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Communication

Generalized topological ordered groups

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Abstract: A structured group is a group endowed with a binary internal relation compatible with the group operation, which can be generalized to a type of partially ordered group. In this paper, we investigated two kinds of structured groups: first, those in which the binary internal relation is an equivalence relation, where we showed that the quotient set acquires a group structure, leading to a characterization of its normal subgroups. As a consequence, we obtained that group quotients by congruencies are equivalent to group quotients by normality. Additionally, we proved that the quotient set of a topological group with a compatible equivalence relation is also a topological group whose canonical projection is an open map. The second type of structured groups, we considered, are those with a partial order relation. In this case, we identified nonrestrictive sufficient conditions for the order topology to define a monoid topology, hence, a group topology.

Keywords: topological group; ordered group; order topology

1. Introduction

Throughout this article, we examine a specific algebraic structure: a group endowed with a binary internal relation compatible with the group operation. This structure effectively combines the concept of a group with an additional binary relation, which generalizes the well-known notion of orderable or partially ordered group. When we say the binary relation is expected to be compatible with the group operation, it means that the relation should be preserved under the multiplication (or addition) group operation. However, in this paper, we only consider multiplication group operation. Investigating different types of groups with binary relations compatible with their operations has been a longstanding research topic. For example, orderable groups, which are groups that can be equipped with a total order relation compatible with the group operation. In this case, the order relation is compatible with left and right multiplication by group elements. These groups have been extensively studied since the 1950s (see, for instance, [1–3]); or, partially ordered groups, which are groups endowed with partially ordering compatible with the group operation. In [4], interesting results regarding the connections between partially ordered groups and topological groups have been demonstrated. In general, groups endowed with binary relations compatible with their operations can exhibit unique properties that make them interesting for study and research. These structures have found applications in various areas of mathematics, including group theory, topology, and algebraic structures (see, for instance, [5–8]). With this in mind, in this paper, we establish interesting results regarding the connection between this type of algebraic structure and topological groups, as well as group actions. Moreover, we use the term structured group, which refers to a group endowed with a binary internal relation compatible with the multiplication group operation.

In this paper, we consider two types of structured groups. First, in Subsection 3.1, we concentrate on structured groups by an equivalence relation as their internal binary relation. In this type of structured groups, we characterize normal subgroups, by showing that the quotient set acquires a group structure. Moreover, we will later refer to Theorem 2.3, which provides sufficient conditions for the order topology on an ordered group to be a group topology, and apply it in Theorem 3.18 to prove that the quotient set of a topological group with a compatible equivalence relation is again a topological group, and that the canonical projection is an open map. In Theorem 3.18, we prove that the quotient set of a topological group with a compatible equivalence relation is also a topological group whose canonical projection is an open map.

In Subsection 3.2, we consider those types of structured groups with an order relation. Notice that when a group is endowed with an order relation, Definition 2.1 is consistent with the classical notion of an ordered group (see [9]). In this case, we identify nonrestrictive sufficient conditions for the order topology to define a group topology.

2. Materials and methods

For this reason, for greater generality, the following definition applies to the context of magmas (also known as groupoids).

Definition 2.1 (Structured magma)**.** Let G be a magma and $\mathcal{R}$ be a binary internal relation on G. Then,

- We say $\mathcal{R}$ is left compatible with G if it is left-translation invariant, that is, for all $g_1, g_2, h \in G$, $(g_1, g_2) \in \mathcal{R}$ implies that $(hg_1, hg_2) \in \mathcal{R}$.
- If $\mathcal{R}$ is right-translation invariant, then we say $\mathcal{R}$ is right compatible with G. In simple terms, for all $g_1, g_2, h \in G$, $(g_1, g_2) \in \mathcal{R}$ implies that $(g_1h, g_2h) \in \mathcal{R}$.
- $\mathcal{R}$ is compatible with G if it is left and right compatible.

Note that, if G is a monoid, compatibility with multiplication implies compatibility with inversion. Indeed, if $(g_1, g_2) \in \mathcal{R}$, then by multiplying g_1^{-1} from the left and g_2^{-1} from the right, we have $(g_2^{-1}, g_1^{-1}) \in \mathcal{R}$. In addition, it is not necessarily true that a left compatibility implies a right compatibility, and vice versa. For instance, at the following example, a right compatible relation on a monoid is not a left compatible. Before proceeding with the example, recall that if $(X, \mathcal{R})$ and $(Y, \mathcal{S})$ are sets endowed with binary internal relations $\mathcal{R}, \mathcal{S}$, respectively, then a relator is a function $h : X \to Y$ satisfying that $(h(x), h(y)) \in \mathcal{S}$ whenever $(x, y) \in \mathcal{R}$. They are precisely the morphisms of the category of sets endowed with a binary internal relation. For instance, if $\mathcal{R}, \mathcal{S}$ are order relations, then the relators are exactly increasing functions.

Example 2.2. Let X be a set endowed with a binary internal relation $\mathcal{S}$, and the monoid X^X be all maps on X endowed with the composition. Consider the following binary internal relation:

$$\mathcal{R} := \left\{(f,g) \in X^X \times X^X : \forall x \in X,\ (f(x), g(x)) \in \mathcal{S}\right\}.$$

Clearly, if $\mathcal{S}$ is an order relation or an equivalence relation, then $\mathcal{R}$ retains the same respective property. Moreover, $\mathcal{R}$ is right compatible with X^X. As a matter of fact, if $f, g, h \in X^X$ and $(f,g) \in \mathcal{R}$, then $(f(x), g(x)) \in \mathcal{S}$, for all $x \in X$. In particular, if $(f(h(x)), g(h(x))) \in \mathcal{S}$, for all $x \in X$, then it means that $(f \circ h, g \circ h) \in \mathcal{R}$. In addition, if $\mathcal{M}$ denotes the submonoid of X^X of relators on X, then the inherited relation on $\mathcal{M}$ is compatible with $\mathcal{M}$.

The following classical characterization of group topologies will be used in order to demonstrate that our sufficient conditions work for the order topology to turn into a group topology.

Theorem 2.3 ([10, 11])**.** *Let G be a group. If τ is a group topology on G and $\mathcal{B}$ is a base of neighborhoods of* 1*, then the following is verified:*

(1). For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $UU \subseteq V$.

(2). For every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ with $U^{-1} \subseteq V$.

(3). For every $V \in \mathcal{B}$ and every $g \in G$, there exists $U \in \mathcal{B}$ with $gU \subseteq Vg$.

Conversely, if $\mathcal{B}$ is a filter base of $\mathcal{P}(G)$ verifying the above properties, then there exists a unique group topology on G for which $\mathcal{B}$ is a base of neighborhoods of 1*. This topology is given by*

$$\tau := \{A \subseteq G : \forall a \in A \ \ \exists U \in \mathcal{B} \quad aU \subseteq A\} \cup \{\emptyset\}.$$

It is worth to mention that, typically, the order topology is defined for totally ordered sets with at least two points (see [12]). However, it can also be extended to posets using a more subtle construction (see [13]). Let X be a set and $\mathcal{P}(X)$ be the power set of X. If $\mathcal{S}$ is a nonempty subset of $\mathcal{P}(X)$, then the set of finite intersections of $\mathcal{S}$, denoted as $\mathcal{B}(\mathcal{S}) := \{\bigcap_{T \in \mathcal{T}} T : \mathcal{T} \subseteq \mathcal{S} \text{ finite}\}$, is trivially closed under finite intersections, and so $\mathcal{B}(\mathcal{S})$ is a base for a topology on X if, and only if, $\bigcup_{S \in \mathcal{S}} S = X$.

3. Results

3.1. Structured groups by an equivalence relation

In this section, we concentrate on structured groups in which the internal binary relation is an equivalence relation. We begin by introducing the new notation as congruent groups. To provide a broader context for this discussion, we define the concept of magma as an underlying structure, which serves as a more general framework for the definitions and properties we will explore.

Definition 3.1 (Congruent magma)**.** A left-congruent magma is a magma endowed with a left-compatible equivalence relation. In a similar manner, we consider the notion of a right-congruent magma. In addition, a congruent magma will be a magma endowed with a compatible equivalent relation.

We are now prepared to demonstrate our results. However, we first provide some necessary technical results.

Lemma 3.2. *Let G be a magma, and $\mathcal{R}$ be a transitive and compatible binary internal relation on G. If $(g_1, g_2), (h_1, h_2) \in \mathcal{R}$, for all $g_1, g_2, h_1, h_2 \in G$, then $(g_1h_1, g_2h_2) \in \mathcal{R}$.*

Proof. Using the left compatibility, we have $(g_1h_1, g_1h_2) \in \mathcal{R}$, and by the right one, $(g_1h_2, g_2h_2) \in \mathcal{R}$, for all $g_1, g_2, h_1, h_2 \in G$. Now, the proof is straightforward by applying transitivity.

In the forthcoming lemmas, we remind that if $\mathcal{R}$ is an equivalence relation on a set X, then $[x]_{\mathcal{R}}$ stands for the equivalence class of an element $x \in X$.

Lemma 3.3. *Let G be a magma with an equivalence relation $\mathcal{R}$. If $\mathcal{R}$ is compatible with G, then $[g]_{\mathcal{R}}[h]_{\mathcal{R}} \subseteq [gh]_{\mathcal{R}}$.*

Proof. If $g' \in [g]_{\mathcal{R}}$ and $h' \in [h]_{\mathcal{R}}$, then, by Lemma 3.2, we have $(gh, g'h') \in \mathcal{R}$, which means that $g'h' \in [gh]_{\mathcal{R}}$, as required.

Moving forward, we recall that for a monoid G, submonoid $G^\times$ denotes the invertible elements in G.

Lemma 3.4. *Let G be a monoid with a compatible equivalence relation $\mathcal{R}$. For every $g \in G^\times$ and $h \in G$, the following statements hold.*

(1). $\left[g^{-1}\right]_{\mathcal{R}} = [g]_{\mathcal{R}}^{-1}$.

(2). $[gh]_{\mathcal{R}} = [g]_{\mathcal{R}}[h]_{\mathcal{R}}$ *and* $[hg]_{\mathcal{R}} = [h]_{\mathcal{R}}[g]_{\mathcal{R}}$.

(3). $[g]_{\mathcal{R}} = g[1]_{\mathcal{R}} = [1]_{\mathcal{R}}g$.

Proof. The proof will be itemized as the statement of the lemma.

(1). Let $h \in G$ and $\left(g^{-1}, h\right) \in \mathcal{R}$. Then, by right compatibility, $(1, hg) \in \mathcal{R}$, and by left compatibility, $\left(h^{-1}, g\right) \in \mathcal{R}$, which means that $h^{-1} \in [g]_{\mathcal{R}}$. Therefore, $\left[g^{-1}\right]_{\mathcal{R}} \subseteq [g]_{\mathcal{R}}^{-1}$. With the same argument, $[g]_{\mathcal{R}} \subseteq \left[g^{-1}\right]_{\mathcal{R}}^{-1}$, and so $[g]_{\mathcal{R}}^{-1} = \left[g^{-1}\right]_{\mathcal{R}}$.

(2). According to Lemma 3.3, we have $[g]_{\mathcal{R}}[h]_{\mathcal{R}} \subseteq [gh]_{\mathcal{R}}$. Let $k \in G$ with $(gh, k) \in \mathcal{R}$. By left compatibility, $\left(h, g^{-1}k\right) \in \mathcal{R}$, which means that $g^{-1}k \in [h]_{\mathcal{R}}$. Therefore, $k = g\left(g^{-1}k\right) \in [g]_{\mathcal{R}}[h]_{\mathcal{R}}$, and so $[gh]_{\mathcal{R}} \subseteq [g]_{\mathcal{R}}[h]_{\mathcal{R}}$. In a similar way, $[hg]_{\mathcal{R}} = [h]_{\mathcal{R}}[g]_{\mathcal{R}}$.

(3). We know that $g[1]_{\mathcal{R}} \subseteq [g]_{\mathcal{R}}[1]_{\mathcal{R}} = [g1]_{\mathcal{R}} = [g]_{\mathcal{R}}$. Moreover, for every $g' \in [g]_{\mathcal{R}}$, by left compatibility, we have $\left(1, g^{-1}g'\right) \in \mathcal{R}$. Hence, $g' = g\left(g^{-1}g'\right) \in g[1]_{\mathcal{R}}$, and so $[g]_{\mathcal{R}} \subseteq g[1]_{\mathcal{R}}$. In a dual way, we conclude that $[g]_{\mathcal{R}} = [1]_{\mathcal{R}}g$.

Remark 3.5. Note that, by the proof of Lemma 3.4, left compatibility of $\mathcal{R}$ is sufficient to ensure $[g]_{\mathcal{R}} = g[1]_{\mathcal{R}}$. Dually, right compatibility of $\mathcal{R}$ is sufficient to ensure $[g]_{\mathcal{R}} = [1]_{\mathcal{R}}g$.

We now focus on providing the quotient set of a congruent group with structure of group.

Theorem 3.6. *Let G be a group with a compatible equivalence relation $\mathcal{R}$. The quotient set ${}^{G}/_{\mathcal{R}}$ acquires structure of group under the binary internal operation given by the set product $[g]_{\mathcal{R}}[h]_{\mathcal{R}} = [gh]_{\mathcal{R}}$, for all $g, h \in G$.*

Proof. By Lemma 3.4, we only need to prove associativity property. Let $g, h, k \in G$, then $([g]_{\mathcal{R}}[h]_{\mathcal{R}})\,[k]_{\mathcal{R}} = [gh]_{\mathcal{R}}[k]_{\mathcal{R}} = [(gh)k]_{\mathcal{R}} = [g(hk)]_{\mathcal{R}} = [g]_{\mathcal{R}}[hk]_{\mathcal{R}} = [g]_{\mathcal{R}}\,([h]_{\mathcal{R}}[k]_{\mathcal{R}})$, as required.

It is worth it to mention that the compatibility in Theorem 3.6 is necessary. A simple example follows.

Example 3.7. Consider the symmetric group $S_3 = \{e, (12), (13), (23), (123), (132)\}$. Let $H = \{e, (12)\}$. Define an equivalence relation $\mathcal{R}$ on S_3 by $x\mathcal{R}y$ if, and only if, $Hx = Hy$, that is, x and y belong to the same right coset of H. This relation is right compatible, since equality of right cosets is preserved under right multiplication. However, it is not left compatible. Indeed, although $e\mathcal{R}(12)$, we have $(13)e = (13)$ and $(13)(12) = (123)$, and these two elements do not belong to the same right coset of H. Therefore, the induced operation $[g][h] := [gh]$ on ${}^{S_3}/_{\mathcal{R}}$ is not well-defined. Consequently, the quotient set ${}^{S_3}/_{\mathcal{R}} = \{H, H(13), H(23)\}$ does not admit a group structure under this operation.

As a direct consequence of Theorem 3.6, we obtain the classical construction of quotient group via normal subgroups.

Corollary 3.8 (Quotient group)**.** Let G be a group with a normal subgroup H. Considering cosets of H, which is a partition of G, we have an equivalence relation compatible with G. Therefore, ${}^{G}/_{H}$ acquires structure of group with the set product given by $(g_1H)(g_2H) = (g_1g_2)H$, for all $g_1, g_2 \in G$.

Normal subgroups are the only groups for which the partition of cosets is compatible with the group. As expected, there is a dual version of the following theorem for right cosets.

Theorem 3.9. *Let G be a group with a subgroup H. Consider the equivalence relation induced by the partition of left cosets* $\{gH : g \in G\}$*. Then:*

(1). Such equivalence relation is left compatible with G.

(2). If such equivalence relation is right compatible with G, then H is normal.

Proof. First, we show that the equivalence relation is left compatible with G. Let $g_1, g_2, g \in G$ be such that (g_1, g_2) is a related pair. Therefore, there exists a $g_0 \in G$ for which $g_1, g_2 \in g_0H$, and so $gg_1, gg_2 \in (gg_0)H$, which implies that the pair (gg_1, gg_2) is related. This shows that the equivalence relation is left compatible with G, as claimed. Moving forward, assume that the equivalence relation is right compatible with G. We show that H is normal. Let $g \in G$. It is sufficient to check that $gH \subseteq Hg$. Clearly, since $gh \in gH$, the pair (g, gh) is related, for every $h \in H$, and so, by the above discussion, $\left(1, ghg^{-1}\right)$ is a related pair. Since $1 \in H$, we have $ghg^{-1} \in H$, implying that $gh \in Hg$, as desired.

Theorem 3.10. *Let G be a congruent monoid by a compatible equivalence relation $\mathcal{R}$. Then,* $[1]_{\mathcal{R}}$ *is a normal symmetric submonoid of G. As a consequence, if G is a congruent group, then* $[1]_{\mathcal{R}}$ *is a normal subgroup of G satisfying that* ${}^{G}/_{[1]_{\mathcal{R}}} = {}^{G}/_{\mathcal{R}}$.

Proof. For every $g, h \in [1]_{\mathcal{R}}$, we know that $1 \in [1]_{\mathcal{R}} = [1 \cdot 1]_{\mathcal{R}} = [1]_{\mathcal{R}}[1]_{\mathcal{R}} = [g]_{\mathcal{R}}[h]_{\mathcal{R}} \subseteq [gh]_{\mathcal{R}}$, meaning that $gh \in [1]_{\mathcal{R}}$. This shows that $[1]_{\mathcal{R}}$ is a submonoid of G. Next, we already know that $g^{-1} \in [1]_{\mathcal{R}}$ whenever $g \in [1]_{\mathcal{R}} \cap G^{\times}$. Also, $g[1]_{\mathcal{R}} = [g]_{\mathcal{R}} = [1]_{\mathcal{R}}g$ for all $g \in G$. All of these imply that $[1]_{\mathcal{R}}$ is a normal symmetric submonoid of G. As a consequence, if G is a congruent group, then $[1]_{\mathcal{R}}$ is a normal subgroup of G. Finally, it only suffices to observe that the cosets $g[1]_{\mathcal{R}}$ are precisely the elements $[g]_{\mathcal{R}}$ of ${}^{G}/_{\mathcal{R}}$.

Corollary 3.11. Quotients by congruencies are equivalent to quotients by normality.

Consider a group G acting on a set M from the right. The collection of orbits $\{m^G : m \in M\}$ forms a partition of M. This collection induces an equivalence relation on G. Whenever a group acts on itself, we will be assuming that it is a right action and we will use the notation g^h, where h is acting on g.

Lemma 3.12. *Let G be a group acting on itself. The following statements are equivalent:*

(1). $(gh)^G \subseteq g\left(h^G\right)$ *for every* $g, h \in G$.

(2). $(gh)^G \supseteq g\left(h^G\right)$ *for every* $g, h \in G$.

(3). $(gh)^G = g\left(h^G\right)$ *for every* $g, h \in G$.

Proof. Suppose first that 1. holds. We will show 3. Fix arbitrary elements $g, h \in G$. On the the one hand, by hypothesis,

$$(gh)^G \subseteq g\left(h^G\right) \Rightarrow g^{-1}\left((gh)^G\right) \subseteq h^G. \tag{3.1}$$

On the other hand, again by hypothesis,

$$h^G = \left(g^{-1}gh\right)^G \subseteq g^{-1}\left((gh)^G\right). \tag{3.2}$$

By combining (3.1) and (3.2) together, we obtain that

$$g^{-1}\left((gh)^G\right) \subseteq h^G = \left(g^{-1}gh\right)^G \subseteq g^{-1}\left((gh)^G\right). \tag{3.3}$$

This shows that

$$h^G = g^{-1}\left((gh)^G\right) \Rightarrow g\left(h^G\right) = (gh)^G. \tag{3.4}$$

In a dual way, it can be shown that 2. also implies 3.

Inspired by Lemma 3.12, we introduce the following definition.

Definition 3.13 (Intertwining action)**.** An action of a group G on itself is called left intertwining if for every $g, h \in G$, $(gh)^G = g\left(h^G\right)$; and the action is called right intertwining if for every $g, h \in G$, $(hg)^G = \left(h^G\right)g$. The action is called intertwining if it is both left and right intertwining.

Notice that Lemma 3.12 has its dual version for right intertwining.

Theorem 3.14. *Consider a group G with an action on itself. Then, the action is left (right) intertwining if, and only if, the equivalence relation* $\mathcal{R}$ *on G, induced by the collection of orbits, is left (right) compatible with G.*

Proof. We will only detail the proof of the left equivalence, and the right equivalence can be verified using similar arguments. Suppose first that the action is left intertwining. Let us prove that the equivalence relation is left compatible. Fix arbitrary elements $g_1, g_2, g \in G$ such that $(g_1, g_2) \in \mathcal{R}$. It only suffices to prove that $gg_1 \in (gg_2)^G$. Indeed, by assumption, $(g_1, g_2) \in \mathcal{R}$, meaning that $g_1 \in (g_2)^G$. Then, $gg_1 \in g(g_2)^G = (gg_2)^G$, hence, $(gg_1, gg_2) \in \mathcal{R}$. Conversely, assume that the equivalence relation is left compatible. Let us prove that the action is left intertwining. Fix arbitrary elements $g, g_2, h \in G$. Observe that $\left((g_2)^h, g_2\right) \in \mathcal{R}$, thus, by assumption, $\left(g(g_2)^h, gg_2\right) \in \mathcal{R}$, that is, $g(g_2)^h \in (gg_2)^G$. This proves that $(gg_2)^G \supseteq g(g_2)^G$. At this stage, Lemma 3.12 does the rest.

From the proof of Theorem 3.14, it becomes clear under which conditions the assumptions of left and right intertwining and compatibility may be omitted.

Corollary 3.15. Consider a group G with an action on itself, and the equivalence relation $\mathcal{R}$ on G, induced by the collection of orbits. Then, the action is intertwining if, and only if, the equivalence relation is compatible with G. In this situation, by Theorem 3.6, we have that ${}^{G}/_{\mathcal{R}}$ acquires structure of group.

The quotient space of a topological space under a given equivalence relation is a new topological space constructed by endowing the quotient set of the original topological space with the quotient topology, that is, with the finest topology that makes continuous the canonical projection map (the function that maps points to their equivalence classes). In other words, a subset of a quotient space is open if, and only if, its preimage under the canonical projection map is open in the original topological space. The upcoming theorem shows the quotient topology on a quotient group is a group topology. However, the following technical lemma must be addressed first.

Lemma 3.16. *Let G be a group, and $\mathcal{R}$ be a compatible equivalence relation on G. If $A \subseteq G$, then $\pi_{\mathcal{R}}^{-1}(\pi_{\mathcal{R}}(A)) = A[1]_{\mathcal{R}} = [1]_{\mathcal{R}}A$, where $\pi_{\mathcal{R}} : G \to {}^{G}/_{\mathcal{R}}$ is the canonical projection.*

Proof. It is clear that $\pi_{\mathcal{R}}(A[1]_{\mathcal{R}}) = \pi_{\mathcal{R}}(A)$, and so $A[1]_{\mathcal{R}} \subseteq \pi_{\mathcal{R}}^{-1}(\pi_{\mathcal{R}}(A))$. For any $g \in G$, with $\pi_{\mathcal{R}}(g) \in \pi_{\mathcal{R}}(A)$, there exists $a \in A$ such that $[g]_{\mathcal{R}} = [a]_{\mathcal{R}}$, which implies that $g \in [a]_{\mathcal{R}} = a[1]_{\mathcal{R}} \subseteq A[1]_{\mathcal{R}}$, in virtue of Lemma 3.4. In a dual way, $\pi_{\mathcal{R}}^{-1}(\pi_{\mathcal{R}}(A)) = [1]_{\mathcal{R}}A$.

Remark 3.17. Given the conditions established by Lemma 3.16, and considering G as a topological group, notice that, for every open subset $V \subseteq G$, the set $\pi_{\mathcal{R}}(V)$ is open in the quotient topology of ${}^{G}/_{\mathcal{R}}$, by the fact that $\pi_{\mathcal{R}}^{-1}(\pi_{\mathcal{R}}(V)) = V[1]_{\mathcal{R}} = \bigcup_{g\in[1]_{\mathcal{R}}} Vg$. Therefore, $\{\pi_{\mathcal{R}}(V) : V \text{ is a neighborhood of } 1 \text{ in } G\}$ is a base of neighborhoods of $[1]_{\mathcal{R}}$ for the quotient topology.

Theorem 3.18. *Let G be a topological group with a compatible equivalence relation $\mathcal{R}$. Then, the quotient topology is a group topology on the quotient group ${}^{G}/_{\mathcal{R}}$ and the canonical projection $\pi_{\mathcal{R}} : G \to {}^{G}/_{\mathcal{R}}$ is an open map.*

Proof. To provide a proof, we consider the conditions specified in Theorem 2.3 for the group topology G and show their validity for the following collection:

$$\{\pi_{\mathcal{R}}(V) : V \text{ is a neighborhood of } 1 \text{ in } G\}.$$

(1). Let V be an arbitrary neighborhood of 1 in G. Therefore, for some neighborhood U of 1 in G, $UU \subseteq V$. Since $[g]_{\mathcal{R}}[h]_{\mathcal{R}} = [gh]_{\mathcal{R}}$ holds for all $g, h \in G$, we have $\pi_{\mathcal{R}}(U)^{-1} = \pi_{\mathcal{R}}(U^{-1}) \subseteq \pi_{\mathcal{R}}(V)$, as desired.

(2). Considering V as an arbitrary neighborhood of 1 in G, we have $U^{-1} \subseteq V$, for some neighborhood U of 1 in G. In view of the fact that $\left[g^{-1}\right]_{\mathcal{R}} = [g]_{\mathcal{R}}^{-1}$, for all $g \in G$, we obtain $\pi_{\mathcal{R}}(U)^{-1} = \pi_{\mathcal{R}}\left(U^{-1}\right) \subseteq \pi_{\mathcal{R}}(V)$, as required.

(3). Finally, for an arbitrary neighborhood V of 1 in G and $g \in G$, there exists a neighborhood U of 1 in G such that $gU \subseteq Vg$. Given that $[g]_{\mathcal{R}}[h]_{\mathcal{R}} = [gh]_{\mathcal{R}}$ holds for all $g, h \in G$, we obtain $\pi_{\mathcal{R}}(g)\pi_{\mathcal{R}}(U) = \pi_{\mathcal{R}}(gU) \subseteq \pi_{\mathcal{R}}(Vg) = \pi_{\mathcal{R}}(V)\pi_{\mathcal{R}}(g)$, which completes the proof.

3.2. Structured groups by an order relation

Before proceeding with the results, it is necessary to establish a common understanding of the required notation and definitions. Let A be a poset. We define $\uparrow a := [a, \infty) := \{x \in A : a \leq x\}$, $\downarrow b := (-\infty, b] := \{x \in A : x \leq b\}$, $\uparrow_{\times} a := (a, \infty) := \{x \in A : a < x\}$, and $\downarrow^{\times} b := (-\infty, b) := \{x \in A : x < b\}$, for every $a, b \in A$. The established notation for bounded intervals is also used throughout. Moreover, $\{\uparrow_{\times} a, \downarrow^{\times} a : a \in A\}$ is a subbase for a topology on A if, and only if, for every $a \in A$, either $\uparrow_{\times} a \neq \varnothing$ or $\downarrow^{\times} a \neq \varnothing$. In this situation, the order topology on A is the topology generated by the subbase $\{\uparrow_{\times} a, \downarrow^{\times} a : a \in A\}$ and we refer it as a topological poset.

There are different ways to construct the order topology in a poset A containing elements that are incomparable to all other elements (i.e., elements $a \in A$ for which $\uparrow_{\times} a = \downarrow^{\times} a = \varnothing$). For example, one possible construction of the order topology is obtained by taking the family $\{\uparrow_{\times} a, \downarrow^{\times} a : a \in A\} \cup \{A\}$ as a subbase. In this section, we focus exclusively on topological poset A that does not contain elements incomparable with all the rest, that is, for every $a \in A$, either $\uparrow_{\times} a \neq \varnothing$ or $\downarrow^{\times} a \neq \varnothing$, and posets with elements incomparable to all others are excluded from our consideration. It is worth mentioning that open intervals are open in the order topology. However, closed intervals are not necessarily closed in the order topology, unless the order is total, for example.

By Definition 2.1, a left-(right-)ordered magma is a left-(right-)structured magma whose binary internal relation is an order relation. Therefore, an ordered magma is both a left- and right-ordered magma. By applying Lemma 3.2 to an ordered magma G, we obtain the property: if $g \leq g'$ and $h \leq h'$, then $gh \leq g'h'$ for all $g, g', h, h' \in G$.

Notice that a left-(right-)ordered magma is not necessarily an ordered magma, as demonstrated by the following example.

Example 3.19. Let X be a poset. Consider the monoid X^X, which is also a poset when equipped with the pointwise ordering. If $X := \mathbb{R}$, then X^X is a right-ordered monoid which is not left-ordered. However, if we restrict to the submonoid $\mathcal{M}$ of increasing functions on X, then $\mathcal{M}$ is an ordered monoid.

In the subsequent lemma, we present several properties concerning ordered monoids. Let us recall that the poset A is called downward directed, if for all a_1,$a_2 \in A$, there is $b \in A$ such that $b \leq a_1$ and $b \leq a_2$. The dual notion of downward directed is the one of upward directed.

Remark 3.20. Let G be a left ordered monoid. Let $g, g_1, g_2, h_1, h_2, h \in G$, and $g, h \in G^{\times}$. The following statements hold:

- $g_1 < g_2$ if, and only if, $hg_1 < hg_2$.
- $g > 1$ if, and only if, $g^{-1} < 1$. In particular, $(\uparrow_{\times} g \cap G^{\times})^{-1} = \downarrow^{\times}(g^{-1}) \cap G^{\times}$ and $(\downarrow^{\times} g \cap G^{\times})^{-1} = \uparrow_{\times} (g^{-1}) \cap G^{\times}$.
- $\uparrow g = g \uparrow 1$, $\uparrow_{\times} g = g \uparrow_{\times} 1$, $\downarrow g = g \downarrow 1$, and $\downarrow^{\times} g = g \downarrow^{\times} 1$.
- $\uparrow_{\times} g$ is downward directed if, and only if, $\uparrow_{\times} 1$ is downward directed.
- $\downarrow^{\times} g$ is upward directed if, and only if, $\downarrow^{\times} 1$ is upward directed.
- $\uparrow_{\times} 1 \cap G^{\times}$ is coinitial in $\uparrow_{\times} 1$ if, and only if, $\uparrow_{\times} g \cap G^{\times}$ is coinitial in $\uparrow_{\times} g$.
- $\downarrow^{\times} 1 \cap G^{\times}$ is cofinal in $\downarrow^{\times} 1$ if, and only if, $\downarrow^{\times} g \cap G^{\times}$ is cofinal in $\downarrow^{\times} g$.

If G is right ordered, then we have the corresponding dual version of the above. If G is ordered, then:

- If $g_1 < g_2$ and $h_1 < h_2$ in G, and any one of g_1, g_2, h_1, h_2 is invertible, then $h_1 g_1 < h_2 g_2$.

- $g > h$ if, and only if, $g^{-1} < h^{-1}$.
- $\uparrow_{\times} 1$ is upward directed and $\downarrow^{\times} 1$ is downward directed.
- $\uparrow_{\times} 1 \cap G^{\times}$ is downward directed if, and only if, $\downarrow^{\times} 1 \cap G^{\times}$ is upward directed.
- If $g_1, g_2 \in G^{\times}$, $g_1 \leq g$, $g_2 \leq g$, and $\uparrow 1$ is totally ordered, then g_1, g_2 are comparable. Indeed, $1 \leq g_1^{-1} g$ and $1 \leq g_2^{-1} g$, so either $g_1^{-1} g \leq g_2^{-1} g$ or $g_2^{-1} g \leq g_1^{-1} g$, resulting in either $g_2 \leq g_1$ or $g_1 \leq g_2$.

As it was mentioned, for a poset A, the order topology is well defined if, and only if, for all $a \in A$, either $\uparrow_{\times} a \neq \emptyset$ or $\downarrow^{\times} a \neq \emptyset$. In other words, incomparability between elements does not exist across the entire set.

Remark 3.21. Let G be a left (or right) ordered monoid. If $\uparrow_{\times} 1 \neq \emptyset$, then $\uparrow_{\times} g \neq \emptyset$ for all $g \in G^{\times}$. Similarly, if $\downarrow^{\times} 1 \neq \emptyset$, then $\downarrow^{\times} g \neq \emptyset$ for all $g \in G^{\times}$. In particular, if G is an ordered group, then order topology on G is well defined if, and only if, either $\uparrow_{\times} 1 \neq \emptyset$ or $\downarrow^{\times} 1 \neq \emptyset$. In this situation, $\uparrow_{\times} g \neq \emptyset$ and $\downarrow^{\times} g \neq \emptyset$ for all $g \in G$.

Here, we have the following proposition.

Proposition 3.22. Let G be an ordered monoid with the well defined order topology. Suppose that $\uparrow_{\times} 1 \neq \emptyset$, $\downarrow^{\times} 1 \neq \emptyset$, $\uparrow_{\times} 1$ is downward directed, $\downarrow^{\times} 1$ is upward directed, $\uparrow_{\times} 1 \cap G^{\times}$ is coinitial in $\uparrow_{\times} 1$, and $\downarrow^{\times} 1 \cap G^{\times}$ is cofinal in $\downarrow^{\times} 1$. Then:

(1). A base of neighborhoods of 1 is established by considering sets of the form $\uparrow_{\times} k^{-1} \cap \downarrow^{\times} k$, where $k > 1$.

(2). If $1 \in \mathrm{cl}\,(\uparrow_{\times} 1) \cap \mathrm{cl}\,(\downarrow^{\times} 1)$, then for any $g_0, h_0 \in G$ satisfying the inequality $g_0 < 1 < h_0$, invertible elements $g_1, g_2 < 1 < h_1, h_2$ can be found such that $(g_1, h_1)(g_2, h_2) \subseteq (g_0, h_0)$.

(3). The order topology on $G^{\times}$ is well defined and coincides with its relative topology from G.

Proof. The proof will be itemized according to the statement of the proposition.

(1). For every neighborhood U of 1 in G, elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ exist such that $1 \in \uparrow_{\times} g_1 \cap \cdots \cap \uparrow_{\times} g_n \cap \downarrow^{\times} h_1 \cap \cdots \cap \downarrow^{\times} h_m \subseteq U$. Therefore, for all $1 \leq i \leq n$ and $1 \leq j \leq m$, we have $g_i < 1 < h_j$. By the fact that $\uparrow_{\times} 1$ is downward directed and $\downarrow^{\times} 1$ is upward directed, $g_i \leq g_0 < 1 < h_0 \leq h_j$, for some $g_0, h_0 \in G$. By the fact that $\uparrow_{\times} 1 \cap G^{\times}$ is coinitial in $\uparrow_{\times} 1$ and $\downarrow^{\times} 1 \cap G^{\times}$ is cofinal in $\downarrow^{\times} 1$, $g_0 \leq g < 1 < h \leq h_0$ for some $g, h \in G^{\times}$. Since $g^{-1}, h \in \uparrow_{\times} 1$, $\uparrow_{\times} 1$ is downward directed, and $\uparrow_{\times} 1 \cap G^{\times}$ is coinitial in $\uparrow_{\times} 1$, there exists $k \in \uparrow_{\times} 1 \cap G^{\times}$ such that $k \leq g^{-1}$ and $k \leq h$. Since $k \in \uparrow_{\times} 1 \cap G^{\times}$, we have $1 \in \uparrow_{\times} k^{-1} \cap \downarrow^{\times} k$. Moreover, for each i with $1 \leq i \leq n$, we have $g_i \leq g \leq k^{-1}$, which implies $\uparrow_{\times} k^{-1} \subseteq \uparrow_{\times} g_i$. Hence, $\uparrow_{\times} k^{-1} \subseteq \uparrow_{\times} g_1 \cap \cdots \cap \uparrow_{\times} g_n$. By a similar argument, $\downarrow^{\times} k \subseteq \downarrow^{\times} h_1 \cap \cdots \cap \downarrow^{\times} h_m$. Therefore, by combining these inclusions, we obtain that $1 \in \uparrow_{\times} k^{-1} \cap \downarrow^{\times} k \subseteq \uparrow_{\times} g_1 \cap \cdots \cap \uparrow_{\times} g_n \cap \downarrow^{\times} h_1 \cap \cdots \cap \downarrow^{\times} h_m \subseteq U$, as required.

(2). Clearly, $\uparrow_{\times} g_0 \cap \downarrow^{\times} h_0$ is a neighborhood of 1. By applying that $1 \in \mathrm{cl}\,(\uparrow_{\times} 1)$ and $\uparrow_{\times} 1 \cap G^{\times}$ is coinitial in $\uparrow_{\times} 1$, we have $h_1 \in \uparrow_{\times} 1 \cap \downarrow^{\times} h_0$ for some $h_1 \in G^{\times}$, meaning that $1 < h_1 < h_0$. Consequently, $1 < h_1^{-1} h_0$. Now, by repeating the process, we have $h_2 \in \uparrow_{\times} 1 \cap \downarrow^{\times} h_1^{-1} h_0$, for some $h_2 \in G^{\times}$, and so $1 < h_2 < h_1^{-1} h_0$. Consequently, we have $h_1 h_2 < h_0$. By a similar argument, we obtain $g_0 < g_1 g_2$. Combining these results, we conclude that $(g_1, h_1)(g_2, h_2) \subseteq (g_1 g_2, h_1 h_2) \subseteq (g_0, h_0)$.

(3). First off, $\uparrow_\times 1 \cap G^\times \neq \emptyset$ by hypothesis, hence the order topology of $G^\times$ is well defined. By construction of the order topology, the order topology of $G^\times$ is coarser than its relative topology from G. According to [14, Proposition 16], the order topology of $G^\times$ and its relative topology from G coincide if we show that $\uparrow_\times g \cap G^\times$ is coinitial in $\uparrow_\times g$ and $\downarrow^\times g \cap G^\times$ is cofinal in $\downarrow^\times g$ for all $g \in G^\times$, which is precisely a direct consequence of our hypotheses.

The following lemma shows an interesting result as to an ordered group with the well-defined order topology.

Lemma 3.23. *Let G be an ordered group with the well-defined order topology. Then, the inversion map is continuous, consequently, and homeomorphism. Moreover,* $\mathrm{cl}\,(\uparrow_\times g) = \mathrm{cl}\left([\downarrow^\times (g^{-1})]^{-1}\right) = \mathrm{cl}\left(\downarrow^\times g^{-1}\right)^{-1}$*, for all* $g \in G$.

Proof. Let $g \in G$. For every neighborhood U of g^{-1} in G, there are elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ satisfying that $g^{-1} \in \uparrow_\times g_1 \cap \cdots \cap \uparrow_\times g_n \cap \downarrow^\times h_1 \cap \cdots \cap \downarrow^\times h_m \subseteq U$. Therefore, $g_i < g^{-1} < h_j$ for all $i \in \{1, \ldots, n\}$ and all $j \in \{1, \ldots, m\}$. Also, because $h_j^{-1} < g < g_i^{-1}$ for all $i \in \{1, \ldots, n\}$ and all $j \in \{1, \ldots, m\}$, we have that $\uparrow_\times h_1^{-1} \cap \cdots \cap \uparrow_\times h_n^{-1} \cap \downarrow^\times g_1^{-1} \cap \cdots \cap \downarrow^\times g_n^{-1}$ is a neighborhood of g, whose image under the inversion map is precisely $\uparrow_\times g_1 \cap \cdots \cap \uparrow_\times g_n \cap \downarrow^\times h_1 \cap \cdots \cap \downarrow^\times h_m$.

Finally, we are now in a position to state and prove a sufficient condition for the order topology on an ordered group to be a group topology.

Corollary 3.24. Let G be an ordered group with the well-defined order topology. Suppose that $1 \in \mathrm{cl}\,(\uparrow_\times 1)$ and $\uparrow_\times 1$ is downward directed. Then, the order topology is a group topology.

Proof. Let $\mathcal{B}$ be the base of neighborhoods of 1 in the order topology which consists of sets of the form $\uparrow_\times h^{-1} \cap \downarrow^\times h$, for $h > 1$. We show that it satisfies in all three conditions of Theorem 2.3.

(1). By Proposition 3.22(3), it is straightforward.

(2). It is a direct consequence of Lemma 3.23.

(3). We claim that, for every $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$, such that $gU \subseteq Vg$, for all $g \in G$. Let $g \in G$, and $h > 1$ with $\uparrow_\times h^{-1} \cap \downarrow^\times h = V$. Notice that $\downarrow^\times h$ is a neighborhood of 1. Since $1 \in \mathrm{cl}\,(\uparrow_\times 1)$, for some $k \in G$, we have $k \in \downarrow^\times h \cap \uparrow_\times 1$. Hence, $1 < k < h$, and so, for all $g \in G$, $hg > g$. Consequently, $1 < g^{-1}hg$. Again, by the fact that $1 \in \mathrm{cl}\,(\uparrow_\times 1)$, we have $k \in \downarrow^\times (g^{-1}hg) \cap \uparrow_\times 1$, for some $k \in G$. Therefore, $1 < k < g^{-1}hg$, or equivalently, $gk < hg$, which implies that $g\left(\uparrow_\times k^{-1} \cap \downarrow^\times k\right) \subseteq \left(\uparrow_\times h^{-1} \cap \downarrow^\times h\right) g$, as claimed.

In the next theorem, we prove that the sufficient condition given by Corollary 3.24, which is $1 \in \mathrm{cl}\,(\uparrow_\times 1)$, is not compulsory for the given context.

Theorem 3.25. *Let G be a group with nontrivial center. Assume that there exists* $g_0 \in \mathsf{Z}(G)$ *as a torsion-free element. We define an ordering on G by* $g \leq h$ *if, and only if,* $h = g_0^n g$*, for some* $n \geq 0$*. Then, this ordering is compatible with the discrete topology as its corresponding order topology, and so a group topology.*

Proof. Notice that if $g_1, g_2, h \in G$ and $g_1 \leq g_2$, then $g_2 = g_0^n g_1$, for some $n \geq 0$. Keep in mind that if $hg_2 = hg_0^n g_1 = g_0^n(hg_1)$, then $hg_1 \leq hg_2$, and so this ordering is compatible with G. Next, we claim that $\uparrow_\times g_0^{-1} \cap \downarrow^\times g_0 = \{1\}$. Let $g \in \uparrow_\times g_0^{-1} \cap \downarrow^\times g_0$, then $g = g_0^{n_1} g_0^{-1}$ and $g_0 = g_0^{n_2} g$, for some $n_1, n_2 \geq 1$, and so $g_0^{n_1-1} = g_0^{1-n_2}$. Since g_0 is torsion-free, $n_1 - 1 = 1 - n_2$, that is, $n_1 + n_2 = 1 + 1$. By hypothesis, $n_1, n_2 \geq 1$, which means that $n_1 = n_2 = 1$, hence, $g = 1$. Finally, we show that the order topology induced by this ordering is the discrete topology. As a matter of fact, for every $g \in G$, the equality $\{1\} = \uparrow_\times g_0^{-1} \cap \downarrow^\times g_0$ concludes that $\{g\} = \uparrow_\times gg_0^{-1} \cap \downarrow^\times gg_0$, as required.

We finish this manuscript by providing a version of the previous corollary to ordered monoids.

Corollary 3.26. Let G be an ordered monoid with the well-defined order topology. Suppose that $1 \in \mathrm{cl}\,(\uparrow_\times 1) \cap \mathrm{cl}\,(\downarrow^\times 1)$, $\uparrow_\times 1$ is downward directed, $\downarrow^\times 1$ is upward directed, $\uparrow_\times 1 \cap G^\times$ is coinitial in $\uparrow_\times 1$, and $\downarrow^\times 1 \cap G^\times$ is cofinal in $\downarrow^\times 1$. Then, multiplication on G is continuous at the unity and the inversion map on $G^\times$ is continuous.

Proof. Continuity of multiplication at the unity is given by Proposition 3.22(2). Continuity of the inversion map on $G^\times$ is given by plugging together Proposition 3.22(3) and Lemma 3.23.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

F. J. García-Pacheco is the guest editor for [Electronic Research Archive] and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

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