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Theory article

A modified generalized Newton method applied to solving absolute value equations

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Abstract: In this paper, we present a modified generalized Newton method (MGNM) for solving absolute value equation (AVEs) $Ax - |x| = b$. Some known iterative schemes are special cases of our method. We prove that the generated sequence of iterates is well defined and converges linearly under condition $\|A^{-1}\| < \frac{1}{3(2t_0+t_1)+1}$ (t_0 and t_1 are constants greater than or equal to 0, and not both zero). Compared to Newton's method, the MGNM can guarantee the convergence of the iterative method under weaker conditions. Numerical experiments on various types of problems are conducted to illustrate the superior efficiency and accuracy of the proposed method, showing its superiority over some existing methods.

Keywords: absolute value equations; modified generalized Newton method; convergence

1. Introduction

We consider the fundamental absolute value equation (AVE) in its canonical form:

$$Ax - |x| = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are given, $|\cdot|$ represents the element-wise absolute value of a vector, and $x \in \mathbb{R}^n$ is the unknown vector to be solved. Rohn [1] first introduced a more general form of the AVE, $Ax + B|x| = b$ (with $B \in \mathbb{R}^{n\times n}$), and Mangasarian [2] later explored it in a broader mathematical context.

We introduce the piece-wise linear vector function $W(x)$ as set out by AVE(1) as follows:

$$W(x) = Ax - |x| - b. \tag{2}$$

The generalized Jacobian of W at x can be written as

$$\partial W(x) = A - P(x), \tag{3}$$

where $P(x) = \partial|x| = \mathrm{diag}(\mathrm{sign}(x))$; $\mathrm{sign}(x)$ represents a vector whose entries take the value $+1, 0,$ or -1, according to whether the corresponding entry of x is positive, zero, or negative; and $\mathrm{diag}(\mathrm{sign}(x))$ stands for the diagonal matrix associated with the vector $\mathrm{sign}(x)$. Note that $P(x)x = |x|$.

A critical insight linking AVEs to broader optimization problems is that the general linear complementarity problem (LCP), which is a well-known Non-deterministic Polynomial-time hard problem (NP-hard) [3] that encompasses linear programming, quadratic programming, and many other mathematical programming tasks, can be equivalently reformulated as AVE(1). This connection implies that AVE(1) is also an NP-hard problem in its general form, underscoring the importance of developing efficient, scalable solution methods for this class of equations. Meanwhile, it is also a non-smooth problem with non-differentiable characteristics.

Newton iteration [4] is the most classical iterative method for solving nonlinear equations:

$$x^{k+1} = x^k - W'(x^k)^{-1}W(x^k), \quad k = 0, 1, 2, \ldots. \tag{4}$$

Based on method (4), Mangasarian [5] further put forward a generalized Newton method to solve AVE(1):

$$x^{k+1} = (A - P(x^k))^{-1}b, \quad k = 0, 1, 2, \ldots. \tag{5}$$

This method achieves linear convergence under the condition $\|(A - P)^{-1}\| < \frac{1}{3}$ or the weaker condition $\|A^{-1}\| < \frac{1}{4}$.

Based on method (4), Traub [6] proposed a third-order convergent iterative method for the solution of nonlinear equations:

$$\begin{cases} y^k &= x^k - W'(x^k)^{-1}W(x^k), \\ x^{k+1} &= y^k - W'(x^k)^{-1}W(y^k), \end{cases} \quad k = 0, 1, 2, \ldots. \tag{6}$$

Building on method (6), Haghani [7] extended the two-step Traub's method to solve AVE(1):

$$\begin{cases} y^k &= (A - P(x^k))^{-1}b, \\ x^{k+1} &= y^k - (A - P(x^k))^{-1}\left((A - P(y^k))y^k - b\right), \end{cases} \quad k = 0, 1, 2, \ldots. \tag{7}$$

where only one inverse of $(A - P(x^k))$ is required per iteration, and under the same condition of method (5), method (7) is well defined and converges linearly.

Based on methods (4) and (6), Feng and Liu [8] put forward a new two-step iterative method:

$$\begin{cases} y^k &= x^k + W'(x^k)^{-1}W(x^k), \\ x^{k+1} &= x^k - W'(x^k)^{-1}(W(y^k) - W(x^k)), \end{cases} \quad k = 0, 1, 2, \ldots. \tag{8}$$

When this method is applied to solve the AVE(1), the iterative formula is expressed as follows:

$$\begin{cases} y^k = x^k + (A - P(x^k))^{-1}\Big((A - P(x^k))x^k - b\Big), \\ x^{k+1} = x^k - (A - P(x^k))^{-1}\Big(A(y^k - x^k) + P(x^k)x^k - P(y^k)y^k\Big), \end{cases} \quad k = 0, 1, 2, \ldots. \tag{9}$$

The sequence $\{x^k\}$ generated by method (9) is a Cauchy sequence and converges globally.

Su [9] proposed a unified model for solving a system of nonlinear equations:

$$\begin{cases} y^k &= x^k - W'(x^k)^{-1} W\left(x^k\right), \\ x^{k+1} &= x^k - W'(y^k)^{-1}\left(-\frac{3}{2} W\left(y^k\right) + \frac{1}{2} W\left(2x^k - y^k\right)\right), \end{cases} \quad k = 0, 1, 2, \ldots. \tag{10}$$

Based on method (10), when applying this method to solve AVE(1), Gul et al. [10] proposed a novel two-step iterative method, which takes the following form:

$$\begin{cases} y^k &= (A - P(x^k))^{-1} b, \\ x^{k+1} &= x^k - (A - P(y^k))^{-1}\Big(A(x^k - 2y^k) + \dfrac{3}{2} P(y^k) y^k - \dfrac{1}{2} P(2x^k - y^k)(2x^k - y^k) + b\Big), \end{cases} \quad k = 0, 1, 2, \ldots. \tag{11}$$

Under the same condition of method (5), the method (11) is well defined and converges to the unique solution.

Using the three-point closed Newton–Cotes quadrature formula, Khan et al. [11] proposed a Newton-type method:

$$\begin{cases} y^k &= x^k - W'(x^k)^{-1} W\left(x^k\right), \\ x^{k+1} &= x^k - 6\left(W'(x^k) + 4W'\left(\frac{x^k + y^k}{2}\right) + W'(y^k)\right)^{-1} W(x^k), \end{cases} \quad k = 0, 1, 2, \ldots. \tag{12}$$

When applying this method to solve AVE(1), it takes the following form:

$$\begin{cases} y^k = (A - P(x^k))^{-1} b, \\ x^{k+1} = x^k - 6\left(6A - P(x^k) - 4P(x^k + y^k) - P(y^k)\right)^{-1}\left((A - P(x^k))x^k - b\right), \end{cases} \quad k = 0, 1, 2, \ldots. \tag{13}$$

This method has linear convergence under the condition $\|(6A - P(x^k) - 4P(x^k + y^k) - P(y^k))^{-1}\| < \frac{1}{18}$ or the weaker $\|A^{-1}\| < \frac{1}{19}$.

Shi et al. [12] proposed a two-step Newton-type method, also using the three-point closed Newton–Cotes quadrature formula, which takes the following form:

$$\begin{cases} y^k &= x^k - W'(x^k)^{-1} W\left(x^k\right), \\ x^{k+1} &= x^k - 4\left(W'(x^k) + 2W'\left(\frac{x^k + y^k}{2}\right) + W'(y^k)\right)^{-1} W(x^k), \end{cases} \quad k = 0, 1, 2, \ldots. \tag{14}$$

The form of this method for solving the AVE is as follows:

$$\begin{cases} y^k = (A - P(x^k))^{-1} b, \\ x^{k+1} = x^k - 4\left(4A - P(x^k) - 2P(x^k + y^k) - P(y^k)\right)^{-1}\left((A - P(x^k))x^k - b\right), \end{cases} \quad k = 0, 1, 2, \ldots. \tag{15}$$

This method has linear convergence under the condition $\|(4A - P(x^k) - 2P(x^k + y^k) - P(y^k))^{-1}\| < \frac{1}{12}$ or the weaker $\|A^{-1}\| < \frac{1}{13}$.

Wang et al. [13–15] proposed several high-efficiency derivative-based iterative schemes for solving nonlinear systems of equations, which can be applied to solve AVEs. Ke and Ma [16] proposed a successive over-relaxation (SOR)-like iterative method for solving AVEs based on the block nonlinear transformation of AVEs and the idea of matrix splitting. Shang et al. [17–19] proposed several derivative-free multi-point iterative schemes with high efficiency. When the coefficient matrix A in AVE(1) has the Toeplitz structure, Gu et al. [20] suggested the nonlinear circulant and skew-circulant splitting (CSCS)-like method and the Picard–CSCS method for solving this problem. Mangasarian [21] investigated the analytical solutions of AVEs via dual complementarity theory, establishing a novel theoretical framework for analyzing the solvability and solution characteristics of absolute value equations. Feng and Liu [22] designed an improved generalized Newton method for AVEs, which optimizes the iterative framework of traditional Newton-type methods and achieves better numerical performance in terms of convergence efficiency. Edalatpour et al. [23] generalized the classic Gauss–Seidel iterative structure and extended it to the field of absolute value equation solving, enriching the family of matrix splitting iterative methods for AVEs. Furthermore, Feng and Liu [24] proposed a high-precision three-step iterative scheme, which realizes higher-order convergence behavior and effectively improves the accuracy of solving nonlinear absolute value systems. Yu et al. [25] constructed a modified multivariate spectral gradient algorithm, which is suitable for large-scale AVE problems and possesses stable convergence and low computational consumption. Khan et al. [26] developed a new efficient two-step iterative method, which balances computational complexity and convergence speed and exhibits excellent adaptability for various types of AVEs. Guo et al. [27] presented a reliable and efficient iterative approach for absolute value equations, which can effectively handle different dimensional AVE systems with good numerical robustness. More recently, Gul et al. [28] introduced a numerical solving strategy based on Simpson's three-eighths formula, providing a new feasible way for the numerical iteration and approximate solution of AVEs.

In the present paper, we put forward a new, improved generalized Newton method based on methods (12) and (14). The proposed method is independent of specific coefficients and applicable to any coefficients, thus possessing stronger generality and wider applicability for effectively solving AVE(1).

The subsequent sections of this paper are arranged as follows. Sections 2 and 3 detail the method and its convergence analysis, respectively; Section 4 presents numerical experiments and compares this method with existing ones and Section 5 concludes the whole study and proposes directions for future research.

2. A modified generalized Newton method

The method is as follows:

$$\begin{cases} y^k &= x^k - W'(x^k)^{-1} W\left(x^k\right), \\ x^{k+1} &= x^k - (2t_0 + t_1)\Big(t_0 W'(x^k) + t_1 W'\Big(\frac{x^k+y^k}{2}\Big) + t_0 W'(y^k)\Big)^{-1} W(x^k), \end{cases} \qquad k = 0, 1, 2, \dots, \tag{16}$$

where t_0 and t_1 are constants greater than or equal to 0, and not both zero. Method (16) is defined as a modified generalized Newton method (MGNM) in the following text.

Theorem 2.1. *Let $W : I \subset \mathbb{R} \to \mathbb{R}$ be sufficiently differentiable with a simple root $\zeta \in I$, i.e., $W(\zeta) = 0$ and $W'(\zeta) \neq 0$. Let $e^k = x^k - \zeta$ and $c_m = \frac{W^{(m)}(\zeta)}{m!\, W'(\zeta)}$ for all $m \geq 2$. Then the method (16) is locally third-order convergent, and the error equation is*

$$e^{k+1} = \frac{(2t_0 + t_1)c_2^2 + (t_0 - \frac{1}{4}t_1)c_3}{2t_0 + t_1} (e^k)^3 + O((e^k)^4). \tag{17}$$

Proof. Since $W(\zeta) = 0$, the Taylor expansion about ζ of function $W(x^k)$ is

$$W(x^k) = W'(\zeta)\left(e^k + c_2(e^k)^2 + c_3(e^k)^3 + O((e^k)^4)\right), \tag{18}$$

$$W'(x^k) = W'(\zeta)\left(1 + 2c_2 e^k + 3c_3(e^k)^2 + O((e^k)^3)\right). \tag{19}$$

Let

$$t = \frac{W(x^k)}{W'(x^k)} = e^k - c_2(e^k)^2 + 2(c_2^2 - c_3)(e^k)^3 + O((e^k)^4). \tag{20}$$

Then the intermediate error satisfies

$$d^k = y^k - \zeta = e^k - t = c_2(e^k)^2 + 2(c_3 - c_2^2)(e^k)^3 + O((e^k)^4). \tag{21}$$

Let $z^k = \frac{x^k+y^k}{2}$. Then the Taylor expansions about ζ of function $W'(y^k)$ and $W'(z^k)$ are

$$W'(y^k) = W'(\zeta)\left(1 + 2c_2 d^k + 3c_3(d^k)^2 + O((d^k)^3)\right), \tag{22}$$

$$W'(z^k) = W'(\zeta)\left(1 + 2c_2 \cdot \frac{e^k + d^k}{2} + 3c_3 \cdot \left(\frac{e^k + d^k}{2}\right)^2 + O((e^k)^3)\right). \tag{23}$$

Substituting Eqs (19)–(23) to Eq (16), we have

$$e^{k+1} - \zeta = e^k - \zeta - (2t_0 + t_1)\frac{W(x^k)}{t_0 W'(x^k) + t_1 W'(z^k) + t_0 W'(y^k)}. \tag{24}$$

With the simplification of Eq (24), we obtain

$$e^{k+1} = \frac{(2t_0 + t_1)c_2^2 + (t_0 - \frac{1}{4}t_1)c_3}{2t_0 + t_1}(e^k)^3 + O((e^k)^4). \tag{25}$$

This means that method (16) is third-order convergent.

Remark 2.1. *In particular, when the parameter* $t_0 = 0$ *or* $t_1 = 0$*, method (16) requires three function-related evaluations in each iteration, so the efficiency index of this special case is given by* $E = 3^{1/3} \approx 1.442$*, while the classical Newton method has an efficiency index of* $2^{1/2} \approx 1.414$*. Since* $3^{1/3} > 2^{1/2}$*, the proposed iterative method (16) at* $t_0 = 0$ *or* $t_1 = 0$ *achieves superior computational efficiency compared with the Newton iterative method.*

Next, we apply the MGNM to solve AVEs.

Let M denote $t_0 W'(x^k) + t_1 W'\left(\frac{x^k+y^k}{2}\right) + t_0 W'(y^k)$ in the second step of method (16), then

$$\begin{aligned} M &= t_0 W'(x^k) + t_1 W'\left(\frac{x^k + y^k}{2}\right) + t_0 W'(y^k) \\ &= t_0(A - P(x^k)) + t_1(A - P(x^k + y^k)) + t_0(A - P(y^k)) \\ &= t_0 A - t_0 P(x^k) + t_1 A - t_1 P(x^k + y^k) + t_0 A - t_0 P(y^k) \\ &= (2t_0 + t_1)A - t_0 P(x^k) - t_1 P(x^k + y^k) - t_0 P(y^k). \end{aligned} \tag{26}$$

So, the MGNM is as follows:

$$\begin{cases} y^k &= (A - P(x^k))^{-1} b, \\ x^{k+1} &= x^k - (2t_0 + t_1)(M)^{-1} \left((A - P(x^k))x^k - b \right), \end{cases} \quad k = 0, 1, 2, \dots, \tag{27}$$

where $M := (2t_0 + t_1)A - t_0 P(x^k) - t_1 P(x^k + y^k) - t_0 P(y^k)$. We can see that method (13) is a special case of the MGNM when $t_0 = 1$ and $t_1 = 4$; method (15) is a special case of the MGNM when $t_0 = 1$ and $t_1 = 2$.

The algorithm of the MGNM can be written as follows:

Algorithm 1 MGNM

Step 1: Select a preliminary approximation $x^{(0)} \in \mathbb{R}^n$.
Step 2: Start for k; compute

$$y^k = (A - P(x^k))^{-1} b.$$

Step 3: Using y^k, compute

$$x^{k+1} = x^k - (2t_0 + t_1)(M)^{-1} \left((A - P(x^k))x^k - b \right),$$

where $M := (2t_0 + t_1)A - t_0 P(x^k) - t_1 P(x^k + y^k) - t_0 P(y^k)$.
Step 4: If $x^{k+1} = x^k$, then stop. If not, then put $k = k + 1$ and turn back to Step 2.

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3. Convergence of MGNM for solving AVEs

In the present section, we verify the convergence of the MGNM.

Lemma 3.1. *The singular values of the matrix $A \in \mathbb{R}^{n \times n}$ exceed 1 if and only if the minimum eigenvalue of $A^T A$ exceeds 1.*

From Lemma 1, we have that $x^T x < x^T A^T A x$.

The first step of the MGNM is proved to be well defined [5] as the predictor step:

$$y^k = (A - P(x^k))^{-1} b. \tag{28}$$

Next, we apply M in the second step, which is nonsingular.

Lemma 3.2. *M^{-1} exists for any diagonal matrix P with diagonal elements ± 1 or 0, provided that the singular values of $A \in \mathbb{R}^{n \times n}$ exceed 1.*

Proof. We proceed by contradiction. Suppose M is singular, i.e., M^{-1} does not exist. Consequently, a nonzero vector x exists satisfying $Mx = 0$, and thus we have

$$(2t_0 + t_1)Ax = \left[t_0 P(x^k) + t_1 P(x^k + y^k) + t_0 P(y^k) \right] x = Vx, \tag{29}$$

where $V = t_0 P(x^k) + t_1 P(x^k + y^k) + t_0 P(y^k)$.

Meanwhile, multiply both sides of Eq (29) by x^T on the left,

$$x^T(2t_0 + t_1)Ax = x^T Vx, \tag{30}$$

take the transpose of Eq (30),

$$\left(x^T(2t_0 + t_1)Ax\right)^T = \left(x^T Vx\right)^T,$$
$$(2t_0 + t_1)x^T A^T x = x^T V^T x, \tag{31}$$

and multiply Eq (30) by Eq (31) to obtain

$$(2t_0 + t_1)^2 x^T A^T Ax = x^T VV^T x = x^T V^2 x, \tag{32}$$

so

$$x^T A^T Ax = \frac{1}{(2t_0 + t_1)^2} x^T V^2 x. \tag{33}$$

Since $P(x^k)P(x^k) \leq I$, $P(y^k)P(y^k) \leq I$, $P(x^k)P(y^k) \leq I$, $P(x^k + y^k)P(x^k + y^k) \leq I$, $P(x^k + y^k)P(x^k) \leq I$, and $P(x^k + y^k)P(y^k) \leq I$,

$$\begin{aligned}
x^T x < x^T A^T Ax &= \frac{1}{(2t_0 + t_1)^2} x^T \left(t_0 P(x^k) + t_1 P(x^k + y^k) + t_0 P(y^k)\right)^2 x \\
&= \frac{1}{(2t_0 + t_1)^2} x^T \Big(t_0^2 P(x^k)P(x^k) + t_0^2 P(y^k)P(y^k) + 2t_0^2 P(x^k)P(y^k) \\
&\quad + t_1^2 P(x^k + y^k)P(x^k + y^k) + 2t_0 t_1 P(x^k)P(x^k + y^k) \\
&\quad + 2t_0 t_1 P(x^k + y^k)P(y^k)\Big)x \\
&\leq \frac{1}{(2t_0 + t_1)^2} x^T (2t_0 + t_1)^2 Ix = x^T x,
\end{aligned} \tag{34}$$

which is a contradiction, hence M is nonsingular. So, method (27) is well defined.

The proof is completed.

For the second step of method (27), we know

$$\begin{aligned}
Mx^{k+1} &= Mx^k - (2t_0 + t_1)W(x^k) \\
&= Mx^k - (2t_0 + t_1)(Ax^k - |x^k| - b) \\
&= [(2t_0 + t_1)A - V]x^k - (2t_0 + t_1)(Ax^k - P(x^k)x^k - b) \\
&= [(t_0 + t_1)P(x^k) - t_1 P(x^k + y^k) - t_0 P(y^k)]x^k + (2t_0 + t_1)b,
\end{aligned} \tag{35}$$

or

$$[(2t_0 + t_1)A - V]x^{k+1} = [(t_0 + t_1)P(x^k) - t_1 P(x^k + y^k) - t_0 P(y^k)]x^k + (2t_0 + t_1)b. \tag{36}$$

Lemma 3.3. *The MGNM is bounded and well defined, provided that the singular values of $A \in \mathbb{R}^{n\times n}$ exceed* 1.

Proof. The result follows from Mangasarian's Proposition 3 [5], so the proof is omitted.

Hence, an accumulation point $\dot{x}$ exists such that

$$\Big((2t_0 + t_1)A - t_0\dot{P}(\dot{x}) - t_1\dot{P}(\dot{x} + \dot{y}) - t_0\dot{P}(\dot{y})\Big)\dot{x} = \Big((t_1 + t_0)\dot{P}(\dot{x}) - t_1\dot{P}(\dot{x} + \dot{y}) - t_0\dot{P}(\dot{y})\Big)\dot{x} + (2t_0 + t_1)b,$$
$$(A - \dot{P}(\dot{x}))\dot{x} = b. \tag{37}$$

Lemma 3.4. *Lipschitz continuity of AVE(1). Let $x, y \in \mathbb{R}^n$ be given. Then*

$$\left\| |x| - |y| \right\| \leq 2\|x - y\|. \tag{38}$$

Proof. The result follows from Mangasarian's Lemma 5 [5], so the proof is omitted.

Theorem 3.1. *The MGNM converges to a solution θ of AVE(1), if $\|(M)^{-1}\| < \frac{1}{3(2t_0+t_1)}$ for any diagonal matrix P with diagonal entries 0 or ± 1, where t_0 and t_1 are non-negative constants and not both zero.*

Consider $x^{k+1} - \theta = x^k - (2t_0 + t_1)H^{-1}W(x^k) - \theta$, and we have

$$M(x^{k+1} - \theta) = M(x^k - \theta) - (2t_0 + t_1)W(x^k). \tag{39}$$

Since θ is the solution of AVE(1),

$$W(\theta) = A\theta - |\theta| - b = 0. \tag{40}$$

From Eqs (39) and (40), we have

$$\begin{aligned} M(x^{k+1} - \theta) &= M(x^k - \theta) - (2t_0 + t_1)W(x^k) + (2t_0 + t_1)W(\theta) \\ &= M(x^k - \theta) - (2t_0 + t_1)(W(x^k) - W(\theta)) \\ &= M(x^k - \theta) - (2t_0 + t_1)(Ax^k - |x^k| - A\theta + |\theta|) \\ &= (M - (2t_0 + t_1)A)(x^k - \theta) - (2t_0 + t_1)(|\theta| - |x^k|) \\ &= -V(x^k - \theta) + (2t_0 + t_1)(|x^k| - |\theta|), \end{aligned} \tag{41}$$

or

$$x^{k+1} - \theta = M^{-1}\Big[(2t_0 + t_1)(|x^k| - |\theta|) - V(x^k - \theta)\Big]. \tag{42}$$

So,

$$\|x^{k+1} - \theta\| \leq \|M^{-1}\|\Big[2(2t_0 + t_1)\|x^k - \theta\| + \|V\|\|x^k - \theta\|\Big]. \tag{43}$$

Since $t_0P(x^k)$, $t_0P(y^k)$, and $t_1P(x^k + y^k)$ are diagonal matrices with elements equal to ± 1 or 0,

$$\begin{aligned} \|V\| &= \left\| t_0P(x^k) + t_1P(x^k + y^k) + t_0P(y^k) \right\| \\ &\leq \|t_0P(x^k)\| + \|t_1P(x^k + y^k)\| + \|t_0P(y^k)\| \\ &= t_0 + t_1 + t_0 = 2t_0 + t_1. \end{aligned} \tag{44}$$

So, we have

$$\|x^{k+1} - \theta\| \leq 3(2t_0 + t_1) \cdot \|M^{-1}\| \cdot \|x^k - \theta\| < \|x^k - \theta\|. \tag{45}$$

In Eq (43), we have used the condition that $\|(M)^{-1}\| < \frac{1}{3(2t_0+t_1)}$. Hence, linear convergence of the sequence $\{x_k\}$ to θ is proved.

Lemma 3.5. *Let the singular values of $A \in \mathbb{R}^{n\times n}$ exceed 1, and $\|A^{-1}\| < \frac{1}{3(2t_0+t_1)+1}$, $\|t_0W(x^k)\|$, $\|t_0W(y^k)\|$, and $\|t_1W(x^k + y^k)\|$ are nonzero. For any b, AVE(1) is shown to admit a unique solution, and the MGNM is well defined and converges to the unique solution of AVE(1) for any initial guess x_0.*

Proof. The unique solvability of AVE(1) for any b requires that $\|A^{-1}\| < 1$ [5]. If A^{-1} exists, then $\left\|((2t_0 + t_1)A)^{-1}\right\| \left\|t_0P(x^k) + t_1P(x^k + y^k) + t_0P(y^k)\right\| < 1$. By the Banach perturbation lemma [29], we have

$$\left\|\left(t_0W'(x^k) + t_1W'\left(\frac{x^k + y^k}{2}\right) + t_0W'(y^k)\right)^{-1}\right\| = \left\|\left((2t_0 + t_1)A - t_0P(x^k) - t_1P(x^k + y^k) - t_0P(y^k)\right)^{-1}\right\|$$

$$\leq \frac{\left\|((2t_0 + t_1)A)^{-1}\right\| \left\|t_0P(x^k) + t_1P(x^k + y^k) + t_0P(y^k)\right\|}{1 - \left\|((2t_0 + t_1)A)^{-1}\right\| \left\|t_0P(x^k) + t_1P(x^k + y^k) + t_0P(y^k)\right\|}$$

$$\leq \frac{\frac{1}{2t_0+t_1}\|A^{-1}\|(2t_0 + t_1)}{1 - \frac{1}{2t_0+t_1}\|A^{-1}\|(2t_0 + t_1)} < \frac{\frac{1}{3(2t_0+t_1)+1}}{1 - \frac{1}{3(2t_0+t_1)+1}} < \frac{1}{3(2t_0 + t_1)}. \tag{46}$$

Accordingly, from Theorem 3.1, it follows that the MGNM exhibits linear convergence to the solution of AVE(1).

The proof is completed.

Remark 3.1. *From Lemma 3.2, when $\|A^{-1}\| < \frac{1}{3(2t_0+t_1)+1}$, the MGNM is well defined and converges to the unique solution of AVE(1). When t_0 and t_1 are sufficiently small, $\|A^{-1}\|$ is close to* 1. *However, the convergence condition [5] of the Newton method requires that $\|A^{-1}\| < \frac{1}{4}$. Compared to Newton method, the MGNM can guarantee the convergence of the iterative method under weaker conditions.*

4. Numerical experiments

In this section, our MGNM is compared with the SOR-like iteration method (SOR) [16] and the iterative method (IM) [30] for solving three types of AVEs.

The SOR is as follows:

$$\begin{cases} x^{(k+1)} = (1 - \omega)x^{(k)} + \omega A^{-1}(y^{(k)} + b), \\ y^{(k+1)} = (1 - \omega)y^{(k)} + \omega\left|x^{(k+1)}\right|, \end{cases} \qquad k = 0, 1, 2, \ldots, \tag{47}$$

where 0¡ω¡2.

The IM is as follows:

$$\begin{cases} x_k = x_{k-1} + \alpha_k y_k, \\ \alpha_k = -\dfrac{\langle Ax_{k-1} - |x_{k-1}| - b,\ y_k\rangle}{\langle (A - D_{k-1})y_k,\ y_k\rangle}, \end{cases} \qquad k = 0, 1, 2, \ldots. \tag{48}$$

In Tables 1–3, Dim represents the evaluation metrics that include the dimension of the problem, Iters represents the number of iterations, RES represents the 2-norm of the residual, and Times represents CPU times, where the residual is defined as RES := $\|Ax^k - |x^k| - b\|_2$. All numerical experiments in this paper are performed on a Dell laptop equipped with an Intel(R) Core(TM) i7-10750H CPU operating at 2.60 GHz and 24.0 GB of RAM. The machine runs the 64-bit edition of the Windows 11 operating system. All proposed algorithms and numerical comparison tests are implemented and executed within the MATLAB R2024a environment.

Example 1. We select the 500-order symmetric tridiagonal matrix

$$A = \text{Tridiag}(1.2, 16, 1.2) = \begin{pmatrix} 16 & 1.2 & 0 & \cdots & 0 & 0 \\ 1.2 & 16 & 1.2 & \cdots & 0 & 0 \\ 0 & 1.2 & 16 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 16 & 1.2 \\ 0 & 0 & 0 & \cdots & 1.2 & 16 \end{pmatrix} \in \mathbb{R}^{500\times 500},$$

the exact solution is the alternating vector

$$\theta = \begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \\ \vdots \\ 2 \\ -2 \end{pmatrix} \in \mathbb{R}^{500},$$

and $b = A\theta - |\theta|$.

The MGNM, SOR, and IM are employed to solve Example 1, and all obtained numerical outcomes are presented in Table 1. The convergence criterion is that the residual reaches the precision of 10^{-6}.

Table 1. Numerical outcomes for Example 1.

Method	Dim	Iters	RES	Times
MGNM	500	2	1.2455×10^{-13}	0.0156
SOR	500	10	4.2429×10^{-7}	0.0312
IM	500	8	6.8422×10^{-7}	1.8438

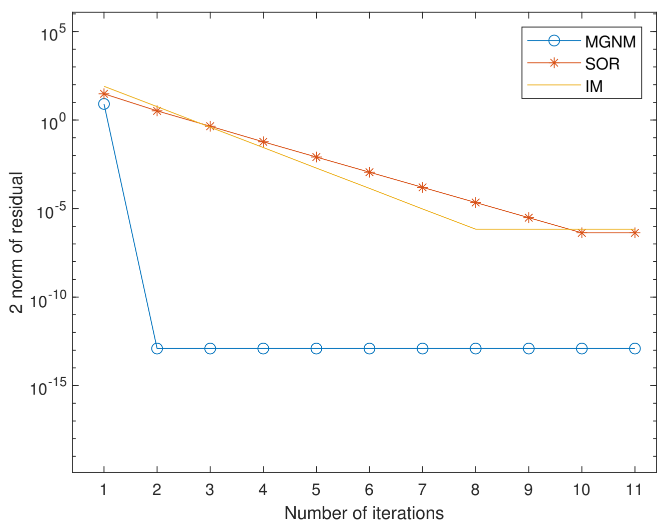

Figure 1. Convergence curves of three methods for Example 1.

From Table 1 and Figure 1, all three methods converge but differ greatly in iterative efficiency and accuracy. The MGNM is highly efficient, converging in only two iterations with a residual norm of 1.2455×10^{-13} and consuming a calculation time of 0.0156. The SOR needs ten iterations to reach convergence with a residual norm of 4.2429×10^{-7} and a calculation time of 0.0312. The IM completes convergence after eight iterations with a residual norm of 6.8422×10^{-7} and a calculation time of 1.8438. Clearly, the MGNM has the fastest convergence speed, optimal calculation accuracy, and lowest time consumption, showing superior numerical stability and overall numerical performance.

Example 2. We select the matrix A and true solution vector θ in the following form:

$$A = 100I - 0.02(2\,\mathrm{rand}(n,n) - 1) \in \mathbb{R}^{n\times n},$$

and $\theta \in \mathbb{R}^{n\times n}$; $b = A\theta - |\theta|$.

The MGNM, SOR, and IM are employed to solve Example 2, and all obtained numerical outcomes are presented in Table 2. The convergence criterion is that the residual reaches the precision of 10^{-8}.

Table 2. Numerical outcomes for Example 2.

Method	Dim	Iters	RES	Times
MGNM	500	2	9.0938×10^{-13}	0.0156
SOR	500	9	8.5612×10^{-9}	0.0625
IM	500	5	1.2450×10^{-10}	1.4062

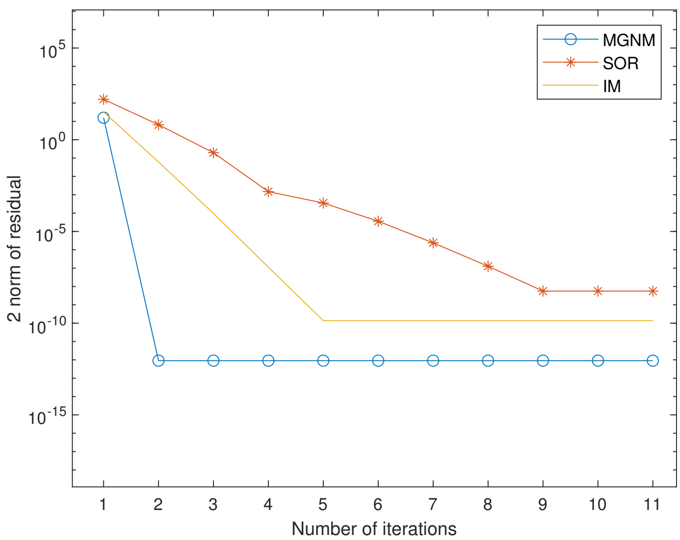

Figure 2. Convergence curves of three methods for Example 2.

From Table 2 and Figure 2, all three methods converge for the 500-dimensional problem. The MGNM still maintains high iterative efficiency, converging in only two iterations with an optimal residual norm of 9.0938×10^{-13} and a time consumption of 0.0156. The SOR needs nine iterations with a residual norm of 8.5612×10^{-9} and a time consumption of 0.0625. The IM needs five iterations with a residual norm of 1.2450×10^{-10} and the longest time consumption of 1.4062.

Example 3. We select

$$A = \begin{pmatrix} 50 & 2 \\ 3 & 60 \end{pmatrix} \in \mathbb{R}^{2\times 2}, \quad \theta = (-1, 1)^T, \quad b = (-49, -56)^T,$$

where $b = A\theta - |\theta|$.

The MGNM, SOR, and IM are employed to solve Example 3, and all obtained numerical outcomes are presented in Table 3. The convergence criterion is that the residual reaches the precision of 10^{-10}.

From Table 3 and Figure 3, all three methods converge for the two-dimensional problem. The MGNM has the optimal iterative efficiency, converging in only two iterations with a residual norm as low as 7.1054×10^{-15} and a time consumption of 0.00015625. The SOR requires nine iterations with a residual norm of 9.2466×10^{-11} and a time consumption of 0.00015625. The IM needs six iterations with a residual norm of 6.2532×10^{-13} and a time consumption of 0.00031250, which fully reflects the superiority of the MGNM.

Table 3. Numerical outcomes for Example 3.

Method	Dim	Iters	RES	Times
MGNM	2	2	7.1054×10^{-15}	0.00015625
SOR	2	9	9.2466×10^{-11}	0.00015625
IM	2	6	6.2532×10^{-13}	0.00031250

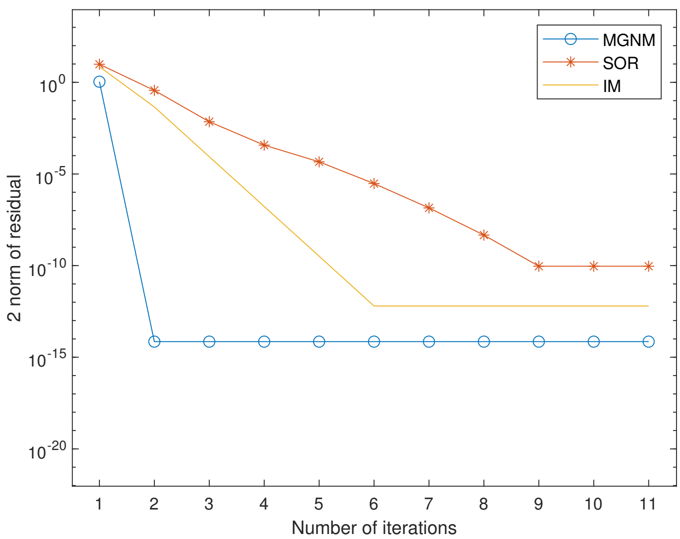

Figure 3. Convergence curves of three methods for Example 3.

5. Conclusions

In this paper, we put forward an MGNM for solving AVE(1). Existing methods (12) and (14) are special cases of the MGNM. Compared to Newton's method, the MGNM can guarantee the convergence of the iterative method under weaker conditions. Numerical results demonstrate that our method achieves excellent numerical performance with faster convergence, fewer iteration steps, lower residual errors, and less computational time consumption. In future work, we will further extend this algorithm to solve large-scale AVE systems and explore its application in practical engineering optimization problems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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