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Research article

Oscillatory and nonoscillatory behavior of nonlinear delay dynamic equations on time scales

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Abstract: This paper is devoted to the qualitative analysis of first-order nonlinear delay dynamic equations on time scales. We establish sufficient conditions for oscillation and nonoscillation of solutions to dynamic equations of the form

$$x^{\Delta}(t) + p(t)\big(x(\tau(t))\big)^{\alpha} = 0, \quad t \geq t_0,$$

where α is a quotient of two odd integers and p, τ satisfy standard regularity and delay conditions on an arbitrary time scale $\mathbb{T}$. Employing comparison principles, the time scales version of L'Hôpital's rule, and integral transformations, we derive new oscillation criteria that generalize several classical results from the differential and difference equation settings to the unified framework of time scales. Furthermore, we provide sufficient conditions ensuring the existence of eventually positive (nonoscillatory) solutions. Our results extend and complement the existing theory in the literature.

Keywords: chain rule; delay dynamic equations; eventually positive solutions; nonoscillations; oscillations; time scales

1. Introduction and preliminaries

Among various qualitative aspects of solutions of difference as well as differential equations, the oscillation theory has been an area that is of great interest to researchers for a long time because of the usefulness it demonstrates in real-life applications. Oscillation theory provides rigorous mathematical tools to understand periodic behavior and predict long-term behaviors such as stability, resonance, and synchronization in complex systems. Recent studies have demonstrated that numerous animal and plant populations exhibit oscillatory behavior due to ecological interactions such as predation, competition, and resource sharing. These interactions often lead to cyclic fluctuations in population sizes, where multiple species or spatially separated populations oscillate in a coordinated manner over time. The study of oscillation helps us to get deeper insights into the behavior of complex biological and social systems. Thus, the problem of establishing sufficient conditions for the oscillation of solutions has been the subject of many investigations in the past several years.

Tang [1] established some sufficient conditions for oscillation and nonoscillation of the solutions of delay difference equations of the form

$$x'(t) + p(t)(x(\tau(t)))^{\alpha} = 0, \quad t \geq t_0, \tag{1.1}$$

where $p \in C([t_0, \infty), [0, \infty))$, $\tau \in C([t_0, \infty), \mathbb{R})$, $\tau(t) < t$, $\lim_{t\to\infty} \tau(t) = \infty$, $\alpha \in (0, \infty)$ is a quotient of odd positive integers; see also [2].

Öcalan [3] presented some results about the oscillatory and nonoscillatory behavior of solutions of the nonlinear delay difference equations of the form

$$\Delta x(n) + p(n)x^{\alpha}(\tau(n)) = 0, \quad n \in \mathbb{N}, \tag{1.2}$$

where $\alpha \in (0, \infty)$ is a ratio of odd positive integers, $\{p(n))\}$ is a sequence of nonnegative real numbers, $\{\tau(n)\}$ is a sequence of integers such that $\tau(n) \leq n$ and $\lim_{n\to\infty} \tau(n) = \infty$, and Δ denotes the forward difference operator given via $\Delta x(n) = x(n+1) - x(n)$; see also [4–6].

The questions of oscillation and nonoscillation of solutions of difference and differential equations are an interesting and important area of study in modern mathematics. This is mainly because oscillation theory enables the prediction of the qualitative behavior of dynamical systems without the necessity of obtaining precise, frequently unattainable, closed-form solutions. Furthermore, within the past four decades, these two related but distinct areas have begun to be combined under a powerful, more robust and general theory titled dynamic equations on time scales. The theory of dynamic equations on time scales, initiated by Hilger [7], provides a unified framework that bridges continuous and discrete analysis. Over the past three decades, significant progress has been made in extending qualitative theories such as stability, oscillation, and periodicity from classical differential and difference equations to time scales. Among the leading contributors to this area, [8, 9] covers early development in the topic, and these two monographs remained standard references for time scales theory till this date. Further, recent advances in the theory and applications of dynamic equations on time scales are excellently presented in [10, 11]. The qualitative theory of functional and delay dynamic systems on arbitrary time scales is systematically developed in [12]. Building upon the foundational results of [13, 14], where the concept of periodicity on time scales was first introduced, an integrated approach to the study of qualitative aspects of boundedness, periodicity, and stability of functional dynamical systems on time scales is provided in [15]. The most recent contributions in the oscillation theory of functional dynamic equations on time scales have been summarized in [16].

In this paper, we wish to generalize and extend the results of [1, 3, 6] for the nonlinear delay dynamic equations on time scales. Indeed, we explore the oscillatory and nonoscillatory behavior of the nonlinear delay dynamic equation of the form

$$x^{\Delta}(t) + p(t)(x(\tau(t)))^{\alpha} = 0, \quad t \geq t_0, \tag{1.3}$$

where the coefficient function p and the delay function τ satisfy the regularity assumptions:

(A1) $p \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[0,\infty)_{\mathbb{T}})$, $\tau \in C([t_0,\infty)_{\mathbb{T}},[0,\infty)_{\mathbb{T}})$, $\tau(t) \leq t$, $\lim_{t\to\infty}\tau(t) = \infty$, $\alpha \in (0,\infty)$ is a quotient of two odd integers, and $t_0 \in \mathbb{T}$.

Throughout this paper, $\mathbb{T}$ denotes time scale (any nonempty closed set of real numbers) with the forward jump operator σ, graininess function μ, and delta derivative Δ, where $\sup\mathbb{T} = \infty$. The intervals with a subscript $\mathbb{T}$ will denote the intersection of the usual interval with $\mathbb{T}$. Further, we shall follow standard notations from time scales theory which are very common in the literature, and we refer the reader to [8, 9] for an excellent review of the topic and [10, 11] for recent advances in the topic.

A solution of Eq (1.3) is said to be oscillatory if it is neither eventually positive nor eventually negative. Dynamic equation (1.3) is said to be oscillatory provided each of its solutions are oscillatory; otherwise, (1.3) is called nonoscillatory. A strong interest in the oscillatory and nonoscillatory behaviors of Eq (1.3) is motivated by the fact that it serves as a unifying model encompassing both the continuous-time delay differential equation (1.1) and the discrete-time delay difference equation (1.2) as special cases when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively. In the recent years, study of oscillatory and nonoscillatory behavior of solutions of first-order delay dynamic equations on time scales

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0, \quad t \in \mathbb{T}, \tag{1.4}$$

which unify the corresponding results for differential and difference equations, has been considered by a large number of authors; we refer the reader to the papers [17–24] and references therein. The study of oscillatory and nonoscillatory behavior of delay dynamic equations on time scales has profound implications in the analysis of population dynamics, control theory, and neural network models, where mixed discrete-continuous behaviors naturally arise.

Our main objective in this paper is to establish new oscillation criteria and comparison results for the nonlinear delay dynamic equation (1.3). By employing suitable transformations and tools from time scale calculus, we derive sufficient conditions under which all solutions of (1.3) either oscillate or remain eventually positive. These results not only extend previously known oscillation criteria in both continuous and discrete cases but also enrich the qualitative theory of dynamic equations on time scales.

2. Main results

We start with the following useful results, which establishes a comparison principle between delay dynamic inequalities and delay dynamic equations.

Lemma 2.1. *Assume* (A1) *and that for large t,*

$$p(s) \neq 0, \quad s \in [t, t^*]_{\mathbb{T}}, \tag{2.1}$$

where t^* satisfies $\tau(t^*) = t$. Then, Eq (1.3) *has an eventually positive solution if and only if the corresponding inequality*

$$x^{\Delta}(t) + p(t)(x(\tau(t)))^{\alpha} \leq 0, \quad t \geq t_0, \tag{2.2}$$

has an eventually positive solution.

Proof. Suppose that (1.3) has an eventually positive solution. Then, clearly, it is an eventually positive solution of (2.2). Now, assume that (2.2) has an eventually positive solution x. Then, there are $t_1 \geq t_2 \geq t_0$ such that $x(t) > 0$ for any $t \geq t_2$ and $\tau(t_1) \geq t_2$. Hence,

$$x^{\Delta}(t) \leq -p(t)(x(\tau(t)))^{\alpha} < 0, \quad t \geq t_1.$$

This means that x is decreasing on $[t_1,\infty)_{\mathbb{T}}$. Let

$$\Omega = \{w \in C_{rd}([t_1,\infty),[0,\infty)) : 0 \le w(t) \le 1, \quad t \ge t_1\}$$

and define the operator $T\colon \Omega \to [0,\infty)$ as

$$T[w](t) = \frac{1}{x(t)} \int_t^\infty p(s)(w(\tau(s)))^\alpha (x(\tau(s)))^\alpha \Delta s, \quad t \ge t_1.$$

Define $w_0(t) = 1$ and $w_{k+1}(t) = T[w_k](t)$, $t \ge t_1$, $k \in \mathbb{N}_0$. Then,

$$\begin{aligned} w_1(t) &= \frac{1}{x(t)} \int_t^\infty p(s)(x(\tau(s)))^\alpha \Delta s \\ &\le 1, \quad t \ge t_1. \end{aligned}$$

Assume that $w_k(t) \le w_{k-1}(t)$, $t \ge t_1$, for some $k \in \mathbb{N}_0$. Then,

$$\begin{aligned} w_{k+1}(t) &= T[w_k](t) \\ &\le \frac{1}{x(t)} \int_t^\infty p(s)(w_{k-1}(\tau(s)))^\alpha (x(\tau(s)))^\alpha \Delta s \\ &= T[w_{k-1}](t) \\ &= w_k(t), \quad t \ge t_1. \end{aligned}$$

Thus, $0 \le w_{n+1}(t) \le w_n(t) \le 1$, $t \ge t_1$, $n \in \mathbb{N}$. This means that $\{w_n(t)\}$ is a monotonically decreasing sequence of positive reals and hence it is convergent. Therefore, there exists $w(t) \in \mathbb{R}$ such that $w(t) = \lim_{n\to\infty} w_n(t)$, $t \ge t_1$ exists, and

$$w(t) = \frac{1}{x(t)} \int_t^\infty p(s)(w(\tau(s)))^\alpha (x(\tau(s)))^\alpha \Delta s, \quad t \ge t_1.$$

Now, set $z(t) = x(t)w(t)$, $t \ge t_1$. Then

$$z(t) = \int_t^\infty p(s)(z(\tau(s)))^\alpha \Delta s, \quad t \ge t_1,$$

and differentiating with respect to t, we arrive at $z^\Delta(t) - p(t)(z(\tau(t)))^\alpha = 0$, $t \ge t_1$, which yields that z is an eventually positive solution of (1.3). This completes the proof.

Theorem 2.1. *Assume* (*A*1) *and* (2.1) *hold, and for large t,*

$$p(t) \le q(t). \tag{2.3}$$

Then, the solution of Eq (1.3) *oscillates if and only if the solution of the equation*

$$x^\Delta(t) + q(t)(x(\tau(t)))^\alpha = 0, \quad t \ge t_0, \tag{2.4}$$

oscillates.

Proof. Suppose that Eq (1.3) is oscillatory. Assume the contrary that Eq (2.4) is not oscillatory and let x be an eventually positive solution of Eq (2.4). Then, there are $t_1 \geq t_2 \geq t_0$ such that $x(t) > 0$, $t \geq t_2$, and $\tau(t_1) \geq t_2$. Hence, using (2.3), we arrive at

$$\begin{aligned} 0 &= x^{\Delta}(t) + q(t)(x(\tau(t)))^{\alpha} \\ &\geq x^{\Delta}(t) + p(t)(x(\tau(t)))^{\alpha}, \quad t \geq t_1. \end{aligned}$$

Applying Lemma 2.1, we conclude that x is an eventually positive solution of Inequality (2.2). Applying Lemma 2.1, we conclude that Eq (1.3) has an eventually positive solution. This is a contradiction. Now, let x be an eventually negative solution of Eq (2.4). Then, there are $t_3 \geq t_4 \geq t_0$ such that $x(t) < 0$, $t \geq t_4$, and $\tau(t_3) \geq t_0$, and

$$\begin{aligned} 0 &= -x^{\Delta}(t) - q(t)(x(\tau(t)))^{\alpha} \\ &= (-x)^{\Delta}(t) + q(t)(-x(\tau(t)))^{\alpha}, \quad t \geq t_0. \end{aligned}$$

Thus, $-x$ is an eventually positive solution of Eq (2.4). As above, we arrive at a contradiction. Consequently, Eq (2.4) is oscillatory.

Conversely, assume that Eq (2.4) is oscillatory. If possible, suppose that Eq (1.3) is not oscillatory and let x be an eventually positive solution of Eq (1.3). Then, there are $t_5 \geq t_6 \geq t_0$ such that $x(t) > 0$, $t \geq t_6$, and $\tau(t_5) \geq t_6$. Hence,

$$\begin{aligned} 0 &= x^{\Delta}(t) + p(t)(x(\tau(t)))^{\alpha} \\ &\leq x^{\Delta}(t) + q(t)(x(\tau(t)))^{\alpha}, \quad t \geq t_5. \end{aligned}$$

Applying Lemma 2.1, we conclude that x is an eventually positive solution of Eq (2.4). This is a contradiction. Now, let x be an eventually negative solution of Eq (1.3). Then, $-x$ is an eventually positive solution of Eq (1.3). As above, we arrive at a contradiction. Hence, Eq (1.3) is oscillatory. This completes the proof.

We derive a sufficient condition for oscillatory solutions of Eq (1.3) in the following theorem.

Theorem 2.2. *Assume* (A1), $\alpha > 1$, $\tau^{\Delta}(t) \geq 0$, $t \geq t_0$, *and* $\mu(t) \to \infty$ *as* $t \to \infty$. *Further, suppose that there exists a nonnegtive function* ϕ *that is classical and delta differentiable such that*

$$\phi^{\Delta}(t) > 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \quad \textit{and} \quad \lim_{t\to\infty} \phi(t) = \infty, \tag{2.5}$$

and

$$\lim_{t\to\infty} \frac{\alpha\Phi(t)}{\phi^{\Delta}(t)} < 1, \tag{2.6}$$

and

$$\liminf_{t\to\infty} \left(p(t) \frac{e^{-\phi(t)}}{\phi^{\Delta}(t)} \right) > 1, \tag{2.7}$$

where

$$\Phi(t) = \left(\int_0^1 \phi'(\tau(t) + h\mu(t)\tau^{\Delta}(t))dh \right) \tau^{\Delta}(t), \quad t \geq t_0.$$

Then, Eq (1.3) *is oscillatory.*

Proof. Since $\lim_{t\to\infty}\mu(t)=\infty$, there is a $t_1\geq t_0$ such that $\mu(t)>1$, $t\geq t_1$. By (2.5) and (2.6), and using the time scales L' Hôpital rule [8, Theorem1.119], we get

$$\limsup_{t\to\infty}\frac{\alpha\phi(\tau(t))}{\phi(t)}\leq\lim_{t\to\infty}\frac{\alpha\Phi(t)}{\phi^{\Delta}(t)}<1. \tag{2.8}$$

Therefore, there exist $k\in(0,1)$ and $t_2\geq t_1$ such that

$$\frac{\alpha\phi(\tau(t))}{\phi(t)}\leq k,\quad t\geq t_2. \tag{2.9}$$

Because of (2.7), there exist $t_3\geq t_2$ and $c>1$ such that

$$p(t)\frac{e^{-\phi(t)}}{\phi^{\Delta}(t)}\geq c,\quad t\geq t_3,$$

and, thus, we have

$$p(t)\geq c\phi^{\Delta}(t)e^{\phi(t)},\quad t\geq t_3.$$

Set

$$q(t)=c\phi^{\Delta}(t)e^{\phi(t)},\quad t\geq t_3.$$

In the view of Theorem 2.1, it is sufficient to prove that Eq (2.4) is oscillatory. Assume the contrary and let x be an eventually positive solution of Eq (2.4). Then, there exist $t_4,t_5\in[t_0,\infty)_{\mathbb{T}}$ such that $t_5\geq t_4\geq t_3$, $\tau(t_5)\geq t_4$, and $x(t)>0$, $t\geq t_4$. Hence,

$$\begin{aligned}x^{\Delta}(t)&=-q(t)(x(\tau(t)))^{\alpha}\\&<0,\quad t\geq t_5,\end{aligned}$$

i.e., x is nonincreasing and has a limit $l\geq 0$. Assume that $l>0$. Then, for $\varepsilon\in(0,l)$, there is a $t_6\geq t_5$ such that

$$0<l-\varepsilon\leq x(t)\leq l+\varepsilon,\quad t\geq t_6.$$

Then, using the nonincreasing nature of x and the fact that ϕ is nonnegative, we arrive at

$$\begin{aligned}0&=x^{\Delta}(t)+c\phi^{\Delta}(t)e^{\phi(t)}(x(\tau(t)))^{\alpha}\\&\geq x^{\Delta}(t)+c\phi^{\Delta}(t)(x(t))^{\alpha}\\&\geq x^{\Delta}(t)+c(l-\varepsilon)^{\alpha}\phi^{\Delta}(t),\quad t\geq t_6.\end{aligned}$$

Integrating the last inequality from t_6 to ∞, we obtain

$$\begin{aligned}0&\geq\lim_{t\to\infty}x(t)-x(t_6)+c(l-\varepsilon)^{\alpha}(\lim_{t\to\infty}\phi(t)-\phi(t_6))\\&=l-x(t_6)+c(l-\varepsilon)^{\alpha}(\infty-\phi(t_6)),\end{aligned}$$

which is a contradiction. Therefore, $l = 0$. Now, let $y(t) = -\ln x(t)$, $t \geq t_6$. Since x is nonincreasing and $\lim_{t\to\infty} x(t) = 0$, it follows that y is nondecreasing and $\lim_{t\to\infty} y(t) = \infty$. Further, note that $x(t) = e^{-y(t)}$, $t \geq t_6$. Then,

$$\begin{aligned} x^\Delta(t) &= \frac{x(\sigma(t)) - x(t)}{\mu(t)} \\ &= \frac{e^{-y(\sigma(t))} - e^{-y(t)}}{\mu(t)} \\ &= -\left(\frac{1 - e^{y(t)-y(\sigma(t))}}{\mu(t)}\right) e^{-y(t)} \\ &= -q(t)e^{-\alpha y(\tau(t))}, \quad t \geq t_6, \end{aligned}$$

whereupon

$$1 - e^{y(t)-y(\sigma(t))} = \mu(t)q(t)e^{y(t)-\alpha y(\tau(t))}, \quad t \geq t_6. \tag{2.10}$$

Assume that $y(t) - \alpha y(\tau(t)) < 0$, $t \geq t_6$. Then, using (2.9), we find

$$\frac{y(t)}{\phi(t)} \leq \frac{\alpha y(\tau(t))}{\phi(t)} = \frac{\alpha\phi(\tau(t))}{\phi(t)} \frac{y(\tau(t))}{\phi(\tau(t))} \leq k \frac{y(\tau(t))}{\phi(\tau(t))}, \quad t \geq t_6.$$

Set $z(t) = y(t)/\phi(t)$, $t \geq t_6$. Then,

$$z(t) \leq kz(\tau(t)), \quad t \geq t_6. \tag{2.11}$$

Since $k \in (0, 1)$, we find $z(t) \leq z(\tau(t))$, $t \geq t_6$, and z is nonincreasing on $[t_6, \infty)_{\mathbb{T}}$. Let $\lim_{t\to\infty} z(t) = b$. If possible, suppose that $b > 0$. Then, in the view of (2.11), we find $b \leq kb$, which is a contradiction. Hence, $b = 0$, which yields that

$$\lim_{t\to\infty} \frac{y(t)}{\phi(t)} = 0. \tag{2.12}$$

Therefore, there exists $t_7 \geq t_6$ such that

$$y(t) < \frac{1}{1+\alpha}\phi(t), \quad t \geq t_7. \tag{2.13}$$

On the other hand, by the Pötzsche chain rule [8, Theorem 1.90], we have

$$\begin{aligned} x^\Delta(t) &= (e^{-y(\cdot)})^\Delta(t) \\ &= -\left(\int_0^1 e^{-(y(t)+h\mu(t)y^\Delta(t))} dh\right) y^\Delta(t) \\ &= -q(t)e^{-\alpha y(\tau(t))}, \quad t \geq t_7, \end{aligned}$$

whereupon

$$\begin{aligned} e^{-y(t)}y^{\Delta}(t) &= \left(\int_0^1 e^{-y(t)}dh\right)y^{\Delta}(t) \\ &\geq \left(\int_0^1 e^{-(y(t)+h\mu(t)y^{\Delta}(t))}dh\right)y^{\Delta}(t) \\ &= q(t)e^{-\alpha y(\tau(t))}, \quad t \geq t_7, \end{aligned}$$

and

$$y^{\Delta}(t) \geq q(t)e^{y(t)-\alpha y(\tau(t))}, \quad t \geq t_7.$$

Thus, keeping in mind (2.13) and the fact that y is nondecreasing, we find

$$\begin{aligned} y^{\Delta}(t) &\geq q(t)e^{y(\tau(t))-\alpha y(\tau(t))} \\ &= q(t)e^{-(\alpha-1)y(\tau(t))} \\ &\geq q(t)e^{-(\alpha-1)\frac{\phi(\tau(t))}{\alpha+1}} \\ &= c\phi^{\Delta}(t)e^{\phi(t)}e^{-(\alpha-1)\frac{\phi(\tau(t))}{\alpha+1}} \\ &\geq c\phi^{\Delta}(t)e^{\phi(\tau(t))}e^{-(\alpha-1)\frac{\phi(\tau(t))}{\alpha+1}} \\ &= c\phi^{\Delta}(t)e^{\frac{\phi(\tau(t))}{\alpha+1}} \\ &\geq c\phi^{\Delta}(t), \end{aligned}$$

i.e., $y^{\Delta}(t) \geq \phi^{\Delta}(t)$, $t \geq t_7$. Now, integrating this inequality from t_7 to t, $t \geq t_7$, we get

$$y(t) - y(t_7) \geq \phi(t) - \phi(t_7), \quad t \geq t_7,$$

which yields

$$\frac{y(t)}{\phi(t)} \geq 1 + \frac{y(t_7) - \phi(t_7)}{\phi(t)}, \quad t \geq t_7.$$

Taking the limit as $t \to \infty$ in the last inequality, we find $\lim_{t\to\infty} z(t) \geq 1$. This is a contradiction. Hence, $y(t) \geq \alpha y(\tau(t))$, $t \geq t_7$. Now, suppose that $t \leq \tau(\sigma(t)) \leq \sigma(t)$, $t \geq t_7$. Then,

$$\begin{aligned} \phi^{\Delta}(t) &= \frac{\phi(\sigma(t)) - \phi(t)}{\mu(t)} \\ &\geq \frac{1}{\mu(t)}\left(\frac{\alpha\phi(\tau(\sigma(t)))}{k} - \phi(t)\right) \\ &\geq \frac{1}{\mu(t)}\left(\frac{\alpha\phi(\tau(\sigma(t)))}{k} - \phi(\tau(\sigma(t)))\right) \\ &= \frac{\phi(\tau(\sigma(t)))}{\mu(t)}\left(\frac{\alpha}{k} - 1\right). \end{aligned}$$

From the hypothesis, we see that the right side of the above inequality approaches to ∞ as $t \to \infty$. So, $\phi^{\Delta}(t) \to \infty$ as $t \to \infty$. Also, for $\tau(\sigma(t)) < t$, taking $\phi(t) = e^{\frac{\alpha}{k}t}$, we find that

$$\begin{aligned}\phi^{\Delta}(t) &= \frac{e^{\frac{\alpha}{k}\sigma(t)} - e^{\frac{\alpha}{k}t}}{\mu(t)} \\ &= \frac{e^{\frac{\alpha}{k}t}}{\mu(t)}(e^{\frac{\alpha}{k}\mu(t)} - 1) \\ &> 0, \quad t \ge t_7,\end{aligned}$$

i.e., the condition (2.5) holds, and, additionally, $\phi^{\Delta}(t) \to \infty$ as $t \to \infty$. Thus, in both cases, we have $\lim_{t\to\infty} \phi^{\Delta}(t) = \infty$. Moreover, we have

$$\begin{aligned}\Phi(t) &= \frac{\alpha}{k}\left(\int_0^1 e^{\frac{\alpha}{k}(\tau(t)+h\mu(t)\tau^{\Delta}(t))}dh\right)\tau^{\Delta}(t) \\ &= \frac{1}{\mu(t)}\left(e^{\frac{\alpha}{k}\tau(\sigma(t))} - e^{\frac{\alpha}{k}\tau(t)}\right) \\ &= \frac{e^{\frac{\alpha}{k}\tau(t)}}{\mu(t)}\left(e^{\frac{\alpha}{k}(\tau(\sigma(t))-\tau(t))} - 1\right), \quad t \ge t_7.\end{aligned}$$

Therefore, $\limsup_{t\to\infty} \frac{\alpha\Phi(t)}{\phi^{\Delta}(t)} > 1$, i.e., (2.6) holds. Now, keeping in mind (2.10), we arrive at $1 \ge q(t) = \phi^{\Delta}(t)e^{\phi(t)}$, which is contradiction since $\phi^{\Delta}(t) \to \infty$ as $t \to \infty$. Hence, x is not an eventually positive solution of Eq (2.4). Now, suppose x is an eventually negative solution of Eq (2.4). Then, $-x$ is an eventually positive solution of Eq (2.4). As discussed above, we again arrive at a contradiction. Therefore, x is not an eventually negative solution of Eq (2.4). This means that Eq (2.4) is oscillatory. Hence, by the virtue of Theorem 2.1, we conclude that Eq (1.3) is oscillatory. This completes the proof.

Remark 2.1. *We observe that Theorem 2.2 is a time scale reformation of* [1, Theorem 1] *and* [3, Theorem 3.1].

Now, we provide a criterion for the existence of nonoscillatory (eventually positive) solutions for Eq (1.3). The following theorem is an extension of [1, Theorem 2] to arbitrary time scales.

Theorem 2.3. *Assume (A1), $\alpha > 0$, $\tau^{\Delta}(t) \ge 0$, $t \ge t_0$. Further, suppose that there exists a nonnegative function ψ that is classical and delta differentiable such that*

$$\psi^{\Delta}(t) > 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \quad \textit{and} \quad \lim_{t\to\infty} \psi(t) = \infty, \tag{2.14}$$

and

$$\liminf_{t\to\infty} \frac{\alpha\Psi(\tau(t))}{\phi^{\Delta}(t)} \ge L > 1, \tag{2.15}$$

for some constant $L > 1$, and

$$\limsup_{t\to\infty}\left(p(t)\frac{e^{-\psi(t)+\frac{L}{L-1}\mu(t)\psi^{\Delta}(t)}}{\psi^{\Delta}(t)}\right) < 1, \tag{2.16}$$

where

$$\Psi(t) = \left(\int_0^1 \psi'(\tau(t) + h\mu(t)\tau^\Delta(t))dh \right) \tau^\Delta(t), \quad t \geq t_0.$$

Then, Eq (1.3) *has a nonoscillatory solution.*

Proof. Keeping in mind (2.15) and the time scale L' Hôpital rule [8, Theoem 1.119], we get

$$\liminf_{t\to\infty} \frac{\alpha\psi(\tau(t))}{\psi(t)} \geq \liminf_{t\to\infty} \frac{\alpha\Psi(t)}{\psi^\Delta(t)} \geq L > 1.$$

Then, there exists a $t_1 \geq t_0$ such that

$$\frac{\alpha\psi(\tau(t))}{\psi(t)} \geq L, \quad t \geq t_1. \tag{2.17}$$

Hence, $\alpha\psi(\tau(t)) \geq L\psi(t)$, $t \geq t_1$. From (2.16), it follows that there is a $t_2 \geq t_1$ such that

$$p(t) \frac{e^{-\psi(t) + \frac{L}{L-1}\mu(t)\psi^\Delta(t)}}{\psi^\Delta(t)} \leq 1, \quad t \geq t_2.$$

Hence,

$$\begin{aligned} p(t) &\leq \psi^\Delta(t) e^{\psi(t) - \frac{L}{L-1}\mu(t)\psi^\Delta(t)} \\ &\leq \frac{L}{L-1} \psi^\Delta(t) e^{L\psi(t) - \frac{L}{L-1}\mu(t)\psi^\Delta(t)}, \quad t \geq t_2. \end{aligned}$$

Set $x(t) = e^{-\frac{L}{L-1}\psi(t)}$, $t \geq t_2$. Then,

$$\begin{aligned} & x^\Delta(t) + p(t)(x(\tau(t)))^\alpha \\ &= -\frac{L}{L-1} \left(\int_0^1 e^{-\frac{L}{L-1}(\psi(t) + h\mu(t)\psi^\Delta(t))} dh \right) \psi^\Delta(t) + p(t) e^{-\alpha\frac{L}{L-1}\psi(\tau(t))} \\ &= e^{-\alpha\frac{L}{L-1}\psi(\tau(t))} \left(p(t) - \frac{L}{L-1} \left(\int_0^1 e^{\alpha\frac{L}{L-1}\psi(\tau(t)) - \frac{L}{L-1}(\psi(t) + h\mu(t)\psi^\Delta(t))} dh \right) \psi^\Delta(t) \right) \\ &= e^{-\alpha\frac{L}{L-1}\psi(\tau(t))} \left(p(t) - \frac{L}{L-1} \left(\int_0^1 e^{\frac{L}{L-1}(\alpha\psi(\tau(t)) - \psi(t)) - \frac{L}{L-1}h\mu(t)\psi^\Delta(t)} dh \right) \psi^\Delta(t) \right) \\ &\leq e^{-\alpha\frac{L}{L-1}\psi(\tau(t))} \left(p(t) - \frac{L}{L-1} \left(\int_0^1 e^{\frac{L}{L-1}(\alpha\psi(\tau(t)) - \psi(t)) - \frac{L}{L-1}\mu(t)\psi^\Delta(t)} dh \right) \psi^\Delta(t) \right) \\ &= e^{-\alpha\frac{L}{L-1}\psi(\tau(t))} \left(p(t) - \frac{L}{L-1} e^{\frac{L}{L-1}(\alpha\psi(\tau(t)) - \psi(t)) - \frac{L}{L-1}\mu(t)\psi^\Delta(t)} \psi^\Delta(t) \right) \\ &= e^{-\alpha\frac{L}{L-1}\psi(\tau(t))} \left(p(t) - \frac{L}{L-1} e^{\frac{L}{L-1}\psi(t)\left(\alpha\frac{\psi(\tau(t))}{\psi(t)} - 1\right) - \frac{L}{L-1}\mu(t)\psi^\Delta(t)} \psi^\Delta(t) \right). \end{aligned}$$

From (2.17), we have $\frac{\alpha\psi(\tau(t))}{\psi(t)} - 1 \geq L - 1$, whereupon

$$\begin{aligned}\frac{L}{L-1}\psi(t)\left(\frac{\alpha\psi(\tau(t))}{\psi(t)} - 1\right) \geq & \frac{L}{L-1}\psi(t)(L-1)\\ \geq & L\psi(t), \quad t \geq t_2.\end{aligned}$$

Then,

$$x^{\Delta}(t) + p(t)(x(\tau(t)))^{\alpha} \leq e^{-\alpha\frac{L}{L-1}\psi(\tau(t))}\left(p(t) - \frac{L}{L-1}\psi^{\Delta}(t)e^{L\psi(t)-\frac{L}{L-1}\mu(t)\psi^{\Delta}(t)}\right),$$

which yields that the inequality $x^{\Delta}(t) + p(t)(x(\tau(t)))^{\alpha} \leq 0$, $t \geq t_2$, has an eventually positive solution. Thus, in the view of Lemma 2.1, we conclude that Eq (1.3) has an eventually positive solution. This completes the proof.

3. Examples

Example 3.1. *Let $\mathbb{T} = [0,1] \cup 2^{\mathbb{N}_0}$. Then, $\sigma(t) = t$, $\mu(t) = 0$ for $t \in [0,1)$, and $\sigma(t) = 2t$, $\mu(t) = t$, $t \in 2^{\mathbb{N}_0}$. Consider the dynamic equation*

$$x^{\Delta}(t) + e^{3t^4}\left(x\left(\frac{t}{3}\right)\right)^{\frac{8}{7}} = 0, \quad t \geq 1. \tag{3.1}$$

Comparing Eq (3.1) *with Eq* (1.3), *we have $\alpha = \frac{8}{7}$, $p(t) = e^{3t^4}$, $\tau(t) = \frac{t}{3}$. Then, $\tau^{\Delta}(t) = \frac{1}{3}$, $t \in \mathbb{T}$, and* $\lim_{t\to\infty}\tau(t) = \infty$. *Take $\phi(t) = t^2$, $t \in \mathbb{T}$. Then, $\phi^{\Delta}(t) = 2t$ for $t \in [0,1)$, $\phi^{\Delta}(t) = \sigma(t) + t = 2t + t = 3t > 0$, $t \in 2^{\mathbb{N}_0}$, $\phi'(t) = 2t$, $t \in \mathbb{R}$,* $\lim_{t\to\infty}\phi(t) = \infty$, *and*

$$\Phi(t) = \frac{2}{3}\left(\int_0^1\left(\frac{1}{3}t + \frac{h}{3}t\right)dh\right) = \frac{2}{9}t + \frac{1}{9}, \quad t \in \mathbb{T}.$$

Hence,

$$\lim_{t\to\infty}\frac{\alpha\Phi(t)}{\phi^{\Delta}(t)} = \lim_{t\to\infty}\frac{\frac{8}{7}\left(\frac{2}{9}t + \frac{1}{9}\right)}{3t} = \frac{16}{189} < 1$$

and

$$\liminf_{t\to\infty}\left(p(t)\frac{e^{-\phi(t)}}{\phi^{\Delta}(t)}\right) = \liminf_{t\to\infty}\left(e^{3t^4}\frac{e^{-t^2}}{3t}\right) > 1.$$

Thus, all conditions of Theorem 2.2 are fulfilled. Hence, Eq (3.1) *is oscillatory.*

Example 3.2. *Let $\mathbb{T} = [0,1] \cup \mathbb{N}$. Then, $\sigma(t) = t$, $\mu(t) = 0$ for $t \in [0,1)$, $\sigma(t) = t + 1$ and $\mu(t) = 1$, $t \in \mathbb{N}$. Consider the dynamic equation*

$$x^{\Delta}(t) + e^{t-3}\left(x\left(\frac{t}{7}\right)\right)^{\frac{4}{3}} = 0, \quad t \geq 0. \tag{3.2}$$

Comparing Eq (3.2) *with Eq* (1.3), *we have $p(t) = e^{t-3}$, $\alpha = \frac{4}{3}$, $\tau(t) = \frac{t}{7}$. Then, $\tau^{\Delta}(t) = \frac{1}{7} > 0$. Take $\psi(t) = t$, $t \in \mathbb{T}$, and $L = 2$. Then, $\psi^{\Delta}(t) = 1 > 0$, $t \in \mathbb{T}$, and $\psi'(t) = 1$, $t \in \mathbb{R}$, and* $\lim_{t\to\infty}\psi(t) = \infty$. *Also, $\Psi(t) = \frac{1}{7}$, $t \in \mathbb{T}$, and*

$$\liminf_{t\to\infty}\frac{\alpha\Psi(\tau(t))}{\phi^{\Delta}(t)} = \liminf_{t\to\infty}\left(\frac{4}{21}t\right) > 2 = L > 1,$$

and

$$\limsup_{t\to\infty}\left(p(t)\frac{e^{-\psi(t)+\frac{L}{L-1}\mu(t)\psi^{\Delta}(t)}}{\psi^{\Delta}(t)}\right)=\limsup_{t\to\infty}\left(e^{t-3}e^{-t+2}\right)=e^{-1}<1.$$

Thus, all conditions of Theorem 2.3 are fulfilled and, hence, Eq (3.2) *has a nonoscillatory solution.*

4. Conclusions

In this paper, we investigate the qualitative behavior regarding the oscillatory and nonoscillatory nature of solutions to the first-order nonlinear delay dynamic equations on arbitrary time scales. The study provides a unified treatment that encompasses both the continuous and discrete cases as special instances. Using comparison principles, integral transformations, and the time scale version of L'Hôpital's rule [8, Theorem 1.119], we establish new sufficient conditions for the oscillation and for the existence of eventually positive (nonoscillatory) solutions. These results generalize and extend several classical oscillation criteria known for differential and difference equations to the broader framework of dynamic equations on time scales. The paper builds upon and complements the foundational work in functional and delay dynamic systems presented in [12], thereby contributing new theoretical tools for analyzing delayed phenomena in hybrid systems that exhibit both discrete and continuous dynamics.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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