

$$+s^{\alpha\eta}_{\beta\kappa}(t)\Psi^\kappa_\gamma(t) + \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\, x^i_{\gamma a}(t)v^{a\eta}_i(t) - L^\alpha_\beta\left(x, x_\xi, x_{\pi\zeta}, \Psi, \Psi_\xi, s, \theta, t\right)\delta^\eta_\gamma, \quad a, \alpha, \beta, \eta, \gamma = 1, \ldots, m.$$

For each pair (α, β), we refer the reader to see $\Theta^{\alpha\eta}_{\beta\gamma}$ as the canonical energy-momentum tensor of the Lagrangian L^α_β.

Theorem 3.1 (Generalized Hamilton PDEs of the anti-trace type) *If the functions $x(\cdot)$, $\Psi(\cdot)$, and $\theta(\cdot)$ are solutions for Euler–Lagrange PDEs of the anti-trace type expressed in* (30)–(32)*, and $u(\cdot) = \left(u^{\alpha\eta}_{\beta i}(\cdot)\right)$, $v(\cdot) = \left(v^{ab}_i(\cdot)\right)$, and $s(\cdot) = \left(s^{\alpha\eta}_{\beta\kappa}(\cdot)\right)$ are defined as above, then the tuple $\left(x(\cdot), x_\xi(\cdot), u(\cdot), x_{\pi\zeta}(\cdot), v(\cdot), \Psi(\cdot), \Psi_\xi(\cdot), s(\cdot), \theta(\cdot)\right)$ is a solution of the Hamiltonian-type differential system*

$$\frac{\partial x^i}{\partial t^\eta}(t) = \frac{\partial}{\partial u^{\alpha\eta}_{\beta i}}H^\alpha_\beta\left(x, x_\xi, u, x_{\pi\zeta}, v, \Psi, \Psi_\xi, s, \theta, t\right) \tag{33}$$

(without summation over repeated indices α, β),

$$\frac{\delta^\alpha_\beta}{\mathcal{Z}(a,b)}\frac{\partial^2 x^i}{\partial t^a \partial t^b}(t) = \frac{\partial}{\partial v^{ab}_i}H^\alpha_\beta\left(x, x_\xi, u, x_{\pi\zeta}, v, \Psi, \Psi_\xi, s, \theta, t\right), \tag{34}$$

$$D_\lambda u^{\alpha\eta}_{\beta i}(t) - \frac{\delta^\alpha_\beta\delta^\eta_\lambda}{m\cdot\mathcal{Z}(a,b)}D^2_{ab}v^{ab}_i(t) = -\frac{\delta^\eta_\lambda}{m}\frac{\partial H^\alpha_\beta}{\partial x^i}\left(x, x_\xi, u, x_{\pi\zeta}, v, \Psi, \Psi_\xi, s, \theta, t\right) \tag{35}$$

(summation over repeated indices a, b is assumed),

$$\frac{\partial \Psi^\kappa}{\partial t^\eta}(t) = \frac{\partial}{\partial s^{\alpha\eta}_{\beta\kappa}}H^\alpha_\beta\left(x, x_\xi, u, x_{\pi\zeta}, v, \Psi, \Psi_\xi, s, \theta, t\right) \tag{36}$$

(without summation over repeated indices α, β),

$$D_\lambda s^{\alpha\eta}_{\beta\kappa}(t) = -\frac{\delta^\eta_\lambda}{m}\frac{\partial H^\alpha_\beta}{\partial \Psi^\kappa}\left(x, x_\xi, u, x_{\pi\zeta}, v, \Psi, \Psi_\xi, s, \theta, t\right), \tag{37}$$

$$\frac{\partial H^\alpha_\beta}{\partial \theta^{\pi\zeta}_{\alpha\beta i}}\left(x, x_\xi, u, x_{\pi\zeta}, v, \Psi, \Psi_\xi, s, \theta, t\right) = 0. \tag{38}$$

Proof. The proof proceeds as in Theorem 2.2. First, we show that

$$\frac{\partial}{\partial x^i}H^\alpha_\beta\left(x, x_\xi, u, x_{\pi\zeta}, v, \Psi, \Psi_\xi, s, \theta, t\right) = -\frac{\partial}{\partial x^i}L^\alpha_\beta\left(x, x_\xi, x_{\pi\zeta}, \Psi, \Psi_\xi, \theta, t\right),$$

$$\frac{\partial}{\partial \Psi^\kappa}H^\alpha_\beta\left(x, x_\xi, u, x_{\pi\zeta}, v, \Psi, \Psi_\xi, s, \theta, t\right) = -\frac{\partial}{\partial \Psi^\kappa}L^\alpha_\beta\left(x, x_\xi, x_{\pi\zeta}, \Psi, \Psi_\xi, \theta, t\right).$$

Now, by direct computation (taking into account that $x, x_\xi, u, x_{\pi\zeta}, v, \Psi, \Psi_\xi, s, \theta, t$ are independent variables), we get

$$\frac{\partial}{\partial u^{\alpha\eta}_{\beta i}}H^\alpha_\beta = x^i_\eta,$$

$$\frac{\partial}{\partial v^{ab}_i}H^\alpha_\beta = \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,b)}x^i_{ab}$$

and

$$\frac{\partial}{\partial s^{\alpha\eta}_{\beta\kappa}}H^\alpha_\beta = \Psi^\kappa_\eta.$$

In addition, the Euler–Lagrange PDEs of the anti-trace type given in (30)–(32) involve

$$D_\lambda u^{\alpha\eta}_{\beta i} - \frac{\delta^\alpha_\beta\delta^\eta_\lambda}{m\cdot\mathcal{Z}(a,b)}D^2_{ab}v^{ab}_i = \frac{\delta^\eta_\lambda}{m}\frac{\partial L^\alpha_\beta}{\partial x^i} = -\frac{\delta^\eta_\lambda}{m}\frac{\partial H^\alpha_\beta}{\partial x^i},$$

$$D_\lambda s^{\alpha\eta}_{\beta\kappa} = \frac{\delta^\eta_\lambda}{m}\frac{\partial L^\alpha_\beta}{\partial \Psi^\kappa} = -\frac{\delta^\eta_\lambda}{m}\frac{\partial H^\alpha_\beta}{\partial \Psi^\kappa}$$

and

$$\frac{\partial H^\alpha_\beta}{\partial \theta^{\pi\zeta}_{\alpha\beta i}} = 0.$$

Finally, we obtained the desired result, and the proof is concluded.

We now state the compatibility conditions under which the generalized anti-trace Hamilton PDEs given in (33)–(38) are formulated.

- L^α_β is of class C^3; $x \in C^4$, $\Psi \in C^2$.
- <u>*Trace.*</u> $\sum_{\alpha=1}^m L^\alpha_\alpha = mL$.
- <u>*Second-order momenta.*</u> The symmetric functions $v^{ab}_i = v^{ba}_i$ exist such that

$$\frac{\partial L^\alpha_\beta}{\partial x^i_{ab}} = \delta^\alpha_\beta\, v^{ab}_i = \delta^\beta_\alpha\, v^{ba}_i = \frac{\partial L^\alpha_\beta}{\partial x^i_{ba}}.$$

In particular, the components with $\alpha \neq \beta$ do not depend on x_{ab} and, together with $\sum_{\alpha=1}^{m} L_{\alpha}^{\alpha} = mL$, give $v_i^{ab} = \dfrac{\partial L}{\partial x_{ab}^i}$.

• *Compatibility of the velocity variables.*

$$x_{\xi}^{i} = \frac{\partial x^i}{\partial t^{\xi}},\ x_{\pi\zeta}^{i} = \frac{\partial^2 x^i}{\partial t^{\pi}\partial t^{\zeta}},\ \Psi_{\xi}^{\kappa} = \frac{\partial \Psi^{\kappa}}{\partial t^{\xi}}$$

involves

$$\frac{\partial}{\partial t^a}\left(x_{\xi}^{i}\right) = \frac{\partial x^i}{\partial t^{\xi a}} = \frac{\partial x^i}{\partial t^{a\xi}} = \frac{\partial}{\partial t^{\xi}}\left(x_{a}^{i}\right),$$

and

$$\frac{\partial}{\partial t^a}\left(\Psi_{\kappa}^{\kappa}\right) = \frac{\partial \Psi^{\kappa}}{\partial t^{\xi a}} = \frac{\partial \Psi^{\kappa}}{\partial t^{a\xi}} = \frac{\partial}{\partial t^{\xi}}\left(\Psi_{a}^{\kappa}\right)$$

(compatibility between the first- and second-order derivatives and commutation of the mixed derivatives).

• *State equations.* State equations are given only for the diagonal tensor components

$$X_{\pi\zeta}^{\alpha\alpha i}\left(x(t), x_{\alpha}(t), x_{\pi\zeta}(t), \Psi(t), \Psi_{\alpha}(t), t\right) = 0.$$

• *Common solution.* The same tuple (x, Ψ, θ) solves the anti-trace Hamilton (33)–(38) (or Euler-Lagrange (30)–(32)) PDEs for every pair (α, β).

• *Complete integrability conditions.* The Lagrange 1-forms $L_{\beta}^{\alpha}\left(x(t), x_{\xi}(t), x_{\pi\zeta}(t), \Psi(t), \Psi_{\xi}(t), \theta(t), t\right)$ are completely integrable, namely $D_{\gamma}L_{\beta}^{\alpha} = D_{\beta}L_{\gamma}^{\alpha}$ for all α, β, γ.

Remark 3.2 The generalized anti-trace Hamilton PDEs (33)–(37) can be rewritten as follows:

$$-\delta_j^i \frac{\partial x^j}{\partial t^{\eta}}(t) + \frac{\partial H_{\beta}^{\alpha}}{\partial u_{\beta i}^{\alpha\eta}}\left(x, x_{\xi}, u, x_{\pi\zeta}, v, \Psi, \Psi_{\xi}, s, \theta, t\right) = 0,$$

$$-\frac{\delta_j^i \delta_{\beta}^{\alpha}}{\mathcal{Z}(a,b)} \frac{\partial^2 x^j}{\partial t^a \partial t^b}(t) + \frac{\partial H_{\beta}^{\alpha}}{\partial v_i^{ab}}\left(x, x_{\xi}, u, x_{\pi\zeta}, v, \Psi, \Psi_{\xi}, s, \theta, t\right) = 0,$$

$$\delta_j^i \frac{\partial u_{\beta j}^{\alpha\eta}}{\partial t^{\lambda}}(t) - \frac{\delta_{\beta}^{\alpha}\delta_j^i\delta_{\lambda}^{\eta}}{m \cdot \mathcal{Z}(a,b)} \frac{\partial^2 v_j^{ab}}{\partial t^a \partial t^b}(t) + \frac{\delta_{\lambda}^{\eta}}{m} \frac{\partial H_{\beta}^{\alpha}}{\partial x^i}\left(x, x_{\xi}, u, x_{\pi\zeta}, v, \Psi, \Psi_{\xi}, s, \theta, t\right) = 0,$$

$$-\delta_r^{\kappa} \frac{\partial \Psi^r}{\partial t^{\eta}}(t) + \frac{\partial}{\partial s_{\beta\kappa}^{\alpha\eta}} H_{\beta}^{\alpha}\left(x, x_{\xi}, u, x_{\pi\zeta}, v, \Psi, \Psi_{\xi}, s, \theta, t\right) = 0,$$

$$\delta_r^{\kappa} \frac{\partial s_{\beta r}^{\alpha\eta}}{\partial t^{\lambda}}(t) + \frac{\delta_{\lambda}^{\eta}}{m} \frac{\partial H_{\beta}^{\alpha}}{\partial \Psi^{\kappa}}\left(x, x_{\xi}, u, x_{\pi\zeta}, v, \Psi, \Psi_{\xi}, s, \theta, t\right) = 0.$$

Using the following block matrices (we recall that $\otimes$ denotes the Kronecker product), we establish, for each fixed pair (α, β), the compact (matrix) form of the generalized anti-trace Hamilton PDEs comprised in Remark 3.3. First, we define the $m^2 \times m^2$-dimensional matrix, Z, as follows:

$$Z := \operatorname{diag}\left(\frac{1}{\mathcal{Z}(a,b)}\right) = \begin{bmatrix} \frac{1}{\mathcal{Z}(1,1)} & 0 & \dots & 0 \\ 0 & \frac{1}{\mathcal{Z}(1,2)} & \dots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{\mathcal{Z}(m,m)} \end{bmatrix}.$$

In addition, we introduce

$${}^{x}J = \begin{pmatrix} -I_n \otimes I_m & O_{nm\times nm^2} & O_{nm\times nm^2} & O_{nm\times n} & O_{nm\times \epsilon m} & O_{nm\times \epsilon m^2} \\ O_{nm^2\times nm} & -\delta^{\alpha}_{\beta}\,(I_n \otimes Z) & O_{nm^2\times nm^2} & O_{nm^2\times n} & O_{nm^2\times \epsilon m} & O_{nm^2\times \epsilon m^2} \\ O_{nm^2\times nm} & O_{nm^2\times nm^2} & I_n \otimes I_{m^2} & -\dfrac{\delta^{\alpha}_{\beta}}{m}\,(I_n \otimes \operatorname{vec}(I_m)) & O_{nm^2\times \epsilon m} & O_{nm^2\times \epsilon m^2} \\ O_{\epsilon m\times nm} & O_{\epsilon m\times nm^2} & O_{\epsilon m\times nm^2} & O_{\epsilon m\times n} & O_{\epsilon m\times \epsilon m} & O_{\epsilon m\times \epsilon m^2} \\ O_{\epsilon m^2\times nm} & O_{\epsilon m^2\times nm^2} & O_{\epsilon m^2\times nm^2} & O_{\epsilon m^2\times n} & O_{\epsilon m^2\times \epsilon m} & O_{\epsilon m^2\times \epsilon m^2} \end{pmatrix},$$

$${}^{\Psi}J = \begin{pmatrix} O_{nm\times nm} & O_{nm\times nm^2} & O_{nm\times nm^2} & O_{nm\times n} & O_{nm\times \epsilon m} & O_{nm\times \epsilon m^2} \\ O_{nm^2\times nm} & O_{nm^2\times nm^2} & O_{nm^2\times nm^2} & O_{nm^2\times n} & O_{nm^2\times \epsilon m} & O_{nm^2\times \epsilon m^2} \\ O_{nm^2\times nm} & O_{nm^2\times nm^2} & O_{nm^2\times nm^2} & O_{nm^2\times n} & O_{nm^2\times \epsilon m} & O_{nm^2\times \epsilon m^2} \\ O_{\epsilon m\times nm} & O_{\epsilon m\times nm^2} & O_{\epsilon m\times nm^2} & O_{\epsilon m\times n} & -I_{\epsilon} \otimes I_m & O_{\epsilon m\times \epsilon m^2} \\ O_{\epsilon m^2\times nm} & O_{\epsilon m^2\times nm^2} & O_{\epsilon m^2\times nm^2} & O_{\epsilon m^2\times n} & O_{\epsilon m^2\times \epsilon m} & I_{\epsilon} \otimes I_{m^2} \end{pmatrix}$$

where $\delta^{\eta}_{\lambda} = I_m$ and $\delta^{i}_{j} = I_n$. Moreover, the operator vec is interpreted as in the Section 2, together with

$$\check{\nu} := (\check{\nu}_i) = \left(\frac{1}{\mathcal{Z}_{(}a,b)}\,\frac{\partial^2 v_i^{ab}}{\partial t^a \partial t^b}\right)$$

(with summation over the repeated indices a, b). The *compact (matrix) form of generalized anti-trace Hamilton PDEs* is given by

$$\left({}^{x}J + {}^{\Psi}J\right)\begin{pmatrix} \dfrac{\partial x}{\partial t} \\ \dfrac{\partial^2 x}{\partial t^2} \\ \dfrac{\partial u}{\partial t} \\ \check{\nu} \\ \dfrac{\partial \Psi}{\partial t} \\ \dfrac{\partial s}{\partial t} \end{pmatrix} = -\begin{pmatrix} \dfrac{\partial H^{\alpha}_{\beta}}{\partial u} \\ \dfrac{\partial H^{\alpha}_{\beta}}{\partial v} \\ \dfrac{1}{m}\dfrac{\partial H^{\alpha}_{\beta}}{\partial x} \otimes \operatorname{vec}(I_m) \\ \dfrac{\partial H^{\alpha}_{\beta}}{\partial s} \\ \dfrac{1}{m}\dfrac{\partial H^{\alpha}_{\beta}}{\partial \Psi} \otimes \operatorname{vec}(I_m) \end{pmatrix}.$$

Table 4 describes the building blocks, Table 5 gives the dimensions of the Kronecker products, and Table 6 gives the dimensions of the column vectors of the compact system.

Table 4. Description of block matrix structures incorporated in ${}^{x}J$ and ${}^{\Psi}J$ (tensor case).

Component	Interpretation	Dimension
I_n	Identity matrix (Kronecker delta)	(n, n)
I_ϵ	Identity matrix (Kronecker delta)	(ϵ, ϵ)
I_m, I_{m^2}	Identity matrices	(m, m), (m^2, m^2)
$\mathrm{vec}(I_m)$	Column vector, vectorization of I_m	$(m^2, 1)$
Z	Diagonal matrix	(m^2, m^2)
δ^α_β	Scalar factor (1 if $\alpha = \beta$, 0 otherwise)	$(1, 1)$

Table 5. Dimensions of the Kronecker products embedded in ${}^{x}J$ and ${}^{\Psi}J$ (tensor case).

Kronecker product	Dimension
$I_n \otimes I_m$	(nm, nm)
$I_n \otimes Z$	(nm^2, nm^2)
$I_n \otimes I_{m^2}$	(nm^2, nm^2)
$I_n \otimes \mathrm{vec}(I_m)$	(nm^2, n)
$I_\epsilon \otimes I_m$	$(\epsilon m, \epsilon m)$
$I_\epsilon \otimes I_{m^2}$	$(\epsilon m^2, \epsilon m^2)$

Table 6. Components of the left- and right-hand sides of the compact form for a fixed pair (α, β) (tensor case).

Block	Dimension
$\dfrac{\partial x}{\partial t}$	$(nm, 1)$
$\dfrac{\partial^2 x}{\partial t^2}$	$(nm^2, 1)$
$\dfrac{\partial u}{\partial t}$	$(nm^2, 1)$
$\check{v}$	$(n, 1)$
$\dfrac{\partial \Psi}{\partial t}$	$(\epsilon m, 1)$
$\dfrac{\partial s}{\partial t}$	$(\epsilon m^2, 1)$
$\dfrac{\partial H^\alpha_\beta}{\partial u}$	$(nm, 1)$
$\dfrac{\partial H^\alpha_\beta}{\partial v}$	$(nm^2, 1)$

Continued on next page

Block	Dimension
$\frac{1}{m}\frac{\partial H_\beta^\alpha}{\partial x}\otimes \mathrm{vec}(I_m)$	$(nm^2, 1)$
$\frac{\partial H_\beta^\alpha}{\partial s}$	$(\epsilon m, 1)$
$\frac{1}{m}\frac{\partial H_\beta^\alpha}{\partial \Psi}\otimes \mathrm{vec}(I_m)$	$(\epsilon m^2, 1)$

Consequently, we derive the following results:

• The matrix $({}^{x}J + {}^{\Psi}J)$ has the dimension $(nm(1+2m)+\epsilon m(1+m),\ nm+2nm^2+n+\epsilon m+\epsilon m^2)$;

• The vector of unknowns has the dimension $(nm+2nm^2+n+\epsilon m+\epsilon m^2,\ 1)$;

• The right-hand side has the dimension $(nm(1+2m)+\epsilon m(1+m),\ 1)$. One can notice that the dimensions are compatible with the matrix product. The dimensions do not depend on (α,β), so the tensor system consists of m^2 systems of the abovementioned form.

Remark 3.3 For autonomous Lagrangian tensor fields L_β^α, the associated Hamiltonians are not conserved, that is, $D_\eta H_\beta^\alpha \neq 0$. However, for the first- and second-order autonomous Lagrangian tensor fields, the divergence of the componentwise energy-momentum tensor $\Theta_{\beta\gamma}^{\alpha\eta}$ is null, as presented in the following. We want to emphasize that the computation of the divergence of the componentwise energy-momentum tensor $\Theta_{\beta\gamma}^{\alpha\eta}$ involve simple index changes the validity of the reported results.

(i) *Euler–Lagrange PDEs of the anti-trace type generated by Lagrangian tensor fields of first-order* Extending the C^2-class multi-dimensional autonomous Lagrangian of first-order $L\Big(x(t), x_\xi(t), \Psi(t), \Psi_\xi(t), \theta(t)\Big)$ (see (i) from Remark 2.5) to the Lagrangian tensor field $L_\beta^\alpha\Big(x(t), x_\xi(t), \Psi(t), \Psi_\xi(t), \theta(t)\Big)$, the associated Hamiltonian tensor field H_β^α becomes

$$H_\beta^\alpha\left(x, x_\xi, u, \Psi, \Psi_\xi, s, \theta\right) = u_{\beta i}^{\alpha\eta}(t)x_\eta^i(t) + s_{\beta\kappa}^{\alpha\eta}(t)\Psi_\eta^\kappa(t)$$

$$-L_\beta^\alpha\left(x(t), x_\xi(t), x_{\pi\zeta}(t), \Psi(t), \Psi_\xi(t), \theta(t)\right),$$

and the corresponding system of generalized anti-trace Hamilton PDEs is the following:

$$\frac{\partial x^i}{\partial t^\eta}(t) = \frac{\partial}{\partial u_{\beta i}^{\alpha\eta}}H_\beta^\alpha\left(x, x_\xi, u, \Psi, \Psi_\xi, s, \theta\right)$$

(without summation over repeated indices α,β),

$$D_\lambda u_{\beta i}^{\alpha\eta}(t) = -\frac{\delta_\lambda^\eta}{m}\frac{\partial H_\beta^\alpha}{\partial x^i}\left(x, x_\xi, u, \Psi, \Psi_\xi, s, \theta\right)$$

(summation over repeated indices a, b is assumed),

$$\frac{\partial \Psi^\kappa}{\partial t^\eta}(t) = \frac{\partial}{\partial s_{\beta\kappa}^{\alpha\eta}}H_\beta^\alpha\left(x, x_\xi, u, \Psi, \Psi_\xi, s, \theta\right)$$

(without summation over repeated indices α, β),

$$D_\lambda s^{\alpha\eta}_{\beta\kappa}(t) = -\frac{\delta^\eta_\lambda}{m}\frac{\partial H^\alpha_\beta}{\partial \Psi^\kappa}\Big(x, x_\xi, u, \Psi, \Psi_\xi, s, \theta\Big),$$

$$\frac{\partial H^\alpha_\beta}{\partial \theta^{\pi\zeta}_{\alpha\beta i}}\Big(x, x_\xi, u, \Psi, \Psi_\xi, s, \theta\Big) = 0.$$

To study the conservation of H^α_β, we calculate

$$D_\ell H^\alpha_\beta = \frac{\partial H^\alpha_\beta}{\partial x^i}\frac{\partial x^i}{\partial t^\ell} + \frac{\partial H^\alpha_\beta}{\partial x^i_\xi}\frac{\partial x^i_\xi}{\partial t^\ell} + \frac{\partial H^\alpha_\beta}{\partial u^{\alpha\eta}_{\beta i}}\frac{\partial u^{\alpha\eta}_{\beta i}}{\partial t^\ell} + \frac{\partial H^\alpha_\beta}{\partial \Psi^\kappa}\frac{\partial \Psi^\kappa}{\partial t^\ell} + \frac{\partial H^\alpha_\beta}{\partial \Psi^\kappa_\xi}\frac{\partial \Psi^\kappa_\xi}{\partial t^\ell} + \frac{\partial H^\alpha_\beta}{\partial s^{\alpha\eta}_{\beta\kappa}}\frac{\partial s^{\alpha\eta}_{\beta\kappa}}{\partial t^\ell} + \frac{\partial H^\alpha_\beta}{\partial \theta^{\pi\zeta}_{\alpha\beta i}}\frac{\partial \theta^{\pi\zeta}_{\alpha\beta i}}{\partial t^\ell},$$

with

$$\frac{\partial H^\alpha_\beta}{\partial x^i_\xi} = u^{\alpha\xi}_{\beta i} - \frac{\partial L^\alpha_\beta}{\partial x^i_\xi} = u^{\alpha\xi}_{\beta i} - u^{\alpha\xi}_{\beta i} = 0,$$

$$\frac{\partial H^\alpha_\beta}{\partial \Psi^\kappa_\xi} = s^{\alpha\xi}_{\beta\kappa} - \frac{\partial L^\alpha_\beta}{\partial \Psi^\kappa} = s^{\alpha\xi}_{\beta\kappa} - s^{\alpha\xi}_{\beta\kappa} = 0.$$

By using the abovementioned generalized anti-trace Hamilton-type PDEs, the expression of $D_\ell H^\alpha_\beta$ becomes

$$D_\ell H^\alpha_\beta = \left[\frac{\partial H^\alpha_\beta}{\partial x^i}\frac{\partial x^i}{\partial t^\ell} - \frac{\delta^\eta_\ell}{m}\frac{\partial x^i}{\partial t^\eta}\frac{\partial H^\alpha_\beta}{\partial x^i}\right] + \left[\frac{\partial H^\alpha_\beta}{\partial \Psi^\kappa}\frac{\partial \Psi^\kappa}{\partial t^\ell} - \frac{\delta^\eta_\ell}{m}\frac{\partial \Psi^\kappa}{\partial t^\eta}\frac{\partial H^\alpha_\beta}{\partial \Psi^\kappa}\right].$$

One can notice that for $m > 1$, $\sum_\alpha D_\alpha H^\alpha_\beta$ is not null. However, we give a conservation law related to $\Theta^{\alpha\eta}_{\beta\gamma}$,

$$\Theta^{\alpha\eta}_{\beta\gamma}\Big(x, x_\xi, u, \Psi, \Psi_\xi, \theta, s\Big) = u^{\alpha\eta}_{\beta i}(t)x^i_\gamma(t) + s^{\alpha\eta}_{\beta\kappa}(t)\Psi^\kappa_\gamma(t) - L^\alpha_\beta\Big(x, x_\xi, \Psi, \Psi_\xi, \theta\Big)\,\delta^\eta_\gamma.$$

Computing $D_\ell \Theta^{\alpha\eta}_{\beta\gamma}$, we get

$$D_\ell \Theta^{\alpha\eta}_{\beta\gamma} = \frac{\partial u^{\alpha\eta}_{\beta i}}{\partial t^\ell}x^i_\gamma + u^{\alpha\eta}_{\beta i}\frac{\partial x^i_\gamma}{\partial t^\ell} + \frac{\partial s^{\alpha\eta}_{\beta\kappa}}{\partial t^\ell}\Psi^\kappa_\gamma + s^{\alpha\eta}_{\beta\kappa}\frac{\partial \Psi^\kappa_\gamma}{\partial t^\ell}$$

$$-\frac{\partial L^\alpha_\beta}{\partial x^i}\frac{\partial x^i}{\partial t^\ell}\delta^\eta_\gamma - \frac{\partial L^\alpha_\beta}{\partial x^i_\xi}\frac{\partial x^i_\xi}{\partial t^\ell}\delta^\eta_\gamma - \frac{\partial L^\alpha_\beta}{\partial \Psi^\kappa}\frac{\partial \Psi^\kappa}{\partial t^\ell}\delta^\eta_\gamma - \frac{\partial L^\alpha_\beta}{\partial \Psi^\kappa_\xi}\frac{\partial \Psi^\kappa_\xi}{\partial t^\ell}\delta^\eta_\gamma - \frac{\partial L^\alpha_\beta}{\partial \theta^{\pi\zeta}_{\alpha\beta i}}\frac{\partial \theta^{\pi\zeta}_{\alpha\beta i}}{\partial t^\ell}$$

$$= -\frac{\delta^\eta_\ell}{m}\frac{\partial H^\alpha_\beta}{\partial x^i}x^i_\gamma + u^{\alpha\eta}_{\beta i}\frac{\partial x^i_\gamma}{\partial t^\ell} - \frac{\delta^\eta_\ell}{m}\frac{\partial H^\alpha_\beta}{\partial \Psi^\kappa}\Psi^\kappa_\gamma + s^{\alpha\eta}_{\beta\kappa}\frac{\partial \Psi^\kappa_\gamma}{\partial t^\ell} - \frac{\partial L^\alpha_\beta}{\partial x^i}\frac{\partial x^i}{\partial t^\ell}\delta^\eta_\gamma - u^{\alpha\xi}_{\beta i}\frac{\partial x^i_\xi}{\partial t^\ell}\delta^\eta_\gamma - \frac{\partial L^\alpha_\beta}{\partial \Psi^\kappa}\frac{\partial \Psi^\kappa}{\partial t^\ell}\delta^\eta_\gamma - s^{\alpha\xi}_{\beta\kappa}\frac{\partial \Psi^\kappa_\xi}{\partial t^\ell}\delta^\eta_\gamma.$$

Now, by contracting with δ^η_ℓ (setting $\eta = \ell$ and summing over η) in the right-hand side expression and using the equalities

$$\frac{\partial H^\alpha_\beta}{\partial x^i} = -\frac{\partial L^\alpha_\beta}{\partial x^i}, \quad \frac{\partial H^\alpha_\beta}{\partial \Psi^\kappa} = -\frac{\partial L^\alpha_\beta}{\partial \Psi^\kappa},$$

we find

$$\sum_{\eta} D_{\eta}\Theta^{\alpha\eta}_{\beta\gamma} = -\frac{\partial H^{\alpha}_{\beta}}{\partial x^{i}}x^{i}_{\gamma} + u^{\alpha\eta}_{\beta i}\frac{\partial x^{i}_{\gamma}}{\partial t^{\eta}} - \frac{\partial H^{\alpha}_{\beta}}{\partial \Psi^{\kappa}}\Psi^{\kappa}_{\gamma} + s^{\alpha\eta}_{\beta\kappa}\frac{\partial \Psi^{\kappa}_{\gamma}}{\partial t^{\eta}} - \frac{\partial L^{\alpha}_{\beta}}{\partial x^{i}}\frac{\partial x^{i}}{\partial t^{\gamma}} - u^{\alpha\xi}_{\beta i}\frac{\partial x^{i}_{\xi}}{\partial t^{\eta}} - \frac{\partial L^{\alpha}_{\beta}}{\partial \Psi^{\kappa}}\frac{\partial \Psi^{\kappa}}{\partial t^{\gamma}} - s^{\alpha\xi}_{\beta\kappa}\frac{\partial \Psi^{\kappa}_{\xi}}{\partial t^{\gamma}}$$

$$\Longrightarrow \sum_{\eta} D_{\eta}\Theta^{\alpha\eta}_{\beta\gamma} = 0,$$

for all state and control functions, $x(t)$, $\Psi(t)$ that satisfy Euler–Lagrange PDEs of the anti-trace type generated by the first-order Lagrangian tensor fields

$$\frac{\delta^{\eta}_{\lambda}}{m}\frac{\partial L^{\alpha}_{\beta}}{\partial x^{i}} - D_{\lambda}\frac{\partial L^{\alpha}_{\beta}}{\partial x^{i}_{\eta}} = 0,$$

$$\frac{\delta^{\eta}_{\lambda}}{m}\frac{\partial L^{\alpha}_{\beta}}{\partial \Psi^{\kappa}} - D_{\lambda}\frac{\partial L^{\alpha}_{\beta}}{\partial \Psi^{\kappa}_{\eta}} = 0.$$

(ii) *Euler-Lagrange PDEs of anti-trace type generated by Lagrangian tensor fields of second-order*
As in the case of first-order autonomous Lagrangian tensor fields, we start by computing the divergence of H^{α}_{β}

$$D_{\ell}H^{\alpha}_{\beta} = \frac{\partial H^{\alpha}_{\beta}}{\partial x^{i}}\frac{\partial x^{i}}{\partial t^{\ell}} + \frac{\partial H^{\alpha}_{\beta}}{\partial x^{i}_{\xi}}\frac{\partial x^{i}_{\xi}}{\partial t^{\ell}} + \frac{\partial H^{\alpha}_{\beta}}{\partial u^{\alpha\eta}_{\beta i}}\frac{\partial u^{\alpha\eta}_{\beta i}}{\partial t^{\ell}} + \frac{1}{\mathcal{Z}(a,b)}\frac{\partial H^{\alpha}_{\beta}}{\partial x^{i}_{ab}}\frac{\partial x^{i}_{ab}}{\partial t^{\ell}} + \frac{1}{\mathcal{Z}(a,b)}\frac{\partial H^{\alpha}_{\beta}}{\partial v^{ab}_{i}}\frac{\partial v^{ab}_{i}}{\partial t^{\ell}}$$

$$+\frac{\partial H^{\alpha}_{\beta}}{\partial \Psi^{\kappa}}\frac{\partial \Psi^{\kappa}}{\partial t^{\ell}} + \frac{\partial H^{\alpha}_{\beta}}{\partial \Psi^{\kappa}_{\xi}}\frac{\partial \Psi^{\kappa}_{\xi}}{\partial t^{\ell}} + \frac{\partial H^{\alpha}_{\beta}}{\partial s^{\alpha\eta}_{\beta\kappa}}\frac{\partial s^{\alpha\eta}_{\beta\kappa}}{\partial t^{\ell}} + \frac{\partial H^{\alpha}_{\beta}}{\partial \theta^{\pi\zeta}_{\alpha\beta i}}\frac{\partial \theta^{\pi\zeta}_{\alpha\beta i}}{\partial t^{\ell}},$$

with

$$\frac{1}{\mathcal{Z}(a,b)}\frac{\partial H^{\alpha}_{\beta}}{\partial x^{i}_{ab}} = \frac{1}{\mathcal{Z}(a,b)}\left(v^{ab}_{i}\delta^{\alpha}_{\beta} - \frac{\partial L^{\alpha}_{\beta}}{\partial x^{i}_{ab}}\right) \Longrightarrow \frac{1}{\mathcal{Z}(a,b)}\frac{\partial H^{\alpha}_{\beta}}{\partial x^{i}_{ab}} = 0 \ \text{ for } \alpha = \beta.$$

Now, following the same stages as in (i), we find

$$D_{\ell}H^{\alpha}_{\beta} = \frac{\partial H^{\alpha}_{\beta}}{\partial x^{i}}\frac{\partial x^{i}}{\partial t^{\ell}} + \frac{\partial H^{\alpha}_{\beta}}{\partial u^{\alpha\eta}_{\beta i}}\frac{\partial u^{\alpha\eta}_{\beta i}}{\partial t^{\ell}} + \frac{1}{\mathcal{Z}(a,b)}\frac{\partial H^{\alpha}_{\beta}}{\partial v^{ab}_{i}}\frac{\partial v^{ab}_{i}}{\partial t^{\ell}} + \frac{\partial H^{\alpha}_{\beta}}{\partial \Psi^{\kappa}}\frac{\partial \Psi^{\kappa}}{\partial t^{\ell}} + \frac{\partial L^{\alpha}_{\beta}}{\partial s^{\alpha\eta}_{\beta\kappa}}\frac{\partial s^{\alpha\eta}_{\beta\kappa}}{\partial t^{\ell}}$$

$$= \left[\frac{\partial H^{\alpha}_{\beta}}{\partial x^{i}}\frac{\partial x^{i}}{\partial t^{\ell}} + \frac{\delta^{\eta}_{\ell}}{m}\frac{\partial x^{i}}{\partial t^{\eta}}\left(\frac{1}{\mathcal{Z}(a,b)}D^{2}_{ab}v^{ab}_{i}\delta^{\alpha}_{\beta} - \frac{\partial H^{\alpha}_{\beta}}{\partial x^{i}}\right) + \frac{1}{\mathcal{Z}(a,b)}\frac{\partial H^{\alpha}_{\beta}}{\partial v^{ab}_{i}}\frac{\partial v^{ab}_{i}}{\partial t^{\ell}} + \frac{1}{\mathcal{Z}(a,b)}\frac{\partial H^{\alpha}_{\beta}}{\partial x^{i}_{ab}}\frac{\partial x^{i}_{ab}}{\partial t^{\ell}}\right]$$

$$+\left[\frac{\partial H^{\alpha}_{\beta}}{\partial \Psi^{\kappa}}\frac{\partial \Psi^{\kappa}}{\partial t^{\ell}} - \frac{\delta^{\eta}_{\ell}}{m}\frac{\partial \Psi^{\kappa}}{\partial t^{\beta}}\frac{\partial H^{\alpha}_{\beta}}{\partial \Psi^{\kappa}}\right].$$

Finally, contracting with δ^{ℓ}_{η} on the right-hand side, the previous relation does not lead us to a setting for which

$$\sum_{\alpha} D_{\alpha}H^{\alpha}_{\beta} = 0$$

remain generally valid. Sufficient conditions for conservation of the Hamiltonian tensor fields constitute a direction for further investigation, to be addressed in future work. However, as in (i), we identify a conservation law involving the component-wise energy-momentum tensor

$$\Theta^{\alpha\eta}_{\beta\gamma}\left(x, x_\xi, u, x_{\pi\zeta}, v, \Psi, \Psi_\xi, s, \theta\right) = \left[u^{\alpha\eta}_{\beta i}(t) - \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\frac{\partial v^{a\eta}_i}{\partial t^a}(t)\right]x^i_\gamma(t)$$

$$+s^{\alpha\eta}_{\beta\kappa}(t)\Psi^\kappa_\gamma(t) + \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\, x^i_{\gamma a}(t)v^{a\eta}_i(t) - L^\alpha_\beta\left(x, x_\xi, x_{\pi\zeta}, \Psi, \Psi_\xi, s, \theta\right)\delta^\eta_\gamma.$$

In this setting, we have

$$D_\ell\Theta^{\alpha\eta}_{\beta\gamma} = \frac{\partial u^{\alpha\eta}_{\beta i}}{\partial t^\ell}x^i_\gamma - \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\frac{\partial}{\partial t^\ell}\left(\frac{\partial v^{a\eta}_i}{\partial t^a}\right)x^i_\gamma + \left[u^{\alpha\eta}_{\beta i}(t) - \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\frac{\partial}{\partial t^a}\left(v^{a\eta}_i\right)\right]\frac{\partial x^i_\gamma}{\partial t^\ell} + \frac{\partial s^{\alpha\eta}_{\beta\kappa}}{\partial t^\ell}\Psi^\kappa_\gamma + s^{\alpha\eta}_{\beta\kappa}\frac{\partial\Psi^\kappa_\gamma}{\partial t^\ell}$$

$$+\frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\frac{\partial x^i_{\gamma a}}{\partial t^\ell}v^{a\eta}_i + \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\frac{\partial v^{a\eta}_i}{\partial t^\ell}x^i_{\gamma a}$$

$$-\frac{\partial L^\alpha_\beta}{\partial x^i}\frac{\partial x^i}{\partial t^\ell}\delta^\eta_\gamma - \frac{\partial L^\alpha_\beta}{\partial x^i_\xi}\frac{\partial x^i_\xi}{\partial t^\ell}\delta^\eta_\gamma - \frac{1}{\mathcal{Z}(a,b)}\frac{\partial L^\alpha_\beta}{\partial x^i_{ab}}\frac{\partial x^i_{ab}}{\partial t^\ell}\delta^\eta_\gamma - \frac{\partial L^\alpha_\beta}{\partial\Psi^\kappa}\frac{\partial\Psi^\kappa}{\partial t^\ell}\delta^\eta_\gamma - \frac{\partial L^\alpha_\beta}{\partial\Psi^\kappa_\xi}\frac{\partial\Psi^\kappa_\xi}{\partial t^\ell}\delta^\eta_\gamma - \frac{\partial L^\alpha_\beta}{\partial\theta^{\pi\zeta}_{\alpha\beta i}}\frac{\partial\theta^{\pi\zeta}_{\alpha\beta i}}{\partial t^\ell}$$

$$= \delta^\eta_\ell\left[\frac{\delta^\alpha_\beta}{m\cdot\mathcal{Z}(a,b)}D^2_{ab}v^{ab}_i - \frac{1}{m}\frac{\partial H^\alpha_\beta}{\partial x^i}\right]x^i_\gamma - \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\frac{\partial}{\partial t^\ell}\left(\frac{\partial v^{a\eta}_i}{\partial t^a}\right)x^i_\gamma + \left[u^{\alpha\eta}_{\beta i}(t) - \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\frac{\partial}{\partial t^a}\left(v^{a\eta}_i\right)\right]\frac{\partial x^i_\gamma}{\partial t^\ell}$$

$$-\frac{\delta^\eta_\ell}{m}\frac{\partial H^\alpha_\beta}{\partial\Psi^\kappa}\Psi^\kappa_\gamma + s^{\alpha\eta}_{\beta\kappa}\frac{\partial\Psi^\kappa_\gamma}{\partial t^\ell} + \frac{\delta^\alpha_\beta}{\mathcal{Z}(\gamma,a)}\frac{\partial x^i_{\gamma a}}{\partial t^\ell}v^{a\eta}_i + \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\frac{\partial v^{a\eta}_i}{\partial t^\ell}x^i_{\gamma a}$$

$$-\frac{\partial L^\alpha_\beta}{\partial x^i}\frac{\partial x^i}{\partial t^\ell}\delta^\eta_\gamma - u^{\alpha\xi}_{\beta i}\frac{\partial x^i_\xi}{\partial t^\ell}\delta^\eta_\gamma - \frac{\delta^\eta_\gamma}{\mathcal{Z}(a,b)}v^{ab}_i\frac{\partial x^i_{ab}}{\partial t^\ell} - \frac{\partial L^\alpha_\beta}{\partial\Psi^\kappa}\frac{\partial\Psi^\kappa}{\partial t^\ell}\delta^\eta_\gamma - s^{\alpha\xi}_{\beta\kappa}\frac{\partial\Psi^\kappa_\xi}{\partial t^\ell}\delta^\eta_\gamma.$$

Finally, by contracting with δ^η_ℓ in the right-hand side expression (setting $\eta = \ell$ and summing over η, as in the Case (i)), together with the equalities

$$\frac{\partial H^\alpha_\beta}{\partial x^i} = -\frac{\partial L^\alpha_\beta}{\partial x^i}, \quad \frac{\partial H^\alpha_\beta}{\partial\Psi^\kappa} = -\frac{\partial L^\alpha_\beta}{\partial\Psi^\kappa},$$

We show

$$\sum_\eta D_\eta\Theta^{\alpha\eta}_{\beta\gamma} = \left[\frac{\delta^\alpha_\beta}{\mathcal{Z}(a,b)}D^2_{ab}v^{ab}_i - \frac{\partial H^\alpha_\beta}{\partial x^i}\right]x^i_\gamma - \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\frac{\partial}{\partial t^\eta}\left(\frac{\partial v^{a\eta}_i}{\partial t^a}\right)x^i_\gamma + \left[u^{\alpha\eta}_{\beta i} - \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\frac{\partial}{\partial t^a}\left(v^{a\eta}_i\right)\right]\frac{\partial x^i_\gamma}{\partial t^\eta}$$

$$-\frac{\partial H^\alpha_\beta}{\partial\Psi^\kappa}\Psi^\kappa_\gamma + s^{\alpha\eta}_{\beta\kappa}\frac{\partial\Psi^\kappa_\gamma}{\partial t^\eta} + \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\frac{\partial x^i_{\gamma a}}{\partial t^\eta}v^{a\eta}_i + \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,\eta)}\frac{\partial v^{a\eta}_i}{\partial t^\eta}x^i_{\gamma a}$$

$$-\frac{\partial L^\alpha_\beta}{\partial x^i}\frac{\partial x^i}{\partial t^\gamma} - u^{\alpha\xi}_{\beta i}\frac{\partial x^i_\xi}{\partial t^\gamma} - \frac{\delta^\alpha_\beta}{\mathcal{Z}(a,b)}v^{ab}_i\frac{\partial x^i_{ab}}{\partial t^\gamma} - \frac{\partial L^\alpha_\beta}{\partial\Psi^\kappa}\frac{\partial\Psi^\kappa}{\partial t^\gamma} - s^{\alpha\xi}_{\beta\kappa}\frac{\partial\Psi^\kappa_\xi}{\partial t^\gamma}.$$

In this way, it results in

$$\sum_\eta D_\eta\Theta^{\alpha\eta}_{\beta\gamma} = 0,$$

for all state and control functions, $x(t)$, $\Psi(t)$, that the satisfy Euler–Lagrange PDEs of the anti-trace type generated by second-order Lagrangian tensor fields

$$\frac{\delta^{\eta}_{\lambda}}{m}\frac{\partial L^{\alpha}_{\beta}}{\partial x^{i}} - D_{\lambda}\frac{\partial L^{\alpha}_{\beta}}{\partial x^{i}_{\eta}} + \frac{\delta^{\eta}_{\lambda}}{m\cdot\mathcal{Z}(a,b)}D^{2}_{ab}\frac{\partial L^{\alpha}_{\beta}}{\partial x^{i}_{ab}} = 0,$$

$$\frac{\delta^{\eta}_{\lambda}}{m}\frac{\partial L^{\alpha}_{\beta}}{\partial \Psi^{\kappa}} - D_{\lambda}\frac{\partial L^{\alpha}_{\beta}}{\partial \Psi^{\kappa}_{\eta}} = 0.$$

4. Illustrative applications

4.1. Application of the results in section 2

To illustrate the anti-trace Euler–Lagrange PDEs developed in Section 2, the associated anti-trace Hamilton PDEs and their compact matrix form, we consider a setting described by two evolution variables $t = (t^1, t^2)$. $\Omega := \Omega_{0,1} = [0,1]^2$ is generated by the diagonally opposite points $(0,0)$ and $(1,1)$; $x : \Omega \to \mathbb{R}$ is the state variable and $\Psi : \Omega \to \mathbb{R}$ denotes the control variable (i.e., $m = 2$, $n = 1$, $\epsilon = 1$). The multiple integral given below is determined by the autonomous controlled Lagrangian of first order

$$W = \frac{1}{2}\Big[(x_{\alpha})^2 + (\Psi_{\alpha})^2 + (x-\Psi)^2\Big],$$

where $x_{\alpha} := \dfrac{\partial x}{\partial t^{\alpha}}$, $\Psi_{\alpha} := \dfrac{\partial \Psi}{\partial t^{\alpha}}$, $\alpha \in \{1,2\}$. Having summation over repeated indices, the Lagrange function W reads

$$W = \frac{1}{2}\Big[\sum_{\alpha=1}^{2}\Big((x_{\alpha})\cdot(x_{\alpha})\Big) + \sum_{\alpha=1}^{2}\Big((\Psi_{\alpha})\cdot(\Psi_{\alpha})\Big) + (x-\Psi)^2\Big]$$

$$= \frac{1}{2}\Big[(x_1)\cdot(x_1) + (x_2)\cdot(x_2) + (\Psi_1)\cdot(\Psi_1) + (\Psi_2)\cdot(\Psi_2) + (x-\Psi)^2\Big]$$

$$= \frac{1}{2}\Big[(x_1)^2 + (x_2)^2 + (\Psi_1)^2 + (\Psi_2)^2 + (x-\Psi)^2\Big].$$

Specifically, it defines the cost functional

$$\mathscr{E}(x(\cdot),\Psi(\cdot)) = \frac{1}{2}\int_{\Omega}\Big[(x_1)^2 + (x_2)^2 + (\Psi_1)^2 + (\Psi_2)^2 + (x-\Psi)^2\Big]dt^1dt^2 \to \text{extremize}$$

subject to the state equations

$$X_{11} := x_{11} + x_{11}x_1 + x - \Psi = 0, \tag{39}$$

$$X_{22} := x_{22} + x_{22}x_2 + x - \Psi = 0, \tag{40}$$

$$X_{12} := x_{12} + \frac{1}{2}(x_{12})^2 + x_1\Psi + x_2\Psi + \Psi = 0, \tag{41}$$

where $x_{ab} := \dfrac{\partial^2 x}{\partial t^a \partial t^b} = \dfrac{\partial^2 x}{\partial t^b \partial t^a}$, $a, b \in \{1, 2\}$. Indeed, we have

$$X_{21} = x_{21} + \frac{1}{2}(x_{21})^2 + x_2\Psi + x_1\Psi + \Psi = x_{12} + \frac{1}{2}(x_{12})^2 + x_1\Psi + x_2\Psi + \Psi = X_{12}$$

as stated in formulation of Problem ($\mathcal{P}$) (Section 2). In addition, the state function x satisfies

$$x|_{\partial\Omega_{0,t_0}} = 0, \ x_\alpha|_{\partial\Omega_{0,t_0}} = 0.$$

Throughout the sequel, we also use $\eta, \beta \in \{1, 2\}$. In the following, we illustrate the results developed in Section 2.

Solution

• **Anti-trace Euler–Lagrange PDEs.** First, we emphasize that in agreement with Remark 2.2, the symmetries related to the second-order partial derivative of x involve $X_{12} = X_{21}$, allowing us to introduce only $nm(m + 1)/2 = 3$ different co-state functions θ^{ab} (associated with X_{11}, X_{22}, X_{12}), that is $\theta^{11}, \theta^{22}, \theta^{12}$. In this setting, the auxiliary Lagrangian is given by

$$L = \frac{1}{2}\Big[(x_1)^2 + (x_2)^2 + (\Psi_1)^2 + (\Psi_2)^2 + (x - \Psi)^2\Big]$$

$$+\theta^{11}\Big[x_{11} + x_{11}x_1 + x - \Psi\Big] + \theta^{22}\Big[x_{22} + x_{22}x_2 + x - \Psi\Big] + \theta^{12}\Big[x_{12} + \frac{1}{2}(x_{12})^2 + x_1\Psi + x_2\Psi + \Psi\Big],$$

where $\theta^{11}, \theta^{22}, \theta^{12} : \Omega \to \mathbb{R}$ are co-state functions (Lagrange multipliers). The partial derivatives of L are

$$L_x = (x - \Psi) + \theta^{11} + \theta^{22},$$

$$L_{x_1} = x_1 + \theta^{11}x_{11} + \theta^{12}\Psi, \quad L_{x_2} = x_2 + \theta^{22}x_{22} + \theta^{12}\Psi,$$
$$L_{x_{11}} = \theta^{11}(1 + x_1), \quad L_{x_{22}} = \theta^{22}(1 + x_2), \quad L_{x_{12}} = \theta^{12}(1 + x_{12})$$

(the derivatives with respect to x_{ab} are taken with the independent coordinates, $a \le b$ (see Remark 2.2)),

$$L_\Psi = -(x - \Psi) - \theta^{11} - \theta^{22} + \theta^{12}(1 + x_1 + x_2),$$

$$L_{\Psi_1} = \Psi_1, \quad L_{\Psi_2} = \Psi_2.$$

In this context, the anti-trace Euler–Lagrange PDEs given in (6)-(8) are derived as follows.

• $\eta = \beta = 1$ in (6), (7) $\implies$

$$\frac{\delta_1^1}{2}\frac{\partial L}{\partial x} - D_1\frac{\partial L}{\partial x_1} + \frac{\delta_1^1}{2}\Big[D_{11}^2\frac{\partial L}{\partial x_{11}} + D_{22}^2\frac{\partial L}{\partial x_{22}} + D_{12}^2\frac{\partial L}{\partial x_{12}}\Big] = 0,$$

$$\frac{\delta_1^1}{2}\frac{\partial L}{\partial \Psi} - D_1\frac{\partial L}{\partial \Psi_1} = 0,$$

equivalently with

$$\frac{1}{2}((x - \Psi) + \theta^{11} + \theta^{22}) - D_1(x_1 + \theta^{11}x_{11} + \theta^{12}\Psi)$$

$$+\frac{1}{2}\Big[D_{11}^2(\theta^{11}(1+x_1))+D_{22}^2(\theta^{22}(1+x_2))+D_{12}^2(\theta^{12}(1+x_{12}))\Big]=0, \tag{42}$$

$$\frac{1}{2}(-(x-\Psi)-\theta^{11}-\theta^{22}+\theta^{12}(1+x_1+x_2))-D_1\Psi_1=0. \tag{43}$$

- $\eta=\beta=2$ in (6), (7) $\Longrightarrow$

$$\frac{\delta_2^2}{2}\frac{\partial L}{\partial x}-D_2\frac{\partial L}{\partial x_2}+\frac{\delta_2^2}{2}\Big[D_{11}^2\frac{\partial L}{\partial x_{11}}+D_{22}^2\frac{\partial L}{\partial x_{22}}+D_{12}^2\frac{\partial L}{\partial x_{12}}\Big]=0,$$

$$\frac{\delta_2^2}{2}\frac{\partial L}{\partial \Psi}-D_2\frac{\partial L}{\partial \Psi_2}=0,$$

equivalently with

$$\frac{1}{2}((x-\Psi)+\theta^{11}+\theta^{22})-D_2(x_2+\theta^{22}x_2+\theta^{12}\Psi)$$

$$+\frac{1}{2}\Big[D_{11}^2(\theta^{11}(1+x_1))+D_{22}^2(\theta^{22}(1+x_2))+D_{12}^2(\theta^{12}(1+x_{12}))\Big]=0, \tag{44}$$

$$\frac{1}{2}(-(x-\Psi)-\theta^{11}-\theta^{22}+\theta^{12}(1+x_1+x_2))-D_2\Psi_2=0. \tag{45}$$

- $\eta=1,\beta=2$ in (6), (7) $\Longrightarrow$

$$\frac{\delta_1^2}{2}\frac{\partial L}{\partial x}-D_2\frac{\partial L}{\partial x_1}+\frac{\delta_1^2}{2}\Big[D_{11}^2\frac{\partial L}{\partial x_{11}}+D_{22}^2\frac{\partial L}{\partial x_{22}}+D_{12}^2\frac{\partial L}{\partial x_{12}}\Big]=0, \tag{46}$$

$$\frac{\delta_1^2}{2}\frac{\partial L}{\partial \Psi}-D_2\frac{\partial L}{\partial \Psi_1}=0, \tag{47}$$

that is

$$-D_2(x_1+\theta^{11}x_{11}+\theta^{12}\Psi)=0, \tag{48}$$

$$-D_2\Psi_1=0. \tag{49}$$

- $\eta=2,\beta=1$ in (6), (7) $\Longrightarrow$

$$\frac{\delta_2^1}{2}\frac{\partial L}{\partial x}-D_1\frac{\partial L}{\partial x_2}+\frac{\delta_2^1}{2}\Big[D_{11}^2\frac{\partial L}{\partial x_{11}}+D_{22}^2\frac{\partial L}{\partial x_{22}}+D_{12}^2\frac{\partial L}{\partial x_{12}}\Big]=0,$$

$$\frac{\delta_2^1}{2}\frac{\partial L}{\partial \Psi}-D_1\frac{\partial L}{\partial \Psi_2}=0,$$

that is

$$-D_1(x_2+\theta^{22}x_{22}+\theta^{12}\Psi)=0,$$

$$-D_1\Psi_2=0.$$

- Finally, (8) $\Longrightarrow$

$$x_{11}+x_{11}x_1+x-\Psi=0,$$

$$x_{22}+x_{22}x_2+x-\Psi=0,$$

$$x_{12} + \frac{1}{2}(x_{12})^2 + x_1\Psi + x_2\Psi + \Psi = 0,$$

involving 11 scalar equations for the five unknown functions $x, \Psi, \theta^{11}, \theta^{22}, \theta^{12}$. We solve these equations numerically in the Python 3.9 environment using *L-BFGS-B (limited-memory Broyden–Fletcher–Goldfarb–Shanno–bound constrained)* method. Figure 1 shows the graphical representation (surface plots) of the state, control, and co-state functions. In addition, to verify the feasibility of the computed solutions, Table 7 reports the maximum and average residual of the PDEs given in (39)–(49).

• **Super-regularity.** To study whether the previous Euler–Lagrange PDEs can be formulated using canonical variables (multi-momenta), we introduce the vector variable y, containing independent velocities $y = (x_1, x_2, x_{11}, x_{22}, x_{12}, \Psi_1, \Psi_2)$; we write $y := (y^h), h \in \{1, \dots, 7\}$. Accordingly, for $\tilde{h}, \hat{h} \in \{1, \dots, 7\}$

$$\left(\frac{\partial^2 L}{\partial y^{\tilde{h}} \partial y^{\hat{h}}}\right) = \begin{pmatrix} \frac{\partial^2 L}{\partial x_1 \partial x_1} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_{11}} & \frac{\partial^2 L}{\partial x_1 \partial x_{22}} & \frac{\partial^2 L}{\partial x_1 \partial x_{12}} & \frac{\partial^2 L}{\partial x_1 \partial \Psi_1} & \frac{\partial^2 L}{\partial x_1 \partial \Psi_2} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2 \partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_{11}} & \frac{\partial^2 L}{\partial x_2 \partial x_{22}} & \frac{\partial^2 L}{\partial x_2 \partial x_{12}} & \frac{\partial^2 L}{\partial x_2 \partial \Psi_1} & \frac{\partial^2 L}{\partial x_2 \partial \Psi_2} \\ \frac{\partial^2 L}{\partial x_{11} \partial x_1} & \frac{\partial^2 L}{\partial x_{11} \partial x_2} & \frac{\partial^2 L}{\partial x_{11} \partial x_{11}} & \frac{\partial^2 L}{\partial x_{11} \partial x_{22}} & \frac{\partial^2 L}{\partial x_{11} \partial x_{12}} & \frac{\partial^2 L}{\partial x_{11} \partial \Psi_1} & \frac{\partial^2 L}{\partial x_{11} \partial \Psi_2} \\ \frac{\partial^2 L}{\partial x_{22} \partial x_1} & \frac{\partial^2 L}{\partial x_{22} \partial x_2} & \frac{\partial^2 L}{\partial x_{22} \partial x_{11}} & \frac{\partial^2 L}{\partial x_{22} \partial x_{22}} & \frac{\partial^2 L}{\partial x_{22} \partial x_{12}} & \frac{\partial^2 L}{\partial x_{22} \partial \Psi_1} & \frac{\partial^2 L}{\partial x_{22} \partial \Psi_2} \\ \frac{\partial^2 L}{\partial x_{12} \partial x_1} & \frac{\partial^2 L}{\partial x_{12} \partial x_2} & \frac{\partial^2 L}{\partial x_{12} \partial x_{11}} & \frac{\partial^2 L}{\partial x_{12} \partial x_{22}} & \frac{\partial^2 L}{\partial x_{12} \partial x_{12}} & \frac{\partial^2 L}{\partial x_{12} \partial \Psi_1} & \frac{\partial^2 L}{\partial x_{12} \partial \Psi_2} \\ \frac{\partial^2 L}{\partial \Psi_1 \partial x_1} & \frac{\partial^2 L}{\partial \Psi_1 \partial x_2} & \frac{\partial^2 L}{\partial \Psi_1 \partial x_{11}} & \frac{\partial^2 L}{\partial \Psi_1 \partial x_{22}} & \frac{\partial^2 L}{\partial \Psi_1 \partial x_{12}} & \frac{\partial^2 L}{\partial \Psi_1 \partial \Psi_1} & \frac{\partial^2 L}{\partial \Psi_1 \partial \Psi_2} \\ \frac{\partial^2 L}{\partial \Psi_2 \partial x_1} & \frac{\partial^2 L}{\partial \Psi_2 \partial x_2} & \frac{\partial^2 L}{\partial \Psi_2 \partial x_{11}} & \frac{\partial^2 L}{\partial \Psi_2 \partial x_{22}} & \frac{\partial^2 L}{\partial \Psi_2 \partial x_{12}} & \frac{\partial^2 L}{\partial \Psi_2 \partial \Psi_1} & \frac{\partial^2 L}{\partial \Psi_2 \partial \Psi_2} \end{pmatrix}.$$

$$= \begin{pmatrix} 1 & 0 & \theta^{11} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \theta^{22} & 0 & 0 & 0 \\ \theta^{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta^{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta^{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

with

$$\det\left(\frac{\partial^2 L}{\partial y^{\tilde{h}} \partial y^{\hat{h}}}\right) = (\theta^{11})^2 (\theta^{22})^2 \theta^{12}.$$

According to Figure 1 c–e, $\theta^{11}, \theta^{22}, \theta^{12} > 0$, involving $\det\left(\frac{\partial^2 L}{\partial y^{\tilde{h}} \partial y^{\hat{h}}}\right) \neq 0, \forall (t^1, t^2) \in \Omega$. Hence, L is super-regular at every point (t^1, t^2).

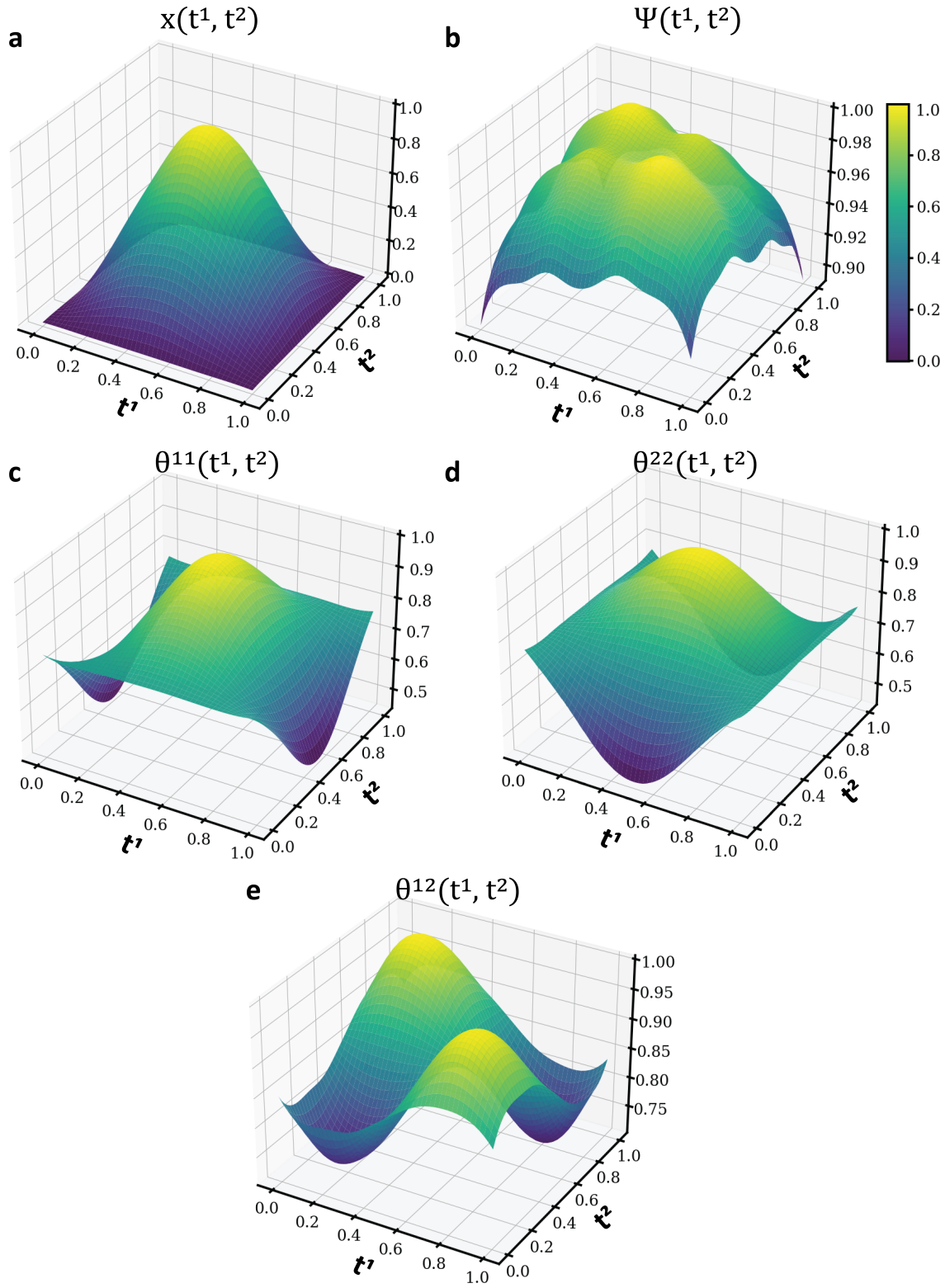


Figure 1. The numerical results (graphical representation) of (a) $x(t^1, t^2)$ (the state function), (b) $\Psi(t^1, t^2)$ (the control function), (c) $\theta^{11}(t^1, t^2)$ (the co-state function associated with with the state equation $X_{11} = 0$), and (d) $\theta^{22}(t^1, t^2)$ (the co-state function associated with the state equation $X_{22} = 0$), (e) $\theta^{12}(t^1, t^2)$ (co-state function associated with with the state equation $X_{12} = 0$).

Table 7. Residual error analysis for the numerical solution of the system.

Equation	Maximum error	Average error
(39)	1.18×10^{-2}	4.02×10^{-3}
(40)	1.45×10^{-2}	1.07×10^{-2}
(41)	1.06×10^{-2}	4.08×10^{-3}
(42)	1.25×10^{-3}	1.07×10^{-3}
(43)	4.05×10^{-3}	1.07×10^{-3}
(44)	1.15×10^{-2}	1.16×10^{-3}
(45)	6.73×10^{-5}	2.07×10^{-5}
(46)	9.72×10^{-6}	3.84×10^{-6}
(47)	6.72×10^{-2}	2.19×10^{-2}
(48)	9.72×10^{-6}	3.84×10^{-6}
(49)	6.72×10^{-2}	2.19×10^{-2}

- **Multi-momenta and Hamiltonian.** For $\theta^{11}\theta^{22}\theta^{12} \neq 0$, the multi-momenta are

$$u^1 = \frac{\partial L}{\partial x_1} = x_1 + \theta^{11}x_{11} + \theta^{12}\Psi, \qquad u^2 = \frac{\partial L}{\partial x_2} = x_2 + \theta^{22}x_{22} + \theta^{12}\Psi,$$

$$v^{11} = \frac{\partial L}{\partial x_{11}} = \theta^{11}(1 + x_1), \qquad v^{22} = \frac{\partial L}{\partial x_{22}} = \theta^{22}(1 + x_2), \quad v^{12} = \frac{\partial L}{\partial x_{12}} = \theta^{12}(1 + x_{12}),$$

$$s^1 = \frac{\partial L}{\partial \Psi_1} = \Psi_1, \quad s^2 = \frac{\partial L}{\partial \Psi_2} = \Psi_2.$$

The controlled Hamiltonian is

$$H(x, u, v, \Psi, s, \theta) = x_1u^1 + x_2u^2 + x_{11}v^{11} + x_{22}v^{22} + x_{12}v^{12} + \Psi_1 s^1 + \Psi_2 s^2 - L,$$

with

$$L := L\big(x, x_\eta\,(x, u, v, \Psi, s, \theta)\,, x_{ab}\,(x, u, v, \Psi, s, \theta)\,, \Psi, \Psi_\eta\,(x, u, v, \Psi, s, \theta)\,, \theta\big).$$

- **Anti-trace Hamilton PDEs.** The system in (24)–(28) reads

$$\frac{\partial x}{\partial t^\beta} - \frac{\partial H}{\partial u^\beta} = 0, \quad \frac{\partial^2 x}{\partial t^a \partial t^b} = \frac{\partial H}{\partial v_i^{ab}}, \quad D_\beta u^\eta - \frac{\delta^\eta_\beta}{2} D^2_{ab} v^{ab}(t) + \frac{\delta^\eta_\beta}{2}\frac{\partial H}{\partial x^i} = 0,$$

$$\frac{\partial \Psi}{\partial t^\beta}(t) - \frac{\partial H}{\partial s^\beta} = 0, \quad D_\beta s^\eta(t) + \frac{\delta^\eta_\beta}{2}\frac{\partial H}{\partial \Psi} = 0.$$

These equations are completed by the state equations formulated in (39)–(41). One can notice that the solutions of the anti-trace Euler–Lagrange PDEs, obtained above, satisfy this Hamilton system of the anti-trace type. The system consists of 15 scalar equations (see Table 8).

Table 8. Groups of scalar equations of the anti-trace Hamilton system for $m = 2$, $n = \epsilon = 1$.

Equations	Number of equations
$\dfrac{\partial x}{\partial t^\beta} - \dfrac{\partial H}{\partial u^\beta} = 0$	2
$\dfrac{\partial^2 x}{\partial t^a \partial t^b} - \dfrac{\partial H}{\partial v_i^{ab}} = 0$	3*
$D_\beta u^\eta - \dfrac{\delta_\beta^\eta}{2} D_{ab}^2 v^{ab}(t) + \dfrac{\delta_\beta^\eta}{2} \dfrac{\partial H}{\partial x^i} = 0$	4
$\dfrac{\partial \Psi}{\partial t^\beta}(t) - \dfrac{\partial H}{\partial s^\beta} = 0$	2
$D_\beta s^\eta(t) + \dfrac{\delta_\beta^\eta}{2} \dfrac{\partial H}{\partial \Psi} = 0$	4

*Note: In agreement with Remarks 2.2 and 2.5.

• **Block-matrix representation.** For $n = \epsilon = 1$, we have $\delta_j^i = \delta_r^\kappa = 1$. According to Remark 2.4, the Hamilton system can be written using block-matrix representation as follows:

$$\underbrace{\begin{pmatrix} -I_2 & O_{2\times3} & O_{2\times4} & O_{2\times1} & O_{2\times2} & O_{2\times4} \\ O_{3\times2} & -I_3 & O_{3\times4} & O_{3\times1} & O_{3\times2} & O_{3\times4} \\ O_{4\times2} & O_{4\times3} & I_4 & -\frac{1}{2}e & O_{4\times2} & O_{4\times4} \\ O_{2\times2} & O_{2\times3} & O_{2\times4} & O_{2\times1} & -I_2 & O_{2\times4} \\ O_{4\times2} & O_{4\times3} & O_{4\times4} & O_{4\times1} & O_{4\times2} & I_4 \end{pmatrix}}_{(15\times16)} \begin{pmatrix} \dfrac{\partial x}{\partial t} \\ \dfrac{\partial^2 x}{\partial t^2} \\ \dfrac{\partial u}{\partial t} \\ \check{v} \\ \dfrac{\partial \Psi}{\partial t} \\ \dfrac{\partial s}{\partial t} \end{pmatrix} = - \begin{pmatrix} \dfrac{\partial H}{\partial u} \\ \dfrac{\partial H}{\partial v} \\ \dfrac{1}{2} e \dfrac{\partial H}{\partial x} \\ \dfrac{\partial H}{\partial s} \\ \dfrac{1}{2} e \dfrac{\partial H}{\partial \Psi} \end{pmatrix},$$

where $e := (1, 0, 0, 1)^T$ and I_2, I_3, I_4 denote the identity matrices.

One can identify

$${}^x J = \begin{pmatrix} -I_2 & O_{2\times3} & O_{2\times4} & O_{2\times1} & O_{2\times2} & O_{2\times4} \\ O_{3\times2} & -I_3 & O_{3\times4} & O_{3\times1} & O_{3\times2} & O_{3\times4} \\ O_{4\times2} & O_{4\times3} & I_4 & -\frac{1}{2}e & O_{4\times2} & O_{4\times4} \\ O_{2\times2} & O_{2\times3} & O_{2\times4} & O_{2\times1} & O_{2\times2} & O_{2\times4} \\ O_{4\times2} & O_{4\times3} & O_{4\times4} & O_{4\times1} & O_{4\times2} & O_{4\times4} \end{pmatrix}$$

$${}^\Psi J = \begin{pmatrix} O_{2\times2} & O_{2\times3} & O_{2\times4} & O_{2\times1} & O_{2\times2} & O_{2\times4} \\ O_{3\times2} & O_{3\times3} & O_{3\times4} & O_{3\times1} & O_{3\times2} & O_{3\times4} \\ O_{4\times2} & O_{4\times3} & O_{4\times4} & O_{4\times1} & O_{4\times2} & O_{4\times4} \\ O_{2\times2} & O_{2\times3} & O_{2\times4} & O_{2\times1} & -I_2 & O_{2\times4} \\ O_{4\times2} & O_{4\times3} & O_{4\times4} & O_{4\times1} & O_{4\times2} & I_4 \end{pmatrix}.$$

The dimension of $\left({}^{x}J + {}^{\Psi}J\right)$ is (15, 16), of the form $(nm + nm(m+1)/2 + nm^2 + \epsilon m + \epsilon m^2, nm + nm(m+1)/2 + nm^2 + n + \epsilon m + \epsilon m^2)$. The unknown vector is

$$(x_1, x_2;\ x_{11}, x_{22}, x_{12};\ D_1u^1, D_2u^1, D_1u^2, D_2u^2;\ w;\ \Psi_1, \Psi_2;\ D_1s^1, D_2s^1, D_1s^2, D_2s^2)^T,$$

with $u = (u^1, u^2)$, $s = (s^1, s^2)$ and (according to Remark 2.3)

$$\check{\nu} = \frac{\partial^2 v^{11}}{\partial t^1 \partial t^1} + \frac{\partial^2 v^{22}}{\partial t^2 \partial t^2} + \frac{\partial^2 v^{12}}{\partial t^1 \partial t^2}.$$

- **Conservation law.** Finally, let us consider the first-order Lagrangian

$$L = \frac{1}{2}(x - \Psi)^2 + (\Psi_1)^2 + (\Psi_2)^2 + \theta^1 (x_1 + x - \Psi) + \theta^2 (x_2 + x - \Psi),$$

for which we verify the conservation law discussed in Remark 2.5 (Point (i)),

$$\sum_{\alpha=1}^{2} D_\alpha \Theta_\beta^\alpha = 0,$$

with

$$\Theta_\beta^\alpha = u^\alpha x_\beta + s^\alpha \Psi_\beta - L\delta_\beta^\alpha.$$

The partial derivatives of the Lagrangian are

$$L_x = (x - \Psi) + \theta^1 + \theta^2, \qquad L_\Psi = -(x - \Psi) - \theta^1 - \theta^2,$$

$$L_{x_1} = \theta^1, \qquad L_{x_2} = \theta^2, \qquad L_{\Psi_1} = 2\Psi_1, \qquad L_{\Psi_2} = 2\Psi_2.$$

The multi-momenta are

$$u^1 = \frac{\partial L}{\partial x_1} = \theta^1, \qquad u^2 = \frac{\partial L}{\partial x_2} = \theta^2,$$

$$s^1 = \frac{\partial L}{\partial \Psi_1} = 2\Psi_1, \qquad s^2 = \frac{\partial L}{\partial \Psi_2} = 2\Psi_2.$$

The controlled Hamiltonian is

$$H(x, u, \Psi, s, \theta) = x_1\theta^1 + x_2\theta^2 + 2(\Psi_1)^2 + 2(\Psi_2)^2 - L$$

and the system in (24)–(28) (taking $\eta = \beta$) becomes

$$\frac{\partial x}{\partial t^\beta} - \frac{\partial H}{\partial u^\beta} = 0,$$

$$D_\beta u^\beta + \frac{1}{2}\frac{\partial H}{\partial x} = 0,$$

$$\frac{\partial \Psi}{\partial t^\beta} - \frac{\partial H}{\partial s^\beta} = 0,$$

$$D_\beta s^\beta + \frac{1}{2}\frac{\partial H}{\partial \Psi} = 0,$$
$$x_1 + x - \Psi = 0,$$
$$x_2 + x - \Psi = 0.$$

Since

$$\frac{\partial H}{\partial w^\beta} = x_\beta, \qquad \frac{\partial H}{\partial s^\beta} = \Psi_\beta, \qquad \frac{\partial H}{\partial x} = -L_x, \qquad \frac{\partial H}{\partial \Psi} = -L_\psi,$$

we obtain the following on solutions of the anti-trace Euler–Lagrange equations:

$$D_1\theta^1 = \frac{1}{2}L_x, \qquad D_2\theta^2 = \frac{1}{2}L_x,$$

and

$$D_1(2\Psi_1) = \frac{1}{2}L_\psi, \qquad D_2(2\Psi_2) = \frac{1}{2}L_\psi,$$

involving

$$D_1\Psi_1 = \frac{1}{4}L_\psi, \qquad D_2\Psi_2 = \frac{1}{4}L_\psi.$$

The components of Θ^α_β are

$$\Theta^1_1 = \theta^1 x_1 + 2(\Psi_1)^2 - L,$$
$$\Theta^2_1 = \theta^2 x_1 + 2\Psi_1\Psi_2,$$
$$\Theta^1_2 = \theta^1 x_2 + 2\Psi_1\Psi_2,$$
$$\Theta^2_2 = \theta^2 x_2 + 2(\Psi_2)^2 - L.$$

Fixing $\beta = 1$, we compute

$$\sum_{\alpha=1}^{2} D_\alpha \Theta^\alpha_1 = D_1\Theta^1_1 + D_2\Theta^2_1,$$

with

$$D_1\Theta^1_1 = (D_1\theta^1)x_1 + \theta^1 x_{11} + 4\Psi_1\Psi_{11} - D_1 L,$$
$$D_2\Theta^2_1 = (D_2\theta^2)x_1 + \theta^2 x_{12} + 2\Psi_2\Psi_{12} + 2\Psi_1\Psi_{22}.$$

Therefore, we have

$$D_1\Theta^1_1 + D_2\Theta^2_1 = (D_1\theta^1)x_1 + (D_2\theta^2)x_1 + \theta^1 x_{11} + \theta^2 x_{12} + 4\Psi_1\Psi_{11} + 2\Psi_2\Psi_{12} + 2\Psi_1\Psi_{22} - D_1 L.$$

Now, using the anti-trace Euler–Lagrange equations, we have

$$(D_1\theta^1)x_1 + (D_2\theta^2)x_1 = L_x x_1,$$

and

$$4\Psi_1\Psi_{11} = L_\psi\Psi_1, \qquad 2\Psi_1\Psi_{22} = 2\Psi_1\Psi_{11}$$
$$\implies D_1\Theta^1_1 + D_2\Theta^2_1 = L_x x_1 + L_\psi\Psi_1 + \theta^1 x_{11} + \theta^2 x_{12} + 2\Psi_2\Psi_{12} + 2\Psi_1\Psi_{22} - D_1 L.$$

On the other hand, we have

$$D_1 L = L_x x_1 + L_{x_1} x_{11} + L_{x_2} x_{12} + L_\psi \Psi_1 + L_{\psi_1} \Psi_{11} + L_{\psi_2} \Psi_{12},$$

that is,

$$D_1 L = L_x x_1 + \theta^1 x_{11} + \theta^2 x_{12} + L_\psi \Psi_1 + 2\Psi_1 \Psi_{11} + 2\Psi_2 \Psi_{12}.$$

Consequently

$$D_1 \Theta_1^1 + D_2 \Theta_1^2 = 2\Psi_1(\Psi_{22} - \Psi_{11}) = 0,$$

because, from the anti-trace Euler–Lagrange equations, we have

$$\Psi_{11} = \Psi_{22} = \frac{1}{4} L_\psi.$$

Similarly, for $\beta = 2$, we have

$$\sum_{\alpha=1}^{2} D_\alpha \Theta_2^\alpha = D_1 \Theta_2^1 + D_2 \Theta_2^2.$$

$$D_1 \Theta_2^1 = (D_1 \theta^1) x_2 + \theta^1 x_{21} + 2\Psi_1 \Psi_{21} + 2\Psi_2 \Psi_{11},$$

and

$$D_2 \Theta_2^2 = (D_2 \theta^2) x_2 + \theta^2 x_{22} + 4\Psi_2 \Psi_{22} - D_2 L$$

$$\implies D_1 \Theta_2^1 + D_2 \Theta_2^2 = (D_1 \theta^1) x_2 + (D_2 \theta^2) x_2 + \theta^1 x_{21} + \theta^2 x_{22} + 2\Psi_1 \Psi_{21} + 2\Psi_2 \Psi_{11} + 4\Psi_2 \Psi_{22} - D_2 L.$$

Using the anti-trace Euler–Lagrange equations again, we get

$$(D_1 \theta^1) x_2 + (D_2 \theta^2) x_2 = L_x x_2,$$

and

$$4\Psi_2 \Psi_{22} = L_\psi \Psi_2, \qquad 2\Psi_2 \Psi_{11} = 2\Psi_2 \Psi_{22}$$

$$\implies D_1 \Theta_2^1 + D_2 \Theta_2^2 = L_x x_2 + L_\psi \Psi_2 + \theta^1 x_{21} + \theta^2 x_{22} + 2\Psi_1 \Psi_{21} + 2\Psi_2 \Psi_{22} - D_2 L.$$

Moreover, we have

$$D_2 L = L_x x_2 + L_{x_1} x_{21} + L_{x_2} x_{22} + L_\psi \Psi_2 + L_{\psi_1} \Psi_{21} + L_{\psi_2} \Psi_{22},$$

that is,

$$D_2 L = L_x x_2 + \theta^1 x_{21} + \theta^2 x_{22} + L_\psi \Psi_2 + 2\Psi_1 \Psi_{21} + 2\Psi_2 \Psi_{22}.$$

Consequently

$$D_1 \Theta_2^1 + D_2 \Theta_2^2 = 2\Psi_2(\Psi_{11} - \Psi_{22}) = 0.$$

Hence, the following conservation law for the first-order Lagrangian is satisfied:

$$\sum_{\alpha=1}^{2} D_\alpha \Theta_\beta^\alpha = 0, \ \beta = 1, 2.$$

4.2. Application of the results in section 3

In the following, we focus on highlighting the effectiveness of the main findings developed in Section 3, where we extended the analysis of the anti-trace Hamilton dynamics to Lagrangian tensor fields. For $m = 2$, $n = 1, \epsilon = 1$, $\Omega_{0,1} = [0, 1]^2$, we consider

$$W = \frac{1}{2}\Big[(x_\alpha)^2 + (\Psi_\alpha)^2 + (x - \Psi)^2\Big] = \frac{1}{2}\Big[(x_1)^2 + (x_2)^2 + (\Psi_1)^2 + (\Psi_2)^2 + (x - \Psi)^2\Big]$$

(as in the application proposed in Section 4.1) together with the state equations imposed only on the diagonal components

$$X_{11} := x_{11} + x - \Psi = 0,$$

$$X_{22} := x_{22} + x - \Psi = 0.$$

The auxiliary scalar Lagrangian reads

$$L = \frac{1}{2}\Big[(x_1)^2 + (x_2)^2 + (\Psi_1)^2 + (\Psi_2)^2 + (x - \Psi)^2\Big] + \theta^{11}(x_{11} + x - \Psi) + \theta^{22}(x_{22} + x - \Psi).$$

Note that in this application, $\theta^{11}, \theta^{22} : \Omega \to \mathbb{R}$ are treated as Lagrange multipliers associated with the scalar Lagrangian, while $\theta^{11}_{11}, \theta^{22}_{11}, \theta^{11}_{22}, \theta^{22}_{22}$ are Lagrange multipliers which incorporate state equations in the diagonal components of the Lagrangian tensor field. Notably, the Lagrangian tensor field has the components

$$L^1_1 = x_1^2 + \Psi_1^2 - \frac{1}{2}(x - \Psi)^2 + \theta^{11}_{11}(x_{11} + x - \Psi) + \theta^{11}_{22}(x_{22} + x - \Psi),$$

$$L^2_2 = x_2^2 + \Psi_2^2 - \frac{1}{2}(x - \Psi)^2 + \theta^{22}_{11}(x_{11} + x - \Psi) + \theta^{22}_{22}(x_{22} + x - \Psi),$$

and

$$L^1_2 = L^2_1 = x_1 x_2 + \Psi_1 \Psi_2.$$

With these mathematical ingredients under study, let us verify the compatibility conditions listed in Section 3.

- **Trace.**

$$\sum_\alpha L^\alpha_\alpha = L^1_1 + L^2_2 = x_1^2 + \Psi_1^2 - \frac{1}{2}(x - \Psi)^2 + \theta^{11}_{11}(x_{11} + x - \Psi) + \theta^{11}_{22}(x_{22} + x - \Psi)$$

$$+x_2^2 + \Psi_2^2 - \frac{1}{2}(x - \Psi)^2 + \theta^{22}_{11}(x_{11} + x - \Psi) + \theta^{22}_{22}(x_{22} + x - \Psi).$$

One can notice that

$$\sum_\alpha L^\alpha_\alpha = 2L$$

if and only if

$$\theta^{11}_{11} + \theta^{22}_{11} = 2\theta^{11}, \quad \theta^{11}_{22} + \theta^{22}_{22} = 2\theta^{22}. \tag{50}$$

• **Momenta.** $u_\beta^{\alpha\eta} = \delta_\alpha^\eta x_\beta + x_\alpha \delta_\beta^\eta$, $s_\beta^{\alpha\eta} = \delta_\alpha^\eta \Psi_\beta + \Psi_\alpha \delta_\beta^\eta$, and,

$$\frac{\partial L_1^1}{\partial x_{11}} = \theta_{11}^{11}, \quad \frac{\partial L_2^2}{\partial x_{11}} = \theta_{11}^{22},$$

$$\frac{\partial L_1^1}{\partial x_{22}} = \theta_{22}^{11}, \quad \frac{\partial L_2^2}{\partial x_{22}} = \theta_{22}^{22},$$

$$\frac{\partial L_2^1}{\partial x_{11}} = \frac{\partial L_1^2}{\partial x_{11}} = 0,$$

$$\frac{\partial L_2^1}{\partial x_{22}} = \frac{\partial L_1^2}{\partial x_{22}} = 0.$$

Satisfying

$$\frac{\partial L_1^1}{\partial x_{11}} = \frac{\partial L}{\partial x_{11}}, \quad \frac{\partial L_1^1}{\partial x_{22}} = \frac{\partial L}{\partial x_{22}}, \quad \frac{\partial L_2^2}{\partial x_{11}} = \frac{\partial L}{\partial x_{11}}, \quad \frac{\partial L_2^2}{\partial x_{22}} = \frac{\partial L}{\partial x_{22}},$$

involve

$$\theta_{11}^{11} = \theta_{11}^{22} = \theta^{11}, \theta_{22}^{11} = \theta_{22}^{22} = \theta^{22},$$

which is in agreement with the compatibility conditions in (50),

$$\theta_{11}^{11} + \theta_{11}^{22} = 2\theta^{11}, \theta_{22}^{11} + \theta_{22}^{22} = 2\theta^{22}.$$

• **Common solution.** Writing

$$L_\beta^\alpha = x_\alpha x_\beta + \Psi_\alpha \Psi_\beta - \frac{1}{2}(x - \Psi)^2 \delta_\beta^\alpha + \delta_\beta^\alpha (\theta^1 X_{11} + \theta^2 X_{22}),$$

for the pair (α, β), the system in (30)–(32) reads

$$\frac{1}{2}\,\delta_\beta^\alpha \left(-x + \Psi + \theta^1 + \theta^2 + D_{11}^2\theta^1 + D_{22}^2\theta^2\right) \delta_\lambda^\eta - \delta_\alpha^\eta x_{\beta\lambda} - \delta_\beta^\eta x_{\alpha\lambda} = 0, \quad \alpha, \beta, \lambda, \eta \in \{1, 2\},$$

$$\frac{1}{2}\,\delta_\beta^\alpha \left(x - \Psi - \theta^1 - \theta^2\right) \delta_\lambda^\eta - \delta_\alpha^\eta \Psi_{\beta\lambda} - \delta_\beta^\eta \Psi_{\alpha\lambda} = 0,$$

$$X_{11} = X_{22} = 0.$$

Let us fix $(\alpha, \beta) = (1, 2)$. For $\eta = 1$, we get $x_{21} = x_{22} = 0$, and for $\eta = 2$, we get $x_{11} = x_{12} = 0$. Hence, all second derivatives of x vanish, involving x is affine: $x = \omega_0 + \omega_1 t^1 + \omega_2 t^2$. In the same way, we show that Ψ is affine. Since $x_{\beta\lambda} = \Psi_{\beta\lambda} = 0$, the equations for $\alpha = \beta$ and $\eta = \lambda$ reduce to $\theta^{11} + \theta^{22} = 0$, $D_{11}^2\theta^1 + D_{22}^2\theta^2 = 0$. Moreover, $X_{11} = X_{22} = 0$ gives $x = \Psi$. The system admits the following solution

$$x = \omega_0 + \omega_1 t^1 + \omega_2 t^2, \qquad \Psi = x, \qquad \theta^1 = -\theta^2 = \phi, \quad \phi_{11} = \phi_{22}.$$

The constants $\omega_0, \omega_1, \omega_2$ are determined by the boundary data, which are the restrictions of x and x_α to $\partial\Omega$.

5. Concluding remarks

The primary aim of this study has been to develop a Hamiltonian framework associated with Euler–Lagrange PDEs of the anti-trace type arising from multi-dimensional controlled Lagrangians. Based on this formulation, the corresponding Hamilton PDEs of the anti-trace type have been derived for controlled Lagrangians involving both first- and second-order derivatives of the state variables. These results establish a unified mathematical framework connecting anti-trace variational equations, second-order controlled Lagrangians, Hamiltonian dynamics, block-matrix representations, and conservation laws. The framework developed here can pave the way for further investigations involving more general higher-order controlled systems, alternative geometric structures, and analytical or numerical methods specifically adapted to the anti-trace Hamiltonian formulation.

The conservation laws established in this paper may also be viewed within the general framework relating conservation laws to pairs of symmetries and adjoint symmetries. In the work of Ma, the characteristics of conservation laws are adjoint symmetries; the conservation laws can be constructed from such pairs even in the absence of a Lagrangian formulation [36]. In our Hamiltonian setting, the conservation laws involve the energy-momentum tensor and the anti-trace Euler–Lagrange equations. Since the systems considered here admit a Lagrangian formulation, this approach is consistent with Noether-type conservation laws. A systematic investigation of the generalized symmetries and adjoint symmetries associated with the conservation laws derived here, as well as their extension to non-Lagrangian formulations, will be the topic of future work.

Moreover, the framework developed in this paper may be naturally related to multicomponent nonlinear Hamiltonian integrable systems formulated through Lax or zero-curvature representations. Recent studies have designed four-component integrable extensions of the Kaup–Newell and derivative nonlinear Schrödinger hierarchies based on higher-dimensional matrix spectral problems, recursion operators and bi-Hamiltonian structures [37,38]. Such systems possess infinitely many generalized symmetries and conservation laws. Specifically, these concluding remarks pave the way for a natural setting for extending the present methodology to higher-order and multicomponent Hamiltonian systems.

Author contributions

Savin Treanţă: Conceptualization, software, validation, formal analysis, investigation, writing–original draft preparation, writing–review and editing, visualization, supervision; Marius-Alin Dragu: Conceptualization, software, validation, formal analysis, investigation, writing–original draft preparation, writing–review and editing, visualization. All authors have read and agreed to the published version of the manuscript.

Use of generative- AI tools declaration

Not applicable.

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Conflicts of interest

The authors declare no conflicts of interest.

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