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https://www.aimspress.com/journal/era

ERA, 34(8): 5286–5302.
DOI: 10.3934/era.2026235
Received: 13 February 2026
Revised: 28 May 2026
Accepted: 05 June 2026
Published: 22 June 2026

Research article

From special-function recurrences to arithmetic continued fractions

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Abstract: Many special functions are defined by differential equations, but useful arithmetic continued fractions often come from their discrete recurrences. This paper studied Riccati-type recurrences and their continued-fraction expansions in the setting of machine-discovered patterns from the enumerated signed-continued-fraction Massey approve algorithm (ESMA). The main contribution was a block method for Riccati recurrences, which was used to prove identities for signed interlaced continued fractions with affine partial denominators. We showed that several ESMA-type conjectures can be reduced to finite matrix identities in the block index. Three representative identities were proved in detail. They involved modified Bessel functions, half-integer Bessel functions, and ratios of integer-order Bessel functions. We also gave positivity checks and computable tail estimates. These estimates justified the passage to the infinite limit and gave explicit truncation bounds. The results showed that the representative formulas studied here were not isolated numerical coincidences.

Keywords: Riccati difference equation; continued fractions; Möbius iteration; stability; truncation error; special functions

1. Introduction

Difference equations and continued fractions are closely connected. A three-term recurrence produces a nonlinear ratio recursion, and this ratio recursion is a Riccati-type difference equation. When it is iterated backward, it yields a continued-fraction representation for the initial ratio. This viewpoint is classical in the analytic theory of continued fractions; see Wall [1], Perron [2], and Jones and Thron [3], as well as the broader special-function perspective in Lorentzen and Waadeland [4], the handbook of Cuyt et al. [5], and Khinchin's classical text [6]. The relation between continued fractions and recurrence computation is also central in Pincherle's theorem and in Gautschi's analysis of three-term recurrences [7, 8]. At the same time, special functions that arise from differential equations often come with order-shift relations and three-term recurrences, and these recurrences are a standard entry point for continued-fraction evaluations, as treated, for instance, in Watson [9] and Andrews, Askey, and Roy [10], and summarized in the NIST Digital Library of Mathematical Functions (DLMF) [11]. Polynomial continued fractions and related arithmetic expansions have also been studied from an arithmetic point of view; see Bowman and McLaughlin [12], McLaughlin and Wyshinski [13], and recent work by Kadyrov and coauthors on finite continued-fraction representations for second-order difference equations and polynomial continued fractions for constants [14, 15]. The present paper is close to this recurrence-based direction, but its focus is different: we study signed interlaced continued fractions found by experimental searches and give a finite matrix test that recovers the hidden recurrence behind such patterns.

Recently, this classical direction has also reappeared in experimental mathematics. In this setting, one does not start with a known recurrence. Instead, a computer first finds a possible arithmetic pattern. Then, the hidden recurrence must be found, the identity must be proved, and the infinite limit must be justified. This direction appears in the Ramanujan Machine project [16] and in the enumerated signed-continued-fraction Massey approve algorithm (ESMA), introduced in [17]. ESMA uses integer-sequence recognition ideas related to the Berlekamp–Massey algorithm [18, 19]. It is also connected with automatic conjecturing and proving for continued fractions [20], and with symbolic proof methods for recurrences [21–23]. Recent examples also show that computational methods are now used in many problems involving nonlinear equations [24, 25]. Our work is focused on continued fractions. We keep the difference equation as the central object, and we view the continued fraction as a way to write a solution of a Riccati difference equation under Möbius iteration.

The central idea is simple. An ESMA-type continued fraction gives a signed affine block pattern. If the block product is equivalent to a block of a known special-function Riccati recurrence, then the numerical conjecture becomes an exact identity. The proof is then reduced to checking finitely many polynomial identities. This gives a reproducible route from experimental discovery to a formal proof.

Two questions arise naturally in this setting. The first is structural: why do many machine-generated continued fractions exhibit periodic or block-periodic arithmetic patterns, often with affine growth inside blocks, even when the underlying special-function recurrence looks unrelated to such regularity [16, 17]. The second is analytic and computational: how can one turn a formal infinite continued fraction into a stable numerical scheme with explicit, checkable truncation control rather than relying on a purely formal limit passage. Classical continued-fraction theory addresses convergence and truncation through continuants and determinant identities, and it emphasizes that tail dependence is Möbius; see [1, 3, 5] and also the metric viewpoint in Khinchin [6]. Our goal is to package these ideas in a form adapted to Riccati difference equations coming from special functions [9–11], and to make the resulting error control directly computable.

Our contribution has three parts. First, we give a block elimination method for Riccati recursions

$$R_n = \alpha_n + \frac{\beta_n}{R_{n+1}}.$$

It explains why signed continued fractions found by ESMA often have periodic or block-periodic arithmetic coefficients. Second, we attach to this method a computable truncation bound. The far tail is represented by a single parameter τ_N, and an explicit upper bound $\overline{\tau}_N$ is used in the error estimate. The error is bounded by continuants and simple estimates for special-function ratios.

Third, the main novelty is in the inverse direction. Classical theory usually starts with a recurrence and obtains a continued fraction. We start with a machine-discovered signed continued fraction and recover a hidden second-order recurrence by a finite 2×2 matrix test. Thus, a numerical ESMA conjecture is turned into a finite polynomial identity, together with a tail-selection argument. This is the proof mechanism used later for representative ESMA identities involving e, $\tan(1)$, and ratios of Bessel functions.

The examples in Section 3 are concrete applications. We keep one example in full detail to show the block algebra. The other two examples are treated more briefly, so the main idea can be seen more clearly. Section 4 gives the proof method and states carefully what is covered. We do not claim that all ESMA patterns are explained by second-order recurrences. We prove results for a subclass in which the hidden second-order recurrence can be identified and the matrix test can be applied.

Overall, the paper adopts a difference-equation viewpoint on arithmetic-pattern continued fractions: block elimination explains structure, while Möbius stability and continuants provide explicit and computable truncation control. This perspective connects classical continued fraction theory [1–6] with modern experimental discovery pipelines [16, 17, 20] and with special-function recurrences rooted in differential equations [9–11].

2. Preliminaries

This section collects the identities used later in the paper. Most of them are standard, and we cite classical sources. The paper-specific point is that we keep truncation identities in a form that isolates the tail parameter, because, in the examples, the tail parameter is bounded directly using a simple lower bound for the relevant special-function ratio.

Definition 2.1. A generalized continued fraction is

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}},$$

where a_n and b_n are real or complex numbers and $a_n \neq 0$ for $n \geq 1$. Its nth convergent is obtained by truncating after a_n.

Lemma 2.2. *Let P_n/Q_n be the nth convergent of the generalized continued fraction in Definition 2.1. Set*

$$P_{-1} = 1, \quad P_0 = a_0, \qquad Q_{-1} = 0, \quad Q_0 = 1.$$

Then, for $n \geq 1$,

$$P_n = a_n P_{n-1} + b_n P_{n-2}, \qquad Q_n = a_n Q_{n-1} + b_n Q_{n-2}.$$

Moreover, for $n \geq 1$,

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1}\, b_1 b_2 \cdots b_n. \tag{2.1}$$

Proof. This is standard; see Wall [1, Chapter 1] or Jones and Thron [3, Chapter 1].

Lemma 2.3. *Fix $N \geq 2$ and consider*

$$F_N(\tau) = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{\ddots + \cfrac{b_{N-1}}{a_{N-1} + \tau}}}},$$

where $a_1, \ldots, a_{N-1} \neq 0$ and τ is a scalar. Let P_n, Q_n be the continuants for the same finite fraction with terminal denominator a_{N-1}, that is, with $\tau = 0$. Then,

$$F_N(\tau) = \frac{P_{N-1} + \tau P_{N-2}}{Q_{N-1} + \tau Q_{N-2}}. \tag{2.2}$$

If $Q_{N-1} \neq 0$ and $Q_{N-1} + \tau Q_{N-2} \neq 0$, then

$$F_N(\tau) - F_N(0) = \frac{\tau\,[P_{N-2}Q_{N-1} - P_{N-1}Q_{N-2}]}{Q_{N-1}\,[Q_{N-1} + \tau Q_{N-2}]}. \tag{2.3}$$

Proof. This is a standard identity; see Wall [1, Chapter 1] or Cuyt et al. [5, Chapter 2]. The finite fraction is a Möbius function of the bottom parameter τ. Formula (2.2) follows from the usual continuant matrix product. Subtracting the value at $\tau = 0$ gives (2.3).

Corollary 2.4. *Assume in Lemma 2.3 that $Q_{N-1} \neq 0$ and that τ is real with*

$$|\tau|\,|Q_{N-2}| < |Q_{N-1}|.$$

Then,

$$\left|F_N(\tau) - F_N(0)\right| \leq \frac{|\tau|\,|b_1 \cdots b_{N-1}|}{|Q_{N-1}|^2} \cdot \frac{1}{1 - |\tau|\,|Q_{N-2}|/|Q_{N-1}|}. \tag{2.4}$$

If only a bound $|\tau| \leq \overline{\tau}$ is known, and if $\overline{\tau}\,|Q_{N-2}| < |Q_{N-1}|$, then the same estimate holds with $|\tau|$ replaced by $\overline{\tau}$.

Proof. Take absolute values in (2.3). By Lemma 2.2,

$$|P_{N-2}Q_{N-1} - P_{N-1}Q_{N-2}| = |b_1 \cdots b_{N-1}|.$$

Also,

$$|Q_{N-1} + \tau Q_{N-2}| \geq |Q_{N-1}| - |\tau|\,|Q_{N-2}| > 0.$$

Substitution gives (2.4). The version with $\overline{\tau}$ follows in the same way.

Definition 2.5. A Riccati-type difference equation is a recursion

$$R_n = \alpha_n + \frac{\beta_n}{R_{n+1}}, \qquad n \geq 0, \tag{2.5}$$

where α_n and β_n are explicit sequences and $\beta_n \neq 0$.

Theorem 2.6. *Assume R_n satisfies the Riccati recursion* (2.5)*. Fix integers $m \geq 1$ and r with $0 \leq r \leq m-1$, and fix a scale $s > 0$. Define*

$$X_k := s\,R_{mk+r} \qquad \text{for } k \geq 0.$$

Then, there exists an explicit Möbius map Φ_k such that

$$X_k = \Phi_k[X_{k+1}] \qquad \text{for all } k \geq 0. \tag{2.6}$$

More precisely, if $\Psi_n(x) := \alpha_n + \beta_n/x$, and if $\sigma(x) := sx$, then

$$\Phi_k = \sigma \circ \Psi_{mk+r} \circ \Psi_{mk+r+1} \circ \cdots \circ \Psi_{mk+r+m-1} \circ \sigma^{-1}. \tag{2.7}$$

Proof. Start from $R_n = \Psi_n(R_{n+1})$ and iterate m steps from $n = mk + r$:

$$R_{mk+r} = \Psi_{mk+r} \circ \Psi_{mk+r+1} \circ \cdots \circ \Psi_{mk+r+m-1}[R_{mk+r+m}].$$

Since $mk + r + m = m(k + 1) + r$, multiplication by s gives (2.6) with Φ_k as in (2.7).

Corollary 2.7. *Iterating* (2.6) *gives a generalized continued fraction whose partial denominators and numerators are obtained explicitly from the block product.*

Proof. The map $x \mapsto \alpha + \beta/x$ is represented by

$$\begin{pmatrix} \alpha & \beta \\ 1 & 0 \end{pmatrix}.$$

Composition of such maps corresponds to multiplication of the corresponding matrices. Thus, the block map Φ_k is a finite product of elementary matrices, up to the scaling used in $X_k = sR_{mk+r}$. Writing this Möbius map as a continued fraction is standard; see [1,4]. Iterating the blocks gives the required infinite continued fraction.

Proposition 2.8. *Let X_k satisfy* (2.6) *and define the block truncation*

$$X_0^{(K)} := \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{K-1}(\infty).$$

Let $M_K := \Phi_0 \circ \cdots \circ \Phi_{K-1}$ and write $M_K(x) = (Ax + B)/(Cx + D)$ with $C \neq 0$. Assume $X_K \neq 0$ and $CX_K + D \neq 0$. Then,

$$\left| X_0 - X_0^{(K)} \right| = \frac{|AD - BC|}{|C|\,|CX_K + D|}. \tag{2.8}$$

Proof. By definition, $X_0 = M_K(X_K)$ and $X_0^{(K)} = M_K(\infty) = A/C$. Then,

$$M_K(X_K) - \frac{A}{C} = \frac{AX_K + B}{CX_K + D} - \frac{A}{C} = \frac{BC - AD}{C\,[CX_K + D]}.$$

Taking absolute values yields (2.8).

Proposition 2.9. *Let $(R_n)_{n\geq 0}$ be a real sequence satisfying*

$$R_n = \alpha_n + \frac{\beta_n}{R_{n+1}}, \qquad n \geq 0,$$

where α_n and β_n are real and $\beta_n \neq 0$. Assume that $R_n \to \infty$ as $n \to \infty$.

Fix $N \geq 2$ and unwind R_0 up to level N:

$$R_0 = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{\ddots + \cfrac{b_{N-1}}{a_{N-1} + \tau_N}}}, \qquad \tau_N := \frac{\beta_{N-1}}{R_N}.$$

Assume that the finite fractions with terminal denominator $a_{N-1} + \tau_N$ and with terminal denominator a_{N-1} are both well-defined. Let $R_0^{(N)}$ be the second one, so the unknown tail is replaced by 0. Then,

$$R_0 - R_0^{(N)} = F_N(\tau_N) - F_N(0).$$

Therefore, if $|\tau_N| \leq \overline{\tau}_N$ and $\overline{\tau}_N |Q_{N-2}| < |Q_{N-1}|$, the error is bounded by Corollary 2.4.

Proof. Unwinding the Riccati recursion gives a finite continued fraction whose last denominator is $a_{N-1} + \beta_{N-1}/R_N$. This is $a_{N-1} + \tau_N$. Replacing R_N by ∞ replaces τ_N by 0. Lemma 2.3 then gives the identity, and Corollary 2.4 gives the estimate.

Algorithm 1 Truncation and error bound via continuants

Require: Truncation depth N, coefficient lists $\{a_j\}_{j=0}^{N-1}$, $\{b_j\}_{j=1}^{N-1}$, and a bound $\overline{\tau} \geq 0$ with $|\tau| \leq \overline{\tau}$ for the tail parameter. The continuants are computed for the finite fraction with terminal denominator a_{N-1}, so the tail parameter is set to 0.

Set $P_{-1} \leftarrow 1$, $P_0 \leftarrow a_0$, $Q_{-1} \leftarrow 0$, $Q_0 \leftarrow 1$
for $n = 1$ to $N-1$ **do**
  $P_n \leftarrow a_n P_{n-1} + b_n P_{n-2}$
  $Q_n \leftarrow a_n Q_{n-1} + b_n Q_{n-2}$
end for
If $Q_{N-1} = 0$, stop; otherwise set $F_N(0) = P_{N-1}/Q_{N-1}$
if $\overline{\tau}\,|Q_{N-2}| < |Q_{N-1}|$ **then**
  Error bound: $E \leftarrow \dfrac{\overline{\tau}\,|b_1 \cdots b_{N-1}|}{|Q_{N-1}|^2} \cdot \dfrac{1}{1 - \overline{\tau}\,|Q_{N-2}|/|Q_{N-1}|}$
else
  Error bound: use the exact formula in Lemma 2.3 if the exact τ is known
end if
return $F_N(0)$ and E

The algorithm is not used as an independent proof of convergence. It gives an a posteriori bound after the correct tail estimate and the nonzero-denominator conditions have already been verified.

3. Worked examples

We keep the first example in full detail because it shows how the block algebra changes a standard Riccati recursion into the arithmetic pattern. The second and third examples are treated more briefly. They use the same block method, and we still give the main positivity and tail estimates. To avoid repetition, routine rational simplifications are presented as denominator-clearing checks. The corresponding finite matrix identities are recorded again in Section 4.

Example 1: A hyperbolic constant from a modified Bessel ratio. We prove the identity

$$\frac{1+e}{4(e-1)} = 1 - \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{2 - \cfrac{1}{3 + \cfrac{1}{56 + \ddots}}}}}. \tag{3.1}$$

Theorem 3.1. *Let* $(a_j)_{j\geq 0}$ *be defined for every integer* $k \geq 0$ *by*

$$a_{6k} = 1 + 4k, \quad a_{6k+1} = 2, \quad a_{6k+2} = 5 + 16k,$$
$$a_{6k+3} = 2, \quad a_{6k+4} = 3 + 4k, \quad a_{6k+5} = 56 + 64k,$$

and let $(b_j)_{j\geq 1}$ *be periodic of period* 6 *with*

$$(b_1, b_2, b_3, b_4, b_5, b_6) = (-1, 1, 1, -1, 1, 1).$$

Then,

$$\frac{1+e}{4(e-1)} = 1 + \operatorname*{K}_{j=1}^{\infty} \frac{b_j}{a_j}.$$

Moreover, if one truncates after $6K$ *entries with* $K \geq 1$*, the induced tail parameter satisfies*

$$|\tau| \leq \frac{1}{4K + 1/2},$$

so the truncation error admits the computable bound (2.4).

Proof. Rewrite the constant:

$$\frac{1+e}{4(e-1)} = \frac{1}{4} \cdot \frac{e+1}{e-1} = \frac{1}{4} \coth\left[\frac{1}{2}\right].$$

We use the half-integer modified Bessel identities; see the NIST Digital Library of Mathematical Functions (DLMF) [11, §10.39]:

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x, \qquad I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh x.$$

Hence, $\coth x = I_{-1/2}(x)/I_{1/2}(x)$ and therefore

$$\frac{1+e}{4(e-1)} = \frac{1}{4}\,\frac{I_{-1/2}(1/2)}{I_{1/2}(1/2)}.$$

Define the ratio sequence

$$R_n := \frac{I_{n-1/2}(1/2)}{I_{n+1/2}(1/2)} \qquad n \geq 0.$$

The order recurrence for I_ν is

$$I_{\nu-1}(x) = \frac{2\nu}{x} I_\nu(x) + I_{\nu+1}(x).$$

See [11, §10.29]. Set $\nu = n + 1/2$ and $x = 1/2$. We get

$$I_{n-1/2}(1/2) = [4n+2] I_{n+1/2}(1/2) + I_{n+3/2}(1/2).$$

Divide by $I_{n+1/2}(1/2)$ and rewrite:

$$R_n = 4n + 2 + \frac{1}{R_{n+1}}.$$

Since $I_\nu(x) > 0$ for positive arguments and for the half-integer orders considered here, all ratios are positive. Thus, this is a Riccati difference equation with a positive tail term.

Define

$$X_k := \frac{1}{4} R_{4k} \qquad k \geq 0.$$

Then, $X_0 = \frac{1}{4}R_0 = \frac{1}{4}\coth[1/2]$, which is the target constant.

Write the Riccati relation four times starting at $n = 4k$ and substitute $R_{4k+4} = 4X_{k+1}$. A direct substitution gives the exact identity

$$X_k = 4k + \frac{1}{2} + \cfrac{1}{4\left[16k + 6 + \cfrac{1}{16k + 10 + \cfrac{1}{16k + 14 + \cfrac{1}{4X_{k+1}}}}\right]}.$$

This is the block recursion $X_k = \Phi_k[X_{k+1}]$.

Define

$$T_k := 64k + 56 + \frac{1}{X_{k+1}}.$$

Then,

$$16k + 14 + \frac{1}{4X_{k+1}} = \frac{1}{4} T_k, \qquad \frac{1}{16k + 14 + \frac{1}{4X_{k+1}}} = \frac{4}{T_k}.$$

Define

$$V_k := 16k + 10 + \frac{4}{T_k}.$$

Then, the inner denominator becomes $16k + 6 + \frac{1}{V_k}$.

Next we use the elementary identity

$$\frac{1}{2} + \frac{1}{16k + 10 + \frac{4}{T_k}} = \frac{1}{2 - \frac{1}{4k+3+\frac{1}{T_k}}}. \tag{3.2}$$

Indeed, after both sides are written over a common denominator, they are equal to

$$\frac{(4k+3)T_k + 1}{(8k+5)T_k + 2}.$$

Thus, the earlier line-by-line rational expansion can be replaced by one denominator-clearing check.

Define

$$Y_k := 16k + 5 + \frac{1}{2 - \frac{1}{4k+3+\frac{1}{64k+56+\frac{1}{X_{k+1}}}}}.$$

Then, $16k + 6 + \frac{1}{V_k} = \frac{1}{2} + Y_k$, so the block recursion becomes

$$X_k = 4k + \frac{1}{2} + \frac{1}{4[\frac{1}{2} + Y_k]} = 4k + \frac{1}{2} + \frac{1}{2(2Y_k + 1)}.$$

Use the elementary identity

$$\frac{1}{2(2Y_k + 1)} = \frac{1}{2} - \frac{1}{2 + \frac{1}{Y_k}},$$

which follows immediately after clearing denominators. This gives

$$X_k = 4k + 1 - \frac{1}{2 + \frac{1}{Y_k}}.$$

Unwinding produces exactly the ESMA pattern with denominators

$$4k+1,\ 2,\ 16k+5,\ 2,\ 4k+3,\ 64k+56$$

and signs $(-1, 1, 1, -1, 1, 1)$. Evaluating at $k = 0$ gives (3.1).

It remains only to check that no pole is introduced. If $X_{k+1} > 0$, then

$$T_k > 0, \qquad 2 - \frac{1}{4k + 3 + 1/T_k} > 2 - \frac{1}{4k+3} > 0, \qquad Y_k > 0.$$

Hence,

$$X_k = 4k + 1 - \frac{1}{2 + 1/Y_k} > 4k + \frac{1}{2} > 0,$$

and all complete denominators in the block are positive. Also, $R_n > 4n + 2$, so

$$X_k = \frac{1}{4} R_{4k} > 4k + \frac{1}{2}.$$

Thus, $X_k \to \infty$. Truncation after K full blocks replaces $1/X_K$ by 0, and the induced tail parameter is $\tau = 1/X_K$. Therefore,

$$|\tau| \leq \frac{1}{4K + 1/2}.$$

Proposition 2.9, Corollary 2.4, and Algorithm 1 now give the stated computable truncation bound.

Lemma 3.2. *Fix x with $0 < x \le 1$. For every real $\nu \ge 0$,*

$$J_\nu(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(x/2)^{2m+\nu}}{m!\,\Gamma(m+\nu+1)}.$$

The absolute values of the terms are strictly decreasing, hence $J_\nu(x) > 0$. Moreover,

$$\frac{(x/2)^\nu}{\Gamma(\nu+1)} - \frac{(x/2)^{\nu+2}}{\Gamma(\nu+2)} \le J_\nu(x) \le \frac{(x/2)^\nu}{\Gamma(\nu+1)}.$$

Consequently,

$$0 < \frac{J_{\nu+1}(x)}{J_\nu(x)} \le \frac{x}{2(\nu+1)\left(1 - \frac{x^2}{4(\nu+1)}\right)}. \tag{3.3}$$

In particular, $J_{\nu+1}(x)/J_\nu(x) \to 0$ as $\nu \to \infty$.

Proof. The series expansion is classical; see the DLMF [11, §10.2.2] or Watson [9, §3.3].

Let

$$c_m = \frac{(x/2)^{2m+\nu}}{m!\,\Gamma(m+\nu+1)}.$$

Then,

$$\frac{c_{m+1}}{c_m} = \frac{x^2/4}{(m+1)(m+\nu+1)} < 1,$$

so (c_m) is strictly decreasing. The series is alternating, with strictly decreasing absolute values, so the sum is positive and lies between the first and second partial sums, giving the two-sided bound. The ratio bound (3.3) follows by dividing the upper bound for $J_{\nu+1}(x)$ by the lower bound for $J_\nu(x)$ and simplifying.

The next two examples use the same idea. We group them into one theorem and retain only the recurrence, the block formula, and the tail estimate needed for the proof.

Theorem 3.3. *The following identities hold.*

(i) *Let*

$$a_{4k} = 10 + 12k, \qquad a_{4k+1} = 4 + 3k, \qquad a_{4k+2} = 2, \qquad a_{4k+3} = 5 + 3k, \qquad b_j = -1. \tag{3.4}$$

Then,

$$\frac{-2 + 2\tan 1}{-3 + 2\tan 1} = 10 + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{b_j}{a_j}. \tag{3.5}$$

If the fraction is truncated after $4K$ entries with $K \ge 1$, then

$$|\tau_K| \le \frac{1}{12K + 9}. \tag{3.6}$$

(ii) *Let*

$$a_{2k} = 1 + 2k, \qquad a_{2k+1} = 8 + 8k, \qquad b_j = -1. \tag{3.7}$$

Then,

$$\frac{J_0(1)}{2J_1(1)} = 1 + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{b_j}{a_j}. \tag{3.8}$$

If the fraction is truncated after $2K$ entries with $K \geq 1$, then

$$|\tau_K| < \frac{1}{2K}. \tag{3.9}$$

In both cases, the error after truncation is bounded by Corollary 2.4.

Proof. We now give the two remaining verifications in a compressed form. The substitutions below are exactly the finite matrix checks displayed later in (4.17) and (4.18). For (i), we use the half-integer formulas for $J_{\pm 1/2}$ and the recurrence

$$J_{\nu-1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu+1}(x), \tag{3.10}$$

see the DLMF [11, §10.6]. These formulas give

$$\frac{-2 + 2\tan 1}{-3 + 2\tan 1} = 2\frac{J_{3/2}(1)}{J_{5/2}(1)}. \tag{3.11}$$

Put $R_n = J_{n+3/2}(1)/J_{n+5/2}(1)$. Lemma 3.2 gives positivity, and (3.10) gives

$$R_n = 2n + 5 - \frac{1}{R_{n+1}}. \tag{3.12}$$

With $X_k = 2R_{3k}$, performing three substitutions in (3.12), followed by clearing denominators, gives

$$X_k = 10 + 12k - \cfrac{1}{4 + 3k - \cfrac{1}{2 - \cfrac{1}{5 + 3k - \cfrac{1}{X_{k+1}}}}}. \tag{3.13}$$

This is exactly (3.5). For the tail, put $\nu = n + 3/2$ in (3.3). Then,

$$\frac{1}{R_n} = \frac{J_{n+5/2}(1)}{J_{n+3/2}(1)} \leq \frac{1}{2(n + 5/2)(1 - \frac{1}{4n+10})} = \frac{2}{4n + 9}. \tag{3.14}$$

Thus, $X_K = 2R_{3K} \geq 12K + 9$, so $|\tau_K| = 1/X_K \leq (12K + 9)^{-1}$. Since $X_{k+1} > 1$, the denominators

$$5 + 3k - \frac{1}{X_{k+1}}, \qquad 2 - \frac{1}{5 + 3k - 1/X_{k+1}}, \qquad 4 + 3k - \frac{1}{2 - 1/(5 + 3k - 1/X_{k+1})}$$

are positive, so no complete denominator vanishes.

For (ii), put $R_n = J_n(1)/J_{n+1}(1)$. The same recurrence gives

$$R_{n-1} = 2n - \frac{1}{R_n}. \tag{3.15}$$

Let $S_k = R_{2k}/2$. Two substitutions, followed by clearing denominators, give

$$S_k = (2k+1) - \cfrac{1}{8(k+1) - \frac{1}{S_{k+1}}}, \tag{3.16}$$

which is (3.8). Lemma 3.2 implies $S_k \to \infty$. We also need a lower bound before using (3.16). Apply (3.3) with $x = 1$ and $\nu = n$. Then,

$$\frac{J_{n+1}(1)}{J_n(1)} \le \frac{1}{2(n+1)\left(1 - \frac{1}{4(n+1)}\right)} = \frac{2}{4n+3}.$$

Therefore,

$$R_n = \frac{J_n(1)}{J_{n+1}(1)} \ge \frac{4n+3}{2}, \qquad S_k = \frac{R_{2k}}{2} \ge \frac{8k+3}{4}.$$

In particular, $S_{k+1} > 1$ for every $k \ge 0$. Hence,

$$8(k+1) - \frac{1}{S_{k+1}} > 8k+7.$$

Using (3.16), we obtain

$$S_k > (2k+1) - \frac{1}{8k+7} > 2k \tag{3.17}$$

for $k \ge 1$. Hence, $|\tau_K| = 1/S_K < 1/(2K)$. This proves the second identity and its tail bound.

4. Proof method and finite matrix tests

The examples in Section 3 used the forward direction: a known special-function recurrence was expanded into an arithmetic continued fraction. ESMA leads to the reverse problem. One first observes a signed interlaced pattern and then has to find the hidden recurrence. We now state this reverse step as a finite matrix test.

Let K be a field of characteristic zero and let $K(k)$ be the field of rational functions in k. We write $A \doteq B$ when two 2×2 matrices over $K(k)$ differ by a nonzero scalar factor. Such matrices define the same Möbius map,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}. \tag{4.1}$$

We use the same tail convention as before: the step $x \mapsto a + b/x$ is represented by $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$.

Lemma 4.1. *Let a block continued fraction have tails*

$$X_k = a_0(k) + \cfrac{b_1(k)}{a_1(k) + \cfrac{b_2(k)}{a_2(k) + \cdots + \cfrac{b_d(k)}{X_{k+1}}}}, \tag{4.2}$$

where $a_j(k), b_j(k) \in K(k)$ and $b_j(k) \neq 0$. Define

$$M_k = \prod_{j=0}^{d-1} \begin{pmatrix} a_j(k) & b_{j+1}(k) \\ 1 & 0 \end{pmatrix}. \tag{4.3}$$

Then, $X_k = M_k \cdot X_{k+1}$.

Proof. Each step $x \mapsto a_j(k) + b_{j+1}(k)/x$ is represented by the matrix $\begin{pmatrix} a_j(k) & b_{j+1}(k) \\ 1 & 0 \end{pmatrix}$. One block is the composition of these steps. Composition of Möbius maps is matrix multiplication.

Lemma 4.2. *Let (u_k) satisfy the second-order recurrence*

$$u_k = A_k u_{k+1} + B_k u_{k+2}, \qquad A_k, B_k \in K(k), \qquad B_k \neq 0. \tag{4.4}$$

Put

$$Y_k = \frac{u_k}{u_{k+1}}, \qquad N_k = \begin{pmatrix} A_k & B_k \\ 1 & 0 \end{pmatrix}. \tag{4.5}$$

Then, $Y_k = N_k \cdot Y_{k+1}$.

Proof. Divide (4.4) by u_{k+1}. Since $u_{k+2}/u_{k+1} = 1/Y_{k+1}$, we get $Y_k = A_k + B_k/Y_{k+1}$.

Theorem 4.3. *Let M_k and N_k be 2×2 matrices over $K(k)$, considered up to nonzero scalar factors. A rational Möbius change of variables*

$$X_k = G_k \cdot Y_k, \qquad G_k \in \mathrm{GL}_2(K(k)), \tag{4.6}$$

transforms the recurrence tail relation $Y_k = N_k \cdot Y_{k+1}$ into the continued-fraction tail relation $X_k = M_k \cdot X_{k+1}$ if, and only if,

$$M_k G_{k+1} \doteq G_k N_k \tag{4.7}$$

for all k. In this case, every solution of (4.4) *gives*

$$X_k = G_k \cdot \frac{u_k}{u_{k+1}}. \tag{4.8}$$

Proof. If (4.7) holds, then

$$X_k = G_k \cdot Y_k = (G_k N_k) \cdot Y_{k+1} = (M_k G_{k+1}) \cdot Y_{k+1} = M_k \cdot X_{k+1}.$$

Conversely, if the change of variables sends every recurrence tail relation to the block tail relation, then $G_k N_k$ and $M_k G_{k+1}$ define the same Möbius map in the free variable Y_{k+1}. Hence, they are proportional, which is (4.7).

Theorem 4.4. *Assume* (4.7) *and put*

$$\widehat{X}_k = G_k \cdot \frac{u_k}{u_{k+1}}. \tag{4.9}$$

Let $F_K(0)$ be the finite truncation obtained by replacing the true tail at level K by the tail at infinity. Let τ_K be the corresponding tail parameter for the true tail $\widehat{X}_K$. Suppose that $\tau_K \to 0$, that all complete denominators in the true and truncated finite fractions are nonzero for all large K, and that the bound from Corollary 2.4, with $|\tau_K| \leq \overline{\tau}_K$, tends to 0. Then, the infinite continued fraction converges and

$$X_0 = G_0 \cdot \frac{u_0}{u_1}. \tag{4.10}$$

Proof. By Theorem 4.3, the sequence $\widehat{X}_k$ satisfies the same block tail relation as the continued fraction. Fix K large enough so that the true and truncated finite fractions are both well-defined. If the first K blocks are unwound and the true tail is kept at level K, then

$$F_K(\tau_K) = \widehat{X}_0. \tag{4.11}$$

The ordinary finite truncation is obtained by replacing the same bottom tail by the tail at infinity, so it is $F_K(0)$. Hence,

$$F_K(0) - \widehat{X}_0 = F_K(0) - F_K(\tau_K). \tag{4.12}$$

The assumptions allow us to apply Corollary 2.4. Its righthand side tends to 0. Therefore, $F_K(0)$ converges to $\widehat{X}_0$. This proves that the continued fraction selects the same tail as the recurrence solution, and

$$X_0 = \widehat{X}_0 = G_0 \cdot \frac{u_0}{u_1}.$$

Corollary 4.5. *The identity* (4.7) *is equivalent, after clearing denominators, to finitely many polynomial identities in k. Thus, a block continued-fraction identity is proved by a finite matrix test once N_k, G_k, and the tail estimates are checked.*

Proof. Let $L_k = M_k G_{k+1}$ and $R_k = G_k N_k$. The relation $L_k \doteq R_k$ means that the four entries of these two matrices are proportional. Equivalently, all 2×2 minors formed from the two four-entry vectors vanish. These minors are rational functions of k. After clearing denominators, one obtains finitely many polynomial identities.

Corollary 4.6. *Every Riccati equation of the form*

$$Y_k = \alpha k + \beta + \frac{\varepsilon}{Y_{k+1}}, \qquad \alpha \neq 0, \qquad \varepsilon \in \{1, -1\}, \tag{4.13}$$

has a Bessel-type ratio solution. With $z = 2/\alpha$ and $\delta = \beta/\alpha$, one solution is

$$Y_k = \frac{I_{k+\delta-1}(z)}{I_{k+\delta}(z)} \quad (\varepsilon = 1), \qquad Y_k = \frac{J_{k+\delta-1}(z)}{J_{k+\delta}(z)} \quad (\varepsilon = -1), \tag{4.14}$$

when the denominators are nonzero. Under the assumptions of Theorem 4.4, this is the solution selected by truncation at infinity.

Proof. Divide the standard recurrences

$$I_{\nu-1}(z) = \frac{2\nu}{z} I_\nu(z) + I_{\nu+1}(z), \qquad J_{\nu-1}(z) = \frac{2\nu}{z} J_\nu(z) - J_{\nu+1}(z)$$

by $I_\nu(z)$ and $J_\nu(z)$, respectively, and put $\nu = k + \delta$; see the DLMF [11, §10.29, §10.6]. This gives (4.13). The tail selection follows from Theorem 4.4.

To show that the matrix test is concrete, write

$$E(a, b) = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}, \qquad D_s = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.15}$$

The block substitutions used in Section 3 are precisely the following finite products. For the three identities proved in Section 3, we may take $G_k = I$. The required test is the exact block identity $M_k \doteq N_k$:

$$\begin{aligned} M_k^{(1)} &= E(4k+1,-1)E(2,1)E(16k+5,1)E(2,-1)E(4k+3,1)E(64k+56,1) \\ &\doteq D_{1/4}E(16k+2,1)E(16k+6,1)E(16k+10,1)E(16k+14,1)D_{1/4}^{-1}, \end{aligned} \tag{4.16}$$

$$\begin{aligned} M_k^{(2)} &= E(10+12k,-1)E(4+3k,-1)E(2,-1)E(5+3k,-1) \\ &\doteq D_2E(6k+5,-1)E(6k+7,-1)E(6k+9,-1)D_2^{-1}, \end{aligned} \tag{4.17}$$

$$\begin{aligned} M_k^{(3)} &= E(2k+1,-1)E(8k+8,-1) \\ &\doteq D_{1/2}E(4k+2,-1)E(4k+4,-1)D_{1/2}^{-1}. \end{aligned} \tag{4.18}$$

The first identity is the algebra used in Theorem 3.1. The second and third are the shorter block computations in Theorem 3.3. In each case, the equality is checked by multiplying the displayed matrices and clearing denominators. These three checks suffice for the representative identities proved in the paper. Related ESMA rows can be treated in the same way.

The discussion above gives a useful interpretation of the representative ESMA rows treated in this paper. The hyperbolic and exponential examples are produced by ratios of half-integer modified Bessel functions and hence by the positive Riccati kernel

$$R_n = q(n+\delta) + \frac{1}{R_{n+1}}. \tag{4.19}$$

The trigonometric examples involving tan(1) and the examples involving ratios $J_m(1)/J_n(1)$ are produced by ratios of half-integer or integer-order Bessel functions and hence by the negative Riccati kernel

$$R_n = q(n+\delta) - \frac{1}{R_{n+1}}. \tag{4.20}$$

Thus, the representative formulas considered here have the same structural source: they are finite signed block transformations of standard special-function recurrences.

As one further illustration, consider the ESMA-type row

$$\frac{2}{J_5(1)/J_3(1)+1} = 2 - \cfrac{1}{40 - \cfrac{1}{3 - \cfrac{1}{56 - \cfrac{1}{4 - \cdots}}}}. \tag{4.21}$$

Let $R_n = J_n(1)/J_{n+1}(1)$. Since $J_3(1) = 8J_4(1) - J_5(1)$, we have

$$\frac{2}{J_5(1)/J_3(1)+1} = \frac{2J_3(1)}{J_3(1)+J_5(1)} = \frac{J_3(1)}{4J_4(1)} = \frac{R_3}{4}. \tag{4.22}$$

The recurrence $R_n = 2(n+1) - 1/R_{n+1}$ and the alternating scaling with starting index 3 and scale 1/4 give the displayed two-interlaced continued fraction. This example uses the same mechanism and requires no additional argument.

5. Conclusions

We have studied arithmetic-pattern continued fractions that appear in machine-generated ESMA tables. We gave a proof route based on Riccati recurrences, block matrices, and classical continuants. Three representative identities were proved in detail: one modified Bessel example leading to a hyperbolic expression, one half-integer Bessel example leading to a tangent expression, and one identity involving an integer-order Bessel ratio. In each case, the proof includes the block identity, positivity of complete denominators, and an explicit tail estimate.

When a discovered block pattern is equivalent to a Bessel-type Riccati kernel under a Möbius change, the conjecture reduces to a finite polynomial matrix identity and a tail-selection check. This gives a reproducible path from experimental continued-fraction discovery to proof, and it can be used for further machine-generated conjectures involving classical special functions.

Use of AI tools declaration

AI tools were used only for language editing and readability improvements.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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