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Research article

Representation of a solution to a class of linear discrete Langevin equations

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Abstract: This paper introduced, for the first time, a novel discrete Langevin equation as a new generalization of the classical Langevin model, involving two distinct integer-order differences. A new special function generated by the system parameters was defined to obtain explicit analytical solutions of the associated initial value problem. Several special cases were discussed, and open problems were proposed. The results established a new theoretical framework for discrete Langevin-type systems and contributed to the qualitative theory of difference equations.

Keywords: discrete Langevin equation; novel discrete exponential; representation of solutions

1. Introduction

Langevin introduced a fundamental model for Brownian motion one century ago, and the Langevin equation has since been widely used to describe stochastic dynamics in fluctuating media [1,2]. However, for many complex physical systems, the classical form may be inadequate, which has motivated several generalized Langevin-type models [3]. Among these, the fractional Langevin-type equation [4–7], obtained by replacing the classical derivative with fractional derivatives, has attracted considerable attention due to its ability to capture memory and hereditary effects.

In parallel with these theoretical developments, significant progress has also been made in the development of efficient analytical and numerical methods for fractional differential equations. In particular, recent approaches based on operational techniques and fractional Taylor expansions have demonstrated high accuracy and stability for solving complex fractional models, including quantum fractional systems (see, e.g., [8]). Such methods highlight the growing importance of constructing reliable solution frameworks for both linear and nonlinear fractional problems.

Although both integer-order and fractional-order Langevin-type equations have been extensively studied in continuous settings [1–6], their discrete counterparts, which can also be viewed as a natural generalization of classical Langevin equations, remain largely unexplored. In particular, explicit solution methods and fundamental qualitative properties for discrete Langevin-type systems have not yet been sufficiently established.

Motivated by this gap, in this paper we focus on a discrete Langevin system with two distinct integer orders and derive explicit representations for both homogeneous and inhomogeneous cases. The main advantage of the proposed representation lies in its explicit and structured form. Unlike classical recursive or iterative approaches, the obtained solution is expressed directly in terms of the discrete exponential function $\mathfrak{e}_m(\lambda, \mu; k)$, thereby avoiding step-by-step computations.

From a computational perspective, this representation reduces the complexity associated with repeated application of difference operators and provides a closed-form expression that is more suitable for numerical evaluation. From a modeling point of view, the introduced function captures the combined influence of the system parameters λ and μ through a double-index structure, enabling a more flexible and accurate description of discrete Langevin-type dynamics compared to standard single-index formulations.

Such models can be interpreted as discrete-time stochastic systems with multi-step memory or multi-lag signal processing structures. In particular, they naturally arise in numerical discretizations of stochastic differential equations, control systems with delayed feedback, and discrete-time models involving multiple temporal scales. The use of two distinct difference orders allows the interaction of these temporal scales to be captured, which cannot be represented within classical single-order discrete Langevin formulations. To the best of our knowledge, such a double-index explicit representation for discrete Langevin systems has not been reported in the literature.

The main contributions of this paper can be summarized as follows:

- A new discrete exponential function $\mathfrak{e}_m(\lambda, \mu; k)$ is introduced, which generalizes classical discrete exponential-type functions through a coupled two-parameter structure.
- An explicit representation of solutions for a class of linear discrete Langevin systems with two distinct integer orders is derived for both homogeneous and inhomogeneous cases.
- The proposed representation provides a closed-form solution, reducing computational complexity compared to recursive approaches.
- The double-index structure of the solution enhances modeling flexibility by capturing the combined effects of the system parameters.

Throughout this paper, we use the notation

$$\mathbb{N}_\ell^r := \{\ell, \ell + 1, \ell + 2, \ldots, r\},$$

where $\ell, r \in \mathbb{Z}$ with $\ell \leq r$, to denote a discrete integer interval. In particular, we write $\mathbb{N}_0 := \mathbb{N}_0^\infty = \{0, 1, 2, \ldots\}$ and $\mathbb{N} := \mathbb{N}_1^\infty = \{1, 2, 3, \ldots\}$. The present work is devoted to the study of a linear discrete Langevin equation with two distinct integer orders, which is described in the following form.

$$\nabla^2 \mathrm{u}(\mathrm{k}) - \mu \nabla \mathrm{u}(\mathrm{k}) - \lambda \mathrm{u}(\mathrm{k}) = f(\mathrm{k}), \ \mathrm{k} \in \mathbb{N}, \tag{1.1}$$

where the operator ∇ denotes the first order backward difference, i.e., $\nabla \mathrm{u}(\mathrm{k}) = \mathrm{u}(\mathrm{k}) - \mathrm{u}(\mathrm{k}-1)$, the operator ∇^2 is the second order backward difference, i.e., $\nabla^2 \mathrm{u}(\mathrm{k}) = \nabla[\nabla \mathrm{u}(\mathrm{k})] = \mathrm{u}(\mathrm{k}) - 2\mathrm{u}(\mathrm{k}-1) + \mathrm{u}(\mathrm{k}-2)$, $\mathrm{u}(\mathrm{k})$ is an unknown real-valued function, and k is a discrete independent variable. The parameters λ, μ are prescribed real constant numbers. By introducing a novel discrete exponential generated by the real coefficients λ, μ, a representation of the solutions for both the homogeneous and nonhomogeneous cases of (1.1) is derived under the following initial condition

$$\mathrm{u}(0) = \mathrm{u}_0 \text{ and } \nabla \mathrm{u}(0) = \nabla \mathrm{u}_0, \tag{1.2}$$

and

$$\mathrm{u}(0) = 0 \text{ and } \nabla \mathrm{u}(0) = 0, \tag{1.3}$$

are given real numbers. Since the conditions guaranteeing the existence and uniqueness of solutions for the problems under consideration are standard and can be verified without difficulty, a detailed discussion of these issues is omitted.

For the reader's convenience, we begin by summarizing several standard notations that will be employed throughout the paper. Empty sum is defined by $\sum_{\mu=\mathrm{u}_1}^{\mathrm{u}_2} \ldots = 0$, whenever u_k, $k = 1, 2$ belongs to integers with $\mathrm{u}_1 > \mathrm{u}_2$ and the dots stand for the summed or undefined terms. Binomial coefficients are described for $\mathrm{u}_1, \mathrm{u}_2 \in \mathbb{Z}$ by

$$\binom{\mathrm{u}_2}{\mathrm{u}_1} = \frac{\mathrm{u}_2!}{\mathrm{u}_1!(\mathrm{u}_2 - \mathrm{u}_1)!} \text{ whenever } \mathrm{u}_1 \leq \mathrm{u}_2, \binom{\mathrm{u}_2}{\mathrm{u}_1} = 0 \text{ otherwise.}$$

In order to ensure that all binomial coefficients appearing in this paper are well-defined, we adopt the generalized binomial coefficient defined for all real (or integer) arguments by

$$\binom{x}{y} := \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)},$$

whenever the righthand side is well-defined. In particular, this definition extends the classical binomial coefficient to cases where the upper argument is not necessarily greater than or equal to the lower argument, which is essential for the validity of the series defining $\mathfrak{e}_m(\lambda, \mu; k)$.

In addition, for consistency with the discrete structure used in this paper, we adopt the following conventions for binomial coefficients with negative indices: $\binom{-1}{-1} := 1$, $\binom{-1}{0} = 1$, while $\binom{m}{-1} = 0$ for all $m \geq 0$.

Definition 1.1. (Generalized rising function [9, Definition 3.4]) The generalized rising function is defined by

$$k^{\overline{r}} := \frac{\Gamma(k+r)}{\Gamma(k)},$$

for all admissible k and r.

Definition 1.2. (Nabla fractional Taylor monomial [9, Definition 3.56]) For $\beta \in \{-1, -2, \ldots\}$, define

$$H_\beta(k, 0) := \frac{k^{\overline{\beta}}}{\Gamma(\beta+1)}.$$

Definition 1.3. (Discrete Mittag–Leffler function [9, Definition 3.98]) For $|p| < 1$, $\alpha > 0$, and $\beta \in \mathbb{R}$, define

$$E_{p,\alpha,\beta}(k) = \sum_{n=0}^{\infty} p^n H_{\alpha n+\beta}(k, 0),$$

or, equivalently,

$$E_{p,\alpha,\beta}(k) = \sum_{n=0}^{\infty} p^n \binom{k + \alpha n + \beta - 1}{\alpha n + \beta}.$$

Lemma 1.4. *[10, Lemma 3.14] The following double-series identity holds:*

$$\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\binom{i_1+i_2}{i_2}b_{i_1,i_2} = b_{0,0} + \sum_{i_1=0}^{\infty}\sum_{i_2=1}^{\infty}\binom{i_1+i_2-1}{i_2-1}b_{i_1,i_2} + \sum_{i_1=1}^{\infty}\sum_{i_2=0}^{\infty}\binom{i_1+i_2-1}{i_2}b_{i_1,i_2},$$

where $b_{i_1,i_2} \in \mathbb{R}$ *for all* $i_1, i_2 \in \mathbb{N}_0$.

This identity given in Lemma 1.4 is used to decompose double sums and to perform index shifting in the proofs.

Theorem 1.5. *[11, Theorem 1] Assume that* $z : \mathbb{N}_a \times \mathbb{N}_{a+1} \to \mathbb{R}$. *Then, for all* $k \in \mathbb{N}_{a+1}$,

$$\nabla\left(\int_a^k z(k,s)\nabla s\right) = \int_a^k \nabla_k z(k,s)\nabla s + z(\rho(k),k),$$

where $\rho(k) = k-1$ *denotes the backward jump operator.*

2. Solution to linear homogeneous initial value problem (1.1) and (1.2)

In this section, we first define the novel discrete exponential function generated by the system coefficients λ and μ which contains the classical discrete exponential (Mittag-Leffler) function. Some properties are discussed and the solution to the linear homogeneous initial value problem is presented via the novel discrete exponential function.

Definition 2.1. Let $m \in \mathbb{N}_0$. The discrete exponential function $\mathfrak{e}_m : \mathcal{D}\times\mathbb{N}_0 \to \mathbb{R}$, $\mathcal{D} := \{(\lambda,\mu) \in \mathbb{R}^2 : |\lambda| + |\mu| < 1\}$, generated by the parameters λ and μ, is defined by

$$\mathfrak{e}_m(\lambda,\mu;k) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\binom{k+2i_1+i_2+m-1}{2i_1+i_2+m}\binom{i_1+i_2}{i_2}\lambda^{i_1}\mu^{i_2}, \quad k \in \mathbb{N}_0.$$

Remark 1. Throughout this paper, binomial coefficients are understood in the generalized sense whenever necessary, allowing for consistent use even when the upper argument is smaller than the lower argument. Under the condition $|\lambda|+|\mu| < 1$, the series representation is absolutely convergent, ensuring numerical stability in practical implementations.

Remark 2. From Definition 2.1, we compute the value at $k = 0$. For $m \geq 1$, since

$$\binom{2i_1+i_2+m-1}{2i_1+i_2+m} = 0,$$

it follows that $\mathfrak{e}_m(0) = 0$. For $m = 0$, only the term $(i_1, i_2) = (0, 0)$ contributes, and, hence, $\mathfrak{e}_0(0) = 1$.

For simplicity, from this point on we use the notation $\mathfrak{e}_m(\mathrm{k})$ instead of $\mathfrak{e}_m(\lambda,\mu;\mathrm{k})$.

Remark 3. The discrete exponential function $\mathfrak{e}_m(\lambda,\mu;\mathrm{k})$ with $\lambda = 0$ and $m = 1$ reduces to the classical discrete exponential (Mittag–Leffler) function given in Definition 1.3, i.e., $\mathfrak{e}_1(0,\mu;\mathrm{k}) = E_{\mu,1,1}(\mathrm{k},0)$.

In contrast to classical discrete Mittag–Leffler–type functions, which are typically generated by a single-index structure, the function $\mathfrak{e}_m(\lambda,\mu;\mathrm{k})$ is defined through a coupled double-index representation. This induces a different binomial kernel and allows the simultaneous incorporation of two system parameters, providing a richer structural framework than standard discrete Mittag–Leffler functions.

Remark 4. To justify that $\mathfrak{e}_m(\lambda, \mu; k)$ is well-defined, we establish absolute convergence of the defining double series. We first estimate

$$\mathfrak{e}_m(\lambda, \mu; k) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \binom{k + 2i_1 + i_2 + m - 1}{2i_1 + i_2 + m} \binom{i_1 + i_2}{i_2} \lambda^{i_1} \mu^{i_2}$$
$$\leq \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \binom{k + 2i_1 + 2i_2 + m - 1}{2i_1 + 2i_2 + m} \binom{i_1 + i_2}{i_2} \lambda^{i_1} \mu^{i_2}.$$

Indeed, since for fixed r the binomial coefficient $\binom{n}{r}$ is increasing in n, and since $2i_1 + i_2 \leq 2(i_1 + i_2)$, we obtain

$$\binom{k + 2i_1 + i_2 + m - 1}{2i_1 + i_2 + m} = \binom{k + 2i_1 + i_2 + m - 1}{k - 1} \leq \binom{k + 2(i_1 + i_2) + m - 1}{k - 1}.$$

Next, we regroup the series by setting $n = i_1 + i_2$. Note that all terms in the above double series are nonnegative after taking absolute values. Therefore, the reindexing

$$(i_1, i_2) \mapsto (n, i_2), \qquad n = i_1 + i_2, \quad 0 \leq i_2 \leq n,$$

is justified, since it defines a bijection between $\mathbb{N}_0 \times \mathbb{N}_0 := {\mathbb{N}_0}^2$ and the set $\{(n, i_2) \in {\mathbb{N}_0}^2 : \ 0 \leq i_2 \leq n\}$, and rearrangement of nonnegative terms does not affect convergence. Hence, the series can be written as

$$\mathfrak{e}_m(\lambda, \mu; k) \leq \sum_{n=0}^{\infty} \sum_{i_1+i_2=n} \binom{k + 2n + m - 1}{2n + m} \binom{n}{i_2} \lambda^{i_1} \mu^{i_2}$$
$$= \sum_{n=0}^{\infty} \binom{k + 2n + m - 1}{2n + m} \sum_{i_2=0}^{n} \binom{n}{i_2} \lambda^{n-i_2} \mu^{i_2}.$$

By the binomial theorem,

$$\sum_{i_2=0}^{n} \binom{n}{i_2} \lambda^{n-i_2} \mu^{i_2} = (\lambda + \mu)^n,$$

and, hence,

$$\mathfrak{e}_m(\lambda, \mu; k) \leq \sum_{n=0}^{\infty} \binom{k + 2n + m - 1}{2n + m} (\lambda + \mu)^n.$$

Taking absolute values yields

$$|\mathfrak{e}_m(\lambda, \mu; k)| \leq \sum_{n=0}^{\infty} \binom{k + 2n + m - 1}{2n + m} (|\lambda| + |\mu|)^n.$$

By choosing $\alpha = 2$ and $\beta = m - 1$ in [9, Theorem 3.97] and Definition 1.3, and assuming

$$|\lambda| + |\mu| < 1,$$

the above series converges. Therefore, the function $\mathfrak{e}_m(\lambda, \mu; k)$ is well-defined for all $(\lambda, \mu) \in \mathcal{D}$ and $k \in \mathbb{N}_0$. We note that all binomial coefficients appearing in Definition 2.1 are interpreted in the generalized sense, ensuring that each term in the series is well-defined.

Lemma 2.2. *For $m, n \in \mathbb{N}_0 := \mathbb{N}$, and $\nabla^n = \nabla^{n-1}(\nabla) = \nabla(\nabla^{n-1})$, $n = 0, 1, 2, \ldots$, one has*

$$\nabla^n \mathfrak{e}_m(\mathrm{k}) = \mathfrak{e}_{m-n}(\mathrm{k}).$$

Proof. It is well-known that $\nabla\binom{\mathrm{k}}{m} = \binom{\mathrm{k}}{m} - \binom{\mathrm{k}-1}{m} = \binom{\mathrm{k}-1}{m-1}$, which follows from standard properties of binomial coefficients. We prove the result by mathematical induction on n. For the base cases $n = 0$ and $n = 1$, the identity follows directly from the definition of $\mathfrak{e}_m(\mathrm{k})$ and the above relation. Assume that the statement holds for some $n \in \mathbb{N}_0$, that is,

$$\nabla^n \mathfrak{e}_m(\mathrm{k}) = \mathfrak{e}_{m-n}(\mathrm{k}).$$

Then,

$$\begin{aligned}
\nabla^{n+1}\mathfrak{e}_m(\mathrm{k}) &= \nabla(\nabla^n \mathfrak{e}_m(\mathrm{k})) = \nabla \mathfrak{e}_{m-n}(\mathrm{k}) \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \nabla\binom{\mathrm{k} + 2i_1 + i_2 + m - n - 1}{2i_1 + i_2 + m - n}\binom{i_1 + i_2}{i_2}\lambda^{i_1}\mu^{i_2} \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \binom{\mathrm{k} + 2i_1 + i_2 + m - n - 2}{2i_1 + i_2 + m - n - 1}\binom{i_1 + i_2}{i_2}\lambda^{i_1}\mu^{i_2} \\
&= \mathfrak{e}_{m-(n+1)}(\mathrm{k}).
\end{aligned}$$

Thus, the result follows for $n + 1$, and the proof is complete.

It is time to present the first main theorem, which expresses a representation of a solution to the linear homogeneous initial value problem (1.1) and (1.2).

Theorem 2.3. *The subsequent function* $\mathrm{u}(\mathrm{k}) = \mathrm{u}_0 + \lambda\mathfrak{e}_2(\mathrm{k})\mathrm{u}_0 + \mathfrak{e}_1(\mathrm{k})\nabla\mathrm{u}_0$ *solves the linear homogeneous initial value problem (1.1) and (1.2).*

Proof. It follows from Lemma 2.2 that $\nabla\mathfrak{e}_2(\mathrm{k}) = \mathfrak{e}_1(\mathrm{k})$ and $\nabla\mathfrak{e}_1(\mathrm{k}) = \mathfrak{e}_0(\mathrm{k})$. For clarity, we explicitly describe the index transformations used below. In particular, we employ the identity

$$\binom{i_1 + i_2}{i_2} = \binom{i_1 + i_2 - 1}{i_2 - 1} + \binom{i_1 + i_2 - 1}{i_2},$$

and the corresponding index shifts $i_2 \mapsto i_2 + 1$ and $i_1 \mapsto i_1 + 1$ in order to rewrite the sums with indices starting from zero.

$$\begin{aligned}
\nabla^2\mathrm{u}(\mathrm{k}) &= \nabla^2(\mathrm{u}_0 + \lambda\mathfrak{e}_2(\mathrm{k})\mathrm{u}_0 + \mathfrak{e}_1(\mathrm{k})\nabla\mathrm{u}_0) \\
&= \lambda\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \nabla^2\binom{\mathrm{k} + 2i_1 + i_2 + 1}{2i_1 + i_2 + 2}\binom{i_1 + i_2}{i_2}\lambda^{i_1}\mu^{i_2}\mathrm{u}_0 \\
&\quad + \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \nabla^2\binom{\mathrm{k} + 2i_1 + i_2}{2i_1 + i_2 + 1}\binom{i_1 + i_2}{i_2}\lambda^{i_1}\mu^{i_2}\nabla\mathrm{u}_0.
\end{aligned}$$

Applying the above decomposition separately to both double sums, and then shifting the indices accordingly, and using Lemma 1.4, which gives the corresponding double-series identity, we obtain

$$\begin{aligned}\nabla^2 \mathrm{u}(\mathrm{k}) = \lambda \sum_{i_1=0}^{\infty}\sum_{i_2=1}^{\infty} \nabla^2 \binom{\mathrm{k}+2i_1+i_2+1}{2i_1+i_2+2}\binom{i_1+i_2-1}{i_2-1}\lambda^{i_1}\mu^{i_2}\mathrm{u}_0 \\ + \lambda \sum_{i_1=1}^{\infty}\sum_{i_2=0}^{\infty} \nabla^2 \binom{\mathrm{k}+2i_1+i_2+1}{2i_1+i_2+2}\binom{i_1+i_2-1}{i_2}\lambda^{i_1}\mu^{i_2}\mathrm{u}_0 + \lambda\nabla^2\binom{k+1}{2}\mathrm{u}_0 \\ + \sum_{i_1=0}^{\infty}\sum_{i_2=1}^{\infty} \nabla^2 \binom{\mathrm{k}+2i_1+i_2}{2i_1+i_2+1}\binom{i_1+i_2-1}{i_2-1}\lambda^{i_1}\mu^{i_2}\nabla\mathrm{u}_0 \\ + \sum_{i_1=1}^{\infty}\sum_{i_2=0}^{\infty} \nabla^2 \binom{\mathrm{k}+2i_1+i_2}{2i_1+i_2+1}\binom{i_1+i_2-1}{i_2}\lambda^{i_1}\mu^{i_2}\nabla\mathrm{u}_0 + \nabla^2\binom{k}{1}\nabla\mathrm{u}_0.\end{aligned}$$

After reindexing, all sums are rewritten so that both indices start again from zero. Then, one gets

$$\begin{aligned}\nabla^2 \mathrm{u}(\mathrm{k}) = \lambda\mu \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \binom{\mathrm{k}+2i_1+i_2}{2i_1+i_2+1}\binom{i_1+i_2}{i_2}\lambda^{i_1}\mu^{i_2}\mathrm{u}_0 \\ + \lambda^2 \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \binom{\mathrm{k}+2i_1+i_2+1}{2i_1+i_2+2}\binom{i_1+i_2}{i_2}\lambda^{i_1}\mu^{i_2}\mathrm{u}_0 + \lambda\mathrm{u}_0 \\ + \mu \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \binom{\mathrm{k}+2i_1+i_2-1}{2i_1+i_2}\binom{i_1+i_2}{i_2}\lambda^{i_1}\mu^{i_2}\nabla\mathrm{u}_0 \\ + \lambda \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \binom{\mathrm{k}+2i_1+i_2}{2i_1+i_2+1}\binom{i_1+i_2}{i_2}\lambda^{i_1}\mu^{i_2}\nabla\mathrm{u}_0.\end{aligned}$$

Using the definitions of $\mathfrak{e}_0(\mathrm{k})$, $\mathfrak{e}_1(\mathrm{k})$, and $\mathfrak{e}_2(\mathrm{k})$, the above sums can be rewritten in terms of these functions.

$$\begin{aligned}\nabla^2 \mathrm{u}(\mathrm{k}) &= \lambda\mathrm{u}_0 + \lambda\mu\mathfrak{e}_1(\mathrm{k})\mathrm{u}_0 + \lambda^2\mathfrak{e}_2(\mathrm{k})\mathrm{u}_0 + \mu\mathfrak{e}_0(\mathrm{k})\nabla\mathrm{u}_0 + \lambda\mathfrak{e}_1(\mathrm{k})\nabla\mathrm{u}_0 \\ &= \lambda\mathrm{u}_0 + \lambda\mu\nabla\mathfrak{e}_2(\mathrm{k})\mathrm{u}_0 + \lambda^2\mathfrak{e}_2(\mathrm{k})\mathrm{u}_0 + \mu\nabla\mathfrak{e}_1(\mathrm{k})\nabla\mathrm{u}_0 + \lambda\mathfrak{e}_1(\mathrm{k})\nabla\mathrm{u}_0 \\ &= \lambda[\mathrm{u}_0 + \lambda\mathfrak{e}_2(\mathrm{k})\mathrm{u}_0 + \mathfrak{e}_1(\mathrm{k})\nabla\mathrm{u}_0] + \mu\nabla[\mathrm{u}_0 + \lambda\mathfrak{e}_2(\mathrm{k})\mathrm{u}_0 + \mathfrak{e}_1(\mathrm{k})\nabla\mathrm{u}_0] \\ &= \lambda\mathrm{u}(\mathrm{k}) + \mu\nabla\mathrm{u}(\mathrm{k}).\end{aligned}$$

The initial conditions in (1.2) are satisfied by using Remark 2. In particular, evaluating the representation at $k = 0$ and using $\mathfrak{e}_1(0) = 0$ and $\mathfrak{e}_0(0) = 1$, we obtain

$$\mathrm{u}(0) = \mathrm{u}_0.$$

Moreover, applying the nabla operator and evaluating at $k = 0$ yields

$$\nabla\mathrm{u}(0) = \nabla\mathrm{u}_0,$$

which confirms that both initial conditions are satisfied. This concludes the proof.

3. Solution to linear nonhomogeneous initial value problem (1.1) and (1.3)

In this section, we present a lemma which is used in the sequent second main theorem. Then, we offer a representation of a solution to linear nonhomogeneous initial value problem (1.1) and (1.3) with $f(\mathrm{k}) \neq 0$ and zero initial conditions. Subsequently, the whole solution to the linear nonhomogeneous Langevin system is presented in the next theorem. Next, we discuss some special cases.

Lemma 3.1. *The following equation always holds*

$$\nabla^2 \mathrm{e}_1(\mathrm{k}) - \mu \nabla \mathrm{e}_1(\mathrm{k}) - \lambda \mathrm{e}_1(\mathrm{k}) = \binom{\mathrm{k}-2}{-1}, \quad \mathrm{k} \in \mathbb{N}_0.$$

Proof. We explicitly indicate the index decomposition used below. In particular, we use the identity

$$\binom{i_1+i_2}{i_2} = \binom{i_1+i_2-1}{i_2-1} + \binom{i_1+i_2-1}{i_2},$$

together with the index shifts $i_2 \mapsto i_2+1$ and $i_1 \mapsto i_1+1$.

$$\nabla^2 \mathrm{e}_1(\mathrm{k}) = \nabla^2 \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \binom{\mathrm{k}+2i_1+i_2}{2i_1+i_2+1} \binom{i_1+i_2}{i_2} \lambda^{i_1} \mu^{i_2}.$$

By applying the above decomposition to each of the double sums separately, then appropriately shifting the indices, and invoking the corresponding double-series identity given in Lemma 1.4, we obtain

$$\begin{aligned}
\nabla^2 \mathrm{e}_1(\mathrm{k}) &= \sum_{i_1=0}^{\infty} \sum_{i_2=1}^{\infty} \nabla^2 \binom{\mathrm{k}+2i_1+i_2}{2i_1+i_2+1} \binom{i_1+i_2-1}{i_2-1} \lambda^{i_1} \mu^{i_2} \\
&\quad + \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{\infty} \nabla^2 \binom{\mathrm{k}+2i_1+i_2}{2i_1+i_2+1} \binom{i_1+i_2-1}{i_2} \lambda^{i_1} \mu^{i_2} + \nabla^2 \binom{k}{1} \\
&= \sum_{i_1=0}^{\infty} \sum_{i_2=1}^{\infty} \binom{\mathrm{k}+2i_1+i_2-2}{2i_1+i_2-1} \binom{i_1+i_2-1}{i_2-1} \lambda^{i_1} \mu^{i_2} \\
&\quad + \sum_{i_1=1}^{\infty} \sum_{i_2=0}^{\infty} \binom{\mathrm{k}+2i_1+i_2-2}{2i_1+i_2-1} \binom{i_1+i_2-1}{i_2} \lambda^{i_1} \mu^{i_2} + \nabla \binom{k-1}{0}.
\end{aligned}$$

After reindexing, all sums are rewritten so that both indices start again from zero. Then, we acquire

$$\begin{aligned}
\nabla^2 \mathrm{e}_1(\mathrm{k}) &= \mu \sum_{i_1=0}^{\infty} \sum_{i_2=1}^{\infty} \binom{\mathrm{k}+2i_1+i_2-1}{2i_1+i_2} \binom{i_1+i_2}{i_2} \lambda^{i_1} \mu^{i_2} \\
&\quad + \lambda \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \binom{\mathrm{k}+2i_1+i_2}{2i_1+i_2+1} \binom{i_1+i_2}{i_2} \lambda^{i_1} \mu^{i_2} + \binom{k-2}{-1} \\
&= \mu \mathrm{e}_0(\mathrm{k}) + \lambda \mathrm{e}_1(\mathrm{k}) + \binom{k-2}{-1} \\
&= \mu \nabla \mathrm{e}_1(\mathrm{k}) + \lambda \mathrm{e}_1(\mathrm{k}) + \binom{k-2}{-1},
\end{aligned}$$

which completes the proof. Here, the term $\binom{k-2}{-1}$ is interpreted under the adopted convention. In particular,

$$\binom{k-2}{-1} = \begin{cases} 1, & k = 1, \\ 0, & k \neq 1, \end{cases}$$

so that this term acts as a discrete selection factor.

Theorem 3.2. *The function* $u(k) = \int_0^k e_1(k - \rho(s))f(s)\nabla s$*, where* $\rho(s) = s - 1$ *solves the linear nonhomogeneous Langevin system (1.1) and (1.3).*

Proof. Using Theorem 1.5, which expresses the discrete backward Leibniz formula, the following can be easily obtained:

$$\begin{aligned}
\nabla u(k) &= \nabla \int_0^k e_1(k - \rho(s))f(s)\nabla s \\
&= \int_0^k \nabla e_1(k - \rho(s))f(s)\nabla s + e_1(\rho(k) - \rho(k))f(k) \\
&= \int_0^k \nabla e_1(k - \rho(s))f(s)\nabla s,
\end{aligned}$$

and

$$\begin{aligned}
\nabla^2 u(k) &= \nabla(\nabla \int_0^k e_1(k - \rho(s))f(s)\nabla s) \\
&= \nabla \int_0^k \nabla e_1(k - \rho(s))f(s)\nabla s \\
&= \int_0^k \nabla^2 e_1(k - \rho(s))f(s)\nabla s + \nabla e_1(\rho(k) - \rho(k))f(k) \\
&= \int_0^k \nabla^2 e_1(k - \rho(s))f(s)\nabla s,
\end{aligned}$$

where $e_1(0) = 0$ has been used and calculated based on the combinatorial coefficients, which are contained in the definition of the discrete exponential. Then, by using Lemma 3.1, one readily gets

$$\begin{aligned}
\nabla^2 u(k) - \mu\nabla u(k) - \lambda u(k) &= \int_0^k [\nabla^2 e_1(k - \rho(s)) - \mu\nabla e_1(k - \rho(s)) - \lambda e_1(k - \rho(s))]f(s)\nabla s \\
&= \int_0^k \binom{k - \rho(s) - 2}{-1} f(s)\nabla s \\
&= \sum_{s=1}^{k} \binom{k - \rho(s) - 2}{-1} f(s) \\
&= \binom{k - \rho(k) - 2}{-1} f(k) + \sum_{s=1}^{k-1} \binom{k - \rho(s) - 2}{-1} f(s) \\
&= f(k),
\end{aligned}$$

because, by the convention $\binom{-1}{-1} = 1$ and $\binom{m}{-1} = 0$ for $m \geq 0$, the term $\binom{k-\rho(s)-2}{-1}$ selects only the contribution at $s = k$.

Theorem 3.3. *The representation of a solution to a class of linear nonhomogeneous discrete Langevin systems (1.1) and (1.2) is given by*

$$\mathrm{u}(\mathrm{k}) = \mathrm{u}_0 + \lambda \mathfrak{e}_2(\mathrm{k})\mathrm{u}_0 + \mathfrak{e}_1(\mathrm{k})\nabla \mathrm{u}_0 + \int_0^{\mathrm{k}} \mathfrak{e}_1(\mathrm{k} - \rho(s)) f(s) \nabla s.$$

Proof. The proof is an immediate result of Theorems 2.3 and 3.2.

Remark 5. If it is taken $\lambda = 0$ or $\mu = 0$, the corresponding systems are new, and so the corresponding discrete exponentials and solutions are also new.

Example 1. Consider the nonhomogeneous discrete Langevin system

$$\nabla^2 \mathrm{u}(\mathrm{k}) - \frac{1}{2}\nabla \mathrm{u}(\mathrm{k}) = 1, \qquad \mathrm{k} \in \mathbb{N}_0,$$

subject to the initial conditions

$$\mathrm{u}(0) = 1, \qquad \nabla \mathrm{u}(0) = 1.$$

Here, $\lambda = 0$, $\mu = \frac{1}{2}$, and $f(\mathrm{k}) \equiv 1$. By Theorem 3.3, the solution is represented as

$$\mathrm{u}(\mathrm{k}) = 1 + \mathfrak{e}_1\left(0, \tfrac{1}{2}; \mathrm{k}\right) + \int_0^{\mathrm{k}} \mathfrak{e}_1\left(0, \tfrac{1}{2}; \mathrm{k} - \rho(s)\right) \nabla s.$$

We now compute the discrete exponential explicitly. From Definition 2.1, for $m = 1$ and $\lambda = 0$,

$$\mathfrak{e}_1\left(0, \tfrac{1}{2}; \mathrm{k}\right) = \sum_{i_2=0}^{\infty} \binom{\mathrm{k} + i_2}{i_2 + 1} \left(\frac{1}{2}\right)^{i_2}.$$

Using the classical generating-function identity

$$\sum_{n=0}^{\infty} \binom{\mathrm{k} + n - 1}{n} x^n = (1 - x)^{-\mathrm{k}}, \qquad |x| < 1,$$

we obtain

$$\begin{aligned} \sum_{i_2=0}^{\infty} \binom{\mathrm{k} + i_2}{i_2 + 1} x^{i_2} &= \frac{1}{x}\left(\sum_{n=0}^{\infty} \binom{\mathrm{k} + n - 1}{n} x^n - 1\right) \\ &= \frac{(1 - x)^{-\mathrm{k}} - 1}{x}. \end{aligned}$$

Hence, by taking $x = \frac{1}{2}$,

$$\mathfrak{e}_1\left(0, \tfrac{1}{2}; \mathrm{k}\right) = 2\left(\left(1 - \frac{1}{2}\right)^{-\mathrm{k}} - 1\right) = 2^{\mathrm{k}+1} - 2.$$

Next, for the convolution term, since $\rho(s) = s - 1$,

$$\mathrm{k} - \rho(s) = \mathrm{k} - s + 1,$$

and, therefore,

$$\int_0^{k} \mathrm{e}_1\Big(0, \tfrac{1}{2}; k - \rho(s)\Big)\, \nabla s = \sum_{s=1}^{k} \mathrm{e}_1\Big(0, \tfrac{1}{2}; k - s + 1\Big) = \sum_{s=1}^{k} \Big(2^{k-s+2} - 2\Big).$$

Setting $r = k - s + 1$, this becomes

$$\sum_{r=1}^{k}(2^{r+1} - 2) = \sum_{r=1}^{k} 2^{r+1} - 2k = 2\sum_{r=1}^{k} 2^{r} - 2k = 2(2^{k+1} - 2) - 2k = 2^{k+2} - 4 - 2k.$$

Consequently,

$$u(k) = 1 + \Big(2^{k+1} - 2\Big) + \Big(2^{k+2} - 4 - 2k\Big) = 3 \cdot 2^{k+1} - 5 - 2k.$$

Thus, the exact solution of the problem is

$$u(k) = 3 \cdot 2^{k+1} - 5 - 2k, \qquad k \in \mathbb{N}_0.$$

This example shows that the representation formula can be evaluated explicitly and provides a concrete computational illustration of the obtained solution.

Example 2. Consider the nonhomogeneous discrete Langevin system

$$\nabla^2 u(k) - \frac{5}{6}\nabla u(k) + \frac{1}{6}u(k) = k, \qquad k \in \mathbb{N}_1,$$

with initial conditions

$$u(0) = 1, \qquad \nabla u(0) = 1.$$

By Theorem 3.3,

$$u(k) = 1 + \lambda \mathrm{e}_2(k) + \mathrm{e}_1(k) + \sum_{s=1}^{k} \mathrm{e}_1(k - s + 1)s.$$

Using the definition,

$$\mathrm{e}_1(k) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \binom{k + 2i_1 + i_2}{2i_1 + i_2 + 1}\binom{i_1 + i_2}{i_2}\left(\frac{1}{6}\right)^{i_1}\left(\frac{5}{6}\right)^{i_2}.$$

We compute the first few contributions. For $(i_1, i_2) = (0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, the corresponding terms are k, $\frac{1}{6}\binom{k+2}{3}$, $\frac{5}{6}\binom{k+1}{2}$, *and* $\frac{10}{36}\binom{k+3}{4}$. Thus,

$$\mathrm{e}_1(k) \approx k + \frac{1}{6}\binom{k + 2}{3} + \frac{5}{6}\binom{k + 1}{2} + \frac{10}{36}\binom{k + 3}{4}.$$

Similarly,

$$\mathrm{e}_2(k) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} \binom{k + 2i_1 + i_2 + 1}{2i_1 + i_2 + 2}\binom{i_1 + i_2}{i_2}\left(\frac{1}{6}\right)^{i_1}\left(\frac{5}{6}\right)^{i_2}.$$

For $(i_1, i_2) = (0,0), (1,0)$, and $(0,1)$, the corresponding ones are $\binom{k+1}{2}$, $\frac{1}{6}\binom{k+3}{4}$, and $\frac{5}{6}\binom{k+2}{3}$. Thus,

$$\mathrm{e}_2(k) \approx \binom{k + 1}{2} + \frac{1}{6}\binom{k + 3}{4} + \frac{5}{6}\binom{k + 2}{3}.$$

This example shows that the solution can be effectively approximated using the series definition of the discrete exponential functions even when both parameters are nonzero.

4. Conclusions and open problems

In this paper, a new class of linear discrete Langevin systems with two different integer orders is investigated. A new discrete exponential function generated by the system coefficients is defined to derive an explicit solution for both the homogeneous and inhomogeneous cases.

The obtained explicit representation provides a suitable framework for further analytical investigations. In particular, a natural direction is to derive necessary and sufficient conditions for the stability of the zero solution in terms of the system parameters λ and μ.

Moreover, the controllability and observability properties of the proposed discrete Langevin system can be studied by exploiting the structure of the discrete exponential function $\mathfrak{e}_m(\lambda, \mu; k)$.

Another promising direction is the extension of the present results to fractional discrete Langevin-type equations, where the integer-order difference operators are replaced by fractional-order operators. In this context, one may aim to construct explicit solution representations and investigate their qualitative properties.

Another important direction is to investigate practical implementations and numerical validation of the proposed model in applications such as stochastic simulation and control systems.

Use of AI tools declaration

No AI tools were used in the development of the scientific content of this paper; however, minor language and stylistic revisions were carried out.

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Conflict of interest

The author declares there is no conflicts of interest.

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