



Research article

A high-order fully actuated system approach to stabilization of discrete-time periodic affine nonlinear systems

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Abstract: This study addresses stabilization for a class of discrete-time periodic affine nonlinear systems within the high-order fully actuated (HOFA) system framework. The nonlinear affine contribution is first handled through an input transformation, while the periodic high-order linear part is retained for controller synthesis. The resulting dynamics are then represented in lifted coordinates, so that the closed-loop behavior can be characterized by its transition over one period. Instead of computing the feedback matrices directly from the monodromy product, a prescribed periodic target model is introduced and related to the closed-loop dynamics through a periodic Sylvester matrix equation. This construction allows the assigned spectrum to be transferred to the actual closed-loop system, and asymptotic stability follows when the prescribed poles lie inside the unit disk. A third-order numerical study, together with a comparison against discrete-time sliding mode control, illustrates the resulting closed-loop behavior.

Keywords: high-order fully actuated system approach; discrete-time periodic systems; pole placement; periodic sylvester matrix equation

Mathematics Subject Classification: 93D15, 93C55

1. Introduction

Many sampled-data processes exhibit operating patterns that repeat after a fixed number of sampling instants. Such behavior occurs, for example, in multirate and event-triggered systems [1], rotor dynamics [2], helicopter vibration suppression [3], and satellite attitude control [4]. For these systems, the matrices governing the dynamics vary with time but repeat periodically, making the evolution across a complete period particularly important for closed-loop analysis.

For a periodic discrete-time model, stability depends on the cumulative state transition rather than on the spectrum of an individual system matrix. Consequently, stabilization and pole placement are naturally formulated in terms of the transition over one period. Czornik et al. [5] studied uniform global asymptotic stabilization of semilinear periodic discrete-time systems. Lv et al. [6] investigated parametric pole assignment for linear periodic systems, while Gao et al. [7] developed an output-feedback pole configuration method for systems with varying state and input dimensions. For linear discrete-time periodic systems with time delays, Chen et al. [8] developed an adaptive Levenberg–Marquardt identification algorithm, addressing the practical identification of periodic model parameters. This complements stabilization studies, where the system model is assumed to be known.

Additional difficulties arise when periodic coefficients coexist with nonlinear terms in a high-order description. The control input, delayed states, and nonlinear dynamics are then coupled within the same model, which complicates both stabilization and spectral configuration. A common treatment is to augment the delayed variables and work with a conventional first-order realization. Although this facilitates standard state-space analysis, the original high-order relation between the state history and the control channel becomes less explicit. Sinha et al. [9] investigated a Koopman-operator approach to data-driven stabilization of discrete-time control-affine nonlinear systems, while Karafyllis and Kotsios [10] studied conditions for global asymptotic stabilization of discrete-time systems. Preserving the original high-order structure while simultaneously addressing periodicity and nonlinearity therefore remains of interest.

The high-order fully actuated (HOFA) system approach provides a framework in which the high-order description itself can be used directly for feedback synthesis. Duan [11] established its basic formulation and subsequently investigated controllability, stabilizability, and parametric design [12]. Duan [13] also developed the discrete-time formulation. Related extensions include frequency-domain control by Duan and Zhou [14], nonlinear-system control by Gu and Wang [15], delay systems by Duan [16], and predictive control by Zhang et al. [17]. More recently, Zhang et al. [18] developed a fast model predictive controller based on the fully actuated approach, while Zhang et al. [19] studied observer-based robust control for mixed-order fully actuated systems. For spacecraft applications, Jin et al. [20] combined a twistor-based model with fully actuated control for integrated attitude–orbit dynamics of flexible spacecraft, while Jia et al. [21] extended the high-order approach to attitude stabilization of underactuated rigid and flexible spacecraft. In practical tracking applications, Liu et al. [22] combined fully actuated spacecraft attitude control with disturbance estimation and parameter optimization under actuator saturation. Tian et al. [23] developed a fully actuated tracking strategy for robot manipulators incorporating model identification, disturbance observation, and constrained gain optimization. These application-oriented developments address tracking and practical constraints, whereas the present work focuses on nominal stabilization and one-period spectral assignment for discrete-time periodic affine nonlinear systems.

The model studied in this paper contains two components that must be treated differently during controller construction. The nonlinear affine term should be compensated so that it does not obstruct the subsequent design, whereas the periodic linear part must remain available because it determines the closed-loop spectral structure. A formulation that removes both components simultaneously would therefore be unsuitable for the intended pole-assignment procedure. This observation leads to a structure-preserving input transformation followed by a periodic feedback design.

Periodic Sylvester matrix equations provide a convenient algebraic tool for relating two periodic state-transition descriptions. Lv et al. [24] developed an iterative method for constructing least-squares solutions to periodic Sylvester equations. Ma and Yan [25] proposed finite-step iterative algorithms for discrete-time periodic Sylvester equations, while Lv et al. [26] investigated gradient-based neural methods. Li and Ma [27], Wang and Song [28], and Qi and Song [29] presented further neural and iterative schemes for related matrix equations. Lv et al. [30] also studied parametric pole assignment using high-order periodic matrix equations. These studies mainly address matrix-equation computation or linear periodic control. In particular, the parametric pole-assignment methods developed by Lv and coauthors [6, 30] for linear periodic systems differ from the present nonlinear setting. Duan [13, 16] provided related structural foundations through the discrete-time fully actuated formulation and its delay-system extension. The contribution here is to combine input-level compensation of the affine nonlinear term with the preserved periodic high-order linear dynamics, so that the one-period closed-loop spectrum can be assigned through a periodic Sylvester relation under the stated assumptions. No general solvability claim for arbitrary target matrices or robustness to model uncertainty is made.

Accordingly, the controller developed here separates the nonlinear affine contribution from the part of the model used for spectral design. After the input transformation, the retained high-order dynamics are expressed in lifted coordinates and analyzed through their one-period transition. A target periodic model with a prescribed stable spectrum is then introduced. By connecting the target and closed-loop descriptions through a periodic matrix relation, the feedback computation is reduced to a periodic Sylvester matrix equation.

Before concluding this section, the main contributions are summarized as follows.

First, the controller construction distinguishes between the nonlinear affine contribution and the periodic linear dynamics. The former is compensated at the input level, whereas the latter is retained explicitly as the basis for feedback synthesis.

Second, stability and spectral design are formulated through the transition accumulated over one period. A prescribed target model is connected to the actual closed-loop system by a periodic Sylvester matrix equation, from which the periodic feedback matrices are obtained.

Third, the resulting design is assessed on a third-order nonlinear periodic example and is further compared with discrete-time sliding mode control. The numerical study highlights the different closed-loop mechanisms associated with direct spectral assignment and reaching-law-based regulation.

The remainder of this paper is organized as follows. Section 2 introduces the considered system model and the stabilization problem. Section 3 presents the controller design, the monodromy-based stability condition, and the periodic Sylvester-equation formulation. Section 4 provides the numerical study and the comparison with discrete-time sliding mode control. Section 5 concludes the paper.

Notations

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, while $\mathbb{R}^n$ and $\mathbb{R}^{n \times n}$ denote the n -dimensional real vector space and the set of $n \times n$ real matrices, respectively. The identity matrix of order n is denoted by I_n . For compact representation of the delayed states, define

$$\chi_r(t) := \text{col}\{x(t), x(t-1), \dots, x(t-r)\} \in \mathbb{R}^{(r+1)n}, \quad (1.1)$$

where $\text{col}\{\cdot\}$ denotes vertical concatenation of its arguments.

2. Problem description

Consider the following k -th order step-backward model:

$$x(t+1) = \sum_{i=0}^{k-1} A_i(t)x(t-i) + f(\chi_{k-1}(t), t) + B(\chi_{k-1}(t), t)u(t). \quad (2.1)$$

Here, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^n$ denote the state and control input, respectively. The matrices $A_i(t)$, $i = 0, 1, \dots, k-1$, describe the periodic linear part and satisfy

$$A_i(t+T) = A_i(t), \quad i = 0, 1, \dots, k-1, \quad (2.2)$$

where T is a positive integer representing the system period. The mapping $f(\chi_{k-1}(t), t)$ collects the nonlinear terms, and the input matrix $B(\chi_{k-1}(t), t)$ is assumed to be nonsingular.

The control task is to stabilize (2.1) while keeping its periodic high-order linear component available for the spectral design developed below.

3. Main results

The controller for (2.1) is constructed in two stages. The nonlinear affine term is first compensated through the physical input, after which the remaining periodic dynamics are used to determine the feedback action. Introduce an auxiliary signal $v(t)$ and choose

$$u(t) = B^{-1}(\chi_{k-1}(t), t) [v(t) - f(\chi_{k-1}(t), t)]. \quad (3.1)$$

Since the input matrix is nonsingular, substituting (3.1) into (2.1) yields

$$x(t+1) = \sum_{i=0}^{k-1} A_i(t)x(t-i) + v(t). \quad (3.2)$$

Hence, the nonlinear term is compensated through the input transformation, whereas the periodic high-order linear structure is preserved for the subsequent feedback design. This exact compensation assumes that f and B are known and that B is nonsingular for the states under consideration.

For spectral analysis, introduce the lifted state

$$X(t) := \chi_{k-1}(t) \in \mathbb{R}^{kn}. \quad (3.3)$$

Accordingly, (3.2) can be represented as the first-order periodic system

$$X(t+1) = A_c(t)X(t) + B_c v(t). \quad (3.4)$$

To express the lifted matrices compactly, define

$$\mathcal{R}(t) := \begin{bmatrix} A_0(t) & A_1(t) & \cdots & A_{k-1}(t) \end{bmatrix}, \quad (3.5)$$

and

$$\mathcal{S} := \begin{bmatrix} I_{(k-1)n} & 0_{(k-1)n \times n} \end{bmatrix}. \quad (3.6)$$

Using (3.5) and (3.6), the matrices in (3.4) are written as

$$A_c(t) = \begin{bmatrix} \mathcal{R}(t) \\ \mathcal{S} \end{bmatrix}, \quad B_c = \begin{bmatrix} I_n \\ 0_{(k-1)n \times n} \end{bmatrix}. \quad (3.7)$$

The representation in (3.7) is equivalent to the usual block companion form of the lifted high-order system.

Now impose the T -periodic state-feedback law

$$v(t) = K(t)X(t), \quad K(t+T) = K(t). \quad (3.8)$$

The feedback matrix is partitioned according to the delayed-state blocks as

$$K(t) = \begin{bmatrix} K_0(t) & K_1(t) & \cdots & K_{k-1}(t) \end{bmatrix} \in \mathbb{R}^{n \times kn}. \quad (3.9)$$

Substitution of (3.8) into (3.4) gives

$$X(t+1) = A_{ec}(t)X(t), \quad A_{ec}(t) := A_c(t) + B_c K(t). \quad (3.10)$$

The stability of the original high-order system can therefore be examined through the one-period transition of the lifted closed-loop dynamics.

Lemma 1. *Consider system (2.1) subject to the input transformation (3.1) and the periodic feedback law (3.8). Define the one-period closed-loop transition matrix as*

$$\Phi := A_{ec}(T-1)A_{ec}(T-2) \cdots A_{ec}(0). \quad (3.11)$$

Then the resulting closed-loop system is asymptotically stable if and only if

$$\rho(\Phi) < 1. \quad (3.12)$$

Equivalently, all eigenvalues of Φ lie strictly inside the open unit disk.

Proof. For any $t \geq 0$, write

$$t = qT + r, \quad q \in \mathbb{Z}_{\geq 0}, \quad r \in \{0, 1, \dots, T-1\}. \quad (3.13)$$

For the position r within one period, define the transition matrices

$$\mathcal{P}_0 := I_{kn}, \quad \mathcal{P}_r := A_{ec}(r-1)A_{ec}(r-2) \cdots A_{ec}(0), \quad 1 \leq r \leq T-1. \quad (3.14)$$

Using the T -periodicity of $A_{ec}(t)$, together with (3.11) and (3.13), the lifted state satisfies

$$X(qT+r) = \mathcal{P}_r \Phi^q X(0). \quad (3.15)$$

Assume first that condition (3.12) holds. The powers of the monodromy matrix then satisfy

$$\lim_{q \rightarrow \infty} \|\Phi^q\| = 0. \quad (3.16)$$

Since the finite set of matrices defined in (3.14) is bounded, define

$$c_P := \max_{0 \leq r \leq T-1} \|\mathcal{P}_r\| < \infty. \quad (3.17)$$

It follows from (3.15) and (3.17) that

$$\|X(qT + r)\| \leq c_P \|\Phi^q\| \|X(0)\| \rightarrow 0, \quad q \rightarrow \infty. \quad (3.18)$$

Because (3.18) holds for every $r \in \{0, 1, \dots, T-1\}$, it follows that $X(t) \rightarrow 0$. Since $X(t)$ contains the current state together with only a finite number of delayed states, convergence of the lifted state implies asymptotic stability of the original closed-loop system.

For the converse direction, suppose that the closed-loop system is asymptotically stable for every admissible initial condition. Setting $r = 0$ in (3.15) gives

$$X(qT) = \Phi^q X(0). \quad (3.19)$$

Hence, $\Phi^q X(0) \rightarrow 0$ for every admissible $X(0)$. Suppose, contrary to the desired conclusion, that Φ possesses an eigenvalue μ satisfying $|\mu| \geq 1$, with corresponding nonzero eigenvector z . Then

$$\Phi^q z = \mu^q z. \quad (3.20)$$

The right-hand side of (3.20) cannot converge to zero when $|\mu| \geq 1$, which contradicts the assumed asymptotic stability. Hence every eigenvalue of Φ has modulus strictly smaller than one, which implies that condition (3.12) holds. $\square$

Lemma 1 indicates that the stabilization problem can be treated through the spectral behavior accumulated over one complete period. Let the prescribed closed-loop spectrum be

$$\Gamma := \{\gamma_1, \gamma_2, \dots, \gamma_{kn}\}, \quad \max_{\gamma \in \Gamma} |\gamma| < 1. \quad (3.21)$$

Accordingly, the periodic feedback gain $K(t)$ is required to satisfy the spectral assignment condition

$$\text{spec}(\Phi) = \Gamma, \quad (3.22)$$

where $\text{spec}(\cdot)$ denotes the spectrum of a square matrix.

Directly determining the matrices $K(0), K(1), \dots, K(T-1)$ from the product in (3.11) is difficult because these gains enter the one-period transition multiplicatively. To avoid solving this coupled product equation directly, introduce the auxiliary target periodic system

$$\zeta(t+1) = F(t)\zeta(t), \quad F(t+T) = F(t), \quad (3.23)$$

where

$$F(t) \in \mathbb{R}^{kn \times kn}. \quad (3.24)$$

The transition of (3.23) over one complete period is characterized by

$$\Phi_F := F(T-1)F(T-2) \cdots F(1)F(0). \quad (3.25)$$

The target matrices are selected such that

$$\text{spec}(\Phi_F) = \Gamma. \quad (3.26)$$

Thus, (3.22) can be achieved by constructing a relation that transfers the spectrum of Φ_F to the actual closed-loop monodromy matrix Φ .

For this purpose, introduce the matrix sequences

$$V(t) \in \mathbb{R}^{kn \times kn}, \quad W(t) \in \mathbb{R}^{n \times kn}, \quad (3.27)$$

where $V(t)$ is nonsingular for every t . Both sequences are chosen to satisfy the periodicity conditions

$$V(t+T) = V(t), \quad W(t+T) = W(t). \quad (3.28)$$

Their relation with the actual and target periodic systems is imposed through the periodic Sylvester matrix equation

$$V(t+1)F(t) - A_c(t)V(t) = B_c W(t). \quad (3.29)$$

Equation (3.29) is algebraically equivalent to the conventional periodic Sylvester form and establishes the required connection between the lifted system (3.4) and the target system (3.23).

Remark 1. A sufficient construction can be given when $F(t)$ and $A_c(t)$ share the same lower companion shift blocks. For any constant nonsingular $D \in \mathbb{R}^{n \times n}$, set

$$V(t) = I_k \otimes D, \quad W(t) = [DF_0(t) - A_0(t)D, \dots, DF_{k-1}(t) - A_{k-1}(t)D].$$

The lower block rows of $V(t+1)F(t) - A_c(t)V(t)$ vanish, while its first block row equals $W(t)$. Thus (3.29) holds, both sequences are periodic, and $\det V(t) = (\det D)^k \neq 0$. In particular, choosing $D = I_n$ gives $V(t) = I_{kn}$ and $W(t) = [F_0(t) - A_0(t), \dots, F_{k-1}(t) - A_{k-1}(t)]$. The structured target must additionally satisfy the prescribed one-period spectral condition (3.26), as demonstrated by the polynomial construction in Section 4.1. This is a sufficient construction for structured targets, not a general existence result for arbitrary unstructured $F(t)$.

Once $V(t)$ and $W(t)$ satisfying (3.29) and (3.28) have been obtained, the periodic feedback matrix is determined from

$$K(t)V(t) = W(t). \quad (3.30)$$

Because $V(t)$ is nonsingular, equation (3.30) uniquely determines $K(t)$.

The role of (3.29) in transferring the target spectrum (3.26) to the closed-loop system is established in the following theorem.

Theorem 1. *Consider the periodic affine nonlinear system (2.1). Let $F(t)$ be a T -periodic target sequence satisfying the prescribed spectral condition (3.26), where the spectrum Γ obeys the stability requirement in (3.21). Suppose that there exist T -periodic matrices $V(t)$ and $W(t)$ satisfying (3.29) and (3.28), with $V(t)$ nonsingular for every t . Then the feedback matrix determined by (3.30), together with the input transformation (3.1) and the periodic feedback law (3.8), asymptotically stabilizes system (2.1). Moreover, the resulting closed-loop monodromy matrix possesses the prescribed spectrum Γ .*

Proof. From the feedback relation (3.30), one has

$$W(t) = K(t)V(t). \quad (3.31)$$

Substituting (3.31) into the periodic Sylvester equation (3.29) gives

$$V(t+1)F(t) = [A_c(t) + B_c K(t)] V(t). \quad (3.32)$$

Using the definition of the closed-loop matrix in (3.10), equation (3.32) becomes

$$A_{ec}(t)V(t) = V(t+1)F(t). \quad (3.33)$$

Since $V(t)$ is nonsingular, (3.33) can be rewritten as

$$A_{ec}(t) = V(t+1)F(t)V^{-1}(t). \quad (3.34)$$

Applying (3.34) successively over one complete period yields

$$\begin{aligned} \Phi &= A_{ec}(T-1)A_{ec}(T-2) \cdots A_{ec}(0) \\ &= V(T)F(T-1)V^{-1}(T-1)V(T-1)F(T-2)V^{-1}(T-2) \cdots V(1)F(0)V^{-1}(0) \\ &= V(T)[F(T-1)F(T-2) \cdots F(0)]V^{-1}(0). \end{aligned} \quad (3.35)$$

By the definition of Φ_F in (3.25), equation (3.35) reduces to

$$\Phi = V(T)\Phi_F V^{-1}(0). \quad (3.36)$$

The periodicity condition (3.28) implies $V(T) = V(0)$. Therefore,

$$\Phi = V(0)\Phi_F V^{-1}(0). \quad (3.37)$$

Equation (3.37) shows that Φ and Φ_F are similar matrices. Hence their spectra coincide:

$$\text{spec}(\Phi) = \text{spec}(\Phi_F) = \Gamma. \quad (3.38)$$

Combining (3.38) with the stability requirement in (3.21) gives

$$\rho(\Phi) = \max_{\gamma \in \Gamma} |\gamma| < 1. \quad (3.39)$$

Condition (3.39) coincides with the monodromy stability condition (3.12). The asymptotic stability of the closed-loop system therefore follows from Lemma 1. $\square$

The complete controller construction can now be summarized as follows. First, a stable target spectrum satisfying (3.21) is prescribed and realized by the target periodic system (3.23) through (3.26). Next, the periodic matrix sequences $V(t)$ and $W(t)$ are obtained from (3.29) subject to the periodicity conditions (3.28). The feedback matrix is then determined from (3.30), and the physical control input follows from (3.1). The similarity relation (3.37) guarantees that the prescribed target spectrum is transferred to the actual closed-loop monodromy matrix, while the stability condition is ensured by (3.39).

4. Numerical validation

The numerical study considers the $k = 3$, $n = 2$, and $T = 3$ specialization of (2.1). In this case, the delayed-state vector introduced in (1.1) becomes

$$\chi_2(t) = \text{col}\{x(t), x(t-1), x(t-2)\}.$$

The coefficient matrices over the three phases of one period are specified as follows:

$$A_0(t) = \begin{cases} \begin{bmatrix} 0.20 & 0.35 \\ -0.10 & 0.25 \end{bmatrix}, & t = 3\ell, \\ \begin{bmatrix} 0.30 & -0.15 \\ 0.20 & 0.10 \end{bmatrix}, & t = 3\ell + 1, \\ \begin{bmatrix} 0.10 & 0.25 \\ 0.15 & 0.30 \end{bmatrix}, & t = 3\ell + 2, \end{cases}$$

$$A_1(t) = \begin{cases} \begin{bmatrix} 0.15 & -0.10 \\ 0.05 & 0.20 \end{bmatrix}, & t = 3\ell, \\ \begin{bmatrix} 0.20 & 0.05 \\ -0.10 & 0.15 \end{bmatrix}, & t = 3\ell + 1, \\ \begin{bmatrix} 0.10 & 0.15 \\ 0.10 & 0.05 \end{bmatrix}, & t = 3\ell + 2, \end{cases}$$

$$A_2(t) = \begin{cases} \begin{bmatrix} 0.05 & 0.10 \\ -0.05 & 0.08 \end{bmatrix}, & t = 3\ell, \\ \begin{bmatrix} 0.08 & -0.05 \\ 0.06 & 0.10 \end{bmatrix}, & t = 3\ell + 1, \\ \begin{bmatrix} 0.12 & 0.04 \\ -0.08 & 0.06 \end{bmatrix}, & t = 3\ell + 2, \end{cases}$$

where $\ell \in \mathbb{Z}$.

For compact notation, define

$$\vartheta_t = \frac{2\pi t}{3}.$$

The nonlinear and input mappings used in this example are specified by

$$f(\chi_2(t), t) = \begin{bmatrix} (0.025 + 0.005 \cos \vartheta_t) \sin x_1(t) - 0.015 \sin x_2(t-1) \\ (0.020 + 0.004 \sin \vartheta_t) \tanh x_2(t) - 0.012 \sin x_1(t-2) \end{bmatrix},$$

and

$$B(\chi_2(t), t) = \text{diag} \begin{pmatrix} 1.05 + 0.04 \cos \vartheta_t + 0.03 \tanh x_1(t), \\ 1.08 + 0.03 \sin \vartheta_t + 0.025 \tanh x_2(t) \end{pmatrix}.$$

Since

$$\begin{aligned} 1.05 + 0.04 \cos \vartheta_t + 0.03 \tanh x_1(t) &\geq 0.98 > 0, \\ 1.08 + 0.03 \sin \vartheta_t + 0.025 \tanh x_2(t) &\geq 1.025 > 0, \end{aligned}$$

both diagonal entries remain strictly positive, and $B(\chi_2(t), t)$ is nonsingular for every admissible state and time instant. Hence the input transformation (3.1) is well defined.

4.1. HOFA control

For the prescribed spectrum

$$\Gamma = \{0.0144, 0.0252, 0.0396, 0.0504, 0.0594 \pm 0.0126i\},$$

these one-period poles lie inside the unit disk, as required for asymptotic stability. Smaller pole moduli generally favor faster asymptotic decay per period but may affect the required control input; the present choice illustrates the design and is not claimed to be optimal. The control law (3.1), together with (3.3) and (3.8), takes the form

$$u(t) = B^{-1}(\chi_2(t), t) [v(t) - f(\chi_2(t), t)], \quad v(t) = K(t)\chi_2(t).$$

For this example, choose

$$V(t) \equiv I_6, \quad F(t) \equiv F.$$

Since $F(t)$ is constant and $T = 3$, (3.25) gives

$$\Phi_F = F^3.$$

Let μ_j denote the eigenvalues of F , and let $\gamma_j \in \Gamma$ denote the corresponding prescribed one-period eigenvalues. To enforce (3.26), select μ_j so that

$$\mu_j^3 = \gamma_j.$$

The positive real cube roots are taken for the four real poles; for the complex-conjugate pair, the principal cube root and its conjugate are selected. Their approximate values are

$$\begin{aligned} \mu_1 &\approx 0.2433, & \mu_2 &\approx 0.2932, & \mu_3 &\approx 0.3409, \\ \mu_4 &\approx 0.3694, & \mu_{5,6} &\approx 0.3921 \pm 0.0274i. \end{aligned}$$

To define the target matrix without rounding, construct the monic polynomials

$$\begin{aligned} p_1(z) &= \prod_{j=1}^3 (z - \mu_j) = z^3 + a_1 z^2 + b_1 z + c_1, \\ p_2(z) &= \prod_{j=4}^6 (z - \mu_j) = z^3 + a_2 z^2 + b_2 z + c_2. \end{aligned}$$

The constant block-companion target is defined exactly by

$$F = \begin{bmatrix} F_0 & F_1 & F_2 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix},$$

where

$$\begin{aligned}F_0 &= \text{diag}(-a_1, -a_2), \\F_1 &= \text{diag}(-b_1, -b_2), \\F_2 &= \text{diag}(-c_1, -c_2).\end{aligned}$$

For readability, the corresponding block coefficients are displayed to four decimal places:

$$\begin{aligned}F_0 &\approx \text{diag}(0.8773, 1.1536), \\F_1 &\approx \text{diag}(-0.2542, -0.4442), \\F_2 &\approx \text{diag}(0.0243, 0.0571).\end{aligned}$$

All computations use the unrounded coefficients defined by the polynomials above. Hence, for the exact construction,

$$\text{spec}(F^3) = \Gamma.$$

With $V(t) = I_6$, equation (3.29) gives

$$F - A_c(t) = B_c W(t),$$

whereas (3.30) gives

$$W(t) = K(t).$$

Therefore,

$$B_c K(t) = F - A_c(t).$$

Since the lower companion blocks of F and $A_c(t)$ are identical, comparison of their first block rows yields

$$K(t) = [F_0 - A_0(t) \quad F_1 - A_1(t) \quad F_2 - A_2(t)].$$

The periodic feedback gains are computed from the unrounded block coefficients above. Their four-decimal approximations are

$$\begin{aligned}K(0) &\approx \begin{bmatrix} 0.6773 & -0.3500 & -0.4042 & 0.1000 & -0.0257 & -0.1000 \\ 0.1000 & 0.9036 & -0.0500 & -0.6442 & 0.0500 & -0.0229 \end{bmatrix}, \\K(1) &\approx \begin{bmatrix} 0.5773 & 0.1500 & -0.4542 & -0.0500 & -0.0557 & 0.0500 \\ -0.2000 & 1.0536 & 0.1000 & -0.5942 & -0.0600 & -0.0429 \end{bmatrix}, \\K(2) &\approx \begin{bmatrix} 0.7773 & -0.2500 & -0.3542 & -0.1500 & -0.0957 & -0.0400 \\ -0.1500 & 0.8536 & -0.1000 & -0.4942 & 0.0800 & -0.0029 \end{bmatrix}.\end{aligned}$$

For the unrounded gains defined by $K(t) = [F_0 - A_0(t) \quad F_1 - A_1(t) \quad F_2 - A_2(t)]$, the relation $B_c K(t) = F - A_c(t)$ holds exactly and implies

$$F - A_{ec}(t) = F - A_c(t) - B_c K(t) = 0_{6 \times 6}, \quad t = 0, 1, 2.$$

Hence,

$$\Phi = A_{ec}(2)A_{ec}(1)A_{ec}(0) = F^3.$$

Together with the exact target construction, this gives

$$\text{spec}(\Phi) = \text{spec}(F^3) = \Gamma.$$

Using the unrounded matrices, the numerically computed spectra of F^3 and Φ , reported to the displayed precision, both agree with Γ . Matching the computed eigenvalues of Φ to the prescribed poles gives a maximum assignment error of 3.4936×10^{-15} ; the corresponding error for F^3 is 6.8799×10^{-14} . Moreover,

$$\|\Phi - F^3\|_F = 1.5944 \times 10^{-16}, \quad \rho(\Phi) \approx 0.06072166 < 1,$$

which verifies the spectral assignment numerically and confirms the stability requirement (3.12). The four-decimal matrices above are only approximate representations and were not used to compute these numerical results.

The historical states are initialized as

$$x(-2) = x(-1) = x(0) = \begin{bmatrix} 1.2 \\ -1 \end{bmatrix}.$$

Figure 1 displays the closed-loop state trajectories. Both components approach the origin rapidly, in agreement with the assigned one-period spectrum and the stability prediction above.

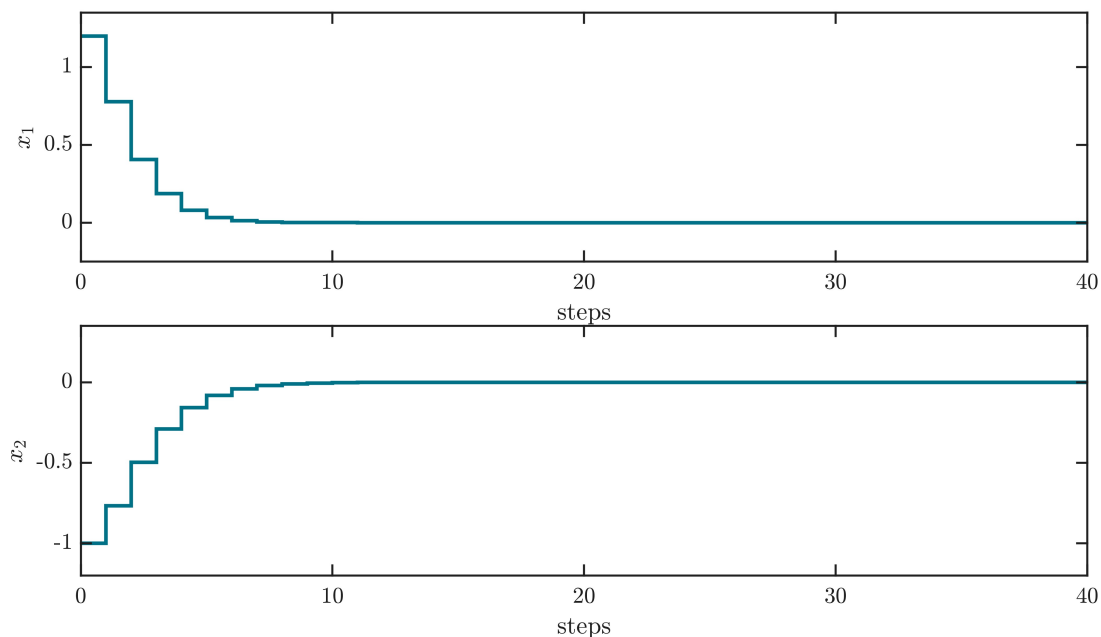


Figure 1. Closed-loop state trajectories obtained with the HOFA controller.

4.2. Comparison with DSMC

For comparison, the same plant is also controlled by a discrete-time sliding mode controller (DSMC). The plant parameters, historical states, and simulation horizon are kept identical to those used for the HOFA controller.

The historical states are set as

$$x(-2) = x(-1) = x(0) = \begin{bmatrix} 1.2 \\ -1 \end{bmatrix}.$$

For the DSMC, the sliding variable is defined as

$$s(t) := x(t) + C_2 x(t-2) - C_1 x(t-1),$$

where

$$C_1 = \text{diag}(1.20, 1.10), \quad C_2 = \text{diag}(0.35, 0.30).$$

The discrete reaching law is selected as

$$s(t+1) = \Lambda s(t) - R \text{sat}\left(\frac{s(t)}{\phi}\right),$$

with

$$\Lambda = \text{diag}(0.45, 0.45), \quad R = \text{diag}(0.02, 0.02), \quad \phi = \begin{bmatrix} 0.02 \\ 0.02 \end{bmatrix}.$$

Solving the one-step sliding relation for the next state gives

$$x(t+1) = s(t+1) + C_1 x(t) - C_2 x(t-1).$$

Define

$$s_r(t) := \Lambda s(t) - R \text{sat}\left(\frac{s(t)}{\phi}\right).$$

Then the DSMC input is

$$u_{\text{smc}}(t) = B^{-1}(\chi_2(t), t) \left[s_r(t) + C_1 x(t) - C_2 x(t-1) - f(\chi_2(t), t) - \sum_{j=0}^2 A_j(t) x(t-j) \right].$$

For the specified historical states, the initial sliding variable is

$$\begin{aligned} s(0) &= (I - C_1 + C_2)x(0) \\ &= \begin{bmatrix} 0.15 & 0 \\ 0 & 0.20 \end{bmatrix} \begin{bmatrix} 1.2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0.18 \\ -0.20 \end{bmatrix}. \end{aligned}$$

Therefore,

$$s_r(0) = 0.45s(0) - 0.02 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.061 \\ -0.070 \end{bmatrix}.$$

The reaching law reduces the sliding variable rapidly during the initial transient. When the i -th sliding variable enters the boundary layer, that is, when $|s_i(t)| \leq \phi_i$, the saturation function becomes linear and the reaching law is reduced to

$$s_i(t+1) = \left(0.45 - \frac{0.02}{0.02}\right) s_i(t) = -0.55s_i(t).$$

Since the coefficient has magnitude $0.55 < 1$, the sliding variable contracts locally within the boundary layer.

The corresponding state responses are shown in Figure 2. For the tested initial history, the complete nonlinear closed-loop simulation indicates that both state components tend toward the origin over the simulated horizon. The local boundary-layer relation alone does not establish global asymptotic stability of the nonlinear closed-loop system.

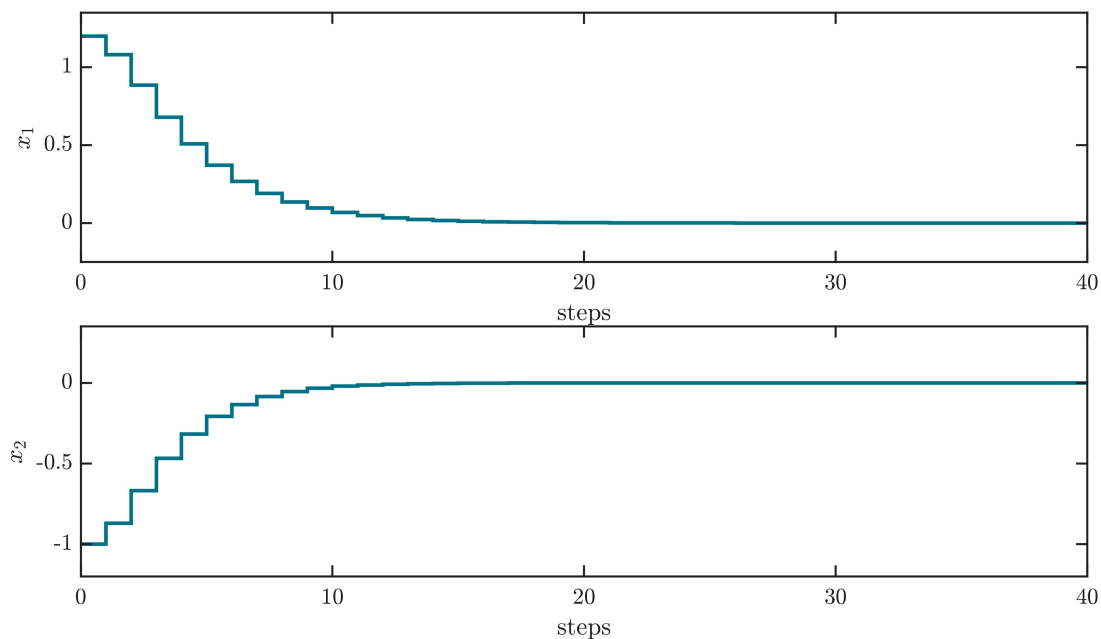


Figure 2. Closed-loop state trajectories obtained with the DSMC.

For a threshold ε , the settling time T_ε is the first sampling step after which $\|x(t)\|_2 < \varepsilon$ holds at every sample through step 40. The peak input is defined as $\max_{0 \leq t \leq 40} \|u(t)\|_2$.

Table 1. State regulation and peak-input comparison over 40 steps.

Measure	HOFA	DSMC
$T_{10^{-2}}$ (steps)	8	16
$T_{10^{-3}}$ (steps)	11	22
$T_{10^{-4}}$ (steps)	14	29
$\ x(40)\ _2$	3.3657×10^{-15}	1.5502×10^{-6}
$\max_{0 \leq t \leq 40} \ u(t)\ _2$	0.5504	0.8366

Comparing Figure 1 with Figure 2, both controllers exhibit convergent state responses under the same simulation conditions. The HOFA controller assigns the one-period closed-loop spectrum through

the periodic Sylvester-equation design, whereas the DSMC uses a sliding surface and reaching law. Table 1 shows that HOFA reaches the prescribed state-error thresholds in fewer steps and uses a smaller peak control input in this example. The adjusted DSMC also exhibits stable numerical convergence for the tested initial history, although its convergence is slower than that of the HOFA controller.

5. Conclusions

This work has presented a structure-preserving feedback design for the considered nonlinear periodic model. The nonlinear affine contribution is handled at the input level, leaving the high-order periodic dynamics available for closed-loop spectral design. Stability is characterized through the transition over one complete period, while the feedback computation is obtained by relating the actual dynamics to a prescribed periodic target through a Sylvester matrix equation. The resulting similarity relation transfers the assigned spectrum to the closed-loop monodromy matrix. Both controllers show state convergence in the numerical example; HOFA reaches the stated state-error thresholds in fewer steps and has a smaller peak control input under the tested conditions. Theoretical analysis of disturbed and uncertain models and higher-dimensional periodic systems, together with output-feedback realization, remains for future work.

Author contributions

Lingling Lv: Conceptualization, Supervision; Ruijie Fu: Methodology, Writing – original draft; Xingyu Xue: Software, Validation; Liyan Kang: Formal analysis, Writing – review & editing; Sunjie Chen: Supervision, Writing – review & editing.

Use of Generative-AI tools declaration

The authors used ChatGPT to assist with language editing and LaTeX formatting. All manuscript content was reviewed and verified by the authors.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. M. Ghodrat, H. J. Marquez, A multirate approach to event-triggered control of periodic sampled-data systems, *IEEE Trans. Control Netw. Syst.*, **12** (2025), 648–660. <https://doi.org/10.1109/TCNS.2024.3395848>

2. T. Choe, K. Ri, C. Yun, K. Kim, K. Kim, Nonlinear vibration analysis and stability analysis of rotor systems with multiple localized nonlinearities, *AIP Adv.*, **12** (2022), 125004. <https://doi.org/10.1063/5.0128600>
3. Y. Chen, Q. Zong, X. Zhang, J. Li, Data-driven active vibration control for helicopter with trailing-edge flaps using adaptive dynamic programming, *Chin. J. Aeronaut.*, **37** (2024), 151–166. <https://doi.org/10.1016/j.cja.2024.02.019>
4. N. Athanasopoulos, M. Lazar, C. Böhm, F. Allgöwer, On stability and stabilization of periodic discrete-time systems with an application to satellite attitude control, *Automatica*, **50** (2014), 3190–3196. <https://doi.org/10.1016/j.automatica.2014.10.028>
5. A. Czornik, E. Makarov, M. Niezabitowski, S. Popova, V. Zaitsev, Uniform global asymptotic stabilization of semilinear periodic discrete-time systems, *IEEE Trans. Autom. Control*, **67** (2022), 3598–3605. <https://doi.org/10.1109/TAC.2021.3105648>
6. L. Lv, G. Duan, B. Zhou, Parametric pole assignment and robust pole assignment for discrete-time linear periodic systems, *SIAM J. Control Optim.*, **48** (2010), 3975–3996. <https://doi.org/10.1137/080730469>
7. J. Gao, C. Han, Y. Zhao, L. Lv, Output feedback based poles configuration for LDP systems with varying state and input dimensions, *IET Control Theory Appl.*, **17** (2023), 318–330. <https://doi.org/10.1049/cth2.12316>
8. Z. Chen, Y. Guo, Z. Zou, Z. Zhang, An adaptive Levenberg–Marquardt identification algorithm for time-delay linear discrete periodic systems, *J. Ind. Manag. Optim.*, **22** (2026), 3023–3049. <https://doi.org/10.3934/jimo.2026111>
9. S. Sinha, S. P. Nandanoori, J. Drgona, D. Vrabie, Data-driven stabilization of discrete-time control-affine nonlinear systems: A Koopman operator approach, in *2022 European Control Conference*, (2022), 552–559. <https://doi.org/10.23919/ECC55457.2022.9837986>
10. I. Karafyllis, S. Kotsios, Conditions for global asymptotic stabilization of discrete-time systems, *IFAC Proc. Vol.*, **37** (2004), 719–724. [https://doi.org/10.1016/S1474-6670\(17\)31390-3](https://doi.org/10.1016/S1474-6670(17)31390-3)
11. G. Duan, High-order fully actuated system approaches: Part I. Models and basic procedure, *Int. J. Syst. Sci.*, **52** (2021a), 422–435. <https://doi.org/10.1080/00207721.2020.1829167>
12. G. Duan, High-order fully actuated system approaches: Part VII. Controllability, stabilisability and parametric designs, *Int. J. Syst. Sci.*, **52** (2021b), 3091–3114. <https://doi.org/10.1080/00207721.2021.1921307>
13. G. Duan, High-order fully actuated system approaches: Part X. Basics of discrete-time systems, *Int. J. Syst. Sci.*, **53** (2022), 810–832. <https://doi.org/10.1080/00207721.2021.1975848>
14. G. Duan, B. Zhou, Fully actuated system approach for linear systems control: A frequency-domain solution, *J. Syst. Sci. Complex.*, **35** (2022), 2046–2061. <https://doi.org/10.1007/s11424-022-1361-8>
15. D. Gu, S. Wang, A high-order fully actuated system approach for a class of nonlinear systems, *J. Syst. Sci. Complex.*, **35** (2022), 714–730. <https://doi.org/10.1007/s11424-022-2041-4>
16. G. Duan, Discrete-time delay systems: Part 1. Global fully actuated case, *Sci. China Inf. Sci.*, **65** (2022), 182201. <https://doi.org/10.1007/s11432-021-3417-3>
17. D. W. Zhang, G. P. Liu, L. Cao, Predictive control of discrete-time high-order fully actuated systems with application to air-bearing spacecraft simulator, *J. Franklin Inst.*, **360** (2023), 5910–5927. <https://doi.org/10.1016/j.jfranklin.2023.04.003>

18. S. Zhang, Y. Qiu, M. Hou, X. Wu, H. Zhang, J. Wang, Design of a fast model predictive controller based on the fully actuated system approach, *Control Eng. Pract.*, **169** (2026), 106762. <https://doi.org/10.1016/j.conengprac.2026.106762>
19. S. Zhang, Y. Qiu, M. Hou, X. Wu, H. Zhang, J. Wang, Multiple filtered high-gain observers-based robust control for mixed-order fully actuated systems, *J. Franklin Inst.*, **363** (2026), 108599. <https://doi.org/10.1016/j.jfranklin.2026.108599>
20. D. Jin, J. Qiu, W. Ji, T. Wang, Attitude–orbit integrated modeling and control for flexible spacecraft based on twistors and fully actuated system approach, *IEEE Trans. Syst. Man Cybern. Syst.*, **56** (2026), 1369–1378. <https://doi.org/10.1109/TSMC.2025.3647684>
21. S. Jia, J. Qiu, T. Wang, High-order fully actuated system approach-based attitude stabilization for underactuated rigid and flexible spacecraft, *IEEE Trans. Autom. Sci. Eng.*, **22** (2025), 15094–15105. <https://doi.org/10.1109/TASE.2025.3567541>
22. G. Liu, K. Zhang, B. Li, Fully-actuated system approach based optimal attitude tracking control of rigid spacecraft with actuator saturation, *J. Syst. Sci. Complex.*, **35** (2022), 688–702. <https://doi.org/10.1007/s11424-022-1515-8>
23. G. Tian, J. Tan, B. Li, G. Duan, Optimal fully actuated system approach-based trajectory tracking control for robot manipulators, *IEEE Trans. Cybern.*, **54** (2024), 7469–7478. <https://doi.org/10.1109/TCYB.2024.3467386>
24. L. Lv, Z. Zhang, L. Zhang, W. Wang, An iterative algorithm for periodic Sylvester matrix equations, *J. Ind. Manag. Optim.*, **14** (2018), 413–425. <https://doi.org/10.3934/jimo.2017053>
25. C. Ma, T. Yan, A finite iterative algorithm for the general discrete-time periodic Sylvester matrix equations, *J. Franklin Inst.*, **359** (2022), 4410–4432. <https://doi.org/10.1016/j.jfranklin.2022.03.047>
26. L. Lv, J. Chen, L. Zhang, F. Zhang, Gradient-based neural networks for solving periodic Sylvester matrix equations, *J. Franklin Inst.*, **359** (2022), 10849–10866. <https://doi.org/10.1016/j.jfranklin.2022.05.023>
27. S. Li, C. Ma, An improved gradient neural network for solving periodic Sylvester matrix equations, *J. Franklin Inst.*, **360** (2023), 4056–4070. <https://doi.org/10.1016/j.jfranklin.2023.02.019>
28. W. Wang, C. Song, Iterative algorithms for discrete-time periodic Sylvester matrix equations and its application in antilinear periodic system, *Appl. Numer. Math.*, **168** (2021), 251–273. <https://doi.org/10.1016/j.apnum.2021.06.006>
29. R. Qi, C. Song, Iterative algorithms for the generalized discrete-time periodic Sylvester transpose matrix equations with application in the periodic state observer design of linear systems, *J. Appl. Anal. Comput.*, **15** (2025), 286–315. <https://doi.org/10.11948/20240085>
30. L. Lv, W. Lai, X. Liu, A parametric poles assignment algorithm for high-order linear discrete periodic systems, *J. Franklin Inst.*, **360** (2023), 11855–11870. <https://doi.org/10.1016/j.jfranklin.2023.09.010>



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