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Research article

The dressing field method in gauge theories - geometric approach

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Abstract: Recently, T. Masson, J. Francois, S. Lazzarini, C. Fournel and J. Attard have introduced a new method of the reduction of gauge symmetries called the dressing field method. In this paper we analyse this method from the fiber bundle point of view and we show the geometric implications for a principal bundle underlying a given gauge theory.We show how the existence of a dressing field satisfying certain conditions naturally leads to the reduction of the principal bundle and, as a consequence, to the reduction of the configuration and phase bundle of the system.

Keywords: dressing field method; gauge theories; jet bundles; principal bundles; reduction
Mathematics Subject Classification: Primary: 53Z05; Secondary: 70S05,70S10, 70S15

1. Introduction

Gauge field theories form a theoretical basis of the modern understanding of fundamental interactions and the research on them is in the very center of theoretical physics. Perhaps the most prominent example of gauge theories are Yang-Mills theories, which provide the description of electroweak and strong interaction between elemetary particles [32]. From the mathematical point of view, gauge field theory found its mathematical formulation in the geometry of principal bundles [5, 22]. Usually, one takes a principal bundle $\pi : P \to M$ with a structure group G, where M represents a physical spacetime. The crucial object here is a connection in the principal bundle, which may be defined by means of the one-form $\omega : \mathsf{T}P \to \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the group G. From the physical point of view connections in P represent gauge fields of the given theory [8, 24]. Therefore, the space of gauge fields is the space of connections in the principal bundle $\pi : P \to M$. On the other hand, the Lagrangian of a system in classical field theory is usually a map $L : \mathsf{J}^1E \to \Omega^n$, where $\tau : E \to M$ is a bundle of fields, J^1E is the first jet bundle of τ and Ω^n is the space of n-covectors on M. Here, we assumed that the dimension of the base manifold M is n. Therefore, one has to consider first jets of the connection form when analysing the structure of gauge theory. These two structures, namely principal bundles and jet bundles, form the geometrical basis of the description of gauge field theories [24, 25]. In our paper we will focus more on the description of kinematics, i.e. on the fibre bundles appearing in gauge theory, than on derivation of the dynamics.

However, despite its numerous successes, the mathematical description of gauge fields still encounters certain significant problems. Perhaps the most serious one is the fact that the existence of a gauge symmetry makes the quantization of such a theory very problematic [31]. Therefore, one of the main problems of gauge theories is the problem of finding a proper way to reduce the gauge symmetry of the system [2].

There are three main aproaches to the reduction of gauge symmetries existing in the literature. The most popular in experimental physics and the simplest one from the conceptual point of view is the so-called gauge fixing [14, 27]. Since the system has a gauge symmetry, all the fields belonging to the same orbit of the gauge transformation describe the same physical state. It means that we can "by hand" choose a particular gauge in such a way that, for example, the calculations take the simplest possible form. The second approach is based on the spontaneous symmetry breaking mechanism. This approach is a basis of the famous Higgs mechanism, which solved the problem of the mass generation for bosons carrying electroweak interaction [15, 16]. The last one, and the most geometric one, is related to the bundle reduction theorem. It turns out that in certain situations, the principal bundle with a given structure group may be reduced to its subbundle with a smaller structure group [28, 30].

In our paper we will focus on a new method of gauge symmetries reduction, which is called the dressing field method. This approach has been recently discovered mainly by J. Attard, J. François, S. Lazzarini and T. Masson [2, 9, 11, 21]. The main idea of this approach is to introduce into a theory certain auxiliary field, which does not belong to the original space of fields of the given gauge theory. In the next step one performs a change of variables in which the original gauge fields are transformed into new fields, which are a combination of the original gauge fields and the dressing field. This procedure is known as a dressing of the gauge field. The new fields, called the dressed fields, are the new variables of the theory. In favorable situations these new fields are invariant under the action of a Lie subgroup of the structure group G. It means that the symmetry related to this subgroup has been erased. Let us stress that the authors approach to the entire method is rather algebraic and they do not delve into the geometric character of the dressing procedure. Our attitude is, in a sharp contrast, strictly geometric with the principal bundle and jet bundle geometry as a basis.

The aim of our paper is to apply the dressing field method to the reduction of the Lagrangian formalism of (classical) gauge field theories in a sense of [24]. In this formulation the space of gauge fields is a bundle $C \to M$, where $C = \mathsf{J}^1P/G$, and the Lagrangian is a map $L : \mathsf{J}^1C \to \Omega^n$ [24]. The task of reduction requires to explore the dressing field method from the geometrical point of view. Since the basis of the mathematical formulation of gauge theories are the principal bundle geometry and jet spaces, we show how the existence of the dressing field affects both geometries. An important tool throughout the paper will be the decomposition of the jet bundle J^1C that may be found in [24]. In our work we proceed in three steps. First, we assume that the structure group contains a Lie subgroup H together with a dressing field $u : P \to H$. In the second step we assume that there is a decomposition $G = JH$, where $H \subset G$ is a normal subgroup of G. In the last step we analyse the case $G = H \times J$, i.e. when the structure group is a direct product of its subgroups. The last step is particularly important in the reduction of the gauge symmetry in the electroweak theory, where the underlying group is $G = SU(2) \times U(1)$. The conclusion from our work is that in favourable situations the bundle C may be reduced to a "smaller" space of fields, which in turn leads to the reduction of the bundle J^1C.

The paper is organised as follows. In section 2 we review the geometric fundamentals of gauge field theories. In particular, we show how the connection in $\pi : P \to M$ may be represented as a suitable section of the first order jet bundle $\mathrm{J}^1P \to P$ and we recall the concept of gauge transformations with three different pictures of this notion. In section 3 we briefly introduce the reader to the dressing field method basing mainly on [2]. Section 4 is the core of our paper. There, we discuss a geometric interpretation of the existence of the dressing field on P. We show that, under suitable assumptions, the existence of the dressing field implies that the original principal bundle may be reduced to a smaller principal bundle with a structure group isomorphic to G/H. This reduction naturally leads to the simplification of the entire structure of the given gauge theory. It turns out that the level of the possible reduction strongly depends on the structure of the structure group G, the issue which is discussed by us in detail.

2. Geometric formulation of gauge theories

In this section we will briefly recall the geometry of principal bundles and the first order jet bundles. We will also fix the notation necessary for our subsequent work. The main element of this chapter is Subsection 2.4 where we show how to represent a connection in the principal bundle as a section of the proper jet bundle. The introduction concerning principal bundles and connections in them is in large based on [8].

2.1. Principal bundles and adjoint bundles

Let G be a Lie group with the Lie algebra $\mathfrak{g}$. We denote by P a smooth manifold such that G acts on it from the right in a smooth, free and proper way. Then, the space $M := P/G$ of orbits of the action of G on P is a smooth manifold as well. The bundle $\pi : P \to M$ is called a principal bundle with the structure group G. Let $U_\alpha \subset M$ be an open subset in M. A local trivialisation of the principal bundle is a G-equivariant diffeomorphism

$$\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G, \qquad \Psi_\alpha(p) = (\pi(p), g_\alpha(p)),$$

where $g_\alpha : \pi^{-1}(U_\alpha) \to G$ is the G-valued function associated with the map Ψ_α. The equivariance condition means that $\Psi_\alpha(pg) = \Psi_\alpha(p)g$, which implies that g_α is also G-equivariant, says $g_\alpha(pg) = g_\alpha(p)g$. Notice that the function g_α uniquely defines a local trivialisation of P. The transition between trivialisations g_α and g_β defined on $\pi^{-1}(U_\alpha \cap U_\beta)$ is realised by the function

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G, \qquad g_{\alpha\beta}(\pi(p)) = g_\alpha(p)g_\beta(p)^{-1}.$$

Let F be a smooth manifold and let G act on F from the left. We introduce an action of G on product $P \times F$ given by

$$g(p, f) = (pg, g^{-1}f).$$

We denote by $N := (P \times F)/G$ the space of orbits of this action. The bundle

$$\xi : N \to M, \quad [(p, f)] \to \pi(p),$$

where $[(p, f)]$ is the orbit of the element $(p, f) \in P \times F$, is called the *associated bundle* of a principal bundle P. Notice that the above projection does not depend on a choice of the representative in $[(p, f)]$, therefore it is well-defined. For F being a vector space, the associated bundle is a linear bundle over M. The most important examples of associated bundles of P in the context of our work are the bundles with fibers $F = \mathfrak{g}$ and $F = G$, i.e. $N = (P \times \mathfrak{g})/G$ and $N = (P \times G)/G$. From now on, we will use the notation

$$\mathrm{ad}(P) := (P \times \mathfrak{g})/G \qquad \text{and} \qquad \mathrm{Ad}(P) := (P \times G)/G.$$

The action of G on $\mathfrak{g}$ and the action of G on G is given by the adjoint map, namely

$$\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}, \quad (g, X) \longmapsto \mathrm{Ad}_g(X),$$

$$\mathrm{Ad} : G \times G \to G, \quad (g, h) \longmapsto \mathrm{Ad}_g(h),$$

respectively. To simplify the notation we have denoted both actions by the same symbol Ad.

Denote by $\Omega^k(M, \mathfrak{g})$ the bundle of $\mathfrak{g}$-valued k-forms on M. Let $\{U_\alpha\}$ be an open covering of M and let $\{\xi_\alpha\}$ be a family of local k-forms on M such that $\xi_\alpha \in \Omega^k(U_\alpha, \mathfrak{g})$ for each $\alpha \in I \subset \mathbb{R}$. We also require that for each overlapping $U_\alpha \cap U_\beta$ the condition

$$\xi_\alpha(m) = \mathrm{Ad}_{g_{\alpha\beta}(m)} \circ \xi_\beta(m), \qquad m \in U_{\alpha\beta}, \quad g_{\alpha\beta} : U_{\alpha\beta} \to G \tag{2.1}$$

is satisfied. We claim that the family of k-forms $\{\xi_\alpha\}$ defines a k-form on M with values in ad$P$. The space of ad$P$-valued k-forms on M will be denoted by $\Omega^k(M, \mathrm{ad}P)$.

2.2. Connection in a principal bundle

Let $\mathsf{V}P$ be the vertical subbundle in $\mathsf{T}P$, i.e. the set of tangent vectors, which are tangent to the fibers of $\pi : P \to M$. A connection in the principal bundle $\pi : P \to M$ is a G-invariant distribution $\mathcal{D}$ in $\mathsf{T}P$, which is complementary to $\mathsf{V}P$ at each point $p \in P$. By definition we have

$$\mathsf{T}_pP = \mathsf{V}_pP \oplus \mathcal{D}_p, \qquad p \in P \tag{2.2}$$

and

$$\mathcal{D}_pg = \mathcal{D}_{pg}, \quad g \in G. \tag{2.3}$$

The above definition is very elegant and general, however, when it comes to applications, it is more convenient to represent a connection in a different way. We will start with introducing some basic mathematical tools. Let X be an element of $\mathfrak{g}$. The group action of G on P defines the vertical vector field σ_X on P associated with the element X, namely

$$\sigma_X(p) := \frac{\mathsf{d}}{\mathsf{d}t}_{|t=0} p \exp(tX).$$

The field σ_X is called the fundamental vector field corresponding to the element X. The fundamental vector field is equivariant in the sense that

$$\sigma_X(pg) = \sigma_{\mathrm{Ad}_{g^{-1}}(X)}(p).$$

A connection form in a principal bundle P is a G-equivariant, $\mathfrak{g}$-valued one-form ω

$$\omega : \mathsf{T}P \to \mathfrak{g},$$

such that $\omega(\sigma_X(p)) = X$ for each $p \in P$ and $X \in \mathfrak{g}$. The G-equivariance means that

$$R_g^*\omega(p) = \mathrm{Ad}_{g^{-1}} \circ \omega(p),$$

where R stands for the right action of G on P. Notice that the distribution $\mathcal{D}_p := \ker\omega(p)$ defines a connection in P. Since a connection form is an identity on vertical vectors, the difference of two connections is a horizontal form. It follows that the space of connections is the space of sections of an affine subbundle $\mathcal{A} \subset \mathsf{T}^*P \otimes \mathfrak{g}$ modelled on the vector bundle of $\mathfrak{g}$-valued, G-equivariant horizontal one-forms on P. It turns out that the space of such horizontal forms may be identified with the space of sections of the bundle $\mathsf{T}^*M \otimes_M \mathrm{ad}P \to M$.

The curvature of a connection is the $\mathfrak{g}$-valued two-form $\Omega_\omega := (\mathrm{d}\omega)^h$, where $(\mathrm{d}\omega)^h$ is the horizontal part of the two-form $\mathrm{d}\omega$. After some basic calculations one can show that

$$\Omega_\omega = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega], \tag{2.4}$$

where $[\omega \wedge \omega]$ is the bracket of forms on P with values in $\mathfrak{g}$ [8]. The curvature form, same as the connection form, is equivariant in the sense that $R_g^*\Omega_\omega = \mathrm{Ad}_{g^{-1}} \circ \Omega_\omega$. The curvature form is horizontal and G-equivariant therefore it defines the $\mathrm{ad}P$-valued two-form

$$F_\omega : M \to \wedge^2\mathsf{T}^*M \otimes_M \mathrm{ad}P. \tag{2.5}$$

From now on we will use the following notation

$$\mathcal{V} := \mathsf{T}^*M \otimes_M \mathrm{ad}P, \tag{2.6}$$

$$\mathcal{F} := \wedge^2\mathsf{T}^*M \otimes_M \mathrm{ad}P. \tag{2.7}$$

2.3. Jet bundles

Let us briefly recall the notion of first order jet spaces, which will be important in our work later on. In our presentation we will follow the notation from [12]. For a more detailed discussion of the jet bundle geometry see e.g. [6, 26].

Let $\pi : E \to M$ be a smooth fibration, where $\dim M = n$ and $\dim E = n + l$. We introduce a local coordinate system $(x^i)_{i=1}^n$ in a domain $U \subset M$. In field theory fields are represented by sections of the fibration π. The total space E is the space of values of the field e.g. a vector field is a section of the π being a tangent bundle, scalar field is a section of the trivial bundle $E = M \times \mathbb{R}$ or $E = M \times \mathbb{C}$, etc. On an open subset $V \subset E$ such that $\pi(V) = U$ we introduce local coordinates (x^i, u^α) adapted to the structure of the bundle. We will also need the dual bundle of the vertical subbundle $\mathsf{V}E \subset \mathsf{T}E$, which we will denote by $\mathsf{V}^*E \to E$.

The space of first jets of sections of the bundle π will be denoted by J^1E. By definition, the first jet $\mathsf{j}^1_m\phi$ of a section ϕ at the point $m \in M$ is an equivalence class of sections having the same value at the point m and such that the spaces tangent to the graphs of the sections at the point $\phi(m)$ coincide. Therefore, there is a natural projection $\pi_{1,0}$ from the space J^1E onto the manifold E

$$\pi_{1,0} : \mathsf{J}^1 E \to E : \quad \mathsf{j}^1_m \phi \longmapsto \phi(m).$$

Moreover, every first jet $\mathsf{j}^1_m\phi$ may be identified with the linear map $\mathsf{T}\phi : \mathsf{T}_m M \to \mathsf{T}_{\phi(m)} E$. Linear maps coming from first jets at the point m form an affine subspace in the vector space $Lin(\mathsf{T}_m M, \mathsf{T}_e E)$ of all linear maps from $\mathsf{T}_m M$ to $\mathsf{T}_e E$, where $e = \phi(m)$. A map belongs to this subspace if composed with $\mathsf{T}\pi$ gives identity. In the tensorial representation we have an inclusion

$$\mathsf{J}^1_e E \subset \mathsf{T}^*_m M \otimes \mathsf{T}_e E.$$

The affine space $\mathsf{J}^1_e E$ is modelled on the vector space $\mathsf{T}^*_m M \otimes \mathsf{V}_e E$. Summarising, the bundle $\mathsf{J}^1 E \to E$ is an affine bundle modelled on the vector bundle

$$\pi^*(\mathsf{T}^* M) \otimes_E \mathsf{V}E \to E.$$

The symbol $\pi^*(\mathsf{T}^* M)$ denotes the pullback of the cotangent bundle $\mathsf{T}^* M$ along to the projection π. In the following we will omit the symbol of the pullback writing simply $\mathsf{T}^* M \otimes_E \mathsf{T}E$ and $\mathsf{T}^* M \otimes_E \mathsf{V}E$.

A connection in the fibration $E \to M$ may be expressed in terms of jet bundles. Indeed, each first jet $\mathsf{j}^1_m\phi$ defines the decomposition of the tangent space

$$\mathsf{T}_{\phi(m)} E = \mathsf{V}_{\phi(m)} E \oplus \mathsf{T}\phi(\mathsf{T}_m M).$$

Notice that $\mathsf{T}\phi(\mathsf{T}_m M)$ does not depend on the choice of a representative in $\mathsf{j}^1_m\phi$. Therefore, each section $\Gamma : E \to \mathsf{J}^1 E$ defines a connection in the bundle $E \to M$.

In the first order field theory the bundle $\mathsf{J}^1 E$ is often called the space of infinitesimal configurations. A Lagrangian of the system is usually a map

$$L : \mathsf{J}^1 E \to \Omega^n, \tag{2.8}$$

where Ω^n is the space of n-covectors on the n-dimensional manifold M [3, 4, 12]. The phase space of the system is the space

$$\mathcal{P} = \mathsf{V}^* E \otimes_E \Omega^{n-1},$$

which is a vector bundle over E. The literature concerning the mathematical formulation of the first and higher order field theory is very rich. One can find for instance the detailed discussion of this topic in the language of the multisymplectic geometry [3, 4, 18], k-symplecitc structures [19, 23], Tulczyjew triples [12, 13] and other approaches [25, 29, 17].

2.4. Principal connection as a section of a jet bundle

Let us consider the first jet bundle of a principal bundle $P \to M$, i.e. the bundle $\mathsf{J}^1 P \to P$. The action of the group G on P may be lifted to the action of G on $\mathsf{J}^1 P$ in a following way

$$\mathsf{J}^1 R_g : \mathsf{J}^1 P \to \mathsf{J}^1 P : \quad \mathsf{j}^1_m\phi \to \mathsf{j}^1_m(\phi g),$$

where ϕg is a section of P such that $(\phi g)(m) := \phi(m) g$. Accordingly, we present the following commutative diagram

$$\begin{array}{ccc} \mathsf{J}^1 P & \xrightarrow{\mathsf{J}^1 R_g} & \mathsf{J}^1 P \\ {\scriptstyle \pi_{1,0}}\downarrow & & \downarrow{\scriptstyle \pi_{1,0}} \\ P & \xrightarrow{R_g} & P \end{array}$$

We will say that a section $\Gamma : P \to \mathsf{J}^1P$ is invariant under the action of G if $\Gamma(pg) = \Gamma(p)g$. Each invariant section of the jet bundle $\mathsf{J}^1P \to P$ defines a connection in the bundle P. The invariance of the section implies invariance of the horizontal distribution defined by this section. Notice that each invariant section $\Gamma : P \to \mathsf{J}^1P$ defines a section

$$\Gamma : M \to C, \qquad \text{where} \qquad C = \mathsf{J}^1P/G.$$

The bundle $C \to M$ is therefore the bundle of principal connections in P. Each section of $C \to M$ defines a connection in the bundle $P \to M$. In gauge theories these sections represent gauge fields of a given theory. From the geometrical point of view $C \to M$ is an affine bundle modelled on the vector bundle $\mathsf{T}^*M \otimes_M \mathrm{ad}P \to M$. The bundle of infinitesimal configurations for a given gauge field theory is the bundle J^1C, while the phase bundle is

$$\mathcal{P} = \mathsf{V}^*C \otimes_C \Omega^{n-1} \simeq C \times_M \mathcal{V}^* \otimes_M \Omega^{n-1} \simeq C \times_M \mathsf{T}M \otimes_M \mathsf{T}M \otimes_M \mathrm{ad}^*P \otimes_M \Omega^n.$$

We will introduce the notation $\overline{\mathcal{P}} := C \times_M \mathsf{T}M \otimes_M \mathsf{T}M \otimes_M \mathrm{ad}^*P \otimes_M \Omega^m$ so that $\mathcal{P} = C \times_M \overline{\mathcal{P}}$. It turns out that the configuration bundle J^1C has a very rich internal structure. One can show that there exists a canonical splitting over C

$$\mathsf{J}^1C = \mathsf{J}^2P/G \oplus_C \mathcal{F}, \tag{2.9}$$

with the natural projections

$$pr_2 : \mathsf{J}^1C \to \mathcal{F},$$
$$(x^i, A^a_j, A^b_{jk}) \longmapsto \left(x^i, A^a_j, \frac{1}{2}(A^l_{jk} - A^l_{kj} + c^l_{ab}A^a_jA^b_k)\right),$$
$$pr_1 : \mathsf{J}^1C \to \mathsf{J}^2P/G,$$
$$(x^i, A^a_j, A^b_{jk}) \longmapsto \left(x^i, A^a_j, \frac{1}{2}(A^l_{jk} + A^l_{kj} - c^l_{ab}A^a_jA^b_k)\right).$$

Here, c^l_{ab} are the structure constants of the Lie algebra of G. Notice that the result of the first projection is just the curvature form

$$F_\omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega],$$

of a given connection. In coordinates F_ω reads

$$F_\omega = \frac{1}{2}F^a_{i\,j}\mathrm{d}x^i \wedge \mathrm{d}x^j \otimes e_a, \qquad F^a_{i\,j} = \partial_jA^a_i - \partial_iA^a_j + c^a_{mn}A^m_iA^n_j.$$

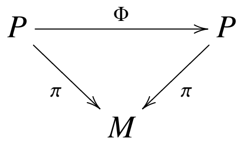
The details concerning above decomposition may be found in [24]. Let us notice that the relation (2.9) implies that the first jet $\mathrm{j}^1_m\omega$ of a section $\omega : M \to C$ at the point m may be decomposed on the curvature of ω at the point m and on some element of J^2P/G. For a more detailed discussion of the mathematical formulation of gauge theories see e.g. [7, 20, 24].

In the end let us see that the decomposition (2.9) leads to a significant simplification of the Lagrangian description of gauge theories. According to (2.8) Lagrangian in gauge field theories is a map $L : \mathsf{J}^1C \to \Omega^n$. It is a common situation in physics that the Lagrangian of a system does not depend on the entire first jet of a connection but only on the value of the connection and the value of its curvature in a given point. In such a case, the Lagrangian may be reduced to a map

$$L : C \times \mathcal{F} \to \Omega^n. \tag{2.10}$$

2.5. *Three pictures of gauge transformations*

We will present now three equivalent pictures of the notion of gauge transformation and discuss basic properties of each of them. Let $\pi : P \to M$ be a principal bundle with a structure group G. A gauge transformation of the bundle P is an equivariant diffeomorphism $\Phi : P \to P$, such that the diagram



is commutative. The equivariance of Φ means that the condition $\Phi(pg) = \Phi(p)g$ is satisfied. Notice that by definition each gauge transformation is an automorphism of P over the identity on M. The set of gauge transformations (automorphisms) of the bundle P will be denoted by $Aut(P)$ and by definition we have

$$Aut(P) = \{\Phi : P \to P, \quad \Phi(pg) = \Phi(p)g, \quad \pi \circ \Phi = \pi, \quad p \in P, \quad g \in G\}.$$

The set $Aut(P)$ is a group with respect to composition of maps. One can check that if Φ_1 and Φ_2 are automorphisms of P then $\Phi_1 \circ \Phi_2$ is an automorphism of P as well. Each automorphism of the principal bundle over the identity on M may be written in a form

$$\Phi(p) = pf(p), \quad f : P \to G,$$

where f is a unique G-valued function on P associated with Φ. The equivariance of the map Φ implies the condition

$$f(pg) = \mathrm{Ad}_{g^{-1}} f(p). \tag{2.11}$$

One can check that if Φ_1, Φ_2 are realised by functions f_1 and f_2 respectively, then

$$\Phi_1 \circ \Phi_2(p) = pf_1(p)f_2(p),$$

which means that the gauge transformation $\Phi_1 \circ \Phi_2$ is realised by the function $f_1 \cdot f_2$. The set of gauge transformations understood as functions on P with values in G form therefore a group with respect to the multiplication of functions. We denote this group by

$$\mathcal{G} := \{f : P \to G, \quad f(pg) = \mathrm{Ad}_{g^{-1}} f(p), \quad p \in P, \quad g \in G\}. \tag{2.12}$$

On the other hand, each function on P with values in G is a section of the trivial bundle $P \times G \to P$ given by the formula

$$\sigma_f : P \to P \times G, \quad p \longmapsto (p, f(p)).$$

From the fact that the function f satisfies the condition (2.11) we obtain that the section σ_f defines a section of $\mathrm{Ad}P \to M$ given by the formula

$$\bar{\sigma}_f : M \to \mathrm{Ad}P, \quad m \longmapsto [p, f(p)], \quad \pi(p) = m, \tag{2.13}$$

where $[p, f(p)]$ is the equivalence class of the element $(p, f(p))$ in $\mathrm{Ad}P$. The set $\Gamma(\mathrm{Ad}P, M)$ of sections of the bundle $\mathrm{Ad}P \to M$ forms a group with respect to the multiplication of sections.

From above considerations we obtain a gauge transformation may be seen in three different ways: as an automorphism of the underlying principal bundle, as a function on P with values in G satisfying (2.11) and as a section of the bundle $\mathrm{Ad}P \to M$. In our subsequent work we will mostly rely on the first and second approach.

2.6. Gauge transformations of connection and curvature

The group of gauge transformations acts in a natural way on a connection, curvature and covariant derivative in the bundle P. Let $\mathcal{D}$ be a principal connection in P, i.e. a distribution in $\mathsf{T}P$ satisfying the conditions (2.2) and (2.3). A gauge transformation of the connection $\mathcal{D}$ is a distribution

$$\mathcal{D}^{\Phi} := \Phi_* \mathcal{D}.$$

One can check that $\mathcal{D}^{\Phi}$ satisfies (2.2) and (2.3), therefore it defines a connection in P as well. In terms of connection forms, the distribution $\mathcal{D}^{\Phi}$ is related to the connection form $\omega^{\Phi} := \Phi^* \omega$. Let us take a closer look on the form ω^{Φ}. Let $v \in \mathsf{T}_p P$ be a tangent vector represented by a curve γ, and $\Phi(p) = pf(p)$ a gauge transformation. It is a matter of computation to check that

$$\Phi^* \omega(p) = \mathrm{Ad}_{f(p)^{-1}} \circ \omega + f(p)^{-1} \mathrm{d} f(p). \tag{2.14}$$

Notice that in the above formula f is a function with values in G, which means that $\mathrm{d}f(p) \in \mathsf{T}^*_p P \otimes \mathsf{T}_{f(p)} G$ and $f(p)^{-1} \mathrm{d} f(p) \in \mathsf{T}^*_p P \otimes \mathfrak{g}$.

The action of a gauge tranformation may be similarly extended to other objects associated with the principal bundle. Let Ω be the curvature form of the connection ω defined as in (2.4), and $D_\omega \alpha$ the covariant derivative of a section $\alpha : M \to \wedge^k \mathsf{T}^* M \otimes_M \mathrm{ad}P$ with respect to the form ω and representation ρ. Performing a similar calculation as for $\Phi^* \omega$ one can derive transformation formulas

$$\Phi^* \Omega = \mathrm{Ad}_{f^{-1}} \circ \Omega,$$

$$\Phi^* D_\omega \alpha = \rho(f^{-1}) D_\omega \alpha.$$

3. Reduction of gauge symmetries - the dressing field method

Let us briefly present main features of the dressing field method approach to the reduction of gauge symmetries. A detailed discussion of the notion of the dressing field together with numerous examples may be found in [1, 2, 9, 11, 21]. The existence of a gauge symmetry means that the Lagrangian is invariant under the action of the group of gauge transformations describing the symmetry. The main idea behind the dressing field method is to introduce a certain auxiliary field, which, in a general case, does not belong to the original space of gauge fields. We call this auxiliary field a *dressing field*. In the next step, we define a transformation of the gauge fields that depends on the dressing field on the one hand and on the original gauge fields on the other. It turns out that in certain situations these new fields are invariant under the action of the gauge group (or its subgroup), which in turns means that the symmetry of the system has been (fully or partially) reduced.

We will move now to the technical aspects of the dressing field method. Let $\pi : P \to M$ be a principal bundle with a structure group G. Assume that there exists a fixed Lie subgroup H in G. We recall that the gauge group of the theory (2.12) is given by

$$\mathcal{G} := \Big\{ f : P \to G, \quad f(pg) = \mathrm{Ad}_{g^{-1}} f(p), \quad g \in G \Big\}.$$

We introduce now a following set of maps associated with the subgroup H

$$\mathcal{H} := \Big\{ f^{\mathcal{H}} : P \to H, \quad f^{\mathcal{H}}(ph) = \mathrm{Ad}_{h^{-1}} f^{\mathcal{H}}(p), \quad h \in H \subset G \Big\}.$$

Notice that each element of $\mathcal{H}$ defines a map

$$\Phi^{\mathcal{H}} : P \to P, \quad \Phi^{\mathcal{H}}(p) = pf^{\mathcal{H}}(p), \tag{3.1}$$

which, despite the similarity of the notation, is not a gauge transformation. Let us introduce a function

$$u : P \to H,$$

which has a following transformation rule with respect to H

$$u(ph) = h^{-1}u(p). \tag{3.2}$$

The function u will be called a *dressing field*. It is a crucial object in the whole dressing field method. Let us stress that the condition (3.2) implies that u is not an element of $\mathcal{H}$. In the next step we use u to define a map

$$\Phi^{u} : P \to P, \quad \Phi^{u}(p) = pu(p), \tag{3.3}$$

which is called a *dressing map*. One can see that it satisfies the relation $\Phi^u(ph) = \Phi^u(p)$. Notice that the above map is not bijective, therefore it does not define a gauge transformation. However, the existence of Φ^u provides a decomposition of the bundle P. Notice that the condition $\Phi^u(ph) = \Phi^u(p)$ implies that Φ^u is constant on the orbits of action of the subgroup H. It means that Φ^u defines a section of the bundle $P \to P/H$. The global section of the principal bundle uniquely defines a global trivialisation of that bundle. Therefore, we obtain that $P = P/H \times H$. It turns out that there exists the opposite implication as well, i.e. the decomposition $P = P/H \times H$ defines a suitable dressing field. Indeed, one can show that the existence of the field $u : P \to H$ is equivalent to the existence of the decomposition $P = P/H \times H$ [10].

The map Φ^u naturally acts on the space of principal connections. Let ω be a connection form on P. We say that the map ω^u given by

$$\omega^u : \mathsf{T}P \to \mathfrak{g}, \qquad \omega^u := (\Phi^u)^*\omega.$$

is a dressing of the connection form ω. It is a matter of straightforward computations to derive the formula

$$\omega^u = \mathrm{Ad}_{u^{-1}} \circ \omega + u^{-1}\mathrm{d}u. \tag{3.4}$$

Using (3.4) one can easily show that the form ω^u is invariant under the action of H. Furthermore, it is also invariant under the transformations given by (3.1). Indeed, for each $f \in \mathcal{H}$ by definition we have

$$(\Phi^{\mathcal{H}})^*\omega^u = (\Phi^u \circ \Phi^{\mathcal{H}})^*\omega.$$

On the other hand

$$\Phi^u \circ \Phi^{\mathcal{H}}(p) = \Phi^u(pf^{\mathcal{H}}(p)) = pf^{\mathcal{H}}(p)u(pf^{\mathcal{H}}(p)) = pf^{\mathcal{H}}(p)f^{\mathcal{H}}(p)^{-1}u(p) = pu(p) = \Phi^u(p),$$

which implies that

$$(\Phi^{\mathcal{H}})^*\omega^u = (\Phi^u)^*\omega = \omega^u.$$

In a similar way we introduce the dressed curvature, which is a $\mathfrak{g}$-valued two-form

$$\Omega^u := (\Phi^u)^*\Omega.$$

One can check that, same as ω^u, it is invariant under the action of H and transformations given by (3.1). Notice that in the above calculations we have used (3.2), which means that it is valid only for functions f with values in H.

The idea behind the dressing field method is as follows. The Lagrangian of the system is a function depending on gauge fields and their curvatures. By introducing a proper dressing field we can define dressed gauge fields and dressed curvatures. It is a natural question then whether it is possible to write down the Lagrangian in terms of dressed fields instead of original variables. If the answer is positive then the part of the gauge symmetry associated with $\mathcal{H}$ has been reduced. Let us emphasize two particular features of the above construction. First of all, the dressing transformation given by (3.3) is not a gauge transformation, despite the algebraic similarity of both maps. It means that the fields ω^u and ω, in a general case, do not belong to the same orbit of the action of $\mathcal{G}$. Therefore, the field ω^u is not, in general, an element of the original space of gauge fields. Secondly, the form of the dressing field has to be deduced ad hoc, basing on the specific form of the Lagrangian.

In the end, let us see the above constructions in application to the example coming from the theory of electroweak interaction. Let ρ be a representation of the group $G = SU(2) \times U(1)$ on $\mathbb{C}^2$ given by

$$\rho : G \to End(\mathbb{C}^2), \quad \rho(b, a)v = bav.$$

We denote the restrictions of ρ to $U(1)$ and $SU(2)$ by ρ_1 and ρ_2, respectively. Let $E \to M$ be the associated bundle of P with respect to the above representation and with a typical fiber $\mathbb{C}^2$. We consider a section $\phi : M \to E$, which, by definition, defines a section of the trivial bundle $\bar{\phi} : P \to P \times \mathbb{C}^2$ satisfying the condition

$$\bar{\phi}(pg) = g^{-1}\phi(p), \qquad g = (b, a) \in SU(2) \times U(1), \tag{3.5}$$

where $g^{-1}\phi(p)$ is the matrix multiplication of a vector $\bar{\phi}(p) \in \mathbb{C}^2$ by the pair of matrices $g^{-1} \in SU(2) \times U(1)$. The group $SU(2)$ is a subgroup in $SU(2) \times U(1)$. Section $\bar{\phi}$ defines a map $u : P \to SU(2)$ given by the formula

$$\bar{\phi}(p) = u(p)\eta, \qquad \text{where} \qquad u : P \to SU(2), \quad \eta = \begin{pmatrix} 0 \\ ||\bar{\phi}|| \end{pmatrix}. \tag{3.6}$$

It is easy to check that the condition (3.5) implies that $u(pb) = b^{-1}u(p)$ for $b \in SU(2)$, which means that u is indeed a dressing field.

4. Geometric approach to the dressing field method

In this section we will present the main result of our paper, which is the geometric interpretation of the dressing field method in the presence of the residual symmetry of u. We will consider the situation when u transforms under J with respect to the adjoint action and when there is a decomposition $G = JH$. In particular we will focus on a case when G is a direct product of H and J. It is one of the two main situations originally considered in [2]. Such a situation occurs for instance in the electroweak theory where the structure group is $SU(2)\times U(1)$. We will show how the existence of such a dressing field leads to the reduction of the underlying principal bundle and, as a consequence, reduction of the configuration and phase bundle of the theory.

4.1. Dressing map as a principal bundle

Let us assume that there exist fixed Lie subgroups J and H in G, such that H is a normal subgroup in G and each element of G may be uniquely written in the form $g = jh$, i.e. $G = JH$ and $J \cap H = e$. Such a situation occurs for instance when G is a direct or a semidirect product of H and J. Then G/H has a structure of a Lie group as well and $G/H \simeq J$. The Lie algebra $\mathfrak{j}$ of the group J is isomorphic to $\mathfrak{g}/\mathfrak{h}$.

Let us stress that such a situation is briefly discussed (without additional assumptions on u) in [2] and afterwards the authors consider more deeply the case when u additionaly satisfies certain transformation conditions with respect to the group J. We will show now that if $G = JH$ then, even without these additional assumptions on u, the dressing map Φ^u defines a principal bundle.

Notice that the bundle

$$G \to J$$

is a principal bundle with a structure group H. Let us introduce the notation $P^J := P/H$. The fibrations

$$\pi_{P^J} : P \to P^J,$$

$$\pi_J : P^J \to M,$$

are principal bundles with the structure groups H and J, respectively.

Let us fix now a dressing field u on P. It defines the embedding of P^J in P given by

$$P \supset P^J = u^{-1}(e).$$

Notice that $u(\Phi^u(p)) = e$, so the preimage $u^{-1}(e)$ is the image of a section $P/H \to P$ defined by Φ^u. The set P^J is therefore a submanifold in P, on which Φ^u acts in a trivial way. The above embedding defines the trivialisation of P given by

$$P \to P^J \times H, \quad p \longmapsto (p_0, h), \quad \text{where} \qquad p_0 = ph, \quad h = u(p). \tag{4.1}$$

Let us notice that $u(p_0) = u(ph) = h^{-1}u(p) = e$, so that $ph \in P^J$. One can check how Φ^u behaves under the above decomposition. Let $p \in P$ be a point, which in the identification (4.1) has a form (p_0, h). Acting Φ^u on p we get

$$\Phi^u(p) = \Phi^u(p_0 h^{-1}) = \Phi^u(p_0) = p_0.$$

From the above calculation we obtain that Φ^u is the projection on the submanifold P^J, i.e.

$$\Phi^u = \pi_{P^J}.$$

Therefore, it turns out that in a case when the structure group has the decomposition $G = JH$, the existence of the dressing field u is equivalent to the existence of the embedding $P^J \simeq P/H$ in P, and, as a consequence, to the existence of the decomposition $P = P^J \times H$ over M with Φ^u being the projection on P^J.

4.2. Residual gauge symmetry and reduced connection form

A gauge symmetry of the dressed field depends on the original gauge field on the one hand and on the dressing field on the other. In practical applications it is usually important how u transforms with respect to the action of J. In the following part of this section we will analyse the case

$$R_j^* u = \mathrm{Ad}_{j^{-1}} u, \tag{4.2}$$

which has applications in the BRST differential algebra. The algebraic discussion of the above case may be found in [2]. Our aim is to analyse the properties of the dressed gauge field under (4.2).
Let us notice that the condition (4.2) implies

$$\Phi^u(pj) = pju(pj) = pu(p)j = \Phi^u(p)j, \qquad p \in P, \quad j \in J,$$

which means that Φ^u commutes with the right action R_j. We recall that the dressed gauge field is a one-form $\omega^u = (\Phi^u)^*\omega$. The connection form ω restricted to tangent vectors $\mathsf{T}P^J \subset \mathsf{T}P$ defines a connection form in the principal bundle $P^J \to M$. Let us denote the restriction of ω to $\mathsf{T}P^J$ by ω^J. From the conclusions of the previous section, i.e. $\Phi^u = \pi_{P^J}$, and the fact that P^J is a submanifold in P we obtain that the dressing of the connection form is just a pull-back of this connection form with respect to the projection π_{P^J}. In the obvious way we have $\pi^*_{P^J}\omega = \pi^*_{P^J}\omega^J$ so that we obtain $\omega^u = \pi^*_{P^J}\omega^J$. Notice that ω^u is also equivariant with respect to the action of J, which comes from the fact that

$$R_j^*\omega^u = R_j^*\pi^*_{P^J}\omega^J = \pi^*_{P^J}R_j^*\omega^J = \pi^*_{P^J}\mathrm{Ad}_{j^{-1}} \circ \omega^J = \mathrm{Ad}_{j^{-1}} \circ \omega^u.$$

From the above discussion we conclude that if u satisfies the condition (4.2), then the form ω^u is a pull-back of the connection one-form in the principal bundle $\pi_J : P^J \to M$.

4.3. Adjoint bundle of the reduced principal bundle

In the following we will assume that P is equipped with a fixed dressing field u, i.e. there exists a decomposition $P = P^J \times H$. For the sake of the clarity of the presentation we will introduce the notation $\widetilde{P} := P^J, \widetilde{\pi} := \pi^J$. The Lie algebras of the groups J and H will be denoted by $\mathfrak{j}$ and $\mathfrak{h}$ respectively. Since the existence of the field u is equivalent to the existence of the embedding $\widetilde{P} \hookrightarrow P$, from now on we will understand the dressing field method rather as a choice of the suitable embdedding of $\widetilde{P}$ in P than as a map $u : P \to H$.

Let us consider the adjoint bundle of the principal bundle $\widetilde{\pi} : \widetilde{P} \to M$. The structure group of $\widetilde{P}$ is the subgroup J, so by definition we have

$$\mathrm{ad}\widetilde{P} := (\widetilde{P} \times \mathfrak{j})/J.$$

The decomposition $G = H \times J$ implies a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{j}$. We will denote the projection from $\mathfrak{g}$ onto $\mathfrak{j}$ by $pr_{\mathfrak{j}}$. There exists a canonical projection

$$\pi_{\widetilde{P}} \times pr_{\mathfrak{j}} : P \times \mathfrak{g} \to \widetilde{P} \times \mathfrak{j},$$

$$(p, X) \to \big(\pi_{\widetilde{P}}(p), pr_{\mathfrak{j}}(X)\big).$$

If we divide both sides of the above projection by G and use the fact that H acts on $\widetilde{P}$ in a trivial way we will obtain a projection

$$\Delta : \mathrm{ad}P \to \mathrm{ad}\widetilde{P}, \qquad [p, X]_G \longmapsto [\pi_{\widetilde{P}}(p), pr_{\mathfrak{j}}(X)]_J, \tag{4.3}$$

where $[p, X]_G$ is the equivalence class of the element (p, X) with respect to the action of G and $[\pi_{\widetilde{P}}(p), pr_{\mathfrak{j}}(X)]_J$ is the equivalence class of the element $(\pi_{\widetilde{P}}(p), pr_{\mathfrak{j}}(X))$ with respect to the action of J. One can easily check that the above projection does not depend on the choice of the representative. Indeed, we have

$$\pi_{P^J} \times pr_{\mathfrak{j}}(pg, \mathrm{Ad}_{g^{-1}} \circ X) = \Big(pr_{P^J}(pg), pr_{\mathfrak{j}}(\mathrm{Ad}_{g^{-1}} \circ X)\Big) = \Big(pr_{P^J}(p)j, \mathrm{Ad}_{j^{-1}} \circ pr_{\mathfrak{j}}(X)\Big)$$

where j is a projection of g onto J. Notice that the equality $pr_{\mathfrak{j}}(\mathrm{Ad}_{g^{-1}} \circ X) = \mathrm{Ad}_{j^{-1}} \circ pr_{\mathfrak{j}}(X)$ requires the existence of the direct product structure in G. Therefore, the map (4.3) is not well-defined in a more general case, e.g. when G is a semi-direct product of H and J. Using above formula we obtain a projection

$$\Delta : \Omega^k(M) \otimes_M \mathrm{ad}P \to \Omega^k(M) \otimes_M \mathrm{ad}\widetilde{P}, \tag{4.4}$$

which for the clarity of the notation we have denoted by the same symbol as (4.3). The above map may be lifted to the map $\mathsf{J}^1\Delta$ represented by the diagram

$$\begin{array}{ccc}
\mathsf{J}^1\Big(\Omega^k(M) \otimes_M \mathrm{ad}P\Big) & \xrightarrow{\mathsf{J}^1\Delta} & \mathsf{J}^1\Big(\Omega^k(M) \otimes_M \mathrm{ad}\widetilde{P}\Big) \\
\downarrow & & \downarrow \\
\Omega^k(M) \otimes_M \mathrm{ad}P & \xrightarrow{\Delta} & \Omega^k(M) \otimes_M \mathrm{ad}\widetilde{P} \\
\downarrow{\scriptstyle pr_M} & & \downarrow{\scriptstyle pr_M} \\
M & \xrightarrow{id} & M
\end{array} \tag{4.5}$$

4.4. Reduction of the connection bundle

Let us consider the bundle of first jets of sections of $\widetilde{\pi}$, i.e. the bundle $\mathsf{J}^1\widetilde{P} \to \widetilde{P}$. The action of J on $\widetilde{P}$ may be lifted to the action on the total space $\mathsf{J}^1\widetilde{P}$. Principal connections in $\widetilde{P}$ are represented by sections of the bundle

$$\widetilde{C} \to M, \tag{4.6}$$

where $\widetilde{C} := \mathsf{J}^1\widetilde{P}/J$. The bundle (4.6) is an affine bundle modelled on a vector bundle $\mathsf{T}^*M \otimes_M \mathrm{ad}\widetilde{P} \to M$. The isomorphism

$$P \to \widetilde{P} \times_M H$$

may be lifted to the isomorphism

$$\mathsf{J}^1P \to \mathsf{J}^1\widetilde{P} \times_M \mathsf{J}^1(M \times H)$$

over M. In particular, the projection $\pi_{\widetilde{P}} : P \to \widetilde{P}$ defines the map $\mathsf{J}^1\pi_{\widetilde{P}} : \mathsf{J}^1 P \to \mathsf{J}^1\widetilde{P}$ between jet bundles over M and represented by the diagram

$$\begin{array}{ccc} \mathsf{J}^1 P & \xrightarrow{\mathsf{J}^1\pi_{PJ}} & \mathsf{J}^1\widetilde{P} \\ \downarrow{\scriptstyle \pi_{1,0}} & & \downarrow{\scriptstyle \pi_{1,0}} \\ P & \xrightarrow{\pi_{PJ}} & \widetilde{P} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\pi}} \\ M & \xrightarrow{id} & M \end{array} \tag{4.7}$$

Let us consider now a section $\omega : P \to \mathsf{J}^1 P$. Notice that the map

$$\widetilde{\omega} : \widetilde{P} \to \mathsf{J}^1\widetilde{P}, \qquad \widetilde{\omega}(\widetilde{\pi}(p)) = \mathsf{J}^1\widetilde{\pi}(\omega(p))$$

is in general not well-defined. It is easy to check that for $p_2 = p_1 h$, where $h \in H$, we have $\widetilde{\pi}(p_2) = \widetilde{\pi}(p_1)$, but $\mathsf{J}^1\widetilde{\pi}(\omega(p_2)) \neq \mathsf{J}^1\widetilde{\pi}(\omega(p_1))$. Let us impose the additional condition that ω is G-invariant, i.e. $\omega(pg) = \omega(p)g$. Then for $p_2 = p_1 g$, where $g = hj$, we have $\widetilde{\pi}(p_2) = \widetilde{\pi}(p_1 g) = \widetilde{\pi}(p_1)j$. Finally, we obtain

$$\widetilde{\omega}(\widetilde{\pi}(p_2)) = \mathsf{J}^1\widetilde{\pi}(\omega(p_2)) = \mathsf{J}^1\widetilde{\pi}(\omega(p_1 g)) = \mathsf{J}^1\widetilde{\pi}(\omega(p_1)g) = \mathsf{J}^1\widetilde{\pi}(\omega(p_1))j,$$

and as a consequence

$$\widetilde{\omega}(\widetilde{\pi}(p_1)j) = \widetilde{\omega}(\widetilde{\pi}(p_1))j.$$

It turns out that for ω being a G-invariant section the map $\widetilde{\omega}$ is a well-defined J-invariant section. The map $\widetilde{\pi}$ defines therefore a projection of the G-invariant section of the bundle $\mathsf{J}^1 P \to P$ onto a J-invariant section of the bundle $\mathsf{J}^1\widetilde{P} \to \widetilde{P}$. Similarly, one can show that each G-invariant section ω defines an H-invariant section of the bundle $\mathsf{J}^1(M \times H) \to M \times H$. In particular, by dividing the left-hand side of the diagram (4.7) by G and the right-hand side by J we obtain the diagram

$$\begin{array}{ccc} C & \xrightarrow{\delta} & \widetilde{C} \\ \downarrow{\scriptstyle pr_M} & & \downarrow{\scriptstyle pr_M} \\ M & \xrightarrow{id} & M \end{array} \tag{4.8}$$

From (4.8) we see that the map δ defines a reduction of the connection bundle. The first jet prolongation of δ

$$\mathsf{J}^1\delta : \mathsf{J}^1 C \to \mathsf{J}^1\widetilde{C}. \tag{4.9}$$

provides a reduction of the configuration bundle. Notice that in the light of the discussion above the map δ defines also a projection

$$\Gamma(C, M) \to \Gamma(\widetilde{C}, M)$$

between modules of sections of bundles $C \to M$ and $\widetilde{C} \to M$. If ω is a section of the bundle $C \to M$, then the map

$$\widetilde{\omega} : M \to \widetilde{C}, \quad \widetilde{\omega} := \delta \circ \omega$$

is a section of the bundle $\widetilde{C} \to M$. Let us emphasize that δ depends on the choice of a dressing field u, which comes from the fact that $\Phi^u = \pi_{\widetilde{P}}$.

4.5. Reduction of the configuration and phase bundle

The reduction of the connection bundle implies the reduction of the configuration and phase bundle. Let us introduce the notation

$$\widetilde{\mathcal{V}} := \mathsf{T}^*M \otimes_M \mathrm{ad}\widetilde{P},$$

$$\widetilde{\mathcal{F}} := \wedge^2\mathsf{T}^*M \otimes_M \mathrm{ad}\widetilde{P}.$$

Using (2.9) and (4.9) we obtain decompositions

$$\mathsf{J}^1C = \mathsf{J}^2P/G \oplus_C (C \times_M \mathcal{F}),$$

$$\mathsf{J}^1\widetilde{C} = \mathsf{J}^2\widetilde{P}/J \oplus_{\widetilde{C}} (\widetilde{C} \times_M \widetilde{\mathcal{F}}),$$

represented by the diagram

$$\begin{array}{ccc} \mathsf{J}^1C & \longrightarrow & \mathsf{J}^2P/G \oplus_C (C \times_M \mathcal{F}) \\ \big\downarrow{\scriptstyle \mathsf{J}^1\delta} & & \big\downarrow{\scriptstyle pr} \\ \mathsf{J}^1\widetilde{C} & \longrightarrow & \mathsf{J}^2\widetilde{P}/J \oplus_{\widetilde{C}} (\widetilde{C} \times_M \widetilde{\mathcal{F}}) \end{array}$$

Here, the projection pr is an immediate consequence of isomorphism (2.9) applied to J^1C and $\mathsf{J}^1\widetilde{C}$ and the map $\mathsf{J}^1\delta$. Above decompositions and the projection $\mathsf{J}^1C \to \mathsf{J}^1\widetilde{C}$ defines a map

$$\mathsf{J}^2P/G \to \mathsf{J}^2\widetilde{P}/J,$$

and

$$\zeta : C \times_M \mathcal{F} \to \widetilde{C} \times_M \widetilde{\mathcal{F}}, \quad (\omega, F) \longmapsto (\widetilde{\omega}, \widetilde{F}). \tag{4.10}$$

In particular, we will be interested in the map (4.10). Let us recall that the phase bundle of the gauge theory is the bundle $C \times \overline{\mathcal{P}} \to C$, where $\overline{\mathcal{P}} := C \times_M \mathsf{T}M \otimes_M \mathsf{T}M \otimes_M \mathrm{ad}^*P \otimes_M \Omega^n$. Applying the map (4.3) to $\overline{\mathcal{P}}$ we obtain

$$\Delta : \overline{\mathcal{P}} \to \widetilde{\mathcal{P}}, \quad \bar{p} \longmapsto \widetilde{p},$$

where

$$\widetilde{\mathcal{P}} := C \times_M \mathsf{T}M \otimes_M \mathsf{T}M \otimes_M \mathrm{ad}^*\widetilde{P} \otimes_M \Omega^n$$

and $\mathrm{ad}^*\widetilde{P} \to M$ is the dual bundle of the bundle $\mathrm{ad}\widetilde{P} \to M$. The reduced phase bundle is therefore the bundle $\widetilde{C} \times \widetilde{\mathcal{P}} \to \widetilde{C}$. Similarly, for the bundle $\mathsf{J}^1\overline{\mathcal{P}}$ we obtain the reduction

$$\mathsf{J}^1\Delta : \mathsf{J}^1\overline{\mathcal{P}} \to \mathsf{J}^1\widetilde{\mathcal{P}}, \quad \mathsf{j}^1_m\bar{p} \longmapsto \mathsf{j}^1_m\widetilde{p}.$$

In the above framework one can also include a reduced Lagrangian. From (2.10) we have that the Lagrangian of the gauge theory has a form

$$L : C \times_M \mathcal{F} \to \Omega^n.$$

Let us assume now that L depends only on the projection on $\widetilde{C} \times_M \widetilde{\mathcal{F}}$. Then, we can introduce the reduced Lagrangian

$$\tilde{L} : \widetilde{C} \times_M \widetilde{\mathcal{F}} \to \Omega^n, \quad L = \tilde{L} \circ \zeta.$$

From the above considerations we conclude that by introducing a dressing field adapted to the form of the Lagrangian of the system, we can partially, or fully, reduce the gauge symmetry of the given theory. Let us notice that if the Lagrangian has the gauge symmetry described by the group G, then the solutions of the equations of motion, by definition will have this symmetry as well. It means that if the field ω is a solution of the equations of motion then each field ω^{Φ} is also such a solution. By performing the reduction of the symmetry with respect to the subgroup $H \subset G$ and by introducing the reduced Lagrangian $\tilde{L}$ we obtain new equations of motion, which are invariant under the gauge transformation described by the subgroup J. The corresponding solutions of the reduced equations of motion will have the symmetry given by J as well.

4.6. Applications and future work

In this paper we have analysed the geometric structure of the dressing field method with a particular focus on the situation, in which u transforms with respect to (4.2). However, our research still has a significant potential for the further development.

First of all, one could consider more general transformation rules than (4.2). For example, in [1, 2] the authors consider the case where

$$(R_j^* u)(p) = j^{-1} u(p) C_p(j). \tag{4.11}$$

Here $C : P \times J \to G'$ is a map such that

$$C_p(jj') = C_p(j) C_p(j'), \qquad \forall j, j' \in J$$

and $G' \supset G$ is a Lie group satisfying certain additional condition. It turns out, that in such a case the dressed field does not longer belong to the original space of gauge fields but to the more general space of twisted-gauge fields [2]. It is a natural, and rather difficult, question how (4.11) affects the geometry of the underlying principal bundle and whether the results of our paper can be extended to this case. Notice that (4.2) is a special case of (4.11) for $G' = G$ and $C_p(j) = j \; \forall j \in J, p \in P$.

On the other hand, one could also extend the above results in a more applied direction. For instance, the authors in [2, 11] analyse how the BRST algebra is modified by the application of the dressing field method. Our results could possibly provide a geometric interpretation of this modification and lead to an interesting link between the algebraic and geometric perspective in that context. Further applications could be found after extending the above formalism to the case (4.11). This transformation rule turns out to be useful in the General Relativity [1, 2, 11] and Cartan geometry [1, 2].

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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