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Research article

On conditional regularity criteria for the 3D generalized MHD system via partial components

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Abstract: This paper focuses on the regularity criteria for the 3D generalized magnetohydrodynamic (MHD) equations, particularly those involving the partial components of the velocity u_3 and the magnetic field (b_1, b_2, b_3). For any $l, i, j, k \in \{1, 2, 3\}$, we establish regularity criteria via $\partial_l u_3$ and $(\partial_i b_1, \partial_j b_2, \partial_k b_3)$, u_3 and $(\partial_i b_1, \partial_j b_2, \partial_k b_3)$, or $\partial_l u_3$ and (b_1, b_2, b_3). Furthermore, a comprehensive regularity condition is obtained via $(\xi u_3, \eta b_1, \zeta b_2, \sigma b_3)$ and $(\xi' \partial_l u_3, \eta' \partial_i b_1, \zeta' \partial_j b_2, \sigma' \partial_k b_3)$ with $\xi + \xi' = \eta + \eta' = \zeta + \zeta' = \sigma + \sigma' = 1$ and $\xi, \eta, \zeta, \sigma, \xi', \eta', \zeta', \sigma' \in \{0, 1\}$.

Keywords: generalized MHD equations; regularity criterion; partial components; weak solutions

1. Introduction

This paper is concerned with the 3D generalized magnetohydrodynamic (MHD) equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - b \cdot \nabla b + \nabla P + (-\Delta)^{\alpha} u = 0, \\ \partial_t b + u \cdot \nabla b - b \cdot \nabla u + (-\Delta)^{\beta} b = 0, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ is the fluid velocity, $b = (b_1, b_2, b_3)$ the magnetic field, and $P(x, t)$ a scalar pressure. $\alpha, \beta \geq 1$ are real parameters, and the operator $(-\Delta)^{\kappa} (\kappa > 0)$ is defined by [1]

$$\widehat{(-\Delta)^{\kappa} f}(\xi) = | \xi |^{2\kappa} \widehat{f},$$

where $\widehat{f}$ denotes the Fourier transform of f. For convenience, we write $(-\Delta)^{1/2}$ as Λ.

The investigation of the generalized MHD equations (1.1) holds profound significance in both the fields of physics and mathematics. These equations are fundamental to understanding the large-scale behavior of electrically conductive, incompressible fluids under the influence of magnetic fields. In their versatile form, they not only capture the core principles of MHD but also seamlessly transform into the renowned Navier-Stokes equations under certain conditions, showcasing a harmonious connection between what may appear to be distinct physical laws. When $\alpha = \beta = 1$, the generalized MHD equations (1.1) reduce to the classic MHD equations. Likewise, by setting $\alpha = 1$ and $b = 0$, the equations transform into the esteemed Navier-Stokes equations. Furthermore, these three systems—the generalized MHD, the classic MHD, and the Navier-Stokes—share a profound similarity in their scaling properties and energy estimations. Consequently, the research on generalized MHD equations (1.1) can enhance and deepen our comprehension of the MHD equations and the Navier-Stokes equations.

For the 3D generalized MHD equations (1.1), Wu [2] established the global existence of a weak solution for any given initial data $(u_0, b_0) \in L^2(R^3)$ and proved that the weak solution is actually a classical solution when $\alpha,\beta \geq \frac{5}{4}$ and the initial data is sufficiently smooth. Furthermore, Wu [3, 4] obtained some regularity criteria only relying on the velocity u when $\alpha,\beta > \frac{1}{2} + \frac{3}{4}$. In [5], Zhou studied the following Serrin-type criteria involving the velocity u:

$$u \in L^p(0, T; L^q(\mathbb{R}^3)) \text{ with } \frac{2\alpha}{p} + \frac{3}{q} \leq 2\alpha - 1, \quad \frac{3}{2\alpha - 1} < q \leq \infty, \quad 1 \leq \alpha = \beta \leq \frac{3}{2}. \tag{1.2}$$

In [6], Yuan extended the condition (1.2) to the case

$$\begin{aligned} u \in L^p(0, T; B^s_{q,\infty}), \quad \text{with } \frac{2\alpha}{p} + \frac{3}{q} \leq 2\alpha - 1 + s, \quad \frac{3}{2\alpha - 1 + s} \leq q \leq \infty, \\ -1 < s < 1, \quad (q, s) \neq (\infty, 1), \quad 1 \leq \alpha = \beta \leq \frac{5}{4}. \end{aligned} \tag{1.3}$$

In [7], Jia considered the conditions on a mixed 3×3 velocity-magnetic gradient tensor $(M_{ij})_{3\times3}$, that is,

$$(M_{ij})_{3\times3} \in L^p(0, T; L^q(R^3)), \text{ with } \frac{2\alpha}{p} + \frac{3}{q} \leq 2\alpha, \ \frac{3}{2\alpha} \leq q \leq \infty, \ 1 \leq \alpha = \beta \leq \frac{5}{4}. \tag{1.4}$$

Recently, Wang et al. [8] improved the condition (1.2) to the one involving partial components of the velocity and the magnetic field:

$$u_3, b \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \text{with } \frac{2\alpha}{p} + \frac{3}{q} \leq \frac{3}{4}(2\alpha - 1) + \frac{3(1-\epsilon)}{4q}, \quad \frac{3+\epsilon}{2\alpha - 1} < q \leq \infty, \quad 0 < \epsilon \leq \frac{1}{3}. \tag{1.5}$$

In addition, readers are referred to a range of regularity criteria for more related models and richer function spaces, for instance the Fourier-Herz space [9], Lei-Lin-Gevrey and Lei-Lin spaces [10], and Besov spaces [11, 12], as well as the related literature [13–15].

The research methods employed in the study of MHD equations and generalized MHD equations exhibit a remarkable degree of similarity. In the extensive body of past research dedicated to MHD equations, the regularity conditions have typically encompassed a wide range of elements. These include the velocity, the partial derivatives of the velocity, and the partial derivatives of partial components of the velocity. Similarly, the magnetic field, which is an essential aspect of the MHD framework, along with its partial derivatives and the partial derivatives of its partial components, have been the focus of numerous studies. In some cases, the partial components of the velocity and the magnetic field have been jointly considered, as their interaction and combined effects are of particular interest.

Extensive studies on this topic can be found in the literature. Specifically, [16–18] establish regularity criteria for the MHD equations via one-directional derivatives of velocity or pressure; [19] considers partial components of velocity and magnetic fields; [20] provides criteria involving the velocity gradient, which is improved in [21] to mixed velocity-magnetic gradient combinations; and [22] obtains a regularity condition in anisotropic spaces for a single directional derivative of velocity. These works have contributed significantly to the understanding and development of the field. Motivated by the valuable findings and insights presented in these studies, especially those in [17, 18], the current paper undertakes an in-depth investigation into the regularity criteria for the 3D generalized MHD equations (1.1). Specifically, the focus is placed on the partial components of the velocity, and the magnetic field. Additionally, various combinations of their partial derivatives are carefully examined, as they are expected to yield important information regarding the regularity and behavior of the equations. For further results and a broader perspective on related topics, readers may refer to [23] on regularity criteria for the 3D magneto-micropolar fluid equations with fractional dissipation, [24, 25] for the 2D MHD equations, [26–28] for the 3D MHD equations, and [29] for the compressible 3D MHD equations, with more relevant studies available in [30, 31].

To understand the significance of the conclusions in this paper, we elaborate on the deeper motivation of the present research from both mathematical and physical perspectives. From a mathematical perspective, the key difference between the generalized MHD equations and the classical MHD equations lies in that their dissipative terms are fractional Laplacians ($(-\Delta)^{\alpha}$ and $(-\Delta)^{\beta}$), which results in more complex coupling behavior between the velocity field and the magnetic field. Choosing specific component combinations (such as the vertical component of the velocity field and certain components of the magnetic field) to perform regularity estimates allows a more refined exploitation of the cancellation effects among different components in the nonlinear terms. Compared with classical MHD equations, such refined analysis is more difficult to achieve in generalized MHD equations. From a physical perspective, the generalized MHD equations have important applications in fields such as plasma physics and astrophysics. In practical physical processes, the velocity or magnetic field often exhibits anisotropic structures (see [32, 33] for more studies on anisotropy). Therefore, the regularity criteria for specific components carry clear physical significance. In particular, regularity criteria based on combined partial derivatives of the velocity and magnetic fields precisely capture the coupling laws in the deformation and rotation behaviors of the velocity and magnetic fields. They reveal that the regularity of generalized MHD flows is determined by their mutual deformation-rotation interaction; for a more detailed discussion, we refer to [21]. Therefore, the regularity criteria for specific components proposed in this paper not only possess rigor in mathematical theory but also have clear practical application value.

Below, we present the main conclusions of this paper, which are the culmination of our extensive research and analysis.

Theorem 1.1 Let $1 \leq \alpha = \beta \leq \frac{3}{2}$. Assume that (u, b) is the local strong solution to (1.1) with initial data $(u_0, b_0) \in H^3(\mathbb{R}^3)$ and div $u_0 =$ div $b_0 = 0$. For any $l, i, j, k \in \{1, 2, 3\}$,

$$(\partial_l u_3, \partial_i b_1, \partial_j b_2, \partial_k b_3) \in L^w(0, T; L^s(\mathbb{R}^3)), \text{ with } \frac{2\alpha}{w} + \frac{3}{s} \le (1 - \frac{1}{4\alpha})(6\alpha - 5) + \frac{1}{s}(6 - 4\alpha - \frac{1}{2\alpha}),\ 2 < s \le \infty, \tag{1.6}$$

then the solution remains smooth on $[0, T]$.

Since the results below depend on Theorem 1.1 in [8], we restate it here for convenience.

Theorem 1.2 [8, Theorem 1.1] Let $1 \le \alpha = \beta \le \frac{3}{2}$. Assume that (u, b) is the local strong solution to (1.1) with initial data $(u_0, b_0) \in H^3(\mathbb{R}^3)$ and div $u_0 =$ div $b_0 = 0$. If

$$u_3, b \in L^w(0, T; L^q(\mathbb{R}^3)), \text{ with } \frac{2\alpha}{w} + \frac{3}{q} \le \frac{3}{4}(2\alpha - 1) + \frac{3(1 - \epsilon)}{4q}, \quad \frac{3 + \epsilon}{2\alpha - 1} < q \le \infty, \quad 0 < \epsilon \le \frac{1}{3}, \tag{1.7}$$

then the solution remains smooth on $[0, T]$.

Combining Theorems 1.1 and 1.2, we obtain the following two results.

Theorem 1.3 Let $1 \le \alpha = \beta \le \frac{3}{2}$. Assume that (u, b) is the local strong solution to (1.1) with initial data $(u_0, b_0) \in H^3(\mathbb{R}^3)$ and div $u_0 =$ div $b_0 = 0$. If

$$u_3 \in L^w(0, T; L^s(\mathbb{R}^3)), \quad \text{with } \frac{2\alpha}{w} + \frac{3}{s} \le \frac{3}{4}(2\alpha - 1) + \frac{3(1 - \epsilon)}{4s}, \quad \frac{3 + \epsilon}{2\alpha - 1} < s \le \infty, \quad 0 < \epsilon \le \frac{1}{3}, \tag{1.8}$$

and for any $i, j, k \in \{1, 2, 3\}$,

$$(\partial_i b_1, \partial_j b_2, \partial_k b_3) \in L^{w'}(0, T; L^{s'}(\mathbb{R}^3)), \text{ with } \frac{2\alpha}{w'} + \frac{3}{s'} \le (1 - \frac{1}{4\alpha})(6\alpha - 5) + \frac{1}{s'}(6 - 4\alpha - \frac{1}{2\alpha}),\ 2 < s' \le \infty, \tag{1.9}$$

then the solution remains smooth on $[0, T]$.

Theorem 1.4 Let $1 \le \alpha = \beta \le \frac{3}{2}$. Assume that (u, b) is the local strong solution to (1.1) with initial data $(u_0, b_0) \in H^3(\mathbb{R}^3)$ and div $u_0 =$ div $b_0 = 0$. For any $l \in \{1, 2, 3\}$,

$$\partial_l u_3 \in L^w(0, T; L^s(\mathbb{R}^3)), \text{ with } \frac{2\alpha}{w} + \frac{3}{s} \le (1 - \frac{1}{4\alpha})(6\alpha - 5) + \frac{1}{s}(6 - 4\alpha - \frac{1}{2\alpha}),\ 2 < s \le \infty, \tag{1.10}$$

and

$$b \in L^{w'}(0, T; L^{s'}(\mathbb{R}^3)), \quad \text{with } \frac{2\alpha}{w'} + \frac{3}{s'} \le \frac{3}{4}(2\alpha - 1) + \frac{3(1 - \epsilon)}{4s'}, \quad \frac{3 + \epsilon}{2\alpha - 1} < s' \le \infty, \quad 0 < \epsilon \le \frac{1}{3}, \tag{1.11}$$

then the solution remains smooth on $[0, T]$.

Furthermore, using Theorems 1.1–1.4, we derive the following comprehensive result.

Theorem 1.5 Let $1 \le \alpha = \beta \le \frac{3}{2}$. Assume that (u, b) is the local strong solution to (1.1) with initial data $(u_0, b_0) \in H^3(\mathbb{R}^3)$ and div $u_0 =$ div $b_0 = 0$. If for any $l, i, j, k \in \{1, 2, 3\}$, the following two conditions are satisfied:

$$\begin{aligned} &(i)\ A^{\xi,\eta,\zeta,\sigma} \in L^w(0, T; L^s(\mathbb{R}^3)), \text{ with } \frac{2\alpha}{w} + \frac{3}{s} \le \frac{3}{4}(2\alpha - 1) + \frac{3(1 - \epsilon)}{4s}, \quad \frac{3 + \epsilon}{2\alpha - 1} < s \le \infty,\ 0 < \epsilon \le \frac{1}{3}; \\ &(ii)\ B^{\xi',\eta',\zeta',\sigma'} \in L^{w'}(0, T; L^{s'}(\mathbb{R}^3)), \text{ with } \frac{2\alpha}{w'} + \frac{3}{s'} \le (1 - \frac{1}{4\alpha})(6\alpha - 5) + \frac{1}{s'}(6 - 4\alpha - \frac{1}{2\alpha}),\ 2 < s' \le \infty, \end{aligned} \tag{1.12}$$

where

$$A^{\xi,\eta,\zeta,\sigma} = (\xi u_3, \eta b_1, \zeta b_2, \sigma b_3), \quad B^{\xi',\eta',\zeta',\sigma'} = (\xi'\partial_l u_3, \eta'\partial_i b_1, \zeta'\partial_j b_2, \sigma'\partial_k b_3), \tag{1.13}$$

and $\xi + \xi' = \eta + \eta' = \zeta + \zeta' = \sigma + \sigma' = 1$, $(\xi, \eta, \zeta, \sigma, \xi', \eta', \zeta', \sigma' \in \{0, 1\})$. Then the solution remains smooth on $[0, T]$.

Remark 1.6 In Theorem 1.5, when $\xi = \eta = \zeta = \sigma = 1, \xi' = \eta' = \zeta' = \sigma' = 0$, this case corresponds to the main conclusion, Theorem 1.1, in [8]. In other words, the conclusion in [8] is a special case of Theorem 1.5, and Theorem 1.5 generalizes the result of [8].

Remark 1.7 In Theorems 1.1–1.5, the results remain valid if u_3 is replaced by either u_1 or u_2 due to the rotation invariance of the MHD system. The choice of u_3 is only for simplicity of presentation and does not affect the generality of the results.

2. Proofs

In this section, we prove Theorems 1.1, and 1.3–1.5. For convenience, we first provide some notations:

$$\nabla_h = (\partial_1, \partial_2), \ \ \Delta_h = \partial_1^2 + \partial_2^2,$$

and

$$\Pi^2(t) = \sup_{0\le\tau\le t}(\|\nabla_h u(\tau)\|_{L^2}^2 + \|\nabla_h b(\tau)\|_{L^2}^2) + \int_0^t (\|\nabla_h \Lambda^\alpha u\|_{L^2}^2 + \|\nabla_h \Lambda^\alpha b\|_{L^2}^2)\mathrm{d}\tau. \tag{2.1}$$

In addition, we present two inequalities required for the subsequent proof:

1) 3D multiplicative Sobolev inequality [34]:

$$\|\nabla u\|_{L^6} \le C\|\partial_1 \nabla u\|_{L^2}^{\frac{1}{3}} \|\partial_2 \nabla u\|_{L^2}^{\frac{1}{3}} \|\partial_3 \nabla u\|_{L^2}^{\frac{1}{3}}. \tag{2.2}$$

2) Interpolation inequality [35–37]: For $f,\ g,\ h \in C_c^\infty(R^3)$, we have

$$|\int_{\mathbb{R}^3} fgh| \le c\|f\|_{L^q}^{\frac{\gamma-1}{\gamma}} \|\partial_i f\|_{L^s}^{\frac{1}{\gamma}} \|g\|_{L^2}^{\frac{\gamma-2}{\gamma}} \|\partial_j g\|_{L^2}^{\frac{1}{\gamma}} \|\partial_k g\|_{L^2}^{\frac{1}{\gamma}} \|h\|_{L^2}, \tag{2.3}$$

where

$$\gamma > 2, \ \ 1 \le q, s \le \infty, \ \ \frac{\gamma - 1}{q} + \frac{1}{s} = 1, \tag{2.4}$$

and $i, j, k \in \{1, 2, 3\}$.

2.1. *Proof of Theorem 1.1*

We first consider the case $s < \infty$ and prove Theorem 1.1 in two steps.

Step 1. We first estimate $\Pi^2(t)$, which plays an important role in the proof of the main theorem.

Multiplying the first two equations of (1.1) by $\Delta_h u$ and $\Delta_h b$, respectively, adding them together, then using integration by parts and noticing the divergence-free condition, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h b\|_{L^2}^2) + \|\nabla_h \Lambda^\alpha u\|_{L^2}^2 + \|\nabla_h \Lambda^\alpha b\|_{L^2}^2 \\
&= \int_{R^3} [(u \cdot \nabla u) \cdot \Delta_h u - (b \cdot \nabla b) \cdot \Delta_h u + (u \cdot \nabla b) \cdot \Delta_h b - (b \cdot \nabla u) \cdot \Delta_h b] \,\mathrm{d}x
\end{aligned}$$

$$=K_1 + K_2 + K_3 + K_4. \tag{2.5}$$

Then we estimate $K_i (i = 1, 2, 3, 4)$ one by one. From [8], we have

$$|K_1| \leq C \int_{R^3} |u_3| \cdot |\nabla u| \cdot |\nabla_h \nabla u| \,\mathrm{d}x, \tag{2.6}$$

and

$$|K_2 + K_3 + K_4| \leq C \int_{R^3} |b| \cdot (|\nabla u| + |\nabla b|) \cdot (|\nabla_h \nabla u| + |\nabla_h \nabla b|) \,\mathrm{d}x. \tag{2.7}$$

For K_1, we can obtain

$$\begin{aligned}
|K_1| \leq & C \int_{R^3} |u_3| \cdot |\nabla u| \cdot |\nabla_h \nabla u| \,\mathrm{d}x \\
\leq & C\|u_3\|_{L^2}^{\frac{2s-2}{3s-2}} \|\partial_l u_3\|_{L^s}^{\frac{s}{3s-2}} \|\nabla u\|_{L^2}^{\frac{s-2}{3s-2}} \|\partial_j \nabla u\|_{L^2}^{\frac{s}{3s-2}} \|\partial_k \nabla u\|_{L^2}^{\frac{s}{3s-2}} \|\nabla_h \nabla u\|_{L^2} \\
& \text{(Interpolation inequality (2.3): } \tfrac{\frac{3s-2}{s}-1}{2} + \tfrac{1}{s} = 1,\ l \in \{1,2,3\}, \text{ and } j,k \in \{1,2\}) \\
\leq & C\|\partial_l u_3\|_{L^s}^{\frac{s}{3s-2}} \|\nabla u\|_{L^2}^{\frac{s-2}{3s-2}} \|\nabla_h \nabla u\|_{L^2}^{\frac{5s-2}{3s-2}} \\
\leq & C\|\partial_l u_3\|_{L^s}^{\frac{s}{3s-2}} \|\nabla u\|_{L^2}^{\frac{s-2}{3s-2}} \|\nabla u\|_{L^2}^{\frac{(\alpha-1)(5s-2)}{\alpha(3s-2)}} \|\nabla_h \Lambda^\alpha u\|_{L^2}^{\frac{5s-2}{\alpha(3s-2)}} \\
& \text{(the Gagliardo-Nirenberg inequality: } 2 - \tfrac{3}{2} = (1 - \tfrac{3}{2})\tfrac{\alpha-1}{\alpha} + ((1+\alpha) - \tfrac{3}{2})\tfrac{1}{\alpha}) \\
\leq & C\|\partial_l u_3\|_{L^s}^{\frac{s}{3s-2}} \|\nabla u\|_{L^2}^{\frac{2\alpha(3s-2)-(5s-2)}{\alpha(3s-2)}} \|\nabla_h \Lambda^\alpha u\|_{L^2}^{\frac{5s-2}{\alpha(3s-2)}} \\
\leq & C\|\partial_l u_3\|_{L^s}^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}} \|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\nabla_h \Lambda^\alpha u\|_{L^2}^2 \qquad (2.8) \\
& \text{(Young's inequality: } \tfrac{1}{2/(\frac{2\alpha(3s-2)-(5s-2)}{\alpha(3s-2)})} + \tfrac{1}{2/(\frac{5s-2}{\alpha(3s-2)})} = 1).
\end{aligned}$$

For $K_2 + K_3 + K_4$, we divide it into three terms to estimate and obtain that

$$\begin{aligned}
& |K_2 + K_3 + K_4| \\
\leq & C \int_{R^3} |b| \cdot (|\nabla u| + |\nabla b|) \cdot (|\nabla_h \nabla u| + |\nabla_h \nabla b|) \,\mathrm{d}x \\
\leq & C \int_{R^3} (|b_1| + |b_2| + |b_3|) \cdot (|\nabla u| + |\nabla b|) \cdot (|\nabla_h \nabla u| + |\nabla_h \nabla b|) \,\mathrm{d}x \\
\leq & C \int_{R^3} |b_1| \cdot (|\nabla u| + |\nabla b|) \cdot (|\nabla_h \nabla u| + |\nabla_h \nabla b|) \,\mathrm{d}x + C \int_{R^3} |b_2| \cdot (|\nabla u| + |\nabla b|) \cdot (|\nabla_h \nabla u| + |\nabla_h \nabla b|) \,\mathrm{d}x \\
& + C \int_{R^3} |b_3| \cdot (|\nabla u| + |\nabla b|) \cdot (|\nabla_h \nabla u| + |\nabla_h \nabla b|) \,\mathrm{d}x \\
\triangleq & I + II + III. \qquad (2.9)
\end{aligned}$$

We calculate I and get that

$$\begin{aligned}
|I| =& C\int_{R^3}|b_1|\cdot(|\nabla u|+|\nabla b|)\cdot(|\nabla_h\nabla u|+|\nabla_h\nabla b|)\,\mathrm{d}x\\
\le& C\|b_1\|_{L^2}^{\frac{2s-2}{3s-2}}\|\partial_i b_1\|_{L^s}^{\frac{s}{3s-2}}\||\nabla u|+|\nabla b|\|_{L^2}^{\frac{s-2}{3s-2}}\|\partial_j(|\nabla u|+|\nabla b|)\|_{L^2}^{\frac{s}{3s-2}}\|\partial_k(|\nabla u|+|\nabla b|)\|_{L^2}^{\frac{s}{3s-2}}\\
&\times\||\nabla_h\nabla u|+|\nabla_h\nabla b|\|_{L^2}\\
&\qquad\text{(Interpolation inequality (2.3): } \frac{\frac{3s-2}{s}-1}{2}+\frac{1}{s}=1,\ i\in\{1,2,3\}, \text{ and } j,k\in\{1,2\})\\
\le& C\|\partial_i b_1\|_{L^s}^{\frac{s}{3s-2}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^{\frac{s-2}{3s-2}}(\|\nabla_h\nabla u\|_{L^2}+\|\nabla_h\nabla b\|_{L^2})^{\frac{5s-2}{3s-2}}\\
\le& C\|\partial_i b_1\|_{L^s}^{\frac{s}{3s-2}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^{\frac{s-2}{3s-2}}(\|\nabla u\|_{L^2}^{1-\frac{1}{\alpha}}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{\frac{1}{\alpha}}+\|\nabla b\|_{L^2}^{1-\frac{1}{\alpha}}\|\nabla_h\Lambda^\alpha b\|_{L^2}^{\frac{1}{\alpha}})^{\frac{5s-2}{3s-2}}\\
&\qquad\text{(the Gagliardo-Nirenberg inequality: } 2-\tfrac{3}{2}=(1-\tfrac{3}{2})\tfrac{\alpha-1}{\alpha}+((1+\alpha)-\tfrac{3}{2})\tfrac{1}{\alpha})\\
\le& C\|\partial_i b_1\|_{L^s}^{\frac{s}{3s-2}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^{\frac{s-2}{3s-2}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^{\frac{(\alpha-1)(5s-2)}{\alpha(3s-2)}}(\|\nabla_h\Lambda^\alpha u\|_{L^2}+\|\nabla_h\Lambda^\alpha b\|_{L^2})^{\frac{5s-2}{\alpha(3s-2)}}\\
=& C\|\partial_i b_1\|_{L^s}^{\frac{s}{3s-2}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^{\frac{2\alpha(3s-2)-(5s-2)}{\alpha(3s-2)}}(\|\nabla_h\Lambda^\alpha u\|_{L^2}+\|\nabla_h\Lambda^\alpha b\|_{L^2})^{\frac{5s-2}{\alpha(3s-2)}}\\
\le& C\|\partial_i b_1\|_{L^s}^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^2+\frac{1}{12}(\|\nabla_h\Lambda^\alpha u\|_{L^2}+\|\nabla_h\Lambda^\alpha b\|_{L^2})^2 \qquad (2.10)\\
&\qquad\text{(Young's inequality: } \frac{1}{2/(\frac{2\alpha(3s-2)-(5s-2)}{\alpha(3s-2)})}+\frac{1}{2/(\frac{5s-2}{\alpha(3s-2)})}=1).
\end{aligned}$$

Similarly, it can be concluded that

$$|II|\le C\|\partial_j b_2\|_{L^s}^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^2+\frac{1}{12}(\|\nabla_h\Lambda^\alpha u\|_{L^2}+\|\nabla_h\Lambda^\alpha b\|_{L^2})^2,\ (j\in\{1,2,3\}), \qquad (2.11)$$

and

$$|III|\le C\|\partial_k b_3\|_{L^s}^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^2+\frac{1}{12}(\|\nabla_h\Lambda^\alpha u\|_{L^2}+\|\nabla_h\Lambda^\alpha b\|_{L^2})^2,\ (k\in\{1,2,3\}). \qquad (2.12)$$

We first substitute (2.10)–(2.12) into (2.9), and then substitute them with (2.8) into (2.5) to obtain that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2)+\|\nabla_h\Lambda^\alpha u\|_{L^2}^2+\|\nabla_h\Lambda^\alpha b\|_{L^2}^2\\
\le& C(\|\partial_l u_3\|_{L^s}+\|\partial_i b_1\|_{L^s}+\|\partial_j b_2\|_{L^s}+\|\partial_k b_3\|_{L^s})^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^2. \qquad (2.13)
\end{aligned}$$

Using Grönwall's inequality, we get

$$\begin{aligned}
\Pi^2(t)=&\sup_{0\le\tau\le t}(\|\nabla_h u(\tau)\|_{L^2}^2+\|\nabla_h b(\tau)\|_{L^2}^2)+\int_0^t(\|\nabla_h\Lambda^\alpha u\|_{L^2}^2+\|\nabla_h\Lambda^\alpha b\|_{L^2}^2)\mathrm{d}\tau\\
\le& CJ_0+C\int_0^t(\|\partial_l u_3\|_{L^s}+\|\partial_i b_1\|_{L^s}+\|\partial_j b_2\|_{L^s}+\|\partial_k b_3\|_{L^s})^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^2\mathrm{d}\tau, \qquad (2.14)
\end{aligned}$$

where $J_0=\|\nabla u(0)\|_{L^2}^2+\|\nabla b(0)\|_{L^2}^2$.

Step 2. We prove that the H^1 norm of the local strong solution (u, b) is uniformly bounded on $[0, T]$.

We multiply the first two equations of (1.1) by Δu and Δb, respectively, and then integrate by parts, noticing the divergence-free condition, to get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2 \\
=&\int_{R^3} [(u\cdot\nabla u)\cdot\Delta u - (b\cdot\nabla b)\cdot\Delta u + (u\cdot\nabla b)\cdot\Delta b - (b\cdot\nabla u)\cdot\Delta b]\,\mathrm{d}x \\
=&K_1' + K_2' + K_3' + K_4'.
\end{aligned} \tag{2.15}$$

Then we estimate $K_i'(i = 1, 2, 3, 4)$ one by one. According to the conclusion in [8], it is obvious to obtain that

$$\begin{aligned}
|K_1'| \leq &\int_{R^3} |\nabla_h u|\cdot|\nabla u|^2\,\mathrm{d}x \\
\leq &C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^4}^2 \quad (\text{Hölder's inequality: } \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} = 1) \\
\leq &C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{2-\frac{3}{2\alpha}}\|\Lambda^\alpha u\|_{L^6}^{\frac{3}{2\alpha}} \\
&\quad (\text{the Gagliardo-Nirenberg inequality: } (1-\tfrac{3}{4})\times 2 = (1-\tfrac{3}{2})\times(2-\tfrac{3}{2\alpha}) + (\alpha-\tfrac{3}{6})\times\tfrac{3}{2\alpha}) \\
\leq &C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{2-\frac{3}{2\alpha}}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{\frac{1}{\alpha}}\|\Lambda^{\alpha+1} u\|_{L^2}^{\frac{1}{2\alpha}} \qquad (2.16) \\
&\quad (\text{the multiplicative Sobolev inequality (2.2): } \tfrac{1}{3}\times\tfrac{3}{2\alpha} + \tfrac{2}{3}\times\tfrac{3}{2\alpha} = \tfrac{3}{2\alpha}).
\end{aligned}$$

Further, similar with (2.9), we obtain

$$\begin{aligned}
&|K_2' + K_3' + K_4'| \\
\leq &C\int_{R^3} |b|\cdot(|\nabla u| + |\nabla b|)\cdot(|\Delta u| + |\Delta b|)\,\mathrm{d}x \\
\leq &C\int_{R^3} (|b_1| + |b_2| + |b_3|)\cdot(|\nabla u| + |\nabla b|)\cdot(|\Delta u| + |\Delta b|)\,\mathrm{d}x \\
\leq &C\int_{R^3} (|b_1|)\cdot(|\nabla u| + |\nabla b|)\cdot(|\Delta u| + |\Delta b|)\,\mathrm{d}x + C\int_{R^3} |b_2|\cdot(|\nabla u| + |\nabla b|)\cdot(|\Delta u| + |\Delta b|)\,\mathrm{d}x \\
&+ C\int_{R^3} |b_3|\cdot(|\nabla u| + |\nabla b|)\cdot(|\Delta u| + |\Delta b|)\,\mathrm{d}x \\
\triangleq &I' + II' + III'.
\end{aligned} \tag{2.17}$$

For I', we estimate it as the following:

$$\begin{aligned}
|I'| =&C\int_{R^3} |b_1|\cdot(|\nabla u| + |\nabla b|)\cdot(|\Delta u| + |\Delta b|)\,\mathrm{d}x \\
\leq &C\|b_1\|_{L^2}^{\frac{2s-2}{3s-2}}\|\partial_i b_1\|_{L^s}^{\frac{s}{3s-2}}\||\nabla u| + |\nabla b|\|_{L^2}^{\frac{s-2}{3s-2}}\|\partial_j(|\nabla u| + |\nabla b|)\|_{L^2}^{\frac{s}{3s-2}}
\end{aligned}$$

$$
\begin{aligned}
&\times \|\partial_k(|\nabla u| + |\nabla b|)\|_{L^2}^{\frac{s}{3s-2}} \||\Delta u| + |\Delta b|\|_{L^2} \\
&\qquad\qquad \text{(Interpolation inequality (2.3): } \frac{\frac{3s-2}{s}-1}{2} + \frac{1}{s} = 1, \quad i, j, k \in \{1, 2, 3\}) \\
\le& C\|\partial_i b_1\|_{L^s}^{\frac{s}{3s-2}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^{\frac{s-2}{3s-2}} (\|\Delta u\|_{L^2} + \|\Delta b\|_{L^2})^{\frac{5s-2}{3s-2}} \\
\le& C\|\partial_i b_1\|_{L^s}^{\frac{s}{3s-2}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^{\frac{s-2}{3s-2}} (\|\nabla u\|_{L^2}^{1-\frac{1}{\alpha}} \|\Lambda^{\alpha+1} u\|_{L^2}^{\frac{1}{\alpha}} \\
&+ \|\nabla b\|_{L^2}^{1-\frac{1}{\alpha}} \|\Lambda^{\alpha+1} b\|_{L^2}^{\frac{1}{\alpha}})^{\frac{5s-2}{3s-2}} \\
&\qquad \text{(the Gagliardo-Nirenberg inequality: } 2 - \tfrac{3}{2} = (1 - \tfrac{3}{2})\tfrac{\alpha-1}{\alpha} + ((1 + \alpha) - \tfrac{3}{2})\tfrac{1}{\alpha}) \\
\le& C\|\partial_i b_1\|_{L^s}^{\frac{s}{3s-2}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^{\frac{s-2}{3s-2}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^{\frac{(\alpha-1)(5s-2)}{\alpha(3s-2)}} \\
&\times (\|\Lambda^{\alpha+1} u\|_{L^2} + \|\Lambda^{\alpha+1} b\|_{L^2})^{\frac{5s-2}{\alpha(3s-2)}} \\
\le& C\|\partial_i b_1\|_{L^s}^{\frac{s}{3s-2}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^{\frac{2\alpha(3s-2)-(5s-2)}{\alpha(3s-2)}} (\|\Lambda^{\alpha+1} u\|_{L^2} + \|\Lambda^{\alpha+1} b\|_{L^2})^{\frac{5s-2}{\alpha(3s-2)}} \\
\le& C\|\partial_i b_1\|_{L^s}^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 + \frac{1}{12}(\|\Lambda^{\alpha+1} u\|_{L^2} + \|\Lambda^{\alpha+1} b\|_{L^2})^2 \qquad (2.18) \\
&\qquad \text{(Young's inequality: } \frac{1}{2/(\frac{2\alpha(3s-2)-(5s-2)}{\alpha(3s-2)})} + \frac{1}{2/(\frac{5s-2}{\alpha(3s-2)})} = 1).
\end{aligned}
$$

In the same way, we can estimate II' and III', and then obtain that

$$
\begin{aligned}
|K_2' + K_3' + K_4'| \le& C(\|\partial_i b_1\|_{L^s} + \|\partial_j b_2\|_{L^s} + \|\partial_k b_3\|_{L^s})^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \\
&+ \frac{1}{4}(\|\Lambda^{\alpha+1} u\|_{L^2} + \|\Lambda^{\alpha+1} b\|_{L^2})^2, \quad (i, j, k \in \{1, 2, 3\}). \qquad (2.19)
\end{aligned}
$$

Substituting (2.16) and (2.19) into (2.15) yields

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \frac{3}{4}(\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) \\
\le& C(\|\partial_i b_1\|_{L^s} + \|\partial_j b_2\|_{L^s} + \|\partial_k b_3\|_{L^s})^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}} (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \\
&+ C\|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{2-\frac{3}{2\alpha}} \|\nabla_h \Lambda^{\alpha} u\|_{L^2}^{\frac{1}{\alpha}} \|\Lambda^{\alpha+1} u\|_{L^2}^{\frac{1}{2\alpha}}. \qquad (2.20)
\end{aligned}
$$

Integrating (2.20) over the interval $(0, t)$ can get

$$
\begin{aligned}
&\frac{1}{2}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \frac{3}{4}\int_0^t (\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) \mathrm{d}\tau \\
\le& C + C\int_0^t (\|\partial_i b_1\|_{L^s} + \|\partial_j b_2\|_{L^s} + \|\partial_k b_3\|_{L^s})^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}} (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \, \mathrm{d}\tau \\
&+ C\int_0^t \|\nabla_h u\|_{L^2} \|\nabla u\|_{L^2}^{2-\frac{3}{2\alpha}} \|\nabla_h \Lambda^{\alpha} u\|_{L^2}^{\frac{1}{\alpha}} \|\Lambda^{\alpha+1} u\|_{L^2}^{\frac{1}{2\alpha}} \mathrm{d}\tau \\
\le& C + C\int_0^t (\|\partial_i b_1\|_{L^s} + \|\partial_j b_2\|_{L^s} + \|\partial_k b_3\|_{L^s})^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}} (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \, \mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ C \sup_{0\le\tau\le t} \|\nabla_h u\|_{L^2} \int_0^t \|\nabla u\|_{L^2}^{2-\frac{3}{2\alpha}} \|\nabla_h \Lambda^\alpha u\|_{L^2}^{\frac{1}{\alpha}} \|\Lambda^{\alpha+1} u\|_{L^2}^{\frac{1}{2\alpha}} \mathrm{d}\tau \\
\le & C + C \int_0^t (\|\partial_i b_1\|_{L^s} + \|\partial_j b_2\|_{L^s} + \|\partial_k b_3\|_{L^s})^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}} (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2)\, \mathrm{d}\tau \\
&+ C \sup_{0\le\tau\le t} \|\nabla_h u\|_{L^2} \Big[\int_0^t \|\nabla_h \Lambda^\alpha u\|_{L^2}^2 \,\mathrm{d}\tau\Big]^{\frac{1}{2\alpha}} \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \,\mathrm{d}\tau\Big]^{\frac{1}{4\alpha}} \qquad (2.21) \\
&\quad \text{(Hölder's inequality: } \Big(\frac{2}{2-\frac{3}{2\alpha}}\Big)^{-1} + \Big(\frac{2}{\frac{1}{\alpha}}\Big)^{-1} + \Big(\frac{2}{\frac{1}{2\alpha}}\Big)^{-1} = 1),
\end{aligned}
$$

where

$$
\begin{aligned}
& C \sup_{0\le\tau\le t} \|\nabla_h u\|_{L^2} \Big[\int_0^t \|\nabla_h \Lambda^\alpha u\|_{L^2}^2 \, d\tau\Big]^{\frac{1}{2\alpha}} \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \, d\tau\Big]^{\frac{1}{4\alpha}} \\
\le & C \sup_{0\le\tau\le t} \|\nabla_h u\|_{L^2} \Big[\Big(\int_0^t \|\nabla_h \Lambda^\alpha u\|_{L^2}^2 \, d\tau\Big)^{\frac{1}{2}} + 1\Big] \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \, d\tau\Big]^{\frac{1}{4\alpha}} \\
&\qquad \text{(Young's inequality: } a^{\frac{1}{2\alpha}} \cdot 1 \le C_1 a^{\frac{1}{2}} + C_2 \cdot 1) \\
\le & C\Pi^2(t) \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \, d\tau\Big]^{\frac{1}{4\alpha}} + C \sup_{0\le\tau\le t} \|\nabla_h u\|_{L^2} \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \, d\tau\Big]^{\frac{1}{4\alpha}} \\
\le & C\Pi^2(t) \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \, d\tau\Big]^{\frac{1}{4\alpha}} + C(\sup_{0\le\tau\le t} \|\nabla_h u\|_{L^2}^2 + 1) \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \, d\tau\Big]^{\frac{1}{4\alpha}} \\
&\qquad \text{(Young's inequality: } a \cdot 1 \le C_1 a^2 + C_2 \cdot 1) \\
\le & C\Pi^2(t) \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \, d\tau\Big]^{\frac{1}{4\alpha}} + C \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \, d\tau\Big]^{\frac{1}{4\alpha}}. \qquad (2.22)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
& \frac{1}{2}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \frac{3}{4} \int_0^t (\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) \mathrm{d}\tau \\
\le & C + C \int_0^t (\|\partial_i b_1\|_{L^s} + \|\partial_j b_2\|_{L^s} + \|\partial_k b_3\|_{L^s})^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}} (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2)\, \mathrm{d}\tau \\
&+ C\Pi^2(t) \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \,\mathrm{d}\tau\Big]^{\frac{1}{4\alpha}} + C \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \,\mathrm{d}\tau\Big]^{\frac{1}{4\alpha}}. \qquad (2.23)
\end{aligned}
$$

Together with (2.14), we obtain

$$
\begin{aligned}
& C\Pi^2(t) \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \, d\tau\Big]^{\frac{1}{4\alpha}} \\
\le & C \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \, d\tau\Big]^{\frac{1}{4\alpha}} + C \int_0^t (\|\partial_l u_3\|_{L^s} + \|\partial_i b_1\|_{L^s} + \|\partial_j b_2\|_{L^s} + \|\partial_k b_3\|_{L^s})^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}}
\end{aligned}
$$

$$\times (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \, d\tau \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \, d\tau \Big]^{\frac{1}{4\alpha}}$$

$$\leq C\Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \, d\tau \Big]^{\frac{1}{4\alpha}} + C\Big[\int_0^t (\|\partial_l u_3\|_{L^s} + \|\partial_i b_1\|_{L^s} + \|\partial_j b_2\|_{L^s} + \|\partial_k b_3\|_{L^s})^{\frac{8\alpha^2 s}{(4\alpha-1)(2\alpha(3s-2)-(5s-2))}}$$

$$\times (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \, d\tau \Big]^{\frac{4\alpha-1}{4\alpha}} \Big[\int_0^t \|\Lambda^{\alpha+1} u\|_{L^2}^2 \, d\tau \Big]^{\frac{1}{4\alpha}}$$

(Since $\int a^p b^2 = \int a^p b^q b^{2-q} \leq | a^p b^q |_{L^m} | b^{2-q} |_{L^{m'}}$ satisfies $qm = 2, (2-q)m' = 2$ and $\frac{1}{m} + \frac{1}{m'} = 1$, it holds that $q = 2 - \frac{1}{2\alpha}$, $m = \frac{4\alpha}{4\alpha-1}$. Here, $p = \frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}$.)

$$\leq C \int_0^t (\|\partial_l u_3\|_{L^s} + \|\partial_i b_1\|_{L^s} + \|\partial_j b_2\|_{L^s} + \|\partial_k b_3\|_{L^s})^{\frac{8\alpha^2 s}{(4\alpha-1)(2\alpha(3s-2)-(5s-2))}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \, d\tau$$

$$+ \frac{1}{8} \int_0^t (\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) d\tau \quad \text{(Young's inequality: } \Big(\frac{1}{\frac{4\alpha-1}{4\alpha}}\Big)^{-1} + \Big(\frac{1}{\frac{1}{4\alpha}}\Big)^{-1} = 1). \tag{2.24}$$

We substitute (2.24) into (2.23) and get that

$$\frac{1}{2}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \frac{3}{4} \int_0^t (\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) \mathrm{d}\tau$$

$$\leq C + C \int_0^t (\|\partial_i b_1\|_{L^s} + \|\partial_j b_2\|_{L^s} + \|\partial_k b_3\|_{L^s})^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}} (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) \, \mathrm{d}\tau$$

$$+ C \int_0^t (\|\partial_l u_3\|_{L^s} + \|\partial_i b_1\|_{L^s} + \|\partial_j b_2\|_{L^s} + \|\partial_k b_3\|_{L^s})^{\frac{8\alpha^2 s}{(4\alpha-1)(2\alpha(3s-2)-(5s-2))}}$$

$$\times (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \, \mathrm{d}\tau + \frac{1}{4} \int_0^t (\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) \mathrm{d}\tau$$

$$\leq C + C \int_0^t (\|\partial_l u_3\|_{L^s} + \|\partial_i b_1\|_{L^s} + \|\partial_j b_2\|_{L^s} + \|\partial_k b_3\|_{L^s})^{\frac{8\alpha^2 s}{(4\alpha-1)(2\alpha(3s-2)-(5s-2))}}$$

$$\times (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \, \mathrm{d}\tau + \frac{1}{4} \int_0^t (\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) \mathrm{d}\tau$$

$$\text{(Young's inequality: } a^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}} \cdot 1 \leq C_1 a^{\frac{8\alpha^2 s}{(4\alpha-1)(2\alpha(3s-2)-(5s-2))}} + C_2 \cdot 1, \tag{2.25}$$

$$\text{with } \frac{8\alpha^2 s}{(4\alpha-1)(2\alpha(3s-2)-(5s-2))} \div \frac{2\alpha s}{2\alpha(3s-2)-(5s-2)} = \frac{4\alpha}{4\alpha-1} > 1).$$

Thus, we obtain

$$\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \int_0^t (\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) \mathrm{d}\tau$$

$$\leq C + C \int_0^t (\|\partial_l u_3\|_{L^s} + \|\partial_i b_1\|_{L^s} + \|\partial_j b_2\|_{L^s} + \|\partial_k b_3\|_{L^s})^{\frac{8\alpha^2 s}{(4\alpha-1)(2\alpha(3s-2)-(5s-2))}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \, \mathrm{d}\tau. \tag{2.26}$$

Employing Grönwall's inequality and noticing the condition (1.6), we finally get

$$\begin{aligned}
&\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\int_0^t(\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\alpha+1}b\|_{L^2}^2)\mathrm{d}\tau\\
\leq &C\exp\Big[C\int_0^t(\|\partial_l u_3\|_{L^s}+\|\partial_i b_1\|_{L^s}+\|\partial_j b_2\|_{L^s}+\|\partial_k b_3\|_{L^s})^{\frac{8\alpha^2 s}{(4\alpha-1)(2\alpha(3s-2)-(5s-2))}}\mathrm{d}\tau\Big]\\
<&\infty. \qquad (2.27)
\end{aligned}$$

If $s=\infty$, we still employ the same approach as in the case $s<\infty$ and carry out the proof in two steps.

Step 1. We estimate (2.5), and by the same method, we obtain that

$$|K_1|\leq C\|\partial_l u_3\|_{L^\infty}^{\frac{2\alpha}{6\alpha-5}}\|\nabla u\|_{L^2}^2+\frac{1}{4}\|\nabla_h\Lambda^\alpha u\|_{L^2}^2, \qquad (2.28)$$

and

$$\begin{aligned}
&|K_2+K_3+K_4|\\
\leq &C(\|\partial_i b_1\|_{L^\infty}+\|\partial_j b_2\|_{L^\infty}+\|\partial_k b_3\|_{L^\infty})^{\frac{2\alpha}{6\alpha-5}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^2\\
&+\frac{1}{4}(\|\nabla_h\Lambda^\alpha u\|_{L^2}+\|\nabla_h\Lambda^\alpha b\|_{L^2})^2,\quad (i,j,k\in\{1,2,3\}). \qquad (2.29)
\end{aligned}$$

Then we can get that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla_h u\|_{L^2}^2+\|\nabla_h b\|_{L^2}^2)+\|\nabla_h\Lambda^\alpha u\|_{L^2}^2+\|\nabla_h\Lambda^\alpha b\|_{L^2}^2\\
\leq &C(\|\partial_l u_3\|_{L^\infty}+\|\partial_i b_1\|_{L^\infty}+\|\partial_j b_2\|_{L^\infty}+\|\partial_k b_3\|_{L^\infty})^{\frac{2\alpha}{6\alpha-5}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^2. \qquad (2.30)
\end{aligned}$$

From Grönwall's inequality, it follows that

$$\Pi^2(t)\leq CJ_0+C\int_0^t(\|\partial_l u_3\|_{L^\infty}+\|\partial_i b_1\|_{L^\infty}+\|\partial_j b_2\|_{L^\infty}+\|\partial_k b_3\|_{L^\infty})^{\frac{2\alpha}{6\alpha-5}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^2\mathrm{d}\tau. \qquad (2.31)$$

Step 2. Estimating (2.15) and repeating the procedure in the case $s<\infty$ yields

$$|K_1'|\leq C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{2-\frac{3}{2\alpha}}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{\frac{1}{\alpha}}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{1}{2\alpha}}, \qquad (2.32)$$

and

$$\begin{aligned}
&|K_2'+K_3'+K_4'|\\
\leq &C(\|\partial_i b_1\|_{L^\infty}+\|\partial_j b_2\|_{L^\infty}+\|\partial_k b_3\|_{L^\infty})^{\frac{2\alpha}{6\alpha-5}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^2\\
&+\frac{1}{4}(\|\Lambda^{\alpha+1}u\|_{L^2}+\|\Lambda^{\alpha+1}b\|_{L^2})^2,\quad (i,j,k\in\{1,2,3\}). \qquad (2.33)
\end{aligned}$$

Substituting (2.32) and (2.33) into (2.15), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\frac{3}{4}(\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\alpha+1}b\|_{L^2}^2)$$

$$
\begin{aligned}
&\le C(\|\partial_i b_1\|_{L^\infty}+\|\partial_j b_2\|_{L^\infty}+\|\partial_k b_3\|_{L^\infty})^{\frac{2\alpha}{6\alpha-5}}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\\
&\quad + C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{2-\frac{3}{2\alpha}}\|\nabla_h\Lambda^\alpha u\|_{L^2}^{\frac{1}{\alpha}}\|\Lambda^{\alpha+1}u\|_{L^2}^{\frac{1}{2\alpha}}.
\end{aligned}
\tag{2.34}
$$

By employing the same approach as in (2.21) and (2.22), we arrive at

$$
\begin{aligned}
&\frac{1}{2}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\frac{3}{4}\int_0^t(\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\alpha+1}b\|_{L^2}^2)\mathrm{d}\tau\\
\le{}& C+C\int_0^t(\|\partial_i b_1\|_{L^\infty}+\|\partial_j b_2\|_{L^\infty}+\|\partial_k b_3\|_{L^\infty})^{\frac{2\alpha}{6\alpha-5}}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}\tau\\
&+C\Pi^2(t)\Big[\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2\,\mathrm{d}\tau\Big]^{\frac{1}{4\alpha}}+C\Big[\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2\,\mathrm{d}\tau\Big]^{\frac{1}{4\alpha}},
\end{aligned}
\tag{2.35}
$$

where

$$
\begin{aligned}
&C\Pi^2(t)\Big[\int_0^t\|\Lambda^{\alpha+1}u\|_{L^2}^2\,d\tau\Big]^{\frac{1}{4\alpha}}\\
\le{}& C\int_0^t(\|\partial_l u_3\|_{L^\infty}+\|\partial_i b_1\|_{L^\infty}+\|\partial_j b_2\|_{L^\infty}+\|\partial_k b_3\|_{L^\infty})^{\frac{8\alpha^2}{(4\alpha-1)(6\alpha-5)}}(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^2\,d\tau\\
&+\frac{1}{8}\int_0^t(\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\alpha+1}b\|_{L^2}^2)d\tau.
\end{aligned}
\tag{2.36}
$$

Combining (2.36) with (2.35), we obtain that

$$
\begin{aligned}
&\frac{1}{2}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+\frac{3}{4}\int_0^t(\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\alpha+1}b\|_{L^2}^2)\mathrm{d}\tau\\
\le{}& C+C\int_0^t(\|\partial_i b_1\|_{L^\infty}+\|\partial_j b_2\|_{L^\infty}+\|\partial_k b_3\|_{L^\infty})^{\frac{2\alpha}{6\alpha-5}}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\,\mathrm{d}\tau\\
&+C\int_0^t(\|\partial_l u_3\|_{L^\infty}+\|\partial_i b_1\|_{L^\infty}+\|\partial_j b_2\|_{L^\infty}+\|\partial_k b_3\|_{L^\infty})^{\frac{8\alpha^2}{(4\alpha-1)(6\alpha-5)}}\\
&\qquad\times(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^2\,\mathrm{d}\tau+\frac{1}{4}\int_0^t(\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\alpha+1}b\|_{L^2}^2)\mathrm{d}\tau\\
\le{}& C+C\int_0^t(\|\partial_l u_3\|_{L^\infty}+\|\partial_i b_1\|_{L^\infty}+\|\partial_j b_2\|_{L^\infty}+\|\partial_k b_3\|_{L^\infty})^{\frac{8\alpha^2}{(4\alpha-1)(6\alpha-5)}}\\
&\qquad\times(\|\nabla u\|_{L^2}+\|\nabla b\|_{L^2})^2\,\mathrm{d}\tau+\frac{1}{4}\int_0^t(\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\alpha+1}b\|_{L^2}^2)\mathrm{d}\tau.
\end{aligned}
\tag{2.37}
$$

Consequently, we have

$$
\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\int_0^t(\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\alpha+1}b\|_{L^2}^2)\mathrm{d}\tau
$$

$$\leq C + C\int_0^t (\|\partial_l u_3\|_{L^\infty} + \|\partial_i b_1\|_{L^\infty} + \|\partial_j b_2\|_{L^\infty} + \|\partial_k b_3\|_{L^\infty})^{\frac{8\alpha^2}{(4\alpha-1)(6\alpha-5)}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \, \mathrm{d}\tau. \tag{2.38}$$

Therefore, using Grönwall's inequality yields

$$\begin{aligned} &\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \int_0^t (\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) \mathrm{d}\tau \\ \leq & C \exp\Big[C\int_0^t (\|\partial_l u_3\|_{L^\infty} + \|\partial_i b_1\|_{L^\infty} + \|\partial_j b_2\|_{L^\infty} + \|\partial_k b_3\|_{L^\infty})^{\frac{8\alpha^2}{(4\alpha-1)(6\alpha-5)}} \mathrm{d}\tau\Big] \\ < & \infty. \end{aligned} \tag{2.39}$$

Thus, the proof of Theorem 1.1 is finished.

2.2. Proof of Theorem 1.3

Theorem 1.3 is a combination of Theorems 1.1 and 1.2. In this case, we repeat the steps in [8] for u_3 and the above steps of Theorem 1.1 for b. For $s, s' < \infty$, we get that

$$\begin{aligned} \Pi^2(t) = & \sup_{0\leq\tau\leq t} (\|\nabla_h u(\tau)\|_{L^2}^2 + \|\nabla_h b(\tau)\|_{L^2}^2) + \int_0^t (\|\nabla_h \Lambda^\alpha u\|_{L^2}^2 + \|\nabla_h \Lambda^\alpha b\|_{L^2}^2) \mathrm{d}\tau \\ \leq & C J_0 + C\int_0^t \|u_3\|_{L^s}^{\frac{2\alpha s}{(2\alpha-1)s-3(1-\epsilon)}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^{\frac{2[(2\alpha-1)s-3]}{(2\alpha-1)s-3(1-\epsilon)}} \\ & \times (\|\Lambda^{\alpha+1} u\|_{L^2} + \|\Lambda^{\alpha+1} b\|_{L^2})^{\frac{6\epsilon}{(2\alpha-1)s-3(1-\epsilon)}} \mathrm{d}\tau \\ & + C\int_0^t (\|\partial_i b_1\|_{L^{s'}} + \|\partial_j b_2\|_{L^{s'}} + \|\partial_k b_3\|_{L^{s'}})^{\frac{2\alpha s'}{2\alpha(3s'-2)-(5s'-2)}} \\ & \times (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \mathrm{d}\tau. \end{aligned} \tag{2.40}$$

Then we finally obtain that

$$\begin{aligned} &\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \int_0^t (\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) d\tau \\ \leq & \, C \exp\Big[C\int_0^t (\|u_3\|_{L^s}^{\frac{8\alpha s}{3(2\alpha-1)s+3(1-\epsilon)-12}} + (\|\partial_i b_1\|_{L^{s'}} + \|\partial_j b_2\|_{L^{s'}} + \|\partial_k b_3\|_{L^{s'}})^{\frac{8\alpha^2 s'}{(4\alpha-1)(2\alpha(3s'-2)-(5s'-2))}}) d\tau\Big] \\ < & \, \infty. \end{aligned} \tag{2.41}$$

Further, for the case $s, s' = \infty$, it is easy to prove that

$$\Pi^2(t) \leq C J_0 + C\int_0^t (\|u_3\|_{L^\infty}^{\frac{2\alpha}{2\alpha-1}} + (\|\partial_i b_1\|_{L^\infty} + \|\partial_j b_2\|_{L^\infty} + \|\partial_k b_3\|_{L^\infty})^{\frac{2\alpha}{6\alpha-5}})(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \mathrm{d}\tau, \tag{2.42}$$

and then

$$\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \int_0^t (\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) \mathrm{d}\tau$$

$$
\begin{aligned}
&\leq C \exp\Big[C \int_0^t \|u_3\|_{L^\infty}^{\frac{8\alpha}{3(2\alpha-1)}} + (\|\partial_i b_1\|_{L^\infty} + \|\partial_j b_2\|_{L^\infty} + \|\partial_k b_3\|_{L^\infty})^{\frac{8\alpha^2}{(4\alpha-1)(6\alpha-5)}} \mathrm{d}\tau\Big] \\
&< \infty. \qquad (2.43)
\end{aligned}
$$

This completes the proof of Theorem 1.3.

2.3. *Proof of Theorem 1.4*

The proof of Theorem 1.4 is similar to that of Theorem 1.3. In this case, we estimate b with the method in [8] and estimate u_3 using the method of Theorem 1.1, and then obtain that, for $s, s' < \infty$,

$$
\begin{aligned}
\Pi^2(t) \leq & C J_0 + C \int_0^t \|b\|_{L^{s'}}^{\frac{2\alpha s'}{(2\alpha-1)s'-3(1-\epsilon)}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^{\frac{2[(2\alpha-1)s'-3]}{(2\alpha-1)s'-3(1-\epsilon)}} \\
& \times (\|\Lambda^{\alpha+1} u\|_{L^2} + \|\Lambda^{\alpha+1} b\|_{L^2})^{\frac{6\epsilon}{(2\alpha-1)s'-3(1-\epsilon)}} \mathrm{d}\tau \\
& + C \int_0^t \|\partial_l u_3\|_{L^s}^{\frac{2\alpha s}{2\alpha(3s-2)-(5s-2)}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \mathrm{d}\tau, \qquad (2.44)
\end{aligned}
$$

and then

$$
\begin{aligned}
& \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \int_0^t (\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) \mathrm{d}\tau \\
\leq & C \exp\Big[C \int_0^t (\|b\|_{L^{s'}}^{\frac{8\alpha s'}{3(2\alpha-1)s'+3(1-\epsilon)-12}} + \|\partial_l u_3\|_{L^s}^{\frac{8\alpha^2 s}{(4\alpha-1)(2\alpha(3s-2)-(5s-2))}}) \mathrm{d}\tau\Big] < \infty. \qquad (2.45)
\end{aligned}
$$

For $s, s' = \infty$, we get that

$$
\Pi^2(t) \leq C J_0 + C \int_0^t (\|b\|_{L^\infty}^{\frac{2\alpha}{2\alpha-1}} + \|\partial_l u_3\|_{L^\infty}^{\frac{2\alpha}{6\alpha-5}})(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \mathrm{d}\tau, \qquad (2.46)
$$

and then

$$
\begin{aligned}
& \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \int_0^t (\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) \mathrm{d}\tau \\
\leq & C \exp\Big[C \int_0^t \|b\|_{L^\infty}^{\frac{8\alpha}{3(2\alpha-1)}} + \|\partial_l u_3\|_{L^\infty}^{\frac{8\alpha^2}{(4\alpha-1)(6\alpha-5)}} \mathrm{d}\tau\Big] < \infty. \qquad (2.47)
\end{aligned}
$$

This finishes the proof of Theorem 1.4.

2.4. *Proof of Theorem 1.5*

In combination with Theorems 1.1–1.4, without loss of generality, we only prove the case where $\xi = \eta = \zeta' = \sigma' = 1$ and $\xi' = \eta' = \zeta = \sigma = 0$. Then condition (1.12) turns to be, for any $j, k \in \{1, 2, 3\}$,

$$
(i) (u_3, b_1) \in L^w(0, T; L^s(\mathbb{R}^3)), \ \text{ with } \frac{2\alpha}{w} + \frac{3}{s} \leq \frac{3}{4}(2\alpha - 1) + \frac{3(1-\epsilon)}{4s}, \ \frac{3+\epsilon}{2\alpha-1} < s \leq \infty, \ 0 < \epsilon \leq \frac{1}{3};
$$

$$(ii)(\partial_j b_2, \partial_k b_3) \in L^{w'}(0,T;L^{s'}(\mathbb{R}^3)),\ \text{with}\ \frac{2\alpha}{w'} + \frac{3}{s'} \leq (1-\frac{1}{4\alpha})(6\alpha-5) + \frac{1}{s'}(6-4\alpha-\frac{1}{2\alpha}),\ 2 < s' \leq \infty. \tag{2.48}$$

For this case, we estimate u_3 and b_1 by using the method in [8], and deal with b_2 and b_3 by applying the method in Theorem 1.1. So, we can directly get that, for $s, s' < \infty$,

$$\begin{aligned}\Pi^2(t) \leq & CJ_0 + C\int_0^t (\|u_3\|_{L^s} + \|b_1\|_{L^s})^{\frac{2\alpha s}{(2\alpha-1)s-3(1-\epsilon)}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^{\frac{2[(2\alpha-1)s-3]}{(2\alpha-1)s-3(1-\epsilon)}} \\ & \times (\|\Lambda^{\alpha+1} u\|_{L^2} + \|\Lambda^{\alpha+1} b\|_{L^2})^{\frac{6\epsilon}{(2\alpha-1)s-3(1-\epsilon)}} \mathrm{d}\tau \\ & + C\int_0^t (\|\partial_j b_2\|_{L^{s'}} + \|\partial_k b_3\|_{L^{s'}})^{\frac{2\alpha s'}{2\alpha(3s'-2)-(5s'-2)}} (\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \mathrm{d}\tau,\end{aligned} \tag{2.49}$$

and then

$$\begin{aligned}& \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \int_0^t (\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) d\tau \\ & \leq C\exp\Big[C\int_0^t ((\|u_3\|_{L^s} + \|b_1\|_{L^s})^{\frac{8\alpha s}{3(2\alpha-1)s+3(1-\epsilon)-12}} + (\|\partial_j b_2\|_{L^{s'}} + \|\partial_k b_3\|_{L^{s'}})^{\frac{8\alpha^2 s'}{(4\alpha-1)(2\alpha(3s'-2)-(5s'-2))}}) d\tau\Big] \\ & < \infty.\end{aligned} \tag{2.50}$$

Further, for the case $s, s' = \infty$, it is easy to prove that

$$\Pi^2(t) \leq CJ_0 + C\int_0^t ((\|u_3\|_{L^\infty} + \|b_1\|_{L^\infty})^{\frac{2\alpha}{2\alpha-1}} + (\|\partial_j b_2\|_{L^\infty} + \|\partial_k b_3\|_{L^\infty})^{\frac{2\alpha}{6\alpha-5}})(\|\nabla u\|_{L^2} + \|\nabla b\|_{L^2})^2 \mathrm{d}\tau, \tag{2.51}$$

and then

$$\begin{aligned}& \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \int_0^t (\|\Lambda^{\alpha+1} u\|_{L^2}^2 + \|\Lambda^{\alpha+1} b\|_{L^2}^2) \mathrm{d}\tau \\ \leq & C\exp\Big[C\int_0^t (\|u_3\|_{L^\infty} + \|b_1\|_{L^\infty})^{\frac{8\alpha}{3(2\alpha-1)}} + (\|\partial_j b_2\|_{L^\infty} + \|\partial_k b_3\|_{L^\infty})^{\frac{8\alpha^2}{(4\alpha-1)(6\alpha-5)}} \mathrm{d}\tau\Big] \\ < & \infty.\end{aligned} \tag{2.52}$$

Thus, we complete the proof of Theorem 1.5.

3. Conclusions

In this paper, we study the regularity criteria for the 3D generalized MHD equations involving the partial components of the velocity u_3 and the magnetic field (b_1, b_2, b_3). In subsequent research, we will commit to finding regularity criteria with fewer components involved, and even to investigating those relying only on a single component.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, Princeton University Press, Princeton, 1970.
2. J. Wu, Generalized MHD equations, *J. Differ. Equations*, **195** (2003), 284–312. https://doi.org/10.1016/j.jde.2003.07.007
3. J. Wu, Regularity criteria for the generalized MHD equations, *Commun. Partial Differ. Equations*, **33** (2008), 285–306. https://doi.org/10.1080/03605300701382530
4. J. Wu, Global regularity for a class of generalized magnetohydrodynamic equations, *J. Math. Fluid Mech*, **13** (2011), 295–305. https://doi.org/10.1007/s00021-009-0017-y
5. Y. Zhou, Regularity criteria for the generalized viscous MHD equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **24** (2007), 491–505. https://doi.org/10.1016/j.anihpc.2006.03.014
6. J. Yuan, Existence theorem and regularity criteria for the generalized MHD equations, *Nonlinear Anal.: Real World Appl.*, **11** (2010), 1640–1649. https://doi.org/10.1016/j.nonrwa.2009.03.017
7. X. Jia, On smoothness of 3D generalized MHD equations, *Appl. Math. Lett.*, **133** (2022), 108260. https://doi.org/10.1016/j.aml.2022.108260
8. J. Wang, W. Tan, Y. Nie, A regularity criterion for the 3D generalized MHD system involving partial components, *J. Math. Phys.*, **64** (2023), 051505. https://doi.org/10.1063/5.0143742
9. J. Li, X. Zheng, The well-posedness of the incompressible magnetohydro dynamic equations in the framework of Fourier-Herz space, *J. Differ. Equations*, **263** (2017), 3419–3459, https://doi.org/10.1016/j.jde.2017.04.027
10. W. G. Melo, T. S. R. Santos, P. R. Zingano, Global well-posedness of the 3D generalized MHD equations in Lei-Lin-Gevrey and Lei-Lin spaces, *Z. Angew. Math. Phys.*, **71** (2020), 195. https://doi.org/10.1007/s00033-020-01421-6

11. Z. Zhang, Generalized MHD system with velocity gradient in Besov spaces of negative order, *Acta Appl. Math.*, **149** (2017), 139–144. https://doi.org/10.1007/s10440-016-0091-0
12. Z. Zhang, Remarks on regularity criteria for the 2D generalized MHD system in Besov spaces, *Rocky Mt. J. Math.*, **48** (2018), 2785–2795. https://doi.org/10.1216/RMJ-2018-48-8-2785
13. J. Fan, J. Li, A logarithmic regularity criterion for the 3D generalized MHD system, *Math. Methods Appl. Sci.*, **38** (2016), 5279–5283. https://doi.org/10.1002/mma.3480
14. G. Wu, Regularity criteria for the 3D generalized MHD equations in terms of vorticity, *Nonlinear Anal.: Theory, Methods Appl.*, **71** (2009), 4251–4258. https://doi.org/10.1016/j.na.2009.02.115
15. Z. Ye, X. Zhao, Global well-posedness of the generalized magnetohydrodynamic equations, *Z. Angew. Math. Phys.*, **69** (2018), 126. https://doi.org/10.1007/s00033-018-1021-y
16. C. Cao, J. Wu, Two regularity criteria for the 3D MHD equations, *J. Differ. Equations*, **248** (2009), 2263–2274. https://doi.org/10.1016/j.jde.2009.09.020
17. X. Jia, Y. Zhou, Regularity criteria for the 3D MHD equations via partial derivatives, *Kinet. Relat. Models*, **5** (2012), 505–516. https://doi.org/10.3934/krm.2012.5.505
18. X. Jia, Y. Zhou, Regularity criteria for the 3D MHD equations via partial derivatives II, *Kinet. Relat. Models*, **7** (2014), 291–304. https://doi.org/10.3934/krm.2014.7.291
19. Z. Guo, Y. Li, Z. Skalák, On conditional regularity for the MHD equations via partial components, *J. Math. Fluid Mech.*, **23** (2021), 51. https://doi.org/10.1007/s00021-021-00579-x
20. C. He, Z. Xin, On the regularity of solutions to the magnetohydrodynamic equations, *J. Differ. Equations*, **213** (2005), 235–254. https://doi.org/10.1016/j.jde.2004.07.002
21. X. Jia, Y. Zhou, Regularity criteria for 3D MHD equations via mixed velocity-magnetic gradient tensors, *Nonlinearity*, **36** (2023), 1279–1301. https://doi.org/10.1088/1361-6544/acafcc
22. Q. Liu, On 3D MHD equations with regularity in one directional derivative of the velocity, *Comput. Math. Appl.*, **76** (2018), 2375–2383. https://doi.org/10.1016/j.camwa.2018.08.031
23. Y. Wang, Y. Wang, Regularity criterion of three dimensional magneto-micropolar fluid equations with fractional dissipation, *Electron. Res. Arch.*, **32** (2024), 4416–4432. https://doi.org/10.3934/era.2024199
24. Y. Du, W. Yang, Y. Zhou, On the exponential stability of a stratified flow to the 2D ideal MHD equations with damping, *SIAM J. Math. Anal.*, **51** (2019), 5077–5102. https://doi.org/10.1137/18M1215931
25. X. Zhai, Stability for the 2D incompressible MHD equations with only magnetic diffusion, *J. Differ. Equations*, **374** (2023), 267–278. https://doi.org/10.1016/j.jde.2023.07.033
26. L. Ni, Z. Guo, Y. Zhou, Some new regularity criteria for the 3D MHD equations, *J. Math. Anal. Appl.*, **396** (2012), 108–118. https://doi.org/10.1016/j.jmaa.2012.05.076
27. E. Ji, J. Lee, Some regularity criteria for the 3D incompressible magnetohydrodynamics, *J. Math. Anal. Appl.*, **369** (2010), 317–322. https://doi.org/10.1016/j.jmaa.2010.03.015
28. R. Pan, Y. Zhou, Y. Zhu, Global classical solutions of three dimensional viscous MHD system without magnetic diffusion on periodic boxes, *Arch. Ration. Mech. Anal.*, **227** (2018), 637–662. https://doi.org/10.1007/s00205-017-1170-8

29. X. Zhai, Global solutions to 3D MHD equations with fractional dissipation, *Appl. Math. Lett.*, **176** (2026), 109873. https://doi.org/10.1016/j.aml.2026.109873

30. Z. Lei, Y. Zhou, BKM's criterion and global weak solutions for magnetohydrodynamics with zero viscosity, *Discrete Contin. Dyn. Syst.*, **25** (2009), 575–583. https://doi.org/10.3934/dcds.2009.25.575

31. J. Li, W. Tan, Z. Yin, Local existence and uniqueness for the non-resistive MHD equations in homogeneous Besov spaces, *Adv. Math.*, **317** (2017), 786–798. https://doi.org/10.1016/j.aim.2017.07.013

32. K. Yamazaki, Global regularity of N-dimensional generalized MHD system with mixed data anisotropic dissipation and diffusion, *Nonlinear Anal.*, **122** (2015), 176–191. http://dx.doi.org/10.1016/j.na.2015.04.006

33. Y. Liu, M. Paicu, P. Zhang, Global well-posedness of 3-D anisotropic Navier-Stokes system with small unidirectional derivative, *Arch. Ration. Mech. Anal.*, **238** (2020), 805–843. https://doi.org/10.1007/s00205-020-01555-x

34. C. Cao, Sufficient conditions for the regularity to the 3D Navier-Stokes equations, *Discrete Contin. Dyn. Syst.*, **26** (2010), 1141–1151. http://doi.org/10.3934/dcds.2010.26.1141

35. K. Yamazaki, Remarks on the regularity criteria of generalized MHD and Navier-Stokes systems, *J. Math. Phys.*, **54** (2013), 011502. https://doi.org/10.1063/1.4773833

36. C. Cao, E. S. Titi, Global regularity criterion for the 3D Navier-Stokes equations involving one entry of the velocity gradient tensor, *Arch. Ration. Mech. Anal.*, **202** (2011), 919–932. https://doi.org/10.1007/s00205-011-0439-6

37. Z. Zhang, A Serrin-type regularity criterion for the Navier-Stokes equations via one velocity component, *Commun. Pure Appl. Anal.*, **12** (2013), 117–124. https://doi.org/10.3934/cpaa.2013.12.117

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