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Research article

Nonlinear ξ-skew-Jordan-type higher derivations on unital $*$-algebras

Xinfeng Liang*, Ying Ning and Kangping Yang

School of Mathematics and Big Data, Anhui University of Science & Technology, Huainan 232001, China

* **Correspondence:** Email: xfliang@aust.edu.cn.

Abstract: Let $\mathscr{A}$ be a unital $*$-algebra over the complex field $\mathbb{C}$, and fix $\xi \in \mathbb{C}\setminus\{0, \pm 1\}$. In this paper, we investigated a sequence $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ of arbitrary (not necessarily additive) self-maps on $\mathscr{A}$ satisfying the equation

$$\phi_m(x_1 \diamond_\xi \cdots \diamond_\xi x_n) = \sum_{i_1+\cdots+i_n=m} \phi_{i_1}(x_1) \diamond_\xi \cdots \diamond_\xi \phi_{i_n}(x_n),$$

for all $x_1, \ldots, x_n \in \mathscr{A}$, where $x_1 \diamond_\xi x_2 = x_1x_2 + \xi x_2x_1^*$ and positive integer $n \geq 3$. We prove that, under certain mild conditions, every such nonlinear ξ-skew-Jordan-type higher derivation is in fact an additive higher $*$-derivation that also satisfies the scalar compatibility condition $\phi_m(\xi A) = \xi\phi_m(A)$ for all $A \in \mathscr{A}$. Conversely, every additive higher $*$-derivation enjoying this property satisfies the displayed identity. As a consequence, our characterization both unifies and extends earlier results on nonlinear ξ-skew Jordan triple (higher) derivations in prime $*$-algebras. In particular, the main theorem yields a complete description of such higher derivations in concrete contexts, including standard operator algebras, von Neumann algebras with no central summand of type I_1, and factor von Neumann algebras, thereby providing a broad framework that subsumes several previously unrelated classes of operator-algebraic structures.

Keywords: nonlinear ξ-skew-Jordan-type higher derivation; additive higher $*$-derivation; unital $*$-algebra; standard operator algebra; von Neumann algebra

1. Introduction

The study of derivable mappings on operator algebras and ring theory has long been a central topic in nonlinear functional analysis. The classical investigation of Jordan-type derivable maps traces back to Herstein's foundational work on Jordan homomorphisms of prime rings, which was subsequently incorporated into the framework of Herstein Lie-type mappings [1]. Around the structures of Jordan derivations and $*$-derivations, numerous scholars have carried out a wealth of fruitful research [2–4], gradually forming a systematic theoretical system. In recent years, increasing attention has been

devoted to derivable mappings associated with parameterized product structures, among which ξ-skew Jordan-type derivable maps are particularly distinguished [5, 6]. Such mappings organically combine parameterized algebraic operations with the rigid structure of nonlinear mappings, thereby unifying classical derivation theories, and, in the higher-order setting, admit concrete realizations on important classes of operator algebras; they thus possess both theoretical depth and applicative merit. Nevertheless, existing results are mostly confined to low-order cases of single mappings, and a unified treatment of the global structure of general higher-order derivable families remains lacking. Motivated by this, the present paper undertakes a systematic investigation of the structure of nonlinear ξ-skew Jordan-type higher derivations on unital $*$-algebras.

Let $\mathfrak{A}$ be a unital associative algebra equipped with an involution $*$ satisfying $(xy)^* = y^*x^*$, $(x^*)^* = x$, and $(x + y)^* = x^* + y^*$, then $\mathfrak{A}$ is called a unital $*$-algebra. For a fixed nonzero complex number ξ, the ξ-skew Jordan product is defined as

$$x \diamond_\xi y = xy + \xi yx^* \quad (\forall x, y \in \mathfrak{A}).$$

In particular, when $\xi = -1$, the operation $x \diamond_{-1} y = xy - yx^*$ reduces to the skew Lie product; when $\xi = 1$, it becomes a $*$-deformation of the Jordan product, also referred to as the skew Jordan product. A mapping $\phi_1 : \mathfrak{A} \to \mathfrak{A}$ is called a nonlinear ξ-skew Jordan n-derivation if it satisfies

$$\phi_1(x_1 \diamond_\xi \cdots \diamond_\xi x_n) = \sum_{k=1}^{n} x_1 \diamond_\xi \cdots \diamond_\xi \phi_1(x_k) \diamond_\xi \cdots \diamond_\xi x_n, \tag{1.1}$$

in the case $n = 3$, it is termed a triple derivation. Such mappings are collectively referred to as nonlinear ξ-skew Jordan-type derivations. The nonlinearity at this point means that the mapping $\phi : \mathfrak{A} \to \mathfrak{A}$ is not assumed beforehand to satisfy equation $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in \mathfrak{A}$.

The investigation of parameter-dependent derivable mappings has garnered significant attention and experienced a series of developments. Huo et al. [7] characterized such mappings on von Neumann algebras, showing that, under real parameter conditions, they are either linear or conjugate linear $*$-isomorphisms. Subsequently, the authors of [8] investigated the additivity of nonlinear ξ-skew Jordan triple derivations δ_1, but under the restriction $|\xi| < 1$ and without covering the case $|\xi| = 1$. Zhang [6] extended the study to prime $*$-algebras, removing the parameter restriction by exploiting the existence of nontrivial idempotents and establishing the derivation property. Moreover, he proved that such a mapping is an additive $*$-derivation satisfying some relation. It is worth noting that the approach in [6] subsumes the results of [8], whereas the converse does not hold. The algebraic structures underlying extended Lie superalgebras and super-integrable hierarchies [4, 9], together with the higher-order flows arising from the Riemann-Hilbert method [10], are intrinsically connected to $*$-algebras through derivation-type maps and recursive symmetry generators, which can be uniformly characterized by the $*$-Jordan higher derivations studied in this paper.

The notion of higher derivations originates from the work of Hasse and Schmidt [11] on higher derivations of fields, and was later extended to ring theory and operator algebras by Ferrero and Haetinger [12], Nowicki [13], and others. Let $\mathbb{N}$ denote the set of nonnegative integers, and let $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ be a family of mappings on $\mathfrak{A}$ (not assumed to be additive a priori) with $\phi_0 = \mathrm{id}_{\mathfrak{A}}$. If for all $x, y \in \mathfrak{A}$, and each $m \in \mathbb{N}$, one has

$$\phi_m(xy) = \sum_{i+j=m} \phi_i(x)\phi_j(y), \quad \phi_m(x + y) = \phi_m(x) + \phi_m(y), \quad \phi_m(x^*) = \phi_m(x)^*,$$

then Ψ is called an additive higher $*$-derivation. If, in addition, it satisfies

$$\phi_m(x_1 \diamond_\xi \cdots \diamond_\xi x_n) = \sum_{i_1+\cdots+i_n=m} \phi_{i_1}(x_1) \diamond_\xi \cdots \diamond_\xi \phi_{i_n}(x_n), \tag{1.2}$$

then the family $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ is called a nonlinear ξ-skew Jordan higher n-derivation, and these are collectively termed nonlinear ξ-skew Jordan-type higher derivations. Here we assume that the positive integer $n \geq 3$.

Higher derivations have been intensively studied on various structures such as triangular algebras, nest algebras, and standard operator algebras. Ashraf et al. [14] proved that nonlinear $*$-Lie higher derivations on standard operator algebras are automatically additive higher derivations. Wani et al. [15] proved that nonlinear $*$-Jordan higher derivations on von Neumann algebras are automatically additive higher derivations. Inspired by this, Liang and his collaborators [16] introduced the notion of nonlinear bi-skew Jordan-type higher derivations on unital $*$-algebras and characterized their structure, thereby extending Zhang's result [6]. More recently, Ren et al. [17] investigated nonlinear mixed bi-skew Jordan-type and skew Jordan higher derivations, establishing their additivity and higher $*$-derivation properties. Building on articles [14, 16, 17], Liang and Ning [18] explored the structure of nonlinear ξ-skew Jordan triple higher derivations on prime $*$-algebras, and proved that they are additive higher $*$-derivations satisfying $\phi_m(\xi A) = \xi\phi_m(A)$, thereby extending Zhang's result [6].

Motivated by the above works [14, 16, 18], it is natural to raise the following question:

Question 1.1. Under an appropriate condition ($\mathbb{Q}$), which will be introduced in Section 2, is every nonlinear ξ-skew Jordan-type higher derivation $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ as defined in equation (1.2) on a unital $*$-algebra necessarily an additive higher $*$-derivation with the property $\phi_m(\xi A) = \xi\phi_m(A)$?

The present paper addresses this question and provides an affirmative answer. The results obtained not only extend several existing conclusions, such as those for nonlinear ξ-skew Jordan triple derivations [6] and triple higher derivations [18], but also yield a clear structural description of nonlinear ξ-skew Jordan higher n-derivations on prime $*$-algebras, standard operator algebras, von Neumann algebras with no central summand of type I_1 and factor von Neumann algebras. Methodologically, in addition to traditional tools such as Peirce decomposition for handling unital $*$-algebra structures, we also employ, in Lemma 3.7, the theory of diagonalization of matrices over the complex field $\mathbb{C}$, which helps to enhance the readability and conciseness of the arguments.

2. Notation, preliminaries and first-order derivations

Let $\mathscr{A}$ be a unital $*$-algebra with identity I, and suppose that $\mathscr{A}$ contains a nontrivial symmetric idempotent P_1. Set $P_2 = I - P_1$, then P_2 is also a nontrivial idempotent and satisfies $P_1 + P_2 = I$. Accordingly, by virtue of the Peirce decomposition, $\mathscr{A}$ admits the direct sum decomposition

$$\mathscr{A} = \mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22},$$

where $\mathscr{A}_{ij} = P_i\mathscr{A}P_j$ for $i, j \in \{1, 2\}$. This decomposition satisfies the following multiplicative and involutive relations:

$$\mathscr{A}_{ij}\mathscr{A}_{kl} \subseteq \delta_{jk}\mathscr{A}_{il}, \qquad (\mathscr{A}_{ij})^* \subseteq \mathscr{A}_{ji},$$

with δ_{jk} denoting the Kronecker symbol for $i, j, k, l \in \{1, 2\}$. Moreover, each $\mathscr{A}_{ij}$ carries a $(\mathscr{A}_{ii}, \mathscr{A}_{jj})$-bimodule structure, meaning that it is a left $\mathscr{A}_{ii}$-module and simultaneously a right $\mathscr{A}_{jj}$-module for $i, j \in \{1, 2\}$.

From this point onward, we shall assume that the unital $*$-algebra $\mathscr{A}$ satisfies condition

$$\mathbb{Q} = \begin{cases} For\ all\ Y \in \mathscr{A}, \quad Y\mathscr{A}P_1 = 0 \ \text{ implies } Y = 0 \\ For\ all\ Y \in \mathscr{A}, \quad Y\mathscr{A}P_2 = 0 \ \text{ implies } Y = 0. \end{cases}$$

Under this assumption, it follows from the literature [6, 16] that prime $*$-algebras, standard operator algebras, von Neumann algebras with no central summand of type I_1, and factor von Neumann algebras, among others, serve as canonical examples of unital $*$-algebras.

Below we introduce a key lemma that will be used in subsequent research. To facilitate computations and streamline the proof in the subsequent derivation, we introduce the symbol $X^{\diamond_\xi(m)} = \underbrace{X \diamond_\xi \cdots \diamond_\xi X}_{m\ copies}$ for all $X \in \mathscr{A}$. Although this result has been discussed in the literature [19, Section 2], the existing approaches rely on rather complicated techniques and do not provide a complete proof. In this paper, we offer a new proof using matrix diagonalization theory. Our argument is not only shorter and more transparent, but also more straightforward, which we believe will help readers better grasp the underlying ideas. Since the conclusion of Lemma 2.1 has already been proved in reference [19, Section 2], it is omitted here.

Lemma 2.1. *Let $\diamond_\xi$ denote the ξ-skew Jordan product. Then:*

1) For every positive integer n,

$$I^{\diamond_\xi(n)} = c_n I,$$

with

$$c_n = \begin{cases} 1 & if\ \xi = 0, \\ \frac{1}{2}\left[\left(1 + \frac{\xi}{|\xi|}\right)(1 + |\xi|)^{n-1} + \left(1 - \frac{\xi}{|\xi|}\right)(1 - |\xi|)^{n-1}\right] & if\ \xi \neq 0. \end{cases}$$

2) If $\xi \neq -1$, then for every integer $n > 2$,

$$c_n = c_{n-1} + \xi c_{n-1}^* \neq 0.$$

This is equivalent to the statement that

$$I^{\diamond_\xi(n)} = I^{\diamond_\xi(n-1)} + \xi I^{\diamond_\xi(n-1)} \neq 0.$$

Proof. We now present the proofs of the above two assertions sequentially.

(1). Set $I^{\diamond_\xi(n)} = \underbrace{I \diamond_\xi I \diamond_\xi \cdots \diamond_\xi I}_{n}$, with $I^{\diamond_\xi(1)} = I$. Thus $I^{\diamond_\xi(n)}$ obeys the recursive formula $I^{\diamond_\xi(n)} = I^{\diamond_\xi(n-1)} \diamond_\xi I$. On the other hand, an application of the ξ-skew Jordan product operation shows that there exists a constant $c_n \in \mathbb{C}$ satisfying $I^{\diamond_\xi(n)} = c_n I$. In view of $I^{\diamond_\xi(n-1)} = c_{n-1} I$, together with the recurrence $I^{\diamond_\xi(n)} = I^{\diamond_\xi(n-1)} \diamond_\xi I$, a simple computation then gives

$$c_n = c_{n-1} + \xi c_{n-1}^* = c_{n-1} + \xi \overline{c_{n-1}}, \quad n \geq 2.$$

Applying the conjugation operation to both sides of the above equality, we obtain

$$\overline{c_n} = \overline{c_{n-1}} + \overline{\xi} c_n.$$

In view of the preceding two equalities, by forming c_n and $\overline{c_n}$ into column vectors, we get

$$\begin{pmatrix} c_n \\ \overline{c_n} \end{pmatrix} = \begin{pmatrix} 1 & \xi \\ \overline{\xi} & 1 \end{pmatrix} \begin{pmatrix} c_{n-1} \\ \overline{c_{n-1}} \end{pmatrix}.$$

Consequently, we obtain

$$\begin{pmatrix} c_n \\ \overline{c_n} \end{pmatrix} = \left(\begin{pmatrix} 1 & \xi \\ \overline{\xi} & 1 \end{pmatrix} \right)^{n-1} \begin{pmatrix} c_{n-1} \\ \overline{c_{n-1}} \end{pmatrix}.$$

In what follows, we determine the powers of the matrix $A = \begin{pmatrix} 1 & \xi \\ \overline{\xi} & 1 \end{pmatrix}$ by invoking the diagonalization theory of matrices over C.

We begin by determining the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{pmatrix} 1 & \xi \\ \overline{\xi} & 1 \end{pmatrix}$. The eigenvalues are then obtained by computing the determinant of the characteristic matrix. Then we have

$$|\lambda E - A| = \begin{vmatrix} \lambda - 1 & -\xi \\ -\overline{\xi} & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - |\xi|^2 = 0,$$

which implies

$$\begin{cases} \lambda_1 = 1 + |\xi|, \\ \lambda_2 = 1 - |\xi|, \end{cases}$$

where E is the 2×2 standard matrix unit.

Next, we determine the nonzero eigenvectors for the eigenvalues obtained above. In the case $\lambda_1 = 1 + |\xi|$, solving the homogeneous linear system

$$(\lambda_1 E - A) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0},$$

whose detailed form is given by

$$\begin{cases} |\xi| x_1 - \xi x_2 = 0, \\ -\overline{\xi} x_1 + |\xi| x_2 = 0, \end{cases}$$

yields relation $x_2 = \dfrac{|\xi|}{\xi} x_1$. Letting $x_1 = \xi$, in relation $x_2 = \dfrac{|\xi|}{\xi} x_1$, and therefore the eigenvector is $\boldsymbol{p}_1 = \begin{pmatrix} \xi \\ |\xi| \end{pmatrix}$.

Applying the same computational procedure and techniques for $\lambda_2 = 1 - |\xi|$, we solve the homogeneous system

$$(\lambda_2 E - A) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0},$$

presented explicitly in

$$\begin{cases} -|\xi|x_1 - \xi x_2 = 0, \\ -\overline{\xi}x_1 - |\xi|x_2 = 0, \end{cases}$$

and obtain relation $x_2 = -\dfrac{|\xi|}{\xi}x_1$. Substituting $x_1 = \xi$ into relation $x_2 = -\dfrac{|\xi|}{\xi}x_1$ yields the eigenvector $\boldsymbol{p}_2 = \begin{pmatrix} \xi \\ -|\xi| \end{pmatrix}$.

We now construct the matrix $P = (\boldsymbol{p}_1, \boldsymbol{p}_2) = \begin{pmatrix} \xi & \xi \\ |\xi| & -|\xi| \end{pmatrix}$ from the eigenvectors $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. Condition $\xi \neq 0$ ensures that P is invertible, so we obtain

$$P^{-1} = \frac{1}{-2\xi|\xi|}\begin{pmatrix} -|\xi| & -\xi \\ -|\xi| & \xi \end{pmatrix} = \frac{1}{2\xi|\xi|}\begin{pmatrix} |\xi| & \xi \\ |\xi| & -\xi \end{pmatrix}.$$

By virtue of the matrix diagonalization theory, relation $P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ is established; proceeding step by step, we obtain equation $A = P\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}P^{-1}$. Then equation $A^{n-1} = P\begin{pmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{pmatrix}P^{-1}$. A direct matrix calculation then yields equality

$$\begin{aligned} A^{n-1} &= \frac{1}{2\xi|\xi|}\begin{pmatrix} \xi & \xi \\ |\xi| & -|\xi| \end{pmatrix}\begin{pmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{pmatrix}\frac{1}{2\xi|\xi|}\begin{pmatrix} |\xi| & \xi \\ |\xi| & -\xi \end{pmatrix} \\ &= \frac{1}{2\xi|\xi|}\begin{pmatrix} \xi|\xi|\lambda_1^{n-1} + \xi|\xi|\lambda_2^{n-1} & \xi^2\lambda_1^{n-1} - \xi^2\lambda_2^{n-1} \\ |\xi|^2\lambda_1^{n-1} - |\xi|^2\lambda_2^{n-1} & \xi|\xi|\lambda_1^{n-1} + \xi|\xi|\lambda_2^{n-1} \end{pmatrix}. \end{aligned}$$

Combining this with $\begin{pmatrix} c_n \\ \tilde{c}_n \end{pmatrix} = A^{n-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we further obtain

$$\begin{aligned} c_n &= \frac{1}{2\xi|\xi|}\left[\xi(|\xi| + \xi)\lambda_1^{n-1} + \xi(|\xi| - \xi)\lambda_2^{n-1}\right] \\ &= \frac{1}{2}\left[\left(1 + \frac{\xi}{|\xi|}\right)(1 + |\xi|)^{n-1} + \left(1 - \frac{\xi}{|\xi|}\right)(1 - |\xi|)^{n-1}\right]. \end{aligned}$$

□

2.1. First-order derivation

This section focuses on presenting the structure of nonlinear ξ-skew Jordan n-derivations on unital $*$-algebras, as derived from the literature [19]. This structure is formally stated in Theorem 2.1. The established characterization serves as the base step for the induction argument employed in Section 3, where we address the more general framework of nonlinear ξ-skew Jordan higher n-derivations.

Theorem 2.2. *[19, Theorem 1] Let $\mathscr{A}$ be a unital $*$-algebra with unit I and a nontrivial symmetric idempotent, and fix $\xi \in \mathbb{C} \setminus \{0, \pm 1\}$. Suppose that $\phi_1 : \mathscr{A} \to \mathscr{A}$ is a nonlinear ξ-skew Jordan n-derivation satisfying*

$$\phi_1(A_1 \diamond_\xi A_2 \diamond_\xi \cdots \diamond_\xi A_n) = \sum_{k=1}^{n} A_1 \diamond_\xi \cdots \diamond_\xi \phi_1(A_k) \diamond_\xi \cdots \diamond_\xi A_n$$

for all $A_1, \cdots, A_n \in \mathscr{A}$ and positive integer $n \geq 3$. Then ϕ_1 is a nonlinear ξ-skew Jordan n-derivation if and only if ϕ_1 is an additive $*$-derivation and $\phi_1(\xi A) = \xi\phi_1(A)$ for all $A \in \mathscr{A}$.

In [19], Wu and her coauthors characterized the additivity of nonlinear ξ-skew Jordan n-derivations, showing that for any $x, y \in \mathscr{A}$, the relation $\phi_1(x + y) = \phi_1(x) + \phi_1(y)$ holds. Based on the Pierce decomposition of the algebra $\mathscr{A}$, we refine this additivity into a series of lemmas as follows.

Lemma 2.3. *[19, Lemma 2] $\phi_1(0) = 0$.*

Lemma 2.4. *[19, Lemma 3] $\phi_1(A_{12} + A_{21}) = \phi_1(A_{12}) + \phi_1(A_{21})$ for all $A_{12} \in \mathscr{A}_{12}$ and $A_{21} \in \mathscr{A}_{21}$.*

Lemma 2.5. *[19, Lemma 2, Lemma 3] For any $i, j, k \in \{1, 2\}$ with $i \neq j$, the identity*

$$\phi_1(A_{kk} + A_{ij}) = \phi_1(A_{kk}) + \phi_1(A_{ij})$$

is valid for all $A_{ij} \in \mathscr{A}_{ij}$ and $A_{kk} \in \mathscr{A}_{kk}$.

Lemma 2.6. *[19, Lemma 5] For all $A_{11} \in \mathscr{A}_{11}$, $A_{12} \in \mathscr{A}_{12}$, $A_{21} \in \mathscr{A}_{21}$, and $A_{22} \in \mathscr{A}_{22}$, we have the additivity relations*

$$\phi_1(A_{11} + A_{12} + A_{21}) = \phi_1(A_{11}) + \phi_1(A_{12}) + \phi_1(A_{21}),$$
$$\phi_1(A_{12} + A_{21} + A_{22}) = \phi_1(A_{12}) + \phi_1(A_{21}) + \phi_1(A_{22}).$$

Lemma 2.7. *[19, Lemma 6] For arbitrary elements $A_{11} \in \mathscr{A}_{11}$, $A_{12} \in \mathscr{A}_{12}$, $A_{21} \in \mathscr{A}_{21}$, and $A_{22} \in \mathscr{A}_{22}$, the additivity of ϕ_1 on the sum of four components is expressed as*

$$\phi_1(A_{11} + A_{12} + A_{21} + A_{22}) = \phi_1(A_{11}) + \phi_1(A_{12}) + \phi_1(A_{21}) + \phi_1(A_{22}).$$

Lemma 2.8. *[19, Lemma 7] Let $i, j \in \{1, 2\}$ with $i \neq j$. Then $\phi_1(A_{ij} + B_{ij}) = \phi_1(A_{ij}) + \phi_1(B_{ij})$ for all $A_{ij}, B_{ij} \in \mathscr{A}_{ij}$.*

Lemma 2.9. *[19, Lemma 8] For each $1 \leq i \leq 2$ and for arbitrary $A_{ii}, B_{ii} \in \mathscr{A}_{ii}$, the additivity of ϕ_1 on diagonal components holds:*

$$\phi_1(A_{ii} + B_{ii}) = \phi_1(A_{ii}) + \phi_1(B_{ii}).$$

Lemma 2.10. *[19, Lemma 10] $\phi_1(I) = \phi_1(I)^*$.*

Lemma 2.11. *[19, Lemma 11] $\phi_1(I) = 0$.*

3. Main results: Nonlinear ξ-skew Jordan-type higher derivation

This section is primarily devoted to the structural description of nonlinear ξ-skew Jordan-type higher derivations on unital $*$-algebras. The argument is based inductively on the results established in Subsection 2.1, and the outcome presented here is the main theorem of the article.

Theorem 3.1. *Let $\mathscr{A}$ be a unital $*$-algebra with identity I and a nontrivial symmetric projection, and fix $\xi \in \mathbb{C} \setminus \{0, \pm 1\}$. Suppose $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ is a family of mappings $\phi_m : \mathscr{A} \to \mathscr{A}$ forming a nonlinear ξ-skew Jordan-type higher derivation that satisfies*

$$\phi_m(A_1 \diamond_\xi \cdots \diamond_\xi A_n) = \sum_{r_1+\cdots+r_n=m} \phi_{r_1}(A_1) \diamond_\xi \cdots \diamond_\xi \phi_{r_n}(A_n)$$

for all $A_1, \ldots, A_n \in \mathscr{A}$ and positive integer $n \geq 3$. Then Ψ is a nonlinear ξ-skew Jordan-type higher derivation if and only if Ψ is an additive higher $*$-derivation and $\phi_m(\xi A) = \xi\phi_m(A)$ holds for every $A \in \mathscr{A}$.

Lemma 3.2. $\phi_m(0) = 0$.

Proof. By Lemma 2.3, Equation $\phi_1(0) = 0$ holds. Assume that Equation $\phi_h(0) = 0$ is true for all $h < m$. We now prove that the conclusion $\phi_h(0) = 0$ holds for $h = m$.

Indeed, choose $A_1 = A_2 = \cdots = A_n = 0$ in (1.2). Then

$$\begin{aligned}\phi_m(0) &= \phi_m(0 \diamond_\xi 0 \diamond_\xi \cdots \diamond_\xi 0)\\ &= \sum_{r_1+r_2+\cdots+r_n=m} \phi_{r_1}(0) \diamond_\xi \phi_{r_2}(0) \cdots \diamond_\xi \phi_{r_{n-2}}(0) \diamond_\xi \phi_{r_{n-1}}(0) \diamond_\xi \phi_{r_n}(0)\\ &= 0.\end{aligned}$$

□

Lemma 3.3. $\phi_m(A_{12} + A_{21}) = \phi_m(A_{12}) + \phi_m(A_{21})$ *for all* $A_{12} \in \mathscr{A}_{12}$ *and* $A_{21} \in \mathscr{A}_{21}$.

Proof. Lemma 2.4 gives equation $\phi_1(A_{12} + A_{21}) = \phi_1(A_{12}) + \phi_1(A_{21})$. Suppose equation $\phi_h(A_{12} + A_{21}) = \phi_h(A_{12}) + \phi_h(A_{21})$ holds whenever $h < m$. It remains to show that the conclusion $\phi_h(A_{12} + A_{21}) = \phi_h(A_{12}) + \phi_h(A_{21})$ is valid for $h = m$.

In fact, set $T = \phi_m(A_{12} + A_{21}) - \phi_m(A_{12}) - \phi_m(A_{21})$. Since $I \diamond_\xi \diamond_\xi \cdots \diamond_\xi I \diamond_\xi (P_1 - \frac{1}{\xi}P_2)/c_{n-1} \diamond_\xi A_{12} = 0$, an application of Lemma 3.2 yields the following chain of equalities:

$$\begin{aligned}&\sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k\text{--th component}} \diamond_\xi \ldots I \diamond_\xi \frac{P_1 - \frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi (A_{12} + A_{21})\\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m(\frac{P_1 - \frac{1}{\xi}P_2}{c_{n-1}}) \diamond_\xi (A_{12} + A_{21}) + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1 - \frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi \phi_m(A_{12} + A_{21})\\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \phi_{r_2}(I) \diamond_\xi \cdots \diamond_\xi \phi_{r_{n-2}}(I) \diamond_\xi \phi_{r_{n-1}}(\frac{P_1 - \frac{1}{\xi}P_2}{c_{n-1}}) \diamond_\xi \phi_{r_n}(A_{12} + A_{21})\\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1 - \frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi (A_{12} + A_{21})\right)\\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1 - \frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi A_{12}\right) + \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1 - \frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi A_{21}\right)\\
&= \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k\text{--th component}} \diamond_\xi \ldots I \diamond_\xi \frac{P_1 - \frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi (A_{12} + A_{21})\\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m(\frac{P_1 - \frac{1}{\xi}P_2}{c_{n-1}}) \diamond_\xi (A_{12} + A_{21}) + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1 - \frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi (\phi_m(A_{12}) + \phi_m(A_{21}))\\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \phi_{r_2}(I) \diamond_\xi \cdots \diamond_\xi \phi_{r_{n-2}}(I) \diamond_\xi \phi_{r_{n-1}}(\frac{P_1 - \frac{1}{\xi}P_2}{c_{n-1}}) \diamond_\xi \phi_{r_n}(A_{12} + A_{21}).\end{aligned}$$

Subtracting the corresponding terms, we arrive at

$$I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1 - \frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi T = 0,$$

which, in view of the relation $(\mathbb{Q})$ satisfied by the algebra $\mathscr{A}$, this forces $T_{11} = T_{22} = 0$.

Analogously, employing the identity $I \diamond_\xi I \diamond_\xi \cdots \diamond_\xi A_{12} \diamond_\xi \frac{P_1}{c_{n-1}} = 0$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n-2} I \diamond_\xi \cdots \underbrace{\phi_m(I)}_{k\text{--th component}} \cdots \diamond_\xi (A_{12} + A_{21}) \diamond_\xi \frac{P_1}{c_{n-1}} + I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m(A_{12} + A_{21}) \diamond_\xi \frac{P_1}{c_{n-1}} \\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi (A_{12} + A_{21}) \diamond_\xi \phi_m(\frac{P_1}{c_{n-1}}) \\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \phi_{r_2}(I) \diamond_\xi \cdots \diamond_\xi \phi_{r_{n-1}}(A_{12} + A_{21}) \diamond_\xi \phi_{r_n}(\frac{P_1}{c_{n-1}}) \\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi (A_{12} + A_{21}) \diamond_\xi \frac{P_1}{c_{n-1}}\right) \\
&\quad = \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi A_{12} \diamond_\xi \frac{P_1}{c_{n-1}}\right) + \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi A_{21} \diamond_\xi \frac{P_1}{c_{n-1}}\right) \\
&\quad = \sum_{k=1}^{n-2} I \diamond_\xi \cdots \underbrace{\phi_m(I)}_{k\text{--th component}} \cdots \diamond_\xi (A_{12} + A_{21}) \diamond_\xi \frac{P_1}{c_{n-1}} \\
&\quad + I^{\diamond_\xi(n-2)} \diamond_\xi (\phi_m(A_{12}) + \phi_m(A_{21})) \diamond_\xi \frac{P_1}{c_{n-1}} \\
&\quad + I^{\diamond_\xi(n-2)} \diamond_\xi (A_{12} + A_{21}) \diamond_\xi \phi_m(\frac{P_1}{c_{n-1}}) \\
&\quad + \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \phi_{r_2}(I) \diamond_\xi \cdots \diamond_\xi \phi_{r_{n-1}}(A_{12} + A_{21}) \diamond_\xi \phi_{r_n}(\frac{P_1}{c_{n-1}}).
\end{aligned}$$

Subtracting the corresponding terms, we arrive at

$$I^{\diamond_\xi(n-2)} \diamond_\xi T \diamond_\xi \frac{P_1}{c_{n-1}} = 0,$$

and consequently, $T_{21} = 0$. A completely parallel argument yields $T_{12} = 0$. Hence $T = 0$, as required. □

Lemma 3.4. *Let $i, j, k \in \{1, 2\}$ with $i \neq j$. Then $\phi_m(A_{kk} + A_{ij}) = \phi_m(A_{kk}) + \phi_m(A_{ij})$ for all $A_{kk} \in \mathscr{A}_{kk}$ and $A_{ij} \in \mathscr{A}_{ij}$.*

Proof. We present the proof for the case $i = k = 1$ and $j = 2$; the remaining cases follow by analogous arguments.

From Lemma 2.5, we have the equation $\phi_m(A_{kk} + A_{ij}) = \phi_m(A_{kk}) + \phi_m(A_{ij})$. Assume as the induction hypothesis that equation $\phi_h(A_{kk} + A_{ij}) = \phi_h(A_{kk}) + \phi_h(A_{ij})$ is satisfied for every $h < m$. We shall prove that conclusion $\phi_h(A_{kk} + A_{ij}) = \phi_h(A_{kk}) + \phi_h(A_{ij})$ holds at $h = m$.

Indeed, set $T = \phi_m(A_{11} + A_{12}) - \phi_m(A_{11}) - \phi_m(A_{12})$.

First, observe that

$$I \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \frac{P_2}{c_{n-1}}. \diamond_\xi A_{11} = 0$$

Applying Lemma 3.2 to this identity, and using the defining relation for ϕ_m, we obtain the following chain of equalities:

$$\begin{aligned}
&\phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi (A_{11} + A_{12})\right) \\
&= \sum_{k=1}^{n-2} I \diamond_\xi \cdots \underbrace{\phi_m(I)}_{k\text{--th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi (A_{12} + A_{11}) \\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m\left(\frac{P_2}{c_{n-1}}\right) \diamond_\xi (A_{11} + A_{12}) + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi \phi_m(A_{11} + A_{12}) \\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{r_{n-2}}(I) \diamond_\xi \phi_{r_{n-1}}\left(\frac{P_2}{c_{n-1}}\right) \diamond_\xi \phi_{r_n}(A_{11} + A_{12})
\end{aligned}$$

and we have

$$\begin{aligned}
&\phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi (A_{11} + A_{12})\right) \\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi A_{11}\right) + \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi A_{12}\right) \\
&= \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k\text{--th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi (A_{12} + A_{11}) \\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m\left(\frac{P_2}{c_{n-1}}\right) \diamond_\xi (A_{11} + A_{12}) + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi (\phi_m(A_{11}) + \phi_m(A_{12})) \\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{r_{n-2}}(I) \diamond_\xi \phi_{r_{n-1}}\left(\frac{P_2}{c_{n-1}}\right) \diamond_\xi \phi_{r_n}(A_{11} + A_{12}).
\end{aligned}$$

Subtracting the common terms on both sides yields

$$I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi T = 0.$$

From the Peirce decomposition and the nondegeneracy of the involved projections, it follows that $T_{12} = T_{21} = T_{22} = 0$.

Next, employing the annihilation relation $I^{\diamond_\xi(n-2)} \diamond_\xi \left(\frac{P_1-\frac{1}{\xi}P_2}{c_{n-1}}\right) \diamond_\xi A_{12} = 0$, we have

$$\begin{aligned}
&\phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1-\frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi (A_{11}+A_{12})\right)\\
&= \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k-\text{-th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \left(\frac{P_1-\frac{1}{\xi}P_2}{c_{n-1}}\right) \diamond_\xi (A_{11}+A_{12})\\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m\left(\frac{P_1-\frac{1}{\xi}P_2}{c_{n-1}}\right) \diamond_\xi (A_{11}+A_{12}) + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1-\frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi \phi_m(A_{11}+A_{12})\\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{r_{n-1}}\left(\frac{P_1-\frac{1}{\xi}P_2}{c_{n-1}}\right) \diamond_\xi (\phi_{r_n}(A_{11}+A_{12}))
\end{aligned}$$

and we have

$$\begin{aligned}
&\phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1-\frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi (A_{11}+A_{12})\right)\\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1-\frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi A_{11}\right) + \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1-\frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi A_{12}\right)\\
&= \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k-\text{-th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \left(\frac{P_1-\frac{1}{\xi}P_2}{c_{n-1}}\right) \diamond_\xi (A_{11}+A_{12})\\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m\left(\frac{P_1-\frac{1}{\xi}P_2}{c_{n-1}}\right) \diamond_\xi (A_{11}+A_{12})\\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1-\frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi (\phi_m(A_{11})+\phi_m(A_{12}))\\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{r_{n-1}}\left(\frac{P_1-\frac{1}{\xi}P_2}{c_{n-1}}\right) \diamond_\xi (\phi_{r_n}(A_{11})+\phi_{r_n}(A_{12})).
\end{aligned}$$

Subtracting the common terms on both sides yields

$$I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1-\frac{1}{\xi}P_2}{c_{n-1}} \diamond_\xi T = 0,$$

which forces $T_{11} = 0$. Combining the vanishing of all Peirce components, we conclude that $T = 0$. □

We now establish the additivity of ϕ_m on the full sums $\mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21}$ and $\mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}$, respectively.

Lemma 3.5. *For all $A_{11} \in \mathscr{A}_{11}$, $A_{12} \in \mathscr{A}_{12}$, $A_{21} \in \mathscr{A}_{21}$, and $A_{22} \in \mathscr{A}_{22}$, we have*

$$\phi_m(A_{11} + A_{12} + A_{21}) = \phi_m(A_{11}) + \phi_m(A_{12}) + \phi_m(A_{21}),$$
$$\phi_m(A_{12} + A_{21} + A_{22}) = \phi_m(A_{12}) + \phi_m(A_{21}) + \phi_m(A_{22}).$$

Proof. We prove the additivity of ϕ_m on $\mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21}$.

Lemma 2.6 yields the equation $\phi_1(A_{11} + A_{12} + A_{21}) = \phi_1(A_{11}) + \phi_1(A_{12}) + \phi_1(A_{21})$. Now assume equation $\phi_h(A_{11} + A_{12} + A_{21}) = \phi_h(A_{11}) + \phi_h(A_{12}) + \phi_h(A_{21})$ is true for all smaller values of h, i.e., $h < m$. We next verify conclusion $\phi_h(A_{11} + A_{12} + A_{21}) = \phi_h(A_{11}) + \phi_h(A_{12}) + \phi_h(A_{21})$ for $h = m$.

To see this, let $T = \phi_m(A_{11} + A_{12} + A_{21}) - \phi_m(A_{11}) - \phi_m(A_{12}) - \phi_m(A_{21})$. Applying Lemma 3.3 and using the defining relation for ϕ_m, we obtain

$$\begin{aligned}
&\phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi (A_{11} + A_{12} + A_{21})\right)\\
&= \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k\text{--th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \cdots \diamond_\xi \left(\frac{P_2}{c_{n-1}}\right) \diamond_\xi (A_{11} + A_{12} + A_{21})\\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m\left(\frac{P_2}{c_{n-1}}\right) \diamond_\xi (A_{11} + A_{12} + A_{21}) + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi \phi_m(A_{11} + A_{12} + A_{21})\\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{r_{n-1}}(\frac{P_2}{c_{n-1}}) \diamond_\xi \phi_{r_n}(A_{11} + A_{12} + A_{21}).
\end{aligned}$$

On the other hand, since Lemma 3.4 has already established the additivity of ϕ_m on each pair of subspaces, we may expand the left-hand side as

$$\begin{aligned}
&\phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi (A_{11} + A_{12} + A_{21})\right)\\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi A_{11}\right) + \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi (A_{12} + A_{21})\right)\\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi A_{11}\right) + \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi A_{12}\right) + \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi A_{21}\right)\\
&= \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k\text{--th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \left(\frac{P_2}{c_{n-1}}\right) \diamond_\xi (A_{11} + A_{12} + A_{21})\\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m\left(\frac{P_2}{c_{n-1}}\right) \diamond_\xi (A_{11} + A_{12} + A_{21})\\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi (\phi_m(A_{11}) + \phi_m(A_{12}) + \phi_m(A_{21}))\\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{r_{n-1}}(\frac{P_2}{c_{n-1}}) \diamond_\xi \phi_{r_n}(A_{11} + A_{12} + A_{21}).
\end{aligned}$$

Comparing the two expansions and canceling identical terms, we arrive at

$$I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_2}{c_{n-1}} \diamond_\xi T = 0.$$

The Peirce decomposition then yields $T_{12} = T_{21} = T_{22} = 0$.

Next, employing the annihilation relation $I^{\diamond_\xi(n-2)} \diamond_\xi (-\frac{1}{\xi}P_1 + P_2)/(c_{n-1}) \diamond_\xi A_{21} = 0$, and applying Lemma 3.4 in the same manner, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k-\text{th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \left(\frac{-\frac{1}{\xi}P_1 + P_2}{c_{n-1}}\right) \diamond_\xi (A_{11} + A_{12} + A_{21}) \\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m\left(\frac{-\frac{1}{\xi}P_1 + P_2}{c_{n-1}}\right) \diamond_\xi (A_{11} + A_{12} + A_{21}) + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{-\frac{1}{\xi}P_1 + P_2}{c_{n-1}} \diamond_\xi \phi_m(A_{11} + A_{12} + A_{21}) \\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{r_{n-1}}(\frac{-\frac{1}{\xi}P_1 + P_2}{c_{n-1}}) \diamond_\xi \phi_{r_n}(A_{11} + A_{12} + A_{21}) \\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{-\frac{1}{\xi}P_1 + P_2}{c_{n-1}} \diamond_\xi (A_{11} + A_{12} + A_{21})\right) \\
&= \phi_m(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{-\frac{1}{\xi}P_1 + P_2}{c_{n-1}} \diamond_\xi (A_{11} + A_{12})) + \phi_m\left(I \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \frac{-\frac{1}{\xi}P_1 + P_2}{c_{n-1}} \diamond_\xi A_{21}\right) \\
&\quad = \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k-\text{th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \left(\frac{-\frac{1}{\xi}P_1 + P_2}{c_{n-1}}\right) \diamond_\xi (A_{11} + A_{12} + A_{21}) \\
&\quad + I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m\left(\frac{-\frac{1}{\xi}P_1 + P_2}{c_{n-1}}\right) \diamond_\xi (A_{11} + A_{12} + A_{21}) \\
&\quad + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{-\frac{1}{\xi}P_1 + P_2}{c_{n-1}} \diamond_\xi (\phi_m(A_{11}) + \phi_m(A_{12}) + \phi_m(A_{21})) \\
&\quad + \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{r_{n-1}}(\frac{-\frac{1}{\xi}P_1 + P_2}{c_{n-1}}) \diamond_\xi \phi_{r_n}(A_{11} + A_{12} + A_{21}).
\end{aligned}$$

Comparing the first and last expressions in the above chain of equalities, we deduce that

$$I^{\diamond_\xi(n-2)} \diamond_\xi \frac{-\frac{1}{\xi}P_1 + P_2}{c_{n-1}} \diamond_\xi T = 0,$$

which forces $T_{11} = 0$. Combining these results gives $T = 0$, and hence

$$\phi_m(A_{11} + A_{12} + A_{21}) = \phi_m(A_{11}) + \phi_m(A_{12}) + \phi_m(A_{21}).$$

The remaining case is proved by a completely analogous argument, and we omit the details. □

We now establish the additivity of ϕ_m on the full Peirce sum $\mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}$.

Lemma 3.6. *$\phi_m(A_{11} + A_{12} + A_{21} + A_{22}) = \phi_m(A_{11}) + \phi_m(A_{12}) + \phi_m(A_{21}) + \phi_m(A_{22})$ for all $A_{11} \in \mathscr{A}_{11}$, $A_{12} \in \mathscr{A}_{12}$, $A_{21} \in \mathscr{A}_{21}$, and $A_{22} \in \mathscr{A}_{22}$.*

Proof. Let $T = \phi_m(A_{11} + A_{12} + A_{21} + A_{22}) - \phi_m(A_{11}) - \phi_m(A_{12}) - \phi_m(A_{21}) - \phi_m(A_{22})$. By Lemma 2.7, $\phi_1(A_{11} + A_{12} + A_{21} + A_{22}) = \phi_1(A_{11}) + \phi_1(A_{12}) + \phi_1(A_{21}) + \phi_1(A_{22})$ holds. Assume $\phi_h(A_{11} + A_{12} + A_{21} + A_{22}) = \phi_h(A_{11}) + \phi_h(A_{12}) + \phi_h(A_{21}) + \phi_h(A_{22})$ for $h < m$. We prove $\phi_h(A_{11} + A_{12} + A_{21} + A_{22}) = \phi_h(A_{11}) + \phi_h(A_{12}) + \phi_h(A_{21}) + \phi_h(A_{22})$ for $h = m$.

Indeed, applying Lemma 3.3 and using the defining relation for ϕ_m, we obtain

$$
\begin{aligned}
&\sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k\text{−−th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \frac{P_1}{c_{n-1}} \diamond_\xi (A_{11} + A_{12} + A_{21} + A_{22}) \\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m\left(\frac{P_1}{c_{n-1}}\right) \diamond_\xi (A_{11} + A_{12} + A_{21} + A_{22}) \\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1}{c_{n-1}} \diamond_\xi \phi_m(A_{11} + A_{12} + A_{21} + A_{22}) \\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \phi_{r_2}(I) \cdots \diamond_\xi \phi_{r_{n-2}}(I) \diamond_\xi \phi_{r_{n-1}}(\frac{P_1}{c_{n-1}}) \diamond_\xi \phi_{r_n}(A_{11} + A_{12} + A_{21} + A_{22}) \\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1}{c_{n-1}} \diamond_\xi (A_{11} + A_{12} + A_{21} + A_{22})\right) \\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1}{c_{n-1}} \diamond_\xi (A_{11} + A_{12} + A_{21})\right) + \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1}{c_{n-1}} \diamond_\xi A_{22}\right) \\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1}{c_{n-1}} \diamond_\xi A_{11}\right) + \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1}{c_{n-1}} \diamond_\xi A_{12}\right) + \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1}{c_{n-1}} \diamond_\xi A_{21}\right) \\
&\quad + \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1}{c_{n-1}} \diamond_\xi A_{22}\right) \\
&= \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k\text{−th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \frac{P_1}{c_{n-1}} \diamond_\xi (A_{11} + A_{12} + A_{21} + A_{22}) \\
&\quad + I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m\left(\frac{P_1}{1+\xi}\right) \diamond_\xi (A_{11} + A_{12} + A_{21} + A_{22}) \\
&\quad + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1}{c_{n-1}} \diamond_\xi (\phi_m(A_{11}) + \phi_m(A_{12}) + \phi_m(A_{21}) + \phi_m(A_{22})) \\
&\quad + \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \phi_{r_2}(I) \cdots \diamond_\xi \phi_{r_{n-2}}(I) \diamond_\xi \phi_{r_{n-1}}(\frac{P_1}{c_{n-1}}) \diamond_\xi \phi_{r_n}(A_{11} + A_{12} + A_{21} + A_{22}).
\end{aligned}
$$

Comparing the first and last expressions in the above chain, and canceling all common terms, we arrive at

$$I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_1}{c_{n-1}} \diamond_\xi T = 0.$$

A direct computation, together with Peirce decomposition, leads to the following conclusion: $T_{11} = T_{12} = T_{21} = 0$. Repeating the argument with P_2 in place of P_1 yields $T_{22} = 0$. Hence $T = 0$, proving the desired additivity. □

Lemma 3.7. *Let $i, j \in \{1, 2\}$ with $i \neq j$. Then $\phi_m(A_{ij} + B_{ij}) = \phi_m(A_{ij}) + \phi_m(B_{ij})$ for all $A_{ij}, B_{ij} \in \mathscr{A}_{ij}$.*

Proof. Equation $\phi_1(A_{ij} + B_{ij}) = \phi_1(A_{ij}) + \phi_1(B_{ij})$ follows from Lemma 2.8. Assume inductively that the equation $\phi_h(A_{ij} + B_{ij}) = \phi_h(A_{ij}) + \phi_h(B_{ij})$ is valid for any $h < m$. We now turn to the proof of conclusion $\phi_h(A_{ij} + B_{ij}) = \phi_h(A_{ij}) + \phi_h(B_{ij})$ in the case where $h = m$.

$$I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_i + A_{ij}}{c_{n-1}} \diamond_\xi (P_j + B_{ij}) = A_{ij} + B_{ij} + \xi A_{ij}^* + \xi B_{ij} A_{ij}^*.$$

Applying the defining relation for ϕ_m to the left-hand side, and using the additivity of ϕ_m on Peirce components established in Lemma 3.3, we obtain two expansions of the same term:

$$\begin{aligned}
&\phi_m(A_{ij} + B_{ij}) + \phi_m(\xi A_{ij}^*) + \phi_m(\xi B_{ij} A_{ij}^*) \\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_i + A_{ij}}{c_{n-1}} \diamond_\xi (P_j + B_{ij})\right) \\
&= \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k-\text{th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \frac{P_i + A_{ij}}{c_{n-1}} \diamond_\xi (P_j + B_{ij}) \\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m\left(\frac{P_i + A_{ij}}{c_{n-1}}\right) \diamond_\xi (P_j + B_{ij}) + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_i + A_{ij}}{c_{n-1}} \diamond_\xi \phi_m(P_j + B_{ij}) \\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m, \\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \phi_{r_2}(I) \cdots \diamond_\xi \phi_{r_{n-1}}(\frac{P_i + A_{ij}}{c_{n-1}}) \diamond_\xi \phi_{r_n}(P_j + B_{ij}).
\end{aligned}$$

On the other hand, using the previously established additivity of ϕ_m on sums of Peirce components, we may expand the same expression as

$$\begin{aligned}
&\phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_i + A_{ij}}{c_{n-1}} \diamond_\xi (P_j + B_{ij})\right) \\
&= \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k-\text{th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \frac{P_i + A_{ij}}{c_{n-1}} \diamond_\xi (P_j + B_{ij}) \\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \left(\phi_m\left(\frac{P_i}{c_{n-1}}\right) + \phi_m\left(\frac{A_{ij}}{c_{n-1}}\right)\right) \diamond_\xi (P_j + B_{ij}) \\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_i + A_{ij}}{c_{n-1}} \diamond_\xi \phi_m(P_j) + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_i + A_{ij}}{c_{n-1}} \diamond_\xi \phi_m(B_{ij}) \\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m, \\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \phi_{r_2}(I) \cdots \diamond_\xi \phi_{r_{n-1}}(\frac{P_i + A_{ij}}{c_{n-1}}) \diamond_\xi \phi_{r_n}(P_j + B_{ij}) \\
&= \phi_m(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_i}{c_{n-1}} \diamond_\xi P_j) + \phi_m(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{A_{ij}}{c_{n-1}} \diamond_\xi P_j) + \phi_m(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_i}{c_{n-1}} \diamond_\xi B_{ij}) \\
&+ \phi_m(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{B_{ij}}{c_{n-1}} \diamond_\xi B_{ij}) \\
&= \phi_m(A_{ij}) + \phi_m(B_{ij}) + (\phi_m(\xi A_{ij}^*) + \phi_m(\xi B_{ij} A_{ij}^*)).
\end{aligned}$$

Comparing the two expansions and canceling the common terms involving $\phi_m(\xi A_{ij}^*)$ and $\phi_m(\xi B_{ij} A_{ij}^*)$, we obtain

$$\phi_m(A_{ij} + B_{ij}) = \phi_m(A_{ij}) + \phi_m(B_{ij}).$$

Thus the additivity of ϕ_m on the off-diagonal subspace $\mathscr{A}_{ij}$ is established. □

Before presenting the key additivity property, we establish the following auxiliary result, which will be frequently used in the subsequent derivation of the main theorem.

Lemma 3.8. *$\phi_m(A_{ii} + B_{ii}) = \phi_m(A_{ii}) + \phi_m(B_{ii})$ for all $A_{ii}, B_{ii} \in \mathscr{A}_{ii}$, $1 \le i \le 2$.*

Proof. Invoking Lemma 2.9, we obtain the equation $\phi_1(A_{ii} + B_{ii}) = \phi_1(A_{ii}) + \phi_1(B_{ii})$. With the induction hypothesis that the equation $\phi_h(A_{ii} + B_{ii}) = \phi_h(A_{ii}) + \phi_h(B_{ii})$ holds for each $h < m$, we proceed to demonstrate the conclusion $\phi_m(A_{ii} + B_{ii}) = \phi_m(A_{ii}) + \phi_m(B_{ii})$ for $h = m$.

In fact, set

$$T := \phi_m(A_{ii} + B_{ii}) - \phi_m(A_{ii}) - \phi_m(B_{ii}).$$

We shall prove that $T = 0$. Fix $j \neq i$, and since

$$I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_j}{c_{n-1}} \diamond_\xi A_{ii} = 0 \quad \text{and} \quad I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_j}{c_{n-1}} \diamond_\xi B_{ii} = 0,$$

by the Peirce orthogonality, applying the defining relation for Ψ yields

$$\begin{aligned}
&\phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_j}{c_{n-1}} \diamond_\xi (A_{ii} + B_{ii})\right) \\
&= \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k-\text{th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \frac{P_j}{c_{n-1}} \diamond_\xi (A_{ii} + B_{ii}) \\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m\left(\frac{P_j}{c_{n-1}}\right) \diamond_\xi (A_{ii} + B_{ii}) + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_j}{c_{n-1}} \diamond_\xi \phi_m(A_{ii} + B_{ii}) \\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \phi_{r_2}(I) \cdots \diamond_\xi \phi_{r_{n-1}}(\frac{P_j}{c_{n-1}}) \diamond_\xi \phi_{r_n}(A_{ii} + B_{ii}).
\end{aligned}$$

On the other hand, using the additivity of ϕ_m on off-diagonal subspaces (Lemma 3.7), we may expand the left-hand side as

$$\begin{aligned}
&\phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_j}{c_{n-1}} \diamond_\xi (A_{ii} + B_{ii})\right) \\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_j}{c_{n-1}} \diamond_\xi A_{ii}\right) + \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_j}{c_{n-1}} \diamond_\xi B_{ii}\right) \\
&= \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k--\text{th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \frac{P_j}{c_{n-1}} \diamond_\xi (A_{ii} + B_{ii}) \\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m\left(\frac{P_j}{c_{n-1}}\right) \diamond_\xi (A_{ii} + B_{ii}) + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_j}{c_{n-1}} \diamond_\xi (\phi_m(A_{ii}) + \phi_m(B_{ii})) \\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \phi_{r_2}(I) \cdots \diamond_\xi \phi_{r_{n-1}}(\frac{P_j}{c_{n-1}}) \diamond_\xi (\phi_{r_n}(A_{ii}) + \phi_{r_n}(B_{ii})).
\end{aligned}$$

Comparing the two expansions and canceling common terms, we arrive at

$$I^{\diamond_\xi(n-2)} \diamond_\xi \frac{P_j}{c_{n-1}} \diamond_\xi T = 0.$$

A standard Peirce component analysis shows that this identity annihilates all components of T except possibly T_{ii}; hence $T_{ij} = T_{ji} = T_{jj} = 0$.

It remains to eliminate T_{ii}. Let C_{ij} be an arbitrary element in $\mathscr{A}_{ij}$ $(i \neq j)$. Applying Lemma 3.7 and the defining relation for Φ, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k\text{--th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \frac{C_{ij}^*}{c_{n-1}} \diamond_\xi (A_{ii} + B_{ii}) \\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m(\frac{C_{ij}^*}{c_{n-1}}) \diamond_\xi (A_{ii} + B_{ii}) + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{C_{ij}^*}{c_{n-1}} \diamond_\xi \phi_m(A_{ii} + B_{ii}) \\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \phi_{r_2}(I) \cdots \diamond_\xi \phi_{r_{n-1}}(\frac{C_{ij}^*}{c_{n-1}}) \diamond_\xi \phi_{r_n}(A_{ii} + B_{ii}) \\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{C_{ij}^*}{c_{n-1}} \diamond_\xi (A_{ii} + B_{ii})\right) \\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{C_{ij}^*}{c_{n-1}} \diamond_\xi A_{ii}\right) + \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi \frac{C_{ij}^*}{c_{n-1}} \diamond_\xi B_{ii}\right) \\
&= \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k\text{--th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi \frac{C_{ij}^*}{c_{n-1}} \diamond_\xi (A_{ii} + B_{ii}) \\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m(\frac{C_{ij}^*}{c_{n-1}}) \diamond_\xi (A_{ii} + B_{ii}) + I^{\diamond_\xi(n-2)} \diamond_\xi \frac{C_{ij}^*}{c_{n-1}} \diamond_\xi (\phi_m(A_{ii}) + \phi_m(B_{ii})) \\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{r_1}(I) \diamond_\xi \phi_{r_2}(I) \cdots \diamond_\xi \phi_{r_{n-1}}(\frac{C_{ij}^*}{c_{n-1}}) \diamond_\xi (\phi_{r_n}(A_{ii}) + \phi_{r_n}(B_{ii})).
\end{aligned}$$

Comparing the two expansions yields

$$I^{\diamond_\xi(n-2)} \diamond_\xi \frac{C_{ij}^*}{c_{n-1}} \diamond_\xi T = 0.$$

Since $C_{ij} \in \mathscr{A}_{ij}$ is arbitrary and the algebra satisfies relation (ℚ), it follows that $T_{ii} = 0$.

Combining the above results, all Peirce components of T vanish, so $T = 0$. Therefore,

$$\phi_m(A_{ii} + B_{ii}) = \phi_m(A_{ii}) + \phi_m(B_{ii}),$$

as required.

□

Lemma 3.9. *For each positive integer m, ϕ_m is additive.*

Proof. The additivity of ϕ_m follows from Lemmas 3.6–3.8. The additivity of nonlinear ξ-skew Jordan-type higher derivations holds. □

The additivity of nonlinear ξ-skew Jordan-type higher derivations on unital $*$-algebras has already been established in Lemma 3.9. On this basis, we now present the main theorem of this section, which states that every nonlinear ξ-skew Jordan-type higher derivation is an additive higher $*$-derivation. This result is obtained by mathematical induction, with the structure established in Theorem 2.2 serving as the inductive foundation.

Proof of Theorem 3.1 Clearly, we only need to prove the necessity. We have proved the additivity of ϕ_m. Now, we prove the result in two cases.

Case 1. $|\xi| = 1$ **and** $\xi \neq 1, -1$

We first show that $\varphi_m(I) = 0$ and $\varphi_m(iI) = 0$, where i denotes the imaginary unit. The proof is by complete induction on m. For $m = 1$, the assertion reduces to Case 1 of Theorem 2.2, from which we have $\varphi_1(I) = 0$ and $\varphi_1(iI) = 0$. We assume that $\varphi_h(I) = 0$ and $\varphi_h(iI) = 0$, where $1 < h < m$.

Since $I^{\diamond_\xi(n-2)} \diamond_\xi iI \diamond_\xi iI = 0$, we obtain

$$\begin{aligned}
0 &= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi iI \diamond_\xi iI\right) \\
&= \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k\text{--th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi iI \diamond_\xi iI + I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m(iI) \diamond_\xi iI \\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{i_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{i_{n-2}}(I) \diamond_\xi \phi_{i_{n-1}}(iI) \diamond_\xi \phi_{i_n}(iI) \\
&+ I^{\diamond_\xi(n-2)} \diamond_\xi iI \diamond_\xi \phi_{i_{n-1}}(iI) \\
&= 2^{n-3} i(1+\xi)\left(\phi_m(iI) + \phi_m(iI)^*\right).
\end{aligned}$$

It follows that

$$\phi_m(iI)^* = -\phi_m(iI).$$

Then we have

$$\begin{aligned}
2^{n-2}\phi_m(i(1-\xi)I) &= \phi_m\left(iI \diamond_\xi I^{\diamond_\xi(n-1)}\right) \\
&= \phi_m(iI) \diamond_\xi I^{\diamond_\xi(n-1)} + \sum_{k=2}^{n} iI \diamond_\xi I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k\text{--th component}} \diamond_\xi \cdots \diamond_\xi I \\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{i_1}(iI) \diamond_\xi \phi_{i_2}(I) \diamond_\xi \cdots \diamond_\xi \phi_{i_{n-1}}(I) \diamond_\xi \phi_{i_n}(I) \qquad (3.1) \\
&= \phi_m(I)((n-2)2^{n-3}i(1-\xi) + 2^{n-2}i(1-\xi)) + \phi_m(I)^*(n-2)2^{n-3}i(1-\xi) \\
&+ \phi_m(iI)2^{n-2}i(1-\xi).
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
&2^{n-2}\phi_m(i(1+\xi)I) = \phi_m\left(I^{\diamond_\xi(n-1)} \diamond_\xi iI\right)\\
&= \sum_{k=1}^{n-1} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k\text{--th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi iI + I^{\diamond_\xi(n-1)} \diamond_\xi \phi_m(iI)\\
&+ \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{i_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{i_{n-1}}(I) \diamond_\xi \phi_{i_n}(iI)\\
&= \phi_m(I)(2^{n-2}i + (n-2)2^{n-3}i(1+\xi)) + \phi_m(I)^*(2^{n-2}i\xi + (n-2)2^{n-3}i(1+\xi))\\
&+ \phi_m(iI)2^{n-2}i(1+\xi).
\end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$\phi(I)^* = \frac{\xi - n}{n-2+\xi}\phi(I). \tag{3.3}$$

Taking the adjoint of (3.3) gives

$$\phi(I) = \frac{\overline{\xi} - n}{n-2+\overline{\xi}}\phi(I)^* = \frac{\overline{\xi} - n}{n-2+\overline{\xi}} \cdot \frac{\xi - n}{n-2+\xi}\phi(I).$$

If $\phi_m(I) \neq 0$, then $\xi = 1$, which is a contradiction. Thus

$$\phi_m(I) = 0. \tag{3.4}$$

For all $A \in \mathscr{A}$, we have

$$\begin{aligned}
&2^{n-2}\phi_m((1+\xi)A)\\
&= \phi_m\left(I^{\diamond_\xi(n-1)} \diamond_\xi A\right)\\
&= \sum_{\substack{r_1+r_2+\cdots+r_n=m,\\ r_1,r_2,\ldots,r_n<m}} \phi_{i_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{i_{n-1}}(I) \diamond_\xi \phi_{i_n}(A) + I^{\diamond_\xi(n-1)} \diamond_\xi \phi_m(A)\\
&= 2^{n-2}(1+\xi)\phi_m(A),
\end{aligned}$$

i.e.,

$$\phi_m((1+\xi)A) = (1+\xi)\phi_m(A). \tag{3.5}$$

For all $A, B \in \mathscr{A}$, we obtain

$$\begin{aligned}
&2^{n-3}\phi_m\left((1+\xi)(AB + BA^*)\right)\\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi A \diamond_\xi B\right)\\
&= \sum_{r_1+r_2+\cdots+r_n=m} \phi_{i_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{i_{n-2}}(I) \diamond_\xi \phi_{i_{n-1}}(A) \diamond_\xi \phi_{i_n}(B)\\
&= \sum_{s+t=m} I^{\diamond_\xi(n-2)} \diamond_\xi \phi_s(A) \diamond_\xi \phi_t(B)\\
&= 2^{n-3}(1+\xi) \sum_{s+t=m} \left(\phi_s(A)\phi_t(B) + \phi_t(B)\phi_s(A)^*\right).
\end{aligned} \tag{3.6}$$

This together with (3.5) and (3.6) indicates that

$$\phi_m(AB + BA^*) = \sum_{s+t=m} \phi_s(A)\phi_t(B) + \phi_t(B)\phi_s(A)^*. \tag{3.7}$$

For all $A \in \mathscr{A}$, from (3.4), we obtain

$$\begin{aligned} 2^{n-2}\phi_m(A) + 2^{n-2}\phi_m(\xi A^*) &= \phi_m\left(A \diamond_\xi I^{\diamond_\xi(n-1)}\right) \\ &= \sum_{i_1+i_2+\cdots+i_n=m} \phi_{i_1}(A) \diamond_\xi \phi_{i_2}(I) \diamond_\xi \cdots \phi_{i_n}(I) \\ &= \phi_m(A) \diamond_\xi I^{\diamond_\xi(n-1)} \\ &= 2^{n-2}\phi_m(A) + 2^{n-2}\xi\phi_m(A)^*, \end{aligned}$$

which indicates

$$\phi_m(\xi A^*) = \xi\phi_m(A)^*.$$

From (3.5), we have $\phi_m(\xi A^*) = \xi\phi_m(A^*)$. Then

$$\phi_m(A^*) = \phi_m(A)^*.$$

It follows from (3.4), (3.5), and

$$\begin{aligned} 2^{n-2}\phi_m((\xi-1)I) &= \phi_m\left(iI \diamond_\xi I^{\diamond_\xi(n-2)} \diamond_\xi iI\right) \\ &= \sum_{i_1+i_2+\cdots+ir_n=m} \phi_{i_1}(iI) \diamond_\xi \phi_{i_2}(I) \diamond_\xi \cdots \diamond_\xi \phi_{i_{n-1}}(I) \diamond_\xi \phi_{i_n}(iI) \\ &= \phi_m(iI) \diamond_\xi I^{\diamond_\xi(n-2)} \diamond_\xi iI \\ &+ iI \diamond_\xi I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m(iI) \\ &= 2^{n-1}i(1-\xi)\phi_m(iI) \end{aligned}$$

that

$$\phi_m(iI) = 0.$$

For all $A = A^* \in \mathscr{A}$, from the above equalities, we have

$$2\phi_m(iA) = \phi_m(A(iI) + (iI)A^*) = \phi_m(A)iI + iI\phi_m(A)^* = 2i\phi_m(A),$$

i.e., $\phi_m(iA) = i\phi_m(A)$. Thus for all $A \in \mathscr{A}$, since $A = A_1 + iA_2$ with $A_1 = \frac{1}{2}(A + A^*)$ and $A_2 = \frac{1}{2i}(A - A^*)$, we obtain

$$\begin{aligned} \phi_m(iA) &= \phi_m(i(A_1 + iA_2)) = \phi_m(iA_1 - A_2) = i\phi_m(A_1) - \phi_m(A_2) = i\phi_m(A_1) + i^2\phi_m(A_2) \\ &= i\phi_m(A_1) + i\phi_m(iA_2) = i\phi_m(A_1 + iA_2) = i\phi_m(A). \end{aligned}$$

For all $A, B \in \mathscr{A}$, from $\phi_m(AB + BA^*) = \sum_{s+t=m}(\phi_s(A)\phi_t(B) + \phi_t(B)\phi_s(A)^*)$, we obtain

$$\begin{aligned} \phi_m(-AB + BA^*) &= \phi_m((iA)(iB) + (iB)(iA)^*) \\ &= \sum_{s+t=m} (\phi_s(iA)\phi_t(iB) + \phi_t(iB)\phi_s(iA)^*) \\ &= \sum_{s+t=m} (-\phi_s(A)\phi_t(B) + \phi_t(B)\phi_s(A)^*). \end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we have $\phi_m(AB) = \sum_{s+t=m}(\phi_s(A)\phi_t(B))$. Then ϕ_m is a $*$-derivation. From (4.5), we obtain $\phi_m(\xi A) = \xi\phi_m(A)$ for all $A \in \mathscr{A}$.

Case 2. $|\xi| \neq 1$

We first show that $\varphi_m(I) = 0$ and $\varphi_m(iI) = 0$, where i denotes the imaginary unit. The proof proceeds by complete induction on m. For $m = 1$, the assertion reduces to Case 2 of Theorem 2.2, which gives $\varphi_1(I) = 0$ and $\varphi_1(iI) = 0$. Suppose now that $\varphi_h(I) = 0$ and $\varphi_h(iI) = 0$ for all h with $1 < h < m$.

we introduce the symbol $X^{\diamond_\xi(m)} = \underbrace{X \diamond_\xi \cdots \diamond_\xi X}_{m\ copies}$ for all $X \in \mathscr{A}$. It is easy to verify that

$$\phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi iI \diamond_\xi iI\right) = \phi_m\left(iI \diamond_\xi iI \diamond_\xi I^{\diamond_\xi(n-2)}\right).$$

Thus

$$\begin{aligned}
&\sum_{\substack{i_1+i_2+\cdots+i_n=m,\\ i_1,i_2,\ldots,i_n<m}} \phi_{i_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{i_{n-2}}(I) \diamond_\xi \phi_{i_{n-1}}(iI) \diamond_\xi \phi_{i_n}(iI)\\
&+ \sum_{k=1}^{n-2} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k-\text{-th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi iI \diamond_\xi iI + I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m(iI) \diamond_\xi iI + I^{\diamond_\xi(n-2)} \diamond_\xi iI \diamond_\xi \phi_m(iI)\\
&= \phi_m(iI) \diamond_\xi iI \diamond_\xi I^{\diamond_\xi(n-2)} + iI \diamond_\xi \phi_m(iI) \diamond_\xi I^{\diamond_\xi(n-2)} + \sum_{k=3}^{n} iI \diamond_\xi iI \diamond_\xi I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k-\text{-th component}} \diamond_\xi \cdots \diamond_\xi I\\
&+ \sum_{\substack{i_1+i_2+\cdots+i_n=m,\\ i_1,i_2,\ldots,i_n<m}} \phi_{i_1}(iI) \diamond_\xi \phi_{i_2}(iI) \diamond_\xi \phi_{i_3}(I) \diamond_\xi \cdots \diamond_\xi \phi_{i_n}(I),
\end{aligned}$$

i.e.,

$$\begin{aligned}
&2iI^{\diamond_\xi(n-2)}\phi(iI) + |\xi|^2 iI^{\diamond_\xi(n-2)}\phi_m(iI)^* + \xi i\phi_m(iI)^*(I^{\diamond_\xi(n-2)})^* - i|\xi|^2\phi_m(iI)I^{\diamond_\xi(n-2)} + i\xi(I^{\diamond_\xi(n-2)})^*\phi(iI)\\
&= 2iI^{\diamond_\xi(n-2)}\phi_m(iI) - \xi iI^{\diamond_\xi(n-2)}\phi_m(iI)^* - \xi i\phi_m(iI)I^{\diamond_\xi(n-2)} - i|\xi|^2 I^{\diamond_\xi(n-2)}\phi_m(iI) + i|\xi|^2 I^{\diamond_\xi(n-2)}\phi_m(iI)^*.
\end{aligned}$$

Then $(\xi iI^{\diamond_\xi(n-2)} + \xi i(I^{\diamond_\xi(n-2)})^*)(\phi_m(iI) + \phi_m(iI)^*) = 0$, which indicates

$$\phi_m(iI)^* = -\phi_m(iI). \tag{3.9}$$

On the other hand, we obtain

$$\phi_m\left(I^{\diamond_\xi(n-3)} \diamond_\xi iI \diamond_\xi iI \diamond_\xi iI\right) = -\phi_m\left(I^{\diamond_\xi(n-3)} \diamond_\xi I \diamond_\xi iI \diamond_\xi I\right).$$

Then

$$\begin{aligned}
&\sum_{\substack{i_1+i_2+\cdots+i_n=m,\\ i_1,i_2,\ldots,i_n<m}} \phi_{i_1}(I)\diamond_\xi\cdots\diamond_\xi\phi_{i_{n-3}}(I)\diamond_\xi\phi_{i_{n-2}}(iI)\diamond_\xi\phi_{i_{n-1}}(iI)\diamond_\xi\phi_{i_n}(iI)\\
&+\sum_{k=1}^{n-3} I\diamond_\xi\cdots\diamond_\xi\underbrace{\phi_m(I)}_{k\text{--th component}}\diamond_\xi\cdots\diamond_\xi I\diamond_\xi iI\diamond_\xi iI\diamond_\xi iI+I^{\diamond_\xi(n-3)}\diamond_\xi\phi_m(iI)\diamond_\xi iI\diamond_\xi iI\\
&+I^{\diamond_\xi(n-3)}\diamond_\xi iI\diamond_\xi\phi_m(iI)\diamond_\xi iI+I^{\diamond_\xi(n-3)}\diamond_\xi iI\diamond_\xi iI\diamond_\xi\phi_m(iI)\\
&=-\sum_{k=1}^{n-2} I\diamond_\xi\cdots\diamond_\xi\underbrace{\phi_m(I)}_{k\text{--th component}}\diamond_\xi\cdots\diamond_\xi I\diamond_\xi iI\diamond_\xi I-I^{\diamond_\xi(n-3)}\diamond_\xi\phi_m(iI)\diamond_\xi iI\diamond_\xi iI\\
&-I^{\diamond_\xi(n-3)}\diamond_\xi I\diamond_\xi\phi_m(iI)\diamond_\xi I-I^{\diamond_\xi(n-3)}\diamond_\xi I\diamond_\xi iI\diamond_\xi\phi_m(I)\\
&-\sum_{\substack{i_1+i_2+\cdots+i_n=m,\\ i_1,i_2,\ldots,i_n<m}} \phi_{i_1}(I)\diamond_\xi\cdots\diamond_\xi\phi_{i_{n-3}}(I)\diamond_\xi\phi_{i_{n-2}}(I)\diamond_\xi\phi_{i_{n-1}}(iI)\diamond_\xi\phi_{i_n}(I).
\end{aligned}$$

This, along with (4.9), indicates that

$$\begin{aligned}
&3(|\xi|^2-1)\phi_m(iI)(I^{\diamond_\xi(n-3)}+\xi(I^{\diamond_\xi(n-3)})^*)\\
&=-2i(1-|\xi|^2)\phi_m(I)(I^{\diamond_\xi(n-3)}+\xi(I^{\diamond_\xi(n-3)})^*)+(|\xi|^2-1)\phi_m(iI)(I^{\diamond_\xi(n-3)}+\xi(I^{\diamond_\xi(n-3)})^*),
\end{aligned}$$

which indicates

$$\phi_m(iI)=i\phi_m(I). \tag{3.10}$$

Hence

$$\phi_m(I)^*=\phi_m(I).$$

Let $n\geq 3$ and

$$\begin{aligned}
&\phi_m(i(1-|\xi|^2)I^{\diamond_\xi(n-3)}-i\xi(1-|\xi|^2)(I^{\diamond_\xi(n-3)})^*)\\
&=\phi_m\left(I^{\diamond_\xi(n-3)}\diamond_\xi iI\diamond_\xi I\diamond_\xi I\right)\\
&=\sum_{k=1}^{n-3} I\diamond_\xi\cdots\diamond_\xi\underbrace{\phi_m(I)}_{k\text{--th component}}\diamond_\xi\cdots\diamond_\xi I\diamond_\xi iI\diamond_\xi I\diamond_\xi I+I^{\diamond_\xi(n-3)}\diamond_\xi\phi_m(iI)\diamond_\xi I\diamond_\xi I\\
&+I^{\diamond_\xi(n-3)}\diamond_\xi iI\diamond_\xi\phi_m(I)\diamond_\xi I+I^{\diamond_\xi(n-3)}\diamond_\xi iI\diamond_\xi I\diamond_\xi\phi_m(I)\\
&+\sum_{\substack{i_1+i_2+\cdots+i_n=m,\\ i_1,i_2,\ldots,i_n<m}} \phi_{i_1}(I)\diamond_\xi\cdots\diamond_\xi\phi_{i_{n-3}}(I)\diamond_\xi\phi_{i_{n-2}}(iI)\diamond_\xi\phi_{i_{n-1}}(I)\diamond_\xi\phi_{i_n}(I)\\
&=n(i(1-|\xi|^2)I^{\diamond_\xi(n-3)}\phi_m(I)+i\xi(1-|\xi|^2)(I^{\diamond_\xi(n-3)})^*\phi_m(I)).
\end{aligned} \tag{3.11}$$

On the other hand, we have

$$\begin{aligned}
&\phi_m(i(1+3|\xi|^2)I^{\diamond_\xi(n-3)} + i\xi(3+|\xi|^2)(I^{\diamond_\xi(n-3)})^*)\\
&= \phi_m\left(I^{\diamond_\xi(n-3)} \diamond_\xi I \diamond_\xi I \diamond_\xi iI\right)\\
&= \sum_{k=1}^{n-3} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k-\text{th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi I \diamond_\xi I \diamond_\xi iI + I^{\diamond_\xi(n-3)} \diamond_\xi \phi_m(I) \diamond_\xi I \diamond_\xi iI\\
&+ I^{\diamond_\xi(n-3)} \diamond_\xi I \diamond_\xi \phi_m(I) \diamond_\xi iI + I^{\diamond_\xi(n-3)} \diamond_\xi I \diamond_\xi I \diamond_\xi \phi_m(iI)\\
&+ \sum_{\substack{i_1+i_2+\cdots+i_n=m,\\ i_1,i_2,\ldots,i_n<m}} \phi_{i_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{i_{n-3}}(I) \diamond_\xi \phi_{i_{n-2}}(I) \diamond_\xi \phi_{i_{n-1}}(I) \diamond_\xi \phi_{i_n}(iI)\\
&= n(i(1+3|\xi|^2)I^{\diamond_\xi(n-3)}\phi_m(I) + i\xi(3+|\xi|^2)(I^{\diamond_\xi(n-3)})^*\phi_m(I))
\end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$\begin{aligned}
&\phi_m(i(1+|\xi|^2)I^{\diamond_\xi(n-3)} + i\xi(1+|\xi|^2)(I^{\diamond_\xi(n-3)})^*)\\
&= n(i(1+|\xi|^2)I^{\diamond_\xi(n-3)}\phi_m(I) + i\xi(1+|\xi|^2)(I^{\diamond_\xi(n-3)})^*\phi_m(I))
\end{aligned}$$

and

$$\begin{aligned}
&\phi_m(i(1-|\xi|^2)I^{\diamond_\xi(n-3)} + i\xi(1-|\xi|^2)(I^{\diamond_\xi(n-3)})^*)\\
&= \phi_m\left(I^{\diamond_\xi(n-3)} \diamond_\xi I \diamond_\xi iI \diamond_\xi I\right)\\
&= \sum_{k=1}^{n-3} I \diamond_\xi \cdots \diamond_\xi \underbrace{\phi_m(I)}_{k--\text{th component}} \diamond_\xi \cdots \diamond_\xi I \diamond_\xi I \diamond_\xi iI \diamond_\xi I + I^{\diamond_\xi(n-3)} \diamond_\xi \phi_m(I) \diamond_\xi iI \diamond_\xi I\\
&+ I^{\diamond_\xi(n-3)} \diamond_\xi I \diamond_\xi \phi_m(iI) \diamond_\xi I + I^{\diamond_\xi(n-3)} \diamond_\xi I \diamond_\xi iI \diamond_\xi \phi_m(I)\\
&+ \sum_{\substack{i_1+i_2+\cdots+i_n=m,\\ i_1,i_2,\ldots,i_n<m}} \phi_{i_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{i_{n-3}}(I) \diamond_\xi \phi_{i_{n-2}}(I) \diamond_\xi \phi_{i_{n-1}}(iI) \diamond_\xi \phi_{i_n}(I)\\
&= n(i(1-|\xi|^2)I^{\diamond_\xi(n-3)}\phi_m(I) + i\xi(1-|\xi|^2)(I^{\diamond_\xi(n-3)})^*\phi_m(I).
\end{aligned}$$

We conclude that $\phi_m(I) = 0$. It follows from (3.10) that

$$\phi_m(iI) = 0. \tag{3.13}$$

For all $A \in \mathscr{A}$, from (3.13), we obtain

$$\phi_m\left(I^{\diamond_\xi(n-3)} \diamond_\xi A \diamond_\xi iI \diamond_\xi iI\right) = I^{\diamond_\xi(n-3)} \diamond_\xi \phi_m(A) \diamond_\xi iI \diamond_\xi iI,$$

which implies

$$\phi_m((|\xi|^2-1)c_{n-2}A) = (|\xi|^2-1)\phi_m(A)c_{n-2}. \tag{3.14}$$

From $\phi_m(I) = 0$, we have

$$\phi_m\left(I^{\diamond_\xi(n-3)} \diamond_\xi A \diamond_\xi I \diamond_\xi I\right) = I^{\diamond_\xi(n-3)} \diamond_\xi \phi_m(A) \diamond_\xi I \diamond_\xi I.$$

It follows from (3.14) that

$$\begin{aligned}
&\phi_m((1+|\xi|^2)A(c_{n-3} + \xi c_{n-3}^*) + 2\xi c_{n-3}^* A^* + 2|\xi|^2 A^* c_{n-3})\\
&= (1+|\xi|^2)c_{n-3}\phi_m(A) + 2\xi c_{n-3}^*\phi_m(A^*) + 2|\xi|^2 c_{n-3}\phi_m(A^*) + \xi(1+|\xi|^2)c_{n-3}^*\phi_m(A).
\end{aligned} \tag{3.15}$$

On the other hand, we obtain

$$\phi_m\left(I^{\diamond_\xi(n-3)} \diamond_\xi I \diamond_\xi I \diamond_\xi A^*\right) = I^{\diamond_\xi(n-3)} \diamond_\xi I \diamond_\xi I \diamond_\xi \phi_m\left(A^*\right),$$

which indicates

$$\begin{aligned}&(1+3\xi^2)\phi_m(I^{\diamond_\xi(n-3)}A^*) + \xi(\xi^2+3)\phi_m((I^{\diamond_\xi(n-3)})^*A^*)\\ &= (1+3\xi^2)I^{\diamond_\xi(n-3)}\phi_m(A^*) + \xi(\xi^2+3)(I^{\diamond_\xi(n-3)})^*\phi_m(A^*).\end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we have

$$\phi_m\left(A^*\right) = \phi_m(A)^*.$$

For all $A \in \mathscr{A}$, from $\phi_m\left(|\xi|^2 A\right) = |\xi|^2\phi_m(A)$, we obtain

$$\begin{aligned}&(1-|\xi|^2)\phi_m(iI^{\diamond_\xi(n-2)}A) + i\xi(1-|\xi|^2)\phi_m(i(I^{\diamond_\xi(n-2)})^*A)\\ &= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi I \diamond_\xi iI \diamond_\xi A\right)\\ &= \sum_{i_1+i_2+\cdots+i_n=m} \phi_{i_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{i_{n-3}}(I) \diamond_\xi \phi_{i_{n-2}}(I) \diamond_\xi \phi_{i_{n-1}}(iI) \diamond_\xi \phi_{i_n}(A)\\ &= I^{\diamond_\xi(n-2)} \diamond_\xi I \diamond_\xi iI \diamond_\xi \phi_m(A)\\ &= (1-|\xi|^2)iI^{\diamond_\xi(n-2)}\phi_m(A) + i\xi(1-|\xi|^2)(I^{\diamond_\xi(n-2)})^*\phi_m(A)\\ &= (1-|\xi|^2)i\phi_m(I^{\diamond_\xi(n-2)}A) + i\xi(1-|\xi|^2)\phi_m((I^{\diamond_\xi(n-2)})^*A).\end{aligned}$$

Hence

$$\phi_m(iA) = i\phi_m(A). \tag{3.17}$$

For all $A, B \in \mathscr{A}$, we obtain

$$\begin{aligned}&\phi_m\left(I^{\diamond_\xi(n-2)}AB + \xi A(I^{\diamond_\xi(n-2)})^*B + \xi BA^*(I^{\diamond_\xi(n-2)})^* + |\xi|^2 BI^{\diamond_\xi(n-2)}A^*\right)\\ &= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi A \diamond_\xi B\right)\\ &= \sum_{i_1+i_2+\cdots+i_n=m} \phi_{i_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{i_{n-2}}(I) \diamond_\xi \phi_{i_{n-1}}(A) \diamond_\xi \phi_{i_n}(B)\\ &= I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m(A) \diamond_\xi B + I^{\diamond_\xi(n-2)} \diamond_\xi A \diamond_\xi \phi_m(B) + \sum_{s+t=m,s,t<m} I^{\diamond_\xi(n-2)} \diamond_\xi \phi_s(A) \diamond_\xi \phi_t(B)\\ &= I^{\diamond_\xi(n-2)}\phi_m(A)B + \xi\phi_m(A)(I^{\diamond_\xi(n-2)})^*B + \xi B\phi_m(A)^*(I^{\diamond_\xi(n-2)})^* + |\xi|^2 BI^{\diamond_\xi(n-2)}\phi_m(A)^*\\ &+ I^{\diamond_\xi(n-2)}A\phi_m(B) + \xi A(I^{\diamond_\xi(n-2)})^*\phi_m(B) + \xi\phi_m(B)A^*(I^{\diamond_\xi(n-2)})^* + |\xi|^2\phi_m(B)I^{\diamond_\xi(n-2)}A^*\\ &+ \sum_{s+t=m,s,t<m} (I^{\diamond_\xi(n-2)}\phi_s(A)\phi_t(B) + \xi\phi_s(A)(I^{\diamond_\xi(n-2)})^*\phi_t(B) + \xi\phi_t(B)\phi_s(A)^*(I^{\diamond_\xi(n-2)})^*\\ &+ |\xi|^2\phi_t(B)I^{\diamond_\xi(n-2)}\phi_s(A)^*).\end{aligned} \tag{3.18}$$

On the other hand, from (3.17), we have

$$\begin{aligned}
&\phi_m\left(I^{\diamond_\xi(n-2)}AB + \xi A(I^{\diamond_\xi(n-2)})^*B - \xi BA^*(I^{\diamond_\xi(n-2)})^* - |\xi|^2 B(I^{\diamond_\xi(n-2)})^*A^*\right)\\
&= \phi_m\left(I^{\diamond_\xi(n-2)} \diamond_\xi iA \diamond_\xi (-iB)\right)\\
&= \sum_{i_1+i_2+\cdots+i_n=m} \phi_{i_1}(I) \diamond_\xi \cdots \diamond_\xi \phi_{i_{n-3}}(I) \diamond_\xi \phi_{i_{n-2}}(I) \diamond_\xi \phi_{i_{n-1}}(iA) \diamond_\xi \phi_{i_n}(-iB)\\
&= I^{\diamond_\xi(n-2)} \diamond_\xi \phi_m(iA) \diamond_\xi (-iB) + I^{\diamond_\xi(n-2)} \diamond_\xi iA \diamond_\xi \phi_m(-iB)\\
&+ \sum_{s+t=m,s,t<m} I^{\diamond_\xi(n-2)} \diamond_\xi \phi_s(iA) \diamond_\xi \phi_t(-iB)\\
&= I^{\diamond_\xi(n-2)}\phi_m(A)B + \xi\phi_m(A)(I^{\diamond_\xi(n-2)})^*B - \xi B\phi_m(A)^*(I^{\diamond_\xi(n-2)})^* - |\xi|^2 BI^{\diamond_\xi(n-2)}\phi_m(A)^* + I^{\diamond_\xi(n-2)}A\phi_m(B)\\
&+ \xi A(I^{\diamond_\xi(n-2)})^*\phi_m(B) - \xi\phi_m(B)A^*(I^{\diamond_\xi(n-2)})^* - |\xi|^2\phi_m(B)A^*I^{\diamond_\xi(n-2)}\\
&+ \sum_{s+t=m,s,t<m} (I^{\diamond_\xi(n-2)}\phi_s(A)\phi_t(B) + \xi\phi_s(A)(I^{\diamond_\xi(n-2)})^*\phi_t(B) - \xi\phi_t(B)\phi_s(A)^*(I^{\diamond_\xi(n-2)})^*\\
&- |\xi|^2\phi_t(B)(I^{\diamond_\xi(n-2)})^*\phi_s(A)^*)
\end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we have

$$\begin{aligned}
&\phi_m((I^{\diamond_\xi(n-2)} + \xi(I^{\diamond_\xi(n-2)})^*)AB)\\
&= (I^{\diamond_\xi(n-2)} + \xi(I^{\diamond_\xi(n-2)})^*)\phi_m(A)B + (I^{\diamond_\xi(n-2)} + \xi(I^{\diamond_\xi(n-2)})^*)A\phi_m(B)\\
&+ \sum_{s+t=m,s,t<m} (I^{\diamond_\xi(n-2)} + \xi(I^{\diamond_\xi(n-2)})^*)\phi_s(A)\phi_t(B)
\end{aligned}$$

and

$$\begin{aligned}
&((I^{\diamond_\xi(n-2)} + \xi(I^{\diamond_\xi(n-2)})^*))\phi_m(AB)\\
&\quad = ((I^{\diamond_\xi(n-2)} + \xi(I^{\diamond_\xi(n-2)})^*))(\phi_m(A)B + A\phi_m(B) + \sum_{s+t=m,s,t<m} \phi_s(A)\phi_t(B)).
\end{aligned}$$

Hence $\phi_m(AB) = \sum_{s+t=m} \phi_s(A)\phi_t(B)$ for all $A, B \in \mathscr{A}$.

4. Applications of the main theorems

We investigate the structure of nonlinear ξ-skew Jordan-type higher derivations $\Psi = \{\varphi_m\}_{m\in\mathbb{N}}$ on unital $*$-algebras $\mathscr{A}$. Our main result establishes that, under condition ($\mathbb{Q}$), every such mapping reduces to an additive higher $*$-derivation and satisfies $\varphi_m(\xi x) = \xi\varphi_m(x)$ for all $x \in \mathscr{A}$. The proof is based on induction combined with matrix diagonalization techniques; the analysis of first-level derivations carried out in Subsection 2.1 provides the base step for the induction argument developed in Section 3. We then discuss several concrete applications of Theorems 2.2 and 3.1.

The notion of nonlinear ξ-skew Jordan-type higher derivations has been introduced in Eq (1.2). In particular, specializing Theorem 3.1 to the case $n = 3$ yields a structural characterization of nonlinear ξ-skew Jordan triple higher derivations on unital $*$-algebras. Since this result appears to be new, we state it explicitly as Theorem 4.2.

It is worth pointing out that Theorem 3.1 involves the condition $\xi \neq \{0, \pm 1\}$. When $n = 3$, we obtain the following proposition.

Proposition 4.1. *Let $\mathscr{A}$ be a unital $*$-algebra with unit I and a nontrivial symmetric idempotent. Suppose that $\phi_1 : \mathscr{A} \to \mathscr{A}$ is a nonlinear ξ-skew Jordan triple derivation satisfying*

$$\phi_1(A_1 \diamond_\xi A_2 \diamond_\xi A_3) = \sum_{r_1+r_2+r_3=1} \phi_{r_1}(A_1) \diamond_\xi \phi_{r_2}(A_2) \diamond_\xi \phi_{r_3}(A_3)$$

for some complex scalar $\xi \notin \{0, \pm 1\}$. Then ϕ is a nonlinear ξ-skew Jordan triple derivation if and only if ϕ is an additive $*$-derivation and $\phi_1(\xi A) = \xi\phi_1(A)$ for all $A \in \mathscr{A}$.

With Proposition 4.1 serving as the basis for induction, the following theorem can be established by following the argument used in [18] (or, equivalently, by applying Theorem 4.1 directly). We record it as Theorem 4.2.

Theorem 4.2. *Let $\mathfrak{A}$ be a unital $*$-algebra satisfying condition $(\mathbb{Q})$ and fix $\xi \in \mathbb{C} \setminus \{0, \pm 1\}$. Let $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ be a family of mappings $\phi_m : \mathfrak{A} \to \mathfrak{A}$ forming a nonlinear ξ-skew Jordan triple higher derivation satisfying the equation*

$$\phi_m(A_1 \diamond_\xi A_2 \diamond_\xi A_3) = \sum_{r_1+r_2+r_3=m} \phi_{r_1}(A_1) \diamond_\xi \phi_{r_2}(A_2) \diamond_\xi \phi_{r_3}(A_3)$$

for all $A_1, A_2, A_3 \in \mathscr{A}$ if and only if $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ is an additive higher $*$-derivation and $\phi_m(\xi A) = \xi\phi_m(A)$ for all $A \in \mathscr{A}$.

As shown in [16], prime $*$-algebras fulfill condition $(\mathbb{Q})$. This observation allows us to interpret the nonlinear derivations and higher derivations studied, respectively, by Zhang [6, Theorem 2.1] and Liang et al. [18, Theorem 2.1] as immediate consequences of Theorem 4.2. Henceforth, all remaining corollaries are stated and proved under the hypothesis that $\xi \in \mathbb{C} \setminus \{0, \pm 1\}$.

As a direct application of Theorems 3.1 and 4.1, we derive analogous conclusions for several well-known classes of unital $*$-algebras: prime $*$-algebras, factor von Neumann algebras, von Neumann algebras with no central summand of type I_1, and standard operator algebras.

From the definition and properties of prime $*$-algebras (see [20, Section 3]), it follows that every such algebra satisfies condition $(\mathbb{Q})$. Therefore we have the following corollary.

Corollary 4.3. *Let $\mathfrak{B}$ be a prime $*$-algebra $\mathscr{A}$. Then every nonlinear ξ-skew Jordan-type higher derivation $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ as defined in Eq* (1.2) *on $\mathfrak{B}$ is an additive higher $*$-derivation and $\phi_m(\xi A) = \xi\phi_m(A)$ for all $A \in \mathscr{A}$.*

Moreover, the above result encompasses and improves the earlier theorem stated in [18, Theorem 2.1].

Corollary 4.4. *[18, Theorem 2.1] Let $\mathfrak{B}$ be a prime $*$-algebra $\mathscr{A}$. Then every nonlinear ξ-skew Jordan triple higher derivation $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ on $\mathfrak{B}$ (corresponding to $n = 3$ in Eq* (1.2)*) is an additive higher $*$-derivation and $\phi_m(\xi A) = \xi\phi_m(A)$ for all $A \in \mathfrak{B}$.*

Let $\mathcal{S}$ be a complex Hilbert space and $\mathfrak{B}(\mathcal{S})$ the full operator algebra. A von Neumann algebra $\mathbb{A} \subseteq \mathfrak{B}(\mathcal{S})$ is called a factor if its center is one-dimensional, i.e., $Z(\mathbb{A}) = \mathbb{C}I$. Since factors are prime, the next corollary is immediate.

Corollary 4.5. *Let $\mathbb{A}$ be a factor von Neumann algebra on a complex Hilbert space with* $\dim(\mathbb{A}) \geq 2$. *Then every nonlinear ξ-skew Jordan-type higher derivation $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ as defined in Eq* (1.2) *on $\mathfrak{B}$ is an additive higher $*$-derivation and $\phi_m(\xi A) = \xi\phi_m(A)$ for all $A \in \mathbb{A}$.*

The case $n = 3$ leads to the following corollary in a straightforward manner.

Corollary 4.6. *Let $\mathbb{A}$ be a factor von Neumann algebra on a complex Hilbert space with* $\dim(\mathbb{A}) \geq 2$. *Then every nonlinear ξ-skew Jordan triple higher derivation $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ on $\mathfrak{B}$ (corresponding to $n = 3$ in Eq* (1.2)*) is an additive higher $*$-derivation and $\phi_m(\xi A) = \xi\phi_m(A)$ for all $A \in \mathbb{A}$.*

It is shown in [21] and [22] that if a von Neumann algebra $\mathbb{M}$ has no central summands of type I_1, then $\mathbb{M}$ satifies ($\mathbb{Q}$). Now we have the following corollary.

Corollary 4.7. *Let $\mathbb{A}$ be a von Neumann algebras with no central summand of type I_1 on a complex Hilbert space with* $\dim(\mathbb{A}) \geq 2$. *Then every nonlinear ξ-skew Jordan-type higher derivation $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ as defined in Eq* (1.2) *on $\mathfrak{B}$ is an additive higher $*$-derivation and $\phi_m(\xi A) = \xi\phi_m(A)$ for all $A \in \mathbb{A}$.*

The case $n = 3$ leads to the following corollary in a straightforward manner.

Corollary 4.8. *Let $\mathbb{A}$ be a von Neumann algebras with no central summand of type I_1 on a complex Hilbert space with* $\dim(\mathbb{A}) \geq 2$. *Then every nonlinear ξ-skew Jordan triple higher derivation $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ on $\mathfrak{B}$ (corresponding to $n = 3$ in Eq* (1.2)*) is an additive higher $*$-derivation and $\phi_m(\xi A) = \xi\phi_m(A)$ for all $A \in \mathbb{A}$.*

Let $\mathfrak{G}(\mathcal{S})$ be the collection of all finite-rank operators in $\mathfrak{B}(\mathcal{S})$. A subalgebra $\mathfrak{R} \subseteq \mathfrak{B}(\mathcal{S})$ is termed "standard" if it contains $\mathfrak{G}(\mathcal{S})$.

Corollary 4.9. *Let $\mathcal{S}$ be an infinite-dimensional complex Hilbert space and $\mathfrak{R}$ a standard operator algebra on $\mathcal{S}$ containing I and closed under the adjoint. Then every nonlinear ξ-skew Jordan-type higher derivation $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ on $\mathfrak{B}$ as defined in Eq* (1.2) *is an additive higher $*$-derivation and $\phi_m(\xi A) = \xi\phi_m(A)$ for all $A \in \mathfrak{R}$.*

For $n = 3$, the desired corollary is an easy consequence of the preceding results.

Corollary 4.10. *Let $\mathcal{S}$ be an infinite-dimensional complex Hilbert space and $\mathfrak{R}$ a standard operator algebra on $\mathcal{S}$ containing I and closed under the adjoint. Then every nonlinear ξ-skew Jordan triple higher derivation $\Psi = \{\phi_m\}_{m\in\mathbb{N}}$ on $\mathfrak{B}$ (corresponding to $n = 3$ in Eq* (1.2)*) is an additive higher $*$-derivation and $\phi_m(\xi A) = \xi\phi_m(A)$ for all $A \in \mathfrak{R}$.*

5. Future research topics

This paper establishes an additive characterization of nonlinear ξ-skew-Jordan-type higher derivations on unital $*$-algebras, but several directions remain worthy of further exploration. First, a natural generalization is to extend the results of this paper from associative algebras to alternative $*$-algebras. Since every associative algebra is alternative, and the Peirce multiplication rules in the alternative setting are genuinely more general, the associative Peirce rules used in this paper must be replaced by the corresponding alternative rules. Existing studies have shown that Peirce methods can force additivity for several nonlinear multiplicative, Jordan-type, and Lie-type maps on alternative rings and

algebras; among them, [23] studied the additivity of n-multiplicative maps on alternative rings, [24] discussed multiplicative Lie-type derivations on alternative rings, and [25,26] supported a broader Peirce decomposition framework from the perspectives of Jordan triple maps and Jordan triple elementary maps. Specifically, one may formulate the following open problem: Let $\mathcal{A}$ be a unital alternative $*$-algebra containing a nontrivial symmetric idempotent and satisfying condition ($\mathbb{Q}$). Under which hypotheses is every nonlinear ξ-skew-Jordan-type higher derivation on $\mathcal{A}$ necessarily an additive higher $*$-derivation satisfying $\phi_m(\xi A) = \xi\phi_m(A)$?

6. Conclusions

The main contribution of this paper is as follows: Under condition ($\mathbb{Q}$), we proved that nonlinear ξ-skew-Jordan-type higher derivations are automatically additive and, moreover, become additive higher $*$-derivations. This result unifies and extends several existing first-order and triple-product cases, while also providing a new starting point for future research.

Use of AI tools declaration

The authors declare that no Artificial Intelligence (AI) tools were used in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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