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Research article

On conformal Ricci collineations associated with the Bott connection on three-dimensional Lorentzian Lie groups

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Abstract: This paper presents a complete classification of conformal Ricci collineations (CRCs) associated with the Bott connection on three-dimensional Lorentzian Lie groups. For each of the seven distinct classes of such groups, we derive explicit conditions under which left-invariant vector fields define CRCs, and we characterize the corresponding solution spaces. Specifically, we find that groups G_1 and G_7 admit rich non-trivial CRCs associated with ∇^{B_1}, highlighting the rich geometric structure induced by this connection.

Keywords: conformal Ricci collineations; Bott connections; Lorentzian Lie groups

1. Introduction

Symmetry principles play a fundamental role in theoretical physics, particularly in the study of spacetime geometry and gravitational theories. The symmetry here refers to a one-parameter group of diffeomorphisms of the spacetime preserving certain mathematical or physical quantities. One may regard them as vector fields ξ preserving some tensor field defined on the spacetime like the metric tensor, the curvature tensor, the Ricci tensor, or the Weyl tensor [1–4]. The investigation of symmetry properties through collineations has proven to be a powerful tool for understanding the structure of spacetime manifolds. Among these, Ricci collineations and their conformal generalizations provide deep insights into the invariance properties of gravitational fields.

The concept of Ricci collineations was first introduced by Katzin et al. [5] in 1969, defining vector fields that preserve the Ricci tensor through vanishing Lie derivatives. This was subsequently generalized by Duggal [6] to conformal Ricci collineations, where the Ricci tensor preserves its conformal class under the flow of a vector field. Further extensions by Kühnel and Rademacher [7] combined conformal symmetries of both the metric and Ricci tensor, establishing a comprehensive framework for studying spacetime symmetries.

In the context of physics, the investigation of conformal Ricci collineations (CRCs) provides a powerful tool for classifying spacetimes and uncovering new solutions to Einstein's field equations [8–11]. Meanwhile, a systematic mathematical framework has been developed on three-dimensional Lorentzian Lie groups. Louzao et al. [12] initiated this direction by determining all left-invariant Ricci collineations. Tao [13] then extended the concept to both canonical and Kobayashi-Nomizu connections. Wang [14] concluded this effort by characterizing conformal Ricci collineations for the Levi-Civita connection.

Research into diverse linear connections [15] has led to the study of geometric structures defined by specific affine connections. An important and distinct research thread concerns the Bott connection [16–18], which is characterized by its natural association with foliations [19]. This connection has itself become a subject of focused research, as seen in works on affine Ricci solitons [20], Killing magnetic curves [21], and algebraic Schouten solitons [22]. In light of these developments, we are motivated to explore the conformal Ricci collineations associated with the Bott connection. We specifically investigate left-invariant CRCs associated with the Bott connection on seven three-dimensional Lorentzian Lie groups. Our classification demonstrates that, in contrast to the Levi-Civita connection [14], the Bott connection admits non-trivial CRCs, thereby highlighting the unique geometric implications of choosing the Bott connection and providing new insights into the symmetry properties it induces.

This paper is structured as follows. Section 2 provides the necessary preliminaries on three-dimensional Lorentzian Lie groups, the Bott connection, and conformal Ricci collineations. We then present our core results: the complete classification of left-invariant CRCs associated with the Bott connection for the first distribution in Section 3, followed by the classifications for the second and third distributions in Sections 4 and 5, respectively.

2. Preliminaries

Based on the foundational work of Milnor [23], the classification of connected, simply connected three-dimensional Lie groups with a left-invariant Lorentzian metric was further developed by Rahmani, Cordero, and Calvaruso [24–26], leading to the standard division into seven types. These are denoted by $\{G_i\}_{i=1}^7$ with corresponding Lie algebras $\{\mathfrak{g}_i\}_{i=1}^7$. For each algebra, we fix a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ where e_3 is timelike. The Levi-Civita connection and curvature tensor on G_i are denoted by ∇^{LC} and R^{LC}, where

$$R^{LC}(X, Y)Z = \nabla^{LC}_X \nabla^{LC}_Y Z - \nabla^{LC}_Y \nabla^{LC}_X Z - \nabla^{LC}_{[X,Y]} Z.$$

The Ricci tensor of (G_i, g) is defined as follows:

$$\rho^{LC}(X, Y) = -g(R^{LC}(X, e_1)Y, e_1) - g(R^{LC}(X, e_2)Y, e_2) + g(R^{LC}(X, e_3)Y, e_3).$$

Moreover, we have the expression for the Ricci operator Ric^{LC}:

$$\rho^{LC}(X, Y) = g(Ric^{LC}(X), Y).$$

Consider a smooth manifold (M, g) that is equipped with the Levi-Civita connection ∇^{LC}, and let TM represent its tangent bundle, spanned by $\{e_1, e_2, e_3\}$. We introduce the following distribution:

1) $D_1 = \text{span}\{e_1, e_2\}$, $D_1^{\perp} = \text{span}\{e_3\}$;

2) $D_2 = \text{span}\{e_1, e_3\}$, $D_2^{\perp} = \text{span}\{e_2\}$;

3) $D_3 = \mathrm{span}\{e_2, e_3\}$, $D_3^{\perp} = \mathrm{span}\{e_1\}$.

The Bott connection ∇^{B_i} associated with distribution D_i is then defined as follows:

$$\nabla^{B_i}_X Y = \begin{cases} \pi_{D_i}(\nabla^{LC}_X Y), & X, Y \in \Gamma^{\infty}(D_i), \\ \pi_{D_i}([X, Y]), & X \in \Gamma^{\infty}(D_i^{\perp}),\ Y \in \Gamma^{\infty}(D_i), \\ \pi_{D_i^{\perp}}([X, Y]), & X \in \Gamma^{\infty}(D_i),\ Y \in \Gamma^{\infty}(D_i^{\perp}), \\ \pi_{D_i^{\perp}}(\nabla^{LC}_X Y), & X, Y \in \Gamma^{\infty}(D_i^{\perp}), \end{cases} \tag{2.1}$$

where π_{D_i} (respectively, $\pi_{D_i^{\perp}}$) denotes the projection onto D_i (respectively, $D_i^{\perp}$). For a more detailed discussion, refer to [16–18, 27].

Remark. *The Bott connection defined by* (2.1) *depends solely on the distribution D_i and the Levi-Civita connection, not on the choice of local frame. We fix a left-invariant frame for convenience in computations, but all results are geometric and frame-independent.*

Denote the curvature tensor of the Bott connection ∇^{B_i} by R^{B_i}, which is defined as follows:

$$R^{B_i}(X, Y)Z = \nabla^{B_i}_X \nabla^{B_i}_Y Z - \nabla^{B_i}_Y \nabla^{B_i}_X Z - \nabla^{B_i}_{[X,Y]} Z. \tag{2.2}$$

The Ricci tensor ρ^{B_i} associated to the connection ∇^{B_i} is defined as:

$$\rho^{B_i}(X, Y) = \frac{B_i(X, Y) + B_i(Y, X)}{2},$$

where $B_i(X, Y) = -g(R^{B_i}(X, e_1)Y, e_1) - g(R^{B_i}(X, e_2)Y, e_2) + g(R^{B_i}(X, e_3)Y, e_3)$.

Definition 1. *Let X be a vector field and ρ^{B_i} the Ricci tensor associated to the connection ∇^{B_i}. If X satisfies:*

$$L_X \rho^{B_i} = 2\lambda g, \tag{2.3}$$

then X is the conformal Ricci collineation, where λ is a scalar function, up to a constant. When $\lambda = 0$, the conformal Ricci collineation reduces to a Ricci collineation.

In this paper, we investigate the conformal Ricci collineations associated with the Bott connection for three different distributions D_1, D_2, and D_3 on each of the seven three-dimensional Lorentzian Lie groups. The results for each distribution are presented in Sections 3–5, respectively. In Section 6, we provide a comparative analysis of these results, highlighting the differences and similarities among the three distributions. This comparison reveals how the choice of distribution influences the existence and structure of conformal Ricci collineations.

3. Conformal Ricci collineations associated with the Bott connection on three-dimensional Lorentzian Lie group with the first distribution

In this section, we present a complete classification of three-dimensional Lorentzian Lie groups that admit conformal Ricci collineations associated with the Bott connection ∇^{B_1}. In the following theorems, V denotes the solution space of left-invariant vector fields satisfying the conformal Ricci collineation condition (2.3).

3.1. *Conformal Ricci collineations associated with* ∇^{B_1} *of* G_1.

According to [28], the Lie algebra $\mathfrak{g}_1$ has the following structure:

$$[e_1, e_2] = Ae_1 - Be_3, \quad [e_1, e_3] = -Ae_1 - Be_2, \quad [e_2, e_3] = Be_1 + Ae_2 + Ae_3, \quad A \neq 0.$$

Theorem 2. *The conformal Ricci collineations associated with the connection* ∇^{B_1} *on the Lie group* G_1 *are characterized as follows:*

1) $B = 0$, $\lambda = 0$, *and* $X_1 = X_2 = X_3 = 0$*;*

2) $B \neq 0$, $B^2 \neq A^2$, *then* $X_1 = -\frac{2B^3\lambda}{A^2(B^4+A^4)}$, $X_2 = -\frac{2\lambda}{A(B^2-A^2)}$, $X_3 = -\frac{2A\lambda}{B^4+A^4}$, *for any* λ *(with* λ *arbitrary) provided that the compatibility condition* $2B^6 - 3A^2B^4 + 4A^4B^2 - 5A^6 = 0$ *holds;*

3) $B \neq 0$, $B^2 \neq A^2$, $\lambda = 0$, $X_1 = X_2 = X_3 = 0$, *for* $2B^6 - 3A^2B^4 + 4A^4B^2 - 5A^6 \neq 0$*;*

4) $B^2 = A^2$, $\lambda = 0$, $X_1 = X_2 = X_3 = 0$.

Proof. The Ricci tensor components for the connection ∇^{B_1} on G_1 are given by [20]

$$\rho^{B_1}(e_i, e_j) = \begin{pmatrix} -(A^2+B^2) & AB & -\frac{1}{2}AB \\ AB & -(A^2+B^2) & \frac{1}{2}A^2 \\ -\frac{1}{2}AB & \frac{1}{2}A^2 & 0 \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_1, the conformal Ricci collineation condition $L_X\rho^{B_1} = 2\lambda g$ yields the system

$$\begin{cases} (-2A^3 + AB^2)X_2 + 2A^3X_3 = 2\lambda, \\ (A^3 + AB^2 - \frac{1}{2}A^2B)X_1 + \frac{1}{2}A^2BX_2 - \frac{1}{2}A^3X_3 = 0, \\ -A^3X_1 + B^3X_3 = 0, \\ -A^2BX_1 - A^3X_3 = 2\lambda, \\ (\frac{1}{2}A^2B - B^3)X_1 + A^3X_2 + (\frac{1}{2}A^3 - \frac{1}{2}AB^2)X_3 = 0, \\ A(B^2 - A^2)X_2 = -2\lambda. \end{cases} \tag{3.1}$$

Case 1: $B = 0$. The system (3.1) reduces to

$$\begin{cases} -2A^3X_2 + 2A^3X_3 = 2\lambda, \\ A^3X_1 - \frac{1}{2}A^3X_3 = 0, \\ -A^3X_1 = 0, \\ -A^3X_3 = 2\lambda, \\ A^3X_2 + \frac{1}{2}A^3X_3 = 0, \\ -A^3X_2 = -2\lambda. \end{cases}$$

Solving this system yields $X_1 = X_2 = X_3 = 0$, and $\lambda = 0$. This leads to case (1).
Case 2: $B \neq 0$ **and** $B^2 \neq A^2$. From the third and sixth equations (3.1), we have

$$X_1 = \frac{B^3}{A^3}X_3, \quad X_2 = -\frac{2\lambda}{A(B^2 - A^2)}.$$

Substituting into the fourth equation gives $X_3 = -\frac{2A\lambda}{B^4+A^4}$. Now we check consistency by substituting these expressions into the remaining equations (the second and fifth). After substitution, both equations reduce to the same polynomial condition:

$$2B^6 - 3A^2B^4 + 4A^4B^2 - 5A^6 = 0. \tag{3.2}$$

If (3.2) holds, then for any λ we have the non-trivial solution in case (2).
Case 3: $B \neq 0$**,** $B^2 \neq A^2$**, and** $2B^6 - 3A^2B^4 + 4A^4B^2 - 5A^6 \neq 0$**.** In this case, the compatibility condition (3.2) fails. We consider $\lambda = 0$. Then, from the third and sixth equations, we still have $X_1 = \frac{B^3}{A^3}X_3$ and $X_2 = 0$. Substituting into the fourth equation yields:

$$\left(\frac{B^4}{A} + A^3\right)X_3 = 0.$$

Since $B \neq 0$ and $A \neq 0$, the coefficient is non-zero, implying $X_3 = 0$, and hence $X_1 = 0$. This leads to case (3).
Case 4: $B^2 = A^2$. The last equation gives $\lambda = 0$. From the third and fourth equations, we obtain $X_1 = X_3 = 0$, the first equation then gives $X_2 = 0$, and then we have case (4). □

3.2. Conformal Ricci collineations associated with ∇^{B_1} *of* G_2*.*

According to [28], the Lie algebra $\mathfrak{g}_2$ has the following structure:

$$[e_1, e_2] = Ce_2 - Be_3, \quad [e_1, e_3] = -Be_2 - Ce_3, \quad [e_2, e_3] = Ae_1, \quad C \neq 0.$$

Theorem 3. *The conformal Ricci collineations associated with the connection* ∇^{B_1} *on the Lie group* G_2 *are characterized as follows:*

1) If $A = 0$*, then* $\lambda = 0$*,* $X_1 = 0$*,* $X_2 = \frac{B}{C}X_3$*, and* $V = \mathrm{span}\{\frac{B}{C}e_2 + e_3\}$.

2) If $A \neq 0$*, then* $\lambda = 0$ *and* $X_1 = X_2 = X_3 = 0$.

Proof. The Ricci tensor components for the connection ∇^{B_1} on G_2 are given by [20]

$$\rho^{B_1}(e_i, e_j) = \begin{pmatrix} -(B^2 + C^2) & 0 & 0 \\ 0 & -(C^2 + AB) & -\frac{1}{2}AC \\ 0 & -\frac{1}{2}AC & 0 \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} X_i e_i$ on G_2, the conformal Ricci collineation condition $L_X\rho^{B_1} = 2\lambda g$ yields the system

$$\begin{cases} 0 = 2\lambda, \\ C\left(\frac{1}{2}AB - (B^2 + C^2)\right)X_2 + (B^3 + BC^2 - AB^2 - \frac{1}{2}AC^2)X_3 = 0, \\ -\frac{1}{2}ABCX_1 - \frac{1}{2}AC^2X_2 + \frac{1}{2}ABCX_3 = 0, \\ (2C^3 + \frac{3}{2}ABC)X_1 = 2\lambda, \\ -B(AB + C^2)X_1 = 0, \\ -ABCX_1 = -2\lambda. \end{cases}$$

From the first equation, $\lambda = 0$. The fourth and sixth equations with $\lambda = 0$ give $(2C^3 + \frac{3}{2}ABC)X_1 = 0$ and $ABCX_1 = 0$. Combining these yields $2C^3X_1 = 0$, and since $C \neq 0$, we have $X_1 = 0$.

Now we consider two cases.

Case 1: $A = 0$. Substituting $A = 0$ into the second equation gives

$$C(-(B^2 + C^2))X_2 + B(B^2 + C^2)X_3 = 0,$$

which simplifies to $X_2 = \frac{B}{C}X_3$. The third equation is identically satisfied. Hence, we obtain a one-parameter family of solutions with $X_1 = 0$, $X_2 = \frac{B}{C}X_3$, and arbitrary X_3. Thus, $V = \text{span}\{\frac{B}{C}e_2 + e_3\}$. This leads to case (1).

Case 2: $A \neq 0$. With $X_1 = 0$, the third equation becomes

$$CX_2 = BX_3.$$

Substituting this into the second equation (and noting that $X_1 = 0$),

$$A(B^2 + C^2)X_3 = 0.$$

Since $A \neq 0$ and $B^2 + C^2 > 0$ (as $C \neq 0$), we obtain $X_3 = 0$. Then, $X_2 = 0$. Therefore, $X_1 = X_2 = X_3 = 0$, and we have (2). □

3.3. *Conformal Ricci collineations associated with* ∇^{B_1} *of* G_3

According to [28], the Lie algebra $\mathfrak{g}_3$ has the following structure:

$$[e_1, e_2] = -Ce_3, \quad [e_1, e_3] = -Be_2, \quad [e_2, e_3] = Ae_1.$$

Theorem 4. *The conformal Ricci collineations associated with the connection ∇^{B_1} on the Lie group G_3 are characterized as follows:*

1) $A = 0$, $\lambda = 0$, and $V = span\{e_1, e_2, e_3\}$;

2) $A \neq 0$, $B \neq 0$, $\lambda = 0$, and $V = span\{e_1, e_2, e_3\}$;

3) $A, B \neq 0$, $C = 0$, $\lambda = 0$, and $V = span\{e_1, e_2, e_3\}$;

4) $A, B, C \neq 0$, $\lambda = 0$, and $V = span\{e_3\}$.

Proof. The Ricci tensor components for the connection ∇^{B_1} on G_3 are given by [20]

$$\rho^{B_1}(e_i, e_j) = \begin{pmatrix} -BC & 0 & 0 \\ 0 & -AC & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_3, the conformal Ricci collineation condition $L_X\rho^{B_1} = 2\lambda g$ yields the system:

$$\begin{cases} 0 = 2\lambda, \\ ABCX_2 = 0, \\ ABCX_1 = 0. \end{cases}$$

We now consider the following cases:

- If $A = 0$, then the equations are satisfied for any X_i with $\lambda = 0$. Thus, $V = span\{e_1, e_2, e_3\}$, which corresponds to case (1).
- If $A \neq 0$ and $B \neq 0$, then the equations are satisfied for any X_i with $\lambda = 0$. Thus, $V = span\{e_1, e_2, e_3\}$, and we obtain case (2).
- If $A, B \neq 0$ and $C = 0$, then the equations are satisfied for any X_i with $\lambda = 0$. Thus, $V = span\{e_1, e_2, e_3\}$, which gives case (3).
- If $A, B, C \neq 0$, then the second and third equations give $X_1 = X_2 = 0$ where X_3 is free. Thus, $V = span\{e_3\}$, and we have case (4).

□

3.4. Conformal Ricci collineations associated with ∇^{B_1} of G_4.

According to [28], the Lie algebra $\mathfrak{g}_4$ has the following structure:

$$[e_1, e_2] = -e_2 + (2\eta - B)e_3, \quad [e_1, e_3] = e_3 - Be_2, \quad [e_2, e_3] = Ae_1, \quad \eta = 1 \; or \; -1.$$

Theorem 5. *The conformal Ricci collineations associated with the connection ∇^{B_1} on the Lie group G_4 are characterized as follows:*

1) $A = 0$, $\lambda = 0$, $X_2 = X_3$, and $V = span\{e_2 + e_3\}$;

2) $A \neq 0$, $B = 0$, $\lambda = 0$, $\eta = 1$, $A = \frac{4}{7}$, $X_1 = 0$, $X_2 = -X_3$, and $V = span\{e_2 - e_3\}$;

3) $A \neq 0$, $B = 0$, $\lambda = 0$, $\eta = -1$, $A = -\frac{4}{13}$, $X_1 = 0$, $X_2 = -X_3$, and $V = span\{e_2 - e_3\}$;

4) $A \neq 0$, $B \neq 0$, $\lambda = 0$, $\eta = 1$, $X_3 = -2((B-\eta)^2 - \frac{1}{2})X_2$, and $V = span\{e_2 - 2((B-1)^2 - \frac{1}{2})e_3\}$ for $A = -\frac{(B-1)^2}{B^4 - 3B^3 + 2B^2 + \frac{5}{4}B - \frac{7}{4}}$;

5) $A \neq 0$, $B \neq 0$, $\lambda = 0$, $\eta = -1$, $X_3 = -2((B-\eta)^2 - \frac{1}{2})X_2$, and $V = span\{e_2 - 2((B+1)^2 - \frac{1}{2})e_3\}$ for $A = -\frac{(B+1)^2}{B^4 + 5B^3 + 10B^2 + \frac{37}{4}B - \frac{13}{4}}$.

Proof. The Ricci tensor components for the connection ∇^{B_1} on G_4 are given by [20]

$$\rho^{B_1}(e_i, e_j) = \begin{pmatrix} -(B-\eta)^2 & 0 & 0 \\ 0 & 2A\eta - AB - 1 & \frac{1}{2}A \\ 0 & \frac{1}{2}A & 0 \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_4, the conformal Ricci collineation condition $L_X \rho^{B_1} = 2\lambda g$ yields the system

$$\begin{cases} 0 = 2\lambda, \\ -(2A\eta - AB - 1 + \frac{1}{2}A(2\eta - B))X_2 + (2A\eta - AB - 1 - \frac{1}{2}A - A(B-\eta)^2)X_3 = 0, \\ (A(B-\eta)^2 - \frac{1}{2}A)X_2 + \frac{1}{2}AX_3 = 0, \\ 2(2A\eta - AB - 1 - \frac{1}{2}A(2\eta - B))X_1 = 2\lambda, \\ (2AB\eta - AB^2 - B - \frac{1}{2}A)X_1 = 0, \\ ABX_1 = -2\lambda. \end{cases} \tag{3.3}$$

From the first equation, we obtain $\lambda = 0$. Then, the last equation becomes $ABX_1 = 0$. Substituting into the fourth and fifth equations gives

$$\begin{cases} 2(A\eta - 1)X_1 = 0, \\ (B + \frac{1}{2}A)X_1 = 0. \end{cases}$$

Hence, in all cases we have $X_1 = 0$. We now discuss the value of A.

Case:1 $A = 0$. The second equation in (3.3) simplifies to $X_2 = X_3$, thus $V = span\{e_2 + e_3\}$, corresponding to case (1).

Case:2 $A \neq 0$**,** $B = 0$**,** $\eta = 1$. The third equation becomes $A(X_2 + X_3) = 0$, which implies $X_2 = -X_3$. Substituting into the second equation yields $A(5\eta - \frac{3}{2})X_2 = 0$. Since $\eta = 1$, we have $A = \frac{4}{7}$, $X_1 = 0$ and $X_2 = -X_3$, which gives case (2).

Case:3 $A \neq 0$**,** $B = 0$**,** $\eta = -1$. Similar to case 2 but with $\eta = -1$, we obtain $A = -\frac{4}{13}$, $X_1 = 0$, and $X_2 = -X_3$, which leads to case (3).

Case:4 $A \neq 0$**,** $B \neq 0$**,** $\eta = 1$. The second and third equations form a homogeneous system in X_2 and X_3:

$$\begin{cases} aX_2 + bX_3 = 0, \\ cX_2 + dX_3 = 0, \end{cases}$$

where the coefficients are

$$\begin{cases} a = -(2A\eta - AB - 1 + \frac{1}{2}A(2\eta - B)), \\ b = (2A\eta - AB - 1 - \frac{1}{2}A - A(B - \eta)^2), \\ c = (A(B - \eta)^2 - \frac{1}{2}A), \\ d = \frac{1}{2}A. \end{cases}$$

Solving this system, we find a non-trivial solution $X_3 = -2((B - \eta)^2 - \frac{1}{2})X_2$ under the following conditions:

- If $\eta = 1$, then A and B satisfy $A = -\frac{(B-1)^2}{B^4 - 3B^3 + 2B^2 + \frac{5}{4}B - \frac{7}{4}}$, and we obtain case (4).
- If $\eta = -1$, then A and B satisfy $A = -\frac{(B+1)^2}{B^4 + 5B^3 + 10B^2 + \frac{37}{4}B - \frac{13}{4}}$, then gives case (5).

Otherwise, the system admits only the trivial solution $X_1 = X_2 = X_3 = 0$. □

3.5. *Conformal Ricci collineations associated with* ∇^{B_1} *of* G_5.

According to [28], the Lie algebra $\mathfrak{g}_5$ has the following structure:

$$[e_1, e_2] = 0, \quad [e_1, e_3] = Ae_1 + Be_2, \quad [e_2, e_3] = Ce_1 + De_2, \quad A + D \neq 0, \quad AC + BD = 0.$$

From [20], we have the ρ^{B_1} concerning connection ∇^{B_1} of G_5: $\rho^{B_1}(e_i, e_j) = 0$. For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_5, the Lie derivative of the Ricci tensor associated with ∇^{B_1} along X is $L_X \rho^{B_1} = 2\lambda g$. Since $\rho^{B_1} = 0$, then for $\lambda = 0$ the equations holds automatically.

3.6. *Conformal Ricci collineations associated with* ∇^{B_1} *of* G_6.

According to [28], the Lie algebra $\mathfrak{g}_6$ has the following structure:

$$[e_1, e_2] = Ae_2 + Be_3, \quad [e_1, e_3] = Ce_2 + De_3, \quad [e_2, e_3] = 0, \quad A + D \neq 0, \quad AC - BD = 0.$$

Theorem 6. ([3]) *The conformal Ricci collineations associated with the connection ∇^{B_1} on the Lie group G_6 are characterized as follows:*

1) $A = 0$, $B = 0$, $\lambda = 0$, and $V = span\{e_1, e_2, e_3\}$;

2) $A \neq 0$, $C = 0$, $X_1 = X_2 = 0$, $\lambda = 0$, and $V = span\{e_3\}$;

3) $A \neq 0$, $C \neq 0$, $X_1 = 0$, $X_2 = -\frac{C}{A}X_3$, $\lambda = 0$, and $V = span\{-\frac{C}{A}e_2 + e_3\}$.

Proof. The Ricci tensor components for the connection ∇^{B_1} on G_6 are given by [20]

$$\rho^{B_1}(e_i, e_j) = \begin{pmatrix} -(A^2 + BC) & 0 & 0 \\ 0 & -A^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_6, the conformal Ricci collineation condition $L_X\rho^{B_1} = 2\lambda g$ yields the system

$$\begin{cases} 2\lambda = 0, \\ A^2(AX_2 + CX_3) = 0, \\ 2A^3X_1 = 0. \end{cases}$$

From the first equation we obtain $\lambda = 0$.

- If $A = 0$, from the conditions $AC - BD = 0$ and $A + D \neq 0$, we have $B = 0$. The first equation gives $\lambda = 0$, and others are automatically satisfied. Thus, $V = \text{span}\{e_1, e_2, e_3\}$, corresponding to case (1).
- If $A \neq 0$ and $C = 0$, the first equation gives $\lambda = 0$, and the second and third equations yield $X_1 = X_2 = 0$, w
ith e_3 free. Thus, $V = \text{span}\{e_3\}$, which leads to case (2).
- If $A \neq 0$ and $C \neq 0$, the first equation gives $\lambda = 0$, and the second and third equations become $X_1 = 0$ and $X_2 = -\frac{C}{A}X_3$. Thus, $V = \text{span}\{-\frac{C}{A}e_2 + e_3\}$, and we have case (3).

□

3.7. Conformal Ricci collineations associated with ∇^{B_1} of G_7.

According to [28], the Lie algebra $\mathfrak{g}_7$ has the following structure:

$$[e_1, e_2] = -Ae_1 - Be_2 - Be_3, \quad [e_1, e_3] = Ae_1 + Be_2 + Be_3, \quad [e_2, e_3] = Ce_1 + De_2 + De_3,$$

where $A + D \neq 0$ and $AC = 0$.

Theorem 7. *The conformal Ricci collineations associated with the connection ∇^{B_1} on the Lie group G_7 are characterized as follows:*

1) $A = B = 0$, $C \neq 0$, $D \neq 0$, $\lambda = 0$, $X_2 = X_3 = 0$, and $V = span\{e_1\}$;

2) $A = 0$, $B \neq 0$, $C \neq 0$, $D \neq 0$, $X_1 = -\frac{4\lambda M}{9B^3D^2(1-k)}$, $X_2 = \frac{2\lambda k}{3B^2D(1-k)}$, $X_3 = \frac{2\lambda}{3B^2D(1-k)}$ for $15B^3C + 9B^2C^2 + 12B^2D^2 + \frac{27}{2}BCD^2 + 18D^4 = 0$;

3) $A = B = C = 0$, $D \neq 0$, $\lambda = 0$, $X_2 = X_3 = 0$, and $V = span\{e_1\}$;

4) $A = C = 0$, $B \neq 0$, $D \neq 0$, $\lambda = 0$, $X_1 = X_2 = X_3 = 0$*;*

5) $A \neq 0$, $B = C = D = 0$, $\lambda = 0$, $X_1 = 0$, $X_2 = X_3$*, and* $V = span\{e_2 + e_3\}$*;*

6) $A \neq 0$, $B \neq 0$, $C = D = 0$, $\lambda = 0$, $X_1 = 0$, $X_2 = X_3$*, and* $V = span\{e_2 + e_3\}$*.*

Proof. The Ricci tensor components for the connection ∇^{B_1} on G_7 are given by [20]

$$\rho^{B_1}(e_i, e_j) = \begin{pmatrix} -A^2 & \frac{1}{2}B(D-A) & B(A+D) \\ \frac{1}{2}B(D-A) & -(A^2+B^2+BC) & D^2+\frac{1}{2}(BC+AD) \\ B(A+D) & D^2+\frac{1}{2}(BC+AD) & 0 \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_7, the conformal Ricci collineation condition $L_X \rho^{B_1} = 2\lambda g$ yields the system

$$\begin{cases} (2A^3 - AB^2 - 3B^2D)X_2 - (2A^3 - AB^2 - 3B^2D)X_3 = 2\lambda, \\ (-A^3 + \frac{1}{2}AB^2 + \frac{3}{2}B^2D)X_1 + B(\frac{3}{2}A^2 - AD + B^2 + \frac{1}{2}BC - D^2)X_2 \\ \quad +(-\frac{3}{2}A^2B + \frac{3}{2}ABD - B^3 - \frac{1}{2}B^2C + \frac{5}{2}BD^2)X_3 = 0, \\ (-A^3 + \frac{1}{2}AB^2 + \frac{3}{2}B^2D)X_1 + (-A^2B - 2ABD - \frac{5}{2}BD^2 - \frac{1}{2}B^2C)X_2 \\ \quad +(A^2B + \frac{3}{2}ABD + BD^2 + \frac{1}{2}B^2C)X_3 = 0, \\ (2ABD - 3A^2B - 2B^3 - B^2C + 2BD^2)X_1 + (-2A^2D - 2B^2D + AD^2 + 2D^3)X_3 = 2\lambda, \\ (-\frac{1}{2}A^2B + \frac{5}{2}ABD + 2BD^2 - B^3)X_1 + (A^2D + B^2D - D^3 - \frac{1}{2}AD^2)X_2 \\ \quad +(\frac{3}{2}BCD + D^3 + \frac{1}{2}AD^2)X_3 = 0, \\ B(A^2 + \frac{3}{2}AD + D^2 + \frac{1}{2}BC)X_1 - (\frac{3}{2}BCD + D^3 + \frac{1}{2}AD^2)X_2 = -2\lambda. \end{cases}$$

Since $AC = 0, A + D \neq 0$, we consider the following three subcases.

Case 1: $A = 0$**,** $C \neq 0$**,** $D \neq 0$. The system reduces to

$$\begin{cases} -3B^2DX_2 + 3B^2DX_3 = 2\lambda, \\ \frac{3}{2}B^2DX_1 + B(B^2 + \frac{1}{2}BC - D^2)X_2 - B(B^2 + \frac{1}{2}BC - \frac{5}{2}D^2)X_3 = 0, \\ \frac{3}{2}B^2DX_1 - B(\frac{5}{2}D^2 + \frac{1}{2}BC)X_2 + B(D^2 + \frac{1}{2}BC)X_3 = 0, \\ -B(2B^2 + BC - 2D^2)X_1 - (2B^2D - 2D^3)X_3 = 2\lambda, \\ B(2D^2 - B^2)X_1 + (B^2D - D^3)X_2 + (\frac{3}{2}BCD + D^3)X_3 = 0, \\ B(D^2 + \frac{1}{2}BC)X_1 - (\frac{3}{2}BCD + D^3)X_2 = -2\lambda. \end{cases}$$

If $B = 0$, the first equation implies $\lambda = 0$. Substituting into the last equation and combining with the fourth yields $X_2 = X_3 = 0$, giving $V = span\{e_1\}$, and we have case (1).

If $B \neq 0$, the second and third equations imply

$$(B^2 + BC + \frac{3}{2}D^2)X_2 + (-B^2 - BC + \frac{3}{2}D^2)X_3 = 0.$$

Let $k = \frac{2B^2+2BC-3D^2}{2B^2+2BC+3D^2}$, so that $X_2 = kX_3$. Substituting into the third equation gives

$$\frac{3}{2}BDX_1 + [-(\frac{5}{2}D^2 + \frac{1}{2}BC)k + D^2 + \frac{1}{2}BC]X_3 = 0.$$

Setting $M = -(\frac{5}{2}D^2 + \frac{1}{2}BC)k + D^2 + \frac{1}{2}BC$, we obtain $X_1 = -\frac{2}{3BD}MX_3$. Substituting into the remaining equations yields the compatibility condition

$$15B^3C + 9B^2C^2 + 12B^2D^2 + \frac{27}{2}BCD^2 + 18D^4 = 0.$$

Then, we formulate the following result:

- If this holds, the system admits a non-trivial solution:

$$X_1 = -\frac{4\lambda M}{9B^3D^2(1-k)}, \quad X_2 = \frac{2\lambda k}{3B^2D(1-k)}, \quad X_3 = \frac{2\lambda}{3B^2D(1-k)}.$$

  This situation is included in case (2)
- Otherwise, only the trivial solution $X_1 = 0, X_2 = 0, X_3 = 0, \lambda = 0$ exists.

Next, for $A = C = 0, D \neq 0$, the system becomes

$$\begin{cases} -3B^2DX_2 + 3B^2DX_3 = 2\lambda, \\ \frac{3}{2}B^2DX_1 + B(B^2 - D^2)X_2 - B(B^2 - \frac{5}{2}D^2)X_3 = 0, \\ \frac{3}{2}B^2DX_1 - \frac{5}{2}BD^2X_2 + BD^2X_3 = 0, \\ -B(2B^2 - 2D^2)X_1 - (2B^2D - 2D^3)X_3 = 2\lambda, \\ B(2D^2 - B^2)X_1 + (B^2D - D^3)X_2 + D^3X_3 = 0, \\ BD^2X_1 - D^3X_2 = -2\lambda. \end{cases}$$

- If $B = 0$, the system simplifies to

$$\begin{cases} 0 = 2\lambda, \\ D^3X_3 = 2\lambda, \\ D^3X_2 - D^3X_3 = 0, \\ D^3X_2 = 2\lambda, \end{cases}$$

  which implies $X_2 = X_3 = 0$, so $V = span\{e_1\}$, and we obtain case (3).
- If $B \neq 0$, the second and third equations give:

$$(B^2 + \frac{3}{2}D^2)X_2 + (-B^2 + \frac{3}{2}D^2)X_3 = 0.$$

  Let $k = \frac{2B^2-3D^2}{2B^2+3D^2}$ so that $X_2 = kX_3$. Substituting into the third equation yields $X_1 = \frac{D}{3B}(5k-2)X_3$. From the first equation, we have $3B^2D(1-k)X_3 = 2\lambda$. The remaining equations imply the compatibility condition

$$B^2 = D^2, \quad 9B^2 = 2D^2,$$

  which is contradictory. Hence, $X_1 = X_2 = X_3 = 0$, then gives case (4).

Finally, if $A \neq 0, C = D = 0$. The system reduces to:

$$\begin{cases} (2A^3 - AB^3)X_2 - (2A^3 - AB^3)X_3 = 2\lambda, \\ (-A^3 + \frac{1}{2}AB^2)X_1 + B(\frac{3}{2}A^2 + B^2)X_2 - (\frac{3}{2}A^2B + B^3)X_3 = 0, \\ (-A^3 + \frac{1}{2}AB^2)X_1 - A^2BX_2 + A^2BX_3 = 0, \\ -(3A^2B + 2B^3)X_1 = 2\lambda, \\ (\frac{1}{2}A^2B + B^3)X_1 = 0, \\ A^2BX_1 = -2\lambda. \end{cases}$$

- If $B = 0$, the last equation gives $\lambda = 0$, and the first two imply $X_1 = 0$, and $X_2 = X_3$ so $V = \text{span}\{e_2 + e_3\}$, this leads to case (5).
- If $B \neq 0$, the fifth equation yields $X_1 = 0$. Substituting into the fourth gives $\lambda = 0$, and the first equation implies $X_2 = X_3$, so again $V = \text{span}\{e_2 + e_3\}$, and we obtain case (6).

□

4. Conformal Ricci collineations associated with the Bott connection ∇^{B_2} on three-dimensional Lorentzian Lie group

In this section, we present a complete classification of three-dimensional Lorentzian Lie groups that admit conformal Ricci collineations associated with the Bott connection ∇^{B_2}. In the following theorems, V denotes the solution space of left-invariant vector fields satisfying the conformal Ricci collineation condition (2.3)

4.1. Conformal Ricci collineations associated with ∇^{B_2} of G_1.

Theorem 8. *The conformal Ricci collineations associated with the connection ∇^{B_2} on the Lie group G_1 are characterized as follows:*

1) $A \neq 0$, $B = 0$, $\lambda = 0$, $X_1 = X_2 = X_3 = 0$;

2) $A^2 = B^2$, $\lambda = 0$, $X_1 = X_2 = X_3 = 0$;

3) $A^2 \neq B^2$, $\lambda = 0$, $X_1 = X_2 = X_3 = 0$.

Proof. The Ricci tensor components for the connection ∇^{B_2} on G_1 are given by [20]

$$\rho^{B_2}(e_i, e_j) = \begin{pmatrix} A^2 - B^2 & \frac{1}{2}AB & -AB \\ \frac{1}{2}AB & 0 & \frac{1}{2}A^2 \\ -AB & \frac{1}{2}A^2 & B^2 - A^2 \end{pmatrix}. \tag{4.1}$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_1, the conformal Ricci collineation condition

$L_X\rho^{B_2} = 2\lambda g$ yields the system

$$\begin{cases} 2A^3X_2 - (2A^3 - AB^2)X_3 = 2\lambda, \\ -A^3X_1 - B^3X_3 = 0, \\ (A^3 - \frac{1}{2}AB^2)X_1 - \frac{1}{2}A^2BX_2 + \frac{1}{2}A^2BX_3 = 0, \\ (A^3 - AB^2)X_3 = 2\lambda, \\ (B^3 + \frac{1}{2}A^2B)X_1 - (\frac{1}{2}A^3 + \frac{1}{2}AB^2)X_2 - \frac{1}{2}A^3X_3 = 0, \\ -A^2BX_1 + AB^2X_2 = -2\lambda. \end{cases}$$

Case 1: $B = 0$. The last equation implies $\lambda = 0$. Substituting into the system yields $X_1 = X_2 = X_3 = 0$, and we have case (1).

Case 2: $A^2 = B^2$. The system reduces to

$$\begin{cases} 2A^3X_2 - A^3X_3 = 2\lambda, \\ AX_1 + BX_3 = 0, \\ \frac{1}{2}AX_1 - \frac{1}{2}BX_2 + \frac{1}{2}BX_3 = 0, \\ 0 = 2\lambda, \\ \frac{3}{2}BX_1 - AX_2 - \frac{1}{2}AX_3 = 0, \\ BX_1 - AX_2 = -2\lambda. \end{cases}$$

Since $\lambda = 0$, the first and last equations imply $2X_2 = X_3$ and $BX_1 = AX_2$. Together with the second equation, this leads to $X_1 = X_2 = X_3 = 0$, which corresponds to case (2).

Case 3: $A^2 \neq B^2$ **and** $B \neq 0$. First, assume $\lambda \neq 0$. Then, from the fourth equation we obtain:

$$X_3 = \frac{2\lambda}{A(A^2 - B^2)}.$$

Substituting into the first two equations gives

$$X_1 = -\frac{2\lambda B^2}{A^4(A^2 - B^2)}, \quad X_2 = \frac{\lambda(3A^2 - 2B^2)}{A^3(A^2 - B^2)}.$$

Combining with the last equation leads to the condition $B^2+2A^2 = 0$, which contradicts the assumption $A, B \neq 0$. Hence, $\lambda = 0$, and the first and fourth equations imply $X_1 = X_2 = X_3 = 0$, and we obtain case (3). □

4.2. Conformal Ricci collineations associated with ∇^{B_2} *of* G_2*.*

Theorem 9. *The conformal Ricci collineations associated with the connection* ∇^{B_2} *on the Lie group* G_2 *are characterized as follows:*

1) $B = 0$, $\lambda = 0$, $X_1 = X_3 = 0$, *and* $V = span\{e_2\}$;

2) $A = 0$, $B \neq 0$, $\lambda = 0$, $X_1 = 0$, $X_2 = -\frac{C}{B}X_3$, *and* $V = span\{-\frac{C}{B}e_2 + e_3\}$;

3) $A, B \neq 0$, $\lambda = 0$, $X_1 = X_2 = X_3 = 0$.

Proof. The Ricci tensor components for the connection ∇^{B_2} on G_2 are given by [20]

$$\rho^{B_2}(e_i, e_j) = \begin{pmatrix} -(B^2+C^2) & 0 & 0 \\ 0 & 0 & -\frac{1}{2}AC \\ 0 & -\frac{1}{2}AC & AB+C^2 \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_2, the conformal Ricci collineation condition $L_X\rho^{B_2} = 2\lambda g$ yields the system

$$\begin{cases} 0 = 2\lambda, \\ \frac{1}{2}ABCX_2 - (\frac{1}{2}AC^2 + AB^2)X_3 = 0, \\ -BC^2X_2 - C^3X_3 = 0, \\ -ABCX_1 = 2\lambda, \\ B(AB+C^2)X_1 = 0, \\ C(AB+2C^2)X_1 = -2\lambda. \end{cases} \tag{4.2}$$

From the first equation in (4.2), we obtain $\lambda = 0$. We now analyze different cases:

Case 1: $B = 0$. System (4.2) reduces to

$$\begin{cases} AC^2X_3 = 0, \\ CX_3 = 0, \\ 2C^2X_1 = 0. \end{cases}$$

Since $C \neq 0$, we get $X_1 = X_3 = 0$ with X_2 free. Thus, $V = span\{e_2\}$, and we obtain case (1).

Case 2: $A = 0$**,** $B \neq 0$. System (4.2) becomes

$$\begin{cases} BX_2 + CX_3 = 0, \\ BC^2X_1 = 0, \\ 2C^2X_1 = 0. \end{cases}$$

From this, we obtain $X_1 = 0$ and $BX_2 + CX_3 = 0$. Thus, $V = span\{-\frac{C}{B}e_2 + e_3\}$, corresponding to case (2).

Case 3: $A \neq 0$**,** $B \neq 0$. From the fourth equation, we have $X_1 = 0$, and (4.2) forms a homogeneous system:

$$\begin{cases} \frac{1}{2}BCX_2 - (\frac{1}{2}C^2 + B^2)X_3 = 0, \\ BX_2 + CX_3 = 0. \end{cases}$$

The determinant of the coefficient matrix is $B(B^2 + C^2)$. Thus, the only solution is $X_2 = X_3 = 0$. Then, the system admits only the trivial solution $X_1 = X_2 = X_3 = 0$, and this leads to case (3). □

4.3. Conformal Ricci collineations associated with ∇^{B_2} of G_3.

Theorem 10. *The conformal Ricci collineations associated with the connection ∇^{B_2} on the Lie group G_3 are characterized as follows:*

1) $ABC = 0$, $\lambda = 0$, and $V = span\{e_1, e_2, e_3\}$;

2) $ABC \neq 0$, $X_1 = X_3 = 0$, $\lambda = 0$, *and* $V = span\{e_2\}$.

Proof. The Ricci tensor components for the connection ∇^{B_2} on G_3 are given by [20]

$$\rho^{B_2}(e_i, e_j) = \begin{pmatrix} -BC & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & AB \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_3, the conformal Ricci collineation condition $L_X \rho^{B_2} = 2\lambda g$ yields the system

$$\begin{cases} 0 = 2\lambda, \\ -ABCX_3 = 0, \\ ABCX_1 = 0. \end{cases}$$

We now analyze the cases:

Case 1: $ABC = 0$. The equations above are automatically satisfied. Thus, all equations are satisfied for any X with $\lambda = 0$, giving $V = span\{e_1, e_2, e_3\}$, and we have case (1).

Case 2: $ABC \neq 0$. The second and third equations give $X_1 = X_3 = 0$ with X_2 free. Thus, $V = span\{e_2\}$, and this leads to case (2). □

4.4. Conformal Ricci collineations associated with ∇^{B_2} *of* G_4.

Theorem 11. *The conformal Ricci collineations associated with the connection* ∇^{B_2} *on the Lie group* G_4 *are characterized as follows:*

1) $A = 0$, $\lambda = 0$, $X_1 = 0$, $(B - 2\eta)X_2 = X_3$, *and* $V = span\{e_2 + (B - 2\eta)e_3\}$;

2) $A = -1$, $\eta = 1$, $B = 2$, $\lambda = 0$, $X_2 = X_3 = 0$, *and* $V = span\{e_1\}$;

3) $A = 1$, $\eta = -1$, $B = -2$, $\lambda = 0$, $X_2 = X_3 = 0$, *and* $V = span\{e_1\}$;

4) $A \neq 0$, $B \neq 2\eta$, $\lambda = 0$, $X_1 = 0$, $X_2 = \frac{2(B-1)^2-1}{2-B} X_3$, *and* $V = span\{\frac{2(B-1)^2-1}{2-B} e_2 + e_3\}$, *for* $4(B-1)^2(B-2) + A(B-3) = 0$;

5) $A \neq 0$, $B \neq 2\eta$, $\lambda = 0$, $X_1 = X_2 = X_3 = 0$, *for* $4(B-1)^2(B-2) + A(B-3) \neq 0$.

Proof. The Ricci tensor components for the connection ∇^{B_2} on G_4 are given by [20]

$$\rho^{B_2}(e_i, e_j) = \begin{pmatrix} -(B-\eta)^2 & 0 & 0 \\ 0 & 0 & \frac{1}{2}A \\ 0 & \frac{1}{2}A & AB+1 \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_4, the conformal Ricci collineation condition $L_X \rho^{B_2} = 2\lambda g$ yields the system

$$\begin{cases} 0 = 2\lambda, \\ \frac{1}{2}A(2\eta - B)X_2 + (\frac{1}{2}A - A(B-\eta)^2)X_3 = 0, \\ (A(B-\eta)^2 - \frac{1}{2}A + (2\eta - B)(AB+1))X_2 + (\frac{1}{2}A + (AB+1))X_3 = 0, \\ -A(2\eta - B)X_1 = 2\lambda, \\ (B - 2\eta)(AB+1)X_1 = 0, \\ -(AB+2)X_1 = -2\lambda. \end{cases} \tag{4.3}$$

From the first equation, we obtain $\lambda = 0$. We now analyze different cases:

Case 1: $A = 0$. System (4.3) reduces to

$$\begin{cases} (2\eta - B)X_2 + X_3 = 0, \\ (B - 2\eta)X_1 = 0, \\ X_1 = 0. \end{cases}$$

This implies $X_1 = 0$ and $(B - 2\eta)X_2 = X_3$, and we obtain case (1).

Case 2: $A \neq 0$ **and** $B = 2\eta$. Then, system (4.3) becomes

$$\begin{cases} (\frac{1}{2}A - A(B - \eta)^2)X_3 = 0, \\ (A(B - \eta)^2 - \frac{1}{2}A)X_2 + (\frac{1}{2}A + (AB + 1))X_3 = 0, \\ (AB + 2)X_1 = 0. \end{cases} \tag{4.4}$$

- For $\eta = 1$, we have $B = 2$, and Eq (4.4) simplify to

$$\begin{cases} -\frac{1}{2}AX_3 = 0, \\ \frac{1}{2}AX_2 + (\frac{1}{2}A + 2A + 1)X_3 = 0, \\ (2A + 2)X_1 = 0. \end{cases}$$

  If $A = -1$, then $X_2 = X_3 = 0$, and $V = span\{e_1\}$. If $A \neq -1$, then $X_1 = X_2 = X_3 = 0$, and this leads to case (2).
- For $\eta = -1$, we have $B = -2$, and system (4.4) becomes

$$\begin{cases} -\frac{1}{2}AX_3 = 0, \\ \frac{1}{2}AX_2 + (1 - \frac{3}{2}A)X_3 = 0, \\ (A - 1)X_1 = 0. \end{cases}$$

  If $A = 1$, then $X_2 = X_3 = 0$, and $V = span\{e_1\}$. If $A \neq 1$, then $X_1 = X_2 = X_3 = 0$, and we have case (3).

Case 3: $A \neq 0$**, and** $2\eta - B \neq 0$. Similar analysis shows that $X_1 = 0$, and the system reduces to

$$\begin{cases} \frac{1}{2}A(2\eta - B)X_2 + (\frac{1}{2}A - A(B - \eta)^2)X_3 = 0, \\ (A(B - \eta)^2 - \frac{1}{2}A + (2\eta - B)(AB + 1))X_2 + (\frac{1}{2}A + (AB + 1))X_3 = 0. \end{cases}$$

This is a homogeneous linear system in X_2 and X_3. The determinant of the coefficient matrix is proportional to $4(B - 1)^2(B - 2) + A(B - 3) = 0$. If this condition holds, the system admits nontrivial solutions satisfying $(2(B - 1)^2 - 1)X_3 = (2 - B)X_2$, which gives case (4). Otherwise, only the trivial solution $X_1 = X_2 = X_3 = 0$ exists, and we obtain case (5). □

4.5. Conformal Ricci collineations associated with ∇^{B_2} *of* G_5.

Theorem 12. *The conformal Ricci collineations associated with the connection* ∇^{B_2} *on the Lie group* G_5 *are characterized as follows:*

1) $A = 0$, $\lambda = 0$, *and* $V = span\{e_1, e_2, e_3\}$*;*

2) $A \neq 0$, $C = 0$, $\lambda = 0$, $X_1 = X_3 = 0$, $V = span\{e_2\}$*;*

3) $A \neq 0$, $C \neq 0$, $\lambda = 0$, $X_3 = 0$, $X_1 = -\frac{C}{A}X_2$, *and* $V = span\{\frac{C}{A}e_1 - e_2\}$.

Proof. The Ricci tensor components for the connection ∇^{B_2} on G_5 are given by [20]

$$\rho^{B_2}(e_i, e_j) = \begin{pmatrix} A^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -(BC + A^2) \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_5, the conformal Ricci collineation condition $L_X \rho^{B_2} = 2\lambda g$ yields the system

$$\begin{cases} 2A^3X_3 = 2\lambda, \\ A^2CX_3 = 0, \\ -A^3X_1 - A^2CX_2 = 0, \\ 0 = 2\lambda. \end{cases}$$

- If $A = 0$, the last equation gives $\lambda = 0$. Thus, any vector field X is a CRC, giving $V = span\{e_1, e_2, e_3\}$, and we have case (1).
- If $A \neq 0$ and $C = 0$, the last equation gives $\lambda = 0$. The other equations give $X_1 = X_3 = 0$ with X_2 free. Thus, $V = span\{e_2\}$, which leads to case (2).
- If $A \neq 0$ and $C \neq 0$, the last equation gives $\lambda = 0$. The other equations give $X_3 = 0$ and $X_1 = -\frac{C}{A}X_2$. Thus, $V = span\{\frac{C}{A}e_1 - e_2\}$, which gives case (3).

□

4.6. Conformal Ricci collineations associated with ∇^{B_2} *of* G_6.

Theorem 13. *The conformal Ricci collineations associated with the connection* ∇^{B_2} *on the Lie group* G_6 *are characterized as follows:*

1) $D = 0$, $\lambda = 0$, *and* $V = span\{e_1, e_2, e_3\}$*;*

2) $D \neq 0$, $B = 0$, $\lambda = 0$, $X_1 = X_3 = 0$, *and* $V = span\{e_2\}$*;*

3) $D \neq 0$, $B \neq 0$, $\lambda = 0$, $X_1 = 0$, $X_2 = -\frac{D}{B}X_3$, *and* $V = span\{\frac{D}{B}e_2 - e_3\}$.

Proof. The Ricci tensor components for the connection ∇^{B_2} on G_6 are given by [20]

$$\rho^{B_2}(e_i, e_j) = \begin{pmatrix} -(D^2 + BC) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D^2 \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_6, the conformal Ricci collineation condition $L_X \rho^{B_2} = 2\lambda g$ yields the system

$$\begin{cases} 0 = 2\lambda, \\ BD^2X_2 + D^3X_3 = 0, \\ BD^2X_1 = 0, \\ -2D^3X_1 = -2\lambda. \end{cases}$$

- If D = 0, the first equation gives $\lambda = 0$, and other equations are automatically satisfied. Thus, any vector field X is a CRC, so $V = span\{e_1, e_2, e_3\}$. This leads to case (1);
- If $D \neq 0$ and $B = 0$, by the second and fourth equations we have $X_1 = X_3 = 0$. Thus, $V = span\{e_2\}$, which corresponds to case (2);
- If $D \neq 0$ and $B \neq 0$, the first equation gives $\lambda = 0$, and then the third equation becomes $X_1 = 0$. From the second equation, we have $X_2 = -\frac{D}{B}X_3$. Thus, $V = span\{-\frac{D}{B}e_2 + e_3\}$, which gives case (3).

□

4.7. Conformal Ricci collineations associated with ∇^{B_2} of G_7.

Theorem 14. *The conformal Ricci collineations associated with the connection ∇^{B_2} on the Lie group G_7 are characterized as follows:*

1) $A = B = 0$, $C \neq 0$, $D \neq 0$, $\lambda = 0$, $X_2 = X_3 = 0$, and $V = span\{e_1\}$;

2) $A = 0$, $B \neq 0$, $C \neq 0$, $D \neq 0$, $\lambda = 0$, $X_1 = X_2 = X_3 = 0$;

3) $A = B = C = 0$, $D \neq 0$, $\lambda = 0$, $X_2 = X_3 = 0$, and $V = span\{e_1\}$;

4) $A = C = 0$, $B \neq 0$, $D \neq 0$, $\lambda = 0$, $X_1 = X_2 = X_3 = 0$;

5) $A \neq 0$, $B = C = D = 0$, $\lambda = 0$, $X_1 = 0$, $X_2 = X_3$, $V = span\{e_2 + e_3\}$;

6) $A \neq 0$, $C = D = 0$, $B = \frac{1}{2}A^2$, $\lambda = 0$, $X_1 = 0$, $X_2 = X_3$, and $V = span\{e_2 + e_3\}$;

7) $A \neq 0$, $B \neq 0$, $C = D = 0$, $\lambda = 0$, $X_1 = 0$, $X_2 = X_3$, and $V = span\{e_2 + e_3\}$, for $B \neq \frac{1}{2}A^2$.

Proof. The Ricci tensor components for the connection ∇^{B_2} on G_7 are given by [20]

$$\rho^{B_2}(e_i, e_j) = \begin{pmatrix} A^2 & B(A+D) & \frac{1}{2}B(D-A) \\ B(A+D) & 0 & D^2 + \frac{1}{2}(BC+AD) \\ \frac{1}{2}B(D-A) & D^2 + \frac{1}{2}(BC+AD) & B^2 - A^2 - BC \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_7, the conformal Ricci collineation condition $L_X\rho^{B_2} = 2\lambda g$ yields the system

$$\begin{cases} -\frac{1}{2}(X_2 - X_3)(4A^3 + 3AB^2 + 5B^2D) = 2\lambda, \\ (A^3 + \frac{1}{2}AB^2 + \frac{3}{2}B^2D)X_1 - (A^2B + BD^2 + \frac{1}{2}B^2C + \frac{3}{2}ABD)X_2 \\ \quad +(A^2B + \frac{5}{2}BD^2 + \frac{1}{2}B^2C + 2ABD)X_3 = 0, \\ (A^3 + \frac{1}{2}AB^2 + \frac{3}{2}B^2D)X_1 + (-B^3 + \frac{3}{2}A^2B - \frac{3}{2}ABD - \frac{5}{2}BD^2 + \frac{1}{2}B^2C)X_2 \\ \quad +(B^3 - \frac{3}{2}A^2B + ABD + BD^2 - \frac{1}{2}B^2C)X_3 = 0, \\ (2A^2B + 3ABD + 2BD^2 + B^2C)X_1 + (3BCD + 2D^3 + AD^2)X_3 = 2\lambda, \\ (B^3 + 2BD^2 + \frac{5}{2}ABD - \frac{1}{2}A^2B)X_1 - (\frac{3}{2}BCD + D^3 + \frac{1}{2}AD^2)X_2 + (D^3 \\ \quad +\frac{1}{2}AD^2 + B^2D - A^2D)X_3 = 0, \\ (2B^3 - 3A^2B + 2ABD + 2BD^2 - B^2C)X_1 + (2A^2D - AD^2 - 2B^2D - 2D^3)X_2 = -2\lambda. \end{cases} \tag{4.5}$$

Since $AC = 0$, and $A + D \neq 0$, we consider the following three subcases.

Case 1: $A = 0$**,** $C \neq 0$**,** $D \neq 0$. System (4.5) reduces to

$$\begin{cases} -\frac{5}{2}B^2DX_2 + \frac{5}{2}B^2DX_3 = 2\lambda, \\ \frac{3}{2}B^2DX_1 - (BD^2 + \frac{1}{2}B^2C)X_2 + (\frac{5}{2}BD^2 + \frac{1}{2}B^2C)X_3 = 0, \\ \frac{3}{2}B^2DX_1 - (B^3 + \frac{5}{2}BD^2 - \frac{1}{2}B^2C)X_2 + (B^3 + BD^2 - \frac{1}{2}B^2C)X_3 = 0, \\ (2BD^2 + B^2C)X_1 + (3BCD + 2D^3)X_3 = 2\lambda, \\ (B^3 + 2BD^2)X_1 - (\frac{3}{2}BCD + D^3)X_2 + (B^2D + D^3)X_3 = 0, \\ (2B^3 + 2BD^2 - B^2C)X_1 - (2B^2D + 2D^3)X_2 = -2\lambda. \end{cases} \tag{4.6}$$

Subcase 1.1: If $B = 0$, we obtain $\lambda = 0$, $X_2 = X_3 = 0$, so $V = span\{e_1\}$, corresponding to case (1).
Subcase 1.2: If $B \neq 0$, from the fourth and last equations in (4.6), we obtain

$$(2B^3 + 4BD^2)X_1 - (2B^2D + 2D^3)X_2 + (3BCD + 2D^3)X_3 = 0.$$

Combining with the fifth equation in (4.6) gives $(3C - 2B)(X_2 + X_3) = 0$.

- If $3C = 2B$, then from the first equation in (4.6), $X_3 - X_2 = \frac{4\lambda}{5B^2D}$. The fifth equation in (4.6) implies $(B^3 + 2BD^2)X_1 + (B^2D + D^3)(X_3 - X_2) = 0$. Substituting the expression for $X_3 - X_2$ yields $X_1 = -\frac{4\lambda(B^2+D^2)}{5B^3(B^2+2D^2)}$. From the second and third equations, $(2B^2 + 9D^2)X_2 = (2B^2 - 9D^2)X_3$. Together with $X_3 - X_2 = \frac{4\lambda}{5B^2D}$, we obtain

$$X_2 = \frac{2\lambda(2B^2 - 9D^2)}{45B^2D^3}, \quad X_3 = \frac{2\lambda(2B^2 + 9D^2)}{45B^2D^3}.$$

  Substituting into the second equation in (4.6) leads to $2B^4 + 5D^2B^2 + 8D^4 = 0$, a contradiction. Hence $\lambda = 0$ and $X_1 = X_2 = X_3 = 0$, and we obtain case (2).
- If $X_2 + X_3 = 0$, then the second and third equations in (4.6) imply $2B^2(B - C)X_3 = 0$. If $B = C$, the first and second equations give

$$X_3 = \frac{2\lambda}{5B^2D}, \quad X_1 = -\frac{2\lambda(7D^2 + 2B^2)}{115B^3D^2}.$$

  Substituting into the fourth equation in (4.6) yields $8D^4 + 17B^2D^2 + 2B^4 = 0$, again a contradiction. Thus, $X_2 = X_3 = 0$, and consequently $X_1 = 0$, which leads to case (2).

Case 2: $A = C = 0$**,** $D \neq 0$. System (4.5) becomes

$$\begin{cases} -\frac{5}{2}B^2D(X_2 - X_3) = 2\lambda, \\ \frac{3}{2}B^2DX_1 - BD^2X_2 + \frac{5}{2}BD^2X_3 = 0, \\ \frac{3}{2}BD^2X_1 - (B^3 + \frac{5}{2}BD^2)X_2 + (B^3 + BD^2)X_3 = 0, \\ 2BD^2X_1 + 2D^3X_2 = 2\lambda, \\ (B^3 + 2BD^2)X_1 - D^3X_2 + (D^3 + B^2D)X_3 = 0, \\ (2B^3 + 2BD^2)X_1 - (2B^2D + 2D^3)X_2 = -2\lambda. \end{cases} \tag{4.7}$$

- If $B = 0$, then $\lambda = 0$. The fourth and fifth equations in (4.7) imply $X_2 = X_3 = 0$, so $V = span\{e_1\}$, and we obtain case (3).

- If $B \neq 0$, the first and fourth equations in (4.7) give

$$BDX_1 + \frac{5}{4}B^2X_2 + (D^2 - \frac{5}{4}B^2)X_3 = 0.$$

From the second and third equations in (4.7), we have

$$(2B^2 + 3D^2)X_2 = (2B^2 - 3D^2)X_3, \quad X_1 = \frac{2D}{3B}X_2 - \frac{5D}{3B}X_3.$$

Substituting into (4.7) yields $(\frac{2D^2}{3} + \frac{5}{4}B^2)(X_2 - X_3) = 0$, so $X_2 = X_3$. Then, $(2B^2 + 3D^2)X_2 = (2B^2 - 3D^2)X_3$ implies $6D^2 = 0$, contradicting $D \neq 0$. Hence, $X_2 = X_3 = 0$, and the system admits only the trivial solution, which corresponds to case (4).

Case 3: $A \neq 0$**,** $C = D = 0$. System (4.5) reduces to

$$\begin{cases} -\frac{1}{2}(X_2 - X_3)A(4A^2 + 3B^3) = 2\lambda, \\ (A^3 + \frac{1}{2}AB^2)X_1 - A^2BX_2 + A^2BX_3 = 0, \\ (A^3 + \frac{1}{2}AB^2)X_1 - (B^3 - \frac{3}{2}A^2B)X_2 + (B^3 - \frac{3}{2}A^2B)X_3 = 0, \\ 2A^2BX_1 = 2\lambda, \\ (B^3 - \frac{1}{2}A^2B)X_1, \\ (2B^3 - 3A^2B)X_1 = -2\lambda. \end{cases} \tag{4.8}$$

From the fifth equation in (4.8), we consider

1) **Subcase 3.1**: $B = 0$. Then, the first and second equations in (4.8) imply $\lambda = 0$, $X_1 = 0$, and $X_2 = X_3$, which gives case (5).

2) **Subcase 3.2**: $B = \frac{1}{2}A^2$. The second and third equations in (4.8) give $5AB^2X_1 = 0$, so $\lambda = 0$, $X_1 = 0$, and $X_2 = X_3$, which leads to case (6).

3) **Subcase 3.3**: $X_1 = 0$, $B \neq \frac{1}{2}A^2$, $B \neq 0$. Then, $\lambda = X_1 = 0$ and $X_2 = X_3$, which corresponds to case (7).

□

5. Conformal Ricci collineations associated with the Bott connection ∇^{B_3} on three-dimensional Lorentzian Lie group

In this section, we present a complete classification of three-dimensional Lorentzian Lie groups that admit conformal Ricci collineations associated with the Bott connection ∇^{B_3}. In the following theorems, V denotes the solution space of left-invariant vector fields satisfying the conformal Ricci collineation condition (2.3).

5.1. Conformal Ricci collineations associated with ∇^{B_3} of G_1.

Theorem 15. *The conformal Ricci collineations associated with the connection ∇^{B_3} on the Lie group G_1 are characterized as follows:*

1) $B = 0$*,* $\lambda = 0$*, and* $V = span\{e_1, e_2, e_3\}$*;*

2) $B \neq 0$*,* $\lambda = 0$*,* $X_1 = X_2 = X_3 = 0$*.*

Proof. The Ricci tensor components for the connection ∇^{B_3} on G_1 are given by [20]

$$\rho^{B_3}(e_i, e_j) = \begin{pmatrix} 0 & \frac{1}{2}AB & -\frac{1}{2}AB \\ \frac{1}{2}AB & -B^2 & 0 \\ -\frac{1}{2}AB & 0 & B^2 \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_1, the conformal Ricci collineation condition $L_X \rho^{B_3} = 2\lambda g$ yields the system

$$\begin{cases} AB^2X_2 - AB^2X_3 = 2\lambda, \\ -\frac{1}{2}AB^2X_1 + \frac{1}{2}A^2BX_2 + (B^3 - \frac{1}{2}A^2B)X_3 = 0, \\ \frac{1}{2}AB^2X_1 - (B^3 + \frac{1}{2}A^2B)X_2 + \frac{1}{2}A^2BX_3 = 0, \\ -A^2BX_1 - AB^2X_3 = 2\lambda, \\ A^2BX_1 + \frac{1}{2}AB^2X_2 - \frac{1}{2}AB^2X_3 = 0, \\ -A^2BX_1 - AB^2X_2 = -2\lambda. \end{cases} \tag{5.1}$$

We now analyze different cases:

Case 1: $B = 0$. The first equation in (5.1) implies $\lambda = 0$. Thus, any vector field X is a CRC, so $V = span\{e_1, e_2, e_3\}$, this leads to case (1).

Case 2: $B \neq 0$. From the first, fourth, and last equations in (5.1), we have $AX_1 + BX_2 = 0$ and $AX_1 + BX_3 = 0$, which implies $X_2 = X_3$. Substituting into the first equation yields $\lambda = 0$. From the fifth equation,

$$2A^2BX_1 + AB^2X_2 - AB^2X_3 = 0.$$

Since $X_2 = X_3$, this simplifies to $A^2BX_1 = 0$, so $X_1 = 0$. Then, from $AX_1 + BX_2 = 0$, we get $X_2 = 0$, and thus $X_3 = 0$. Therefore, $X_1 = X_2 = X_3 = 0$, and we have case (2). □

5.2. Conformal Ricci collineations associated with ∇^{B_3} *of* G_2*.*

Theorem 16. *The conformal Ricci collineations associated with the connection* ∇^{B_3} *on the Lie group* G_2 *are characterized as follows:*

1) $A = 0$*,* $\lambda = 0$*, and* $V = span\{e_1, e_2, e_3\}$*;*

2) $A \neq 0$*,* $B = 0$*,* $\lambda = 0$*,* $X_2 = X_3 = 0$*, and* $V = span\{e_1\}$*;*

3) $A \neq 0$*,* $B \neq 0$*,* $\lambda = 0$*,* $X_3 = 0$*, and* $V = span\{e_1, e_2\}$*, for* $B = C$*;*

4) $A \neq 0$*,* $B \neq 0$*,* $\lambda = 0$*,* $X_2 = X_3 = 0$*, and* $V = span\{e_1\}$*, for* $B \neq C$*.*

Proof. The Ricci tensor components for the connection ∇^{B_3} on G_2 are given by [20]

$$\rho^{B_3}(e_i, e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -AB & -AC \\ 0 & -AC & AB \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_2, the conformal Ricci collineation condition $L_X \rho^{B_3} = 2\lambda g$ yields the system

$$\begin{cases} 0 = 2\lambda, \\ (AC^2 + AB^2)X_3 = 0, \\ (AB^2 - AC^2)X_2 + 2ABCX_3 = 0. \end{cases} \tag{5.2}$$

From the first equation, we have $\lambda = 0$. We now analyze the following cases:

- If $A = 0$, the Ricci tensor is the zero matrix, so any vector field X is a CRC. Thus, $V = span\{e_1, e_2, e_3\}$, and we have case (1);
- If $A \neq 0$ and $B = 0$, the last two equations in (5.2) imply $X_2 = X_3 = 0$ with X_1 free, so $V = span\{e_1\}$, which leads to case (2);
- If $A \neq 0$ and $B \neq 0$, the second equation in (5.2) gives $X_3 = 0$. Substituting into the last equation, we find that if $B^2 = C^2$, then $V = span\{e_1, e_2\}$, and we obtain case (3); if $B^2 \neq C^2$, then $X_2 = X_3 = 0$, and $V = span\{e_1\}$, which gives case (4).

□

5.3. *Conformal Ricci collineations associated with ∇^{B_3} of G_3.*

Theorem 17. *The conformal Ricci collineations associated with the connection ∇^{B_3} on the Lie group G_3 are characterized as follows:*

1) $ABC = 0$, $\lambda = 0$, and $V = span\{e_1, e_2, e_3\}$;

2) $ABC \neq 0$, $\lambda = 0$, $X_2 = X_3 = 0$, and $V = span\{e_1\}$.

Proof. The Ricci tensor components for the connection ∇^{B_3} on G_3 are given by [20]

$$\rho^{B_3}(e_i, e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -AC & 0 \\ 0 & 0 & AB \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_3, the conformal Ricci collineation condition $L_X \rho^{B_3} = 2\lambda g$ yields the system

$$\begin{cases} 0 = 2\lambda, \\ ABCX_3 = 0, \\ -ABCX_2 = 0. \end{cases}$$

If $ABC = 0$, we have $V = span\{e_1, e_2, e_3\}$, which leads to case (1). If $ABC \neq 0$, we have $X_2 = X_3 = 0$, and $V = span\{e_1\}$, which gives case (2). □

5.4. *Conformal Ricci collineations associated with ∇^{B_3} of G_4.*

Theorem 18. *The conformal Ricci collineations associated with the connection ∇^{B_3} on the Lie group G_4 are characterized as follows:*

1) $A = 0$, $\lambda = 0$, *and* $V = span\{e_1, e_2, e_3\}$*;*

2) $A \neq 0$, $B = 2\eta + 1$, $\lambda = 0$, $X_2 = X_3$, *and* $V = span\{e_1, e_2 + e_3\}$*;*

3) $A \neq 0$, $\lambda = 0$, $X_3 = 0$, *and* $V = span\{e_1, e_2\}$ *for* $B(2\eta - B) = 1$, *and* $B \neq 2\eta + 1$*;*

4) $A \neq 0$, $B \neq 2\eta + 1, B(2\eta - B) \neq 1$, $\lambda = 0$, $X_2 = X_3 = 0$, $V = span\{e_1\}$.

Proof. The Ricci tensor components for the connection ∇^{B_3} on G_4 are given by [20]

$$\rho^{B_3}(e_i, e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & A(2\eta - B) & A \\ 0 & A & AB \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_4, the conformal Ricci collineation condition $L_X \rho^{B_3} = 2\lambda g$ yields the system

$$\begin{cases} 0 = 2\lambda, \\ (A + A(2\eta - B))X_3 = 0, \\ (AB(2\eta - B) - A)X_2 + (AB + A)X_3 = 0. \end{cases}$$

From the first equation, we have $\lambda = 0$. We now analyze different cases.

Case 1: $A = 0$. The Ricci tensor is the zero matrix, so any vector field X is a CRC. Thus, $V = span\{e_1, e_2, e_3\}$, and we have case (1).

Case 2: $A \neq 0$**,** $B = 2\eta + 1$. The second equation is automatically satisfied. The last equation becomes $-A(2\eta + 2)X_2 + A(2\eta + 2)X_3 = 0$. Simplifying, we get $X_2 = X_3$. Thus, X_1 is free, and $V = span\{e_1, e_2 + e_3\}$, which leads to case (2).

Case 3: $A \neq 0$**,** $B \neq 2\eta + 1$**,** $B(2\eta - B) = 1$. The third equation becomes $A(B + 1)X_3 = 0$, so $X_3 = 0$. Thus, $V = span\{e_1, e_2\}$, which gives case (3).

Case 4: $A \neq 0$**,** $B(2\eta - B) \neq 1$. It follows that $X_2 = X_3 = 0$, which leads to case (4). □

5.5. Conformal Ricci collineations associated with ∇^{B_3} *of* G_5*.*

Theorem 19. *The conformal Ricci collineations associated with the connection* ∇^{B_3} *on the Lie group* G_5 *are characterized as follows:*

1) $B = D = 0$, $\lambda = 0$, *and* $V = span\{e_1, e_2, e_3\}$*;*

2) $B = 0$, $D \neq 0$, $X_2 = X_3 = 0$, $\lambda = 0$, *and* $V = span\{e_1\}$*;*

3) $B \neq 0$, $D \neq 0$, $X_1 = -\frac{D}{B}X_2$, $X_3 = 0$, $\lambda = 0$, *and* $V = span\{-\frac{D}{B}e_1 + e_2\}$.

Proof. The Ricci tensor components for the connection ∇^{B_3} on G_5 are given by [20]

$$\rho^{B_3}(e_i, e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & D^2 & 0 \\ 0 & 0 & -(BC + D^2) \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_5, the conformal Ricci collineation condition $L_X\rho^{B_3} = 2\lambda g$ yields the system

$$\begin{cases} 0 = 2\lambda, \\ BD^2X_3 = 0, \\ 2D^3X_3 = 2\lambda, \\ -D^2(BX_1 + DX_2) = 0. \end{cases}$$

- If $B = D = 0$, the equations are automatically satisfied, and $V = span\{e_1, e_2, e_3\}$, which leads to case (1);
- If $B = 0$ and $D \neq 0$, the third and last equations imply $X_2 = X_3 = 0$, leading to case (2);
- If $B \neq 0$, $D \neq 0$, the third equation yields $X_3 = 0$, and the last equation gives $X_1 = -\frac{D}{B}X_2$, giving case (3).

□

5.6. Conformal Ricci collineations associated with ∇^{B_3} of G_6.

From [20], we have the ρ^{B_3} concerning connection ∇^{B_3} of G_6: $\rho^{B_3}(e_i, e_j) = 0$. For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_6, the Lie derivative of the Ricci tensor associated with ∇^{B_3} along X is $L_X\rho^{B_3} = 2\lambda g$. Since $\rho^{B_3} = 0$, then for $\lambda = 0$, the equations hold automatically.

5.7. Conformal Ricci collineations associated with ∇^{B_3} of G_7.

Theorem 20. *The conformal Ricci collineations associated with the connection ∇^{B_3} on the Lie group G_7 are characterized as follows:*

1) $BC = 0$, $\lambda = 0$, and $V = span\{e_1, e_2, e_3\}$;

2) $BC \neq 0$, $D = 0$, $X_1 = 0$, $\lambda = 0$, and $V = span\{e_2, e_3\}$;

3) $BC \neq 0$, $D \neq 0$, $\lambda = 0$, $X_2 = \frac{B}{D}X_1$, $X_3 = -\frac{B}{D}X_1$, and $V = span\{e_1 + \frac{B}{D}e_2 - \frac{B}{D}e_3\}$.

Proof. The Ricci tensor components for the connection ∇^{B_3} on G_7 are given by [20]

$$\rho^{B_3}(e_i, e_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -BC & BC \\ 0 & BC & -BC \end{pmatrix}.$$

For a left-invariant vector field $X = \sum_{i=1}^{3} K_i e_i$ on G_7, the conformal Ricci collineation condition $L_X\rho^{B_3} = 2\lambda g$ yields the system

$$\begin{cases} 0 = 2\lambda, \\ -2(BX_1 + DX_3)BC = 2\lambda, \\ 2(DX_2 - BX_1)BC = -2\lambda. \end{cases}$$

- If $BC = 0$, the Ricci tensor is the zero matrix, so any vector field X is a CRC. Thus, $V = span\{e_1, e_2, e_3\}$, and we have case (1);
- If $BC \neq 0$ and $D = 0$, the second equation implies $X_1 = 0$, which leads to case (2);

- If $BC \neq 0$ and $D \neq 0$, the second and last equations yield $X_2 = \frac{B}{D}X_1$, $X_3 = -\frac{B}{D}X_1$, and we obtain case (3).

□

6. Comparison of results for three distributions

In the previous sections, we have classified the left-invariant conformal Ricci collineations (CRCs) associated with the Bott connection for three different distributions D_1, D_2, D_3 on each of the seven three-dimensional Lorentzian Lie groups. Here, we compare these results and discuss the influence of the distribution on the existence of CRCs. Table 1 summarizes the dimension of the space of CRCs for each group and each distribution under generic parameters. The detailed conditions are given in the respective theorems.

Table 1. Dimension of the space of left-invariant conformal Ricci collineations for the Bott connection on three-dimensional Lorentzian Lie groups.

Group	D_1	D_2	D_3
G_1	0 or 1^*	0	0 or 3 (if $B = 0$)
G_2	0 or 1^*	0 or 1^*	1, 2, or 3^*
G_3	1, 3, or 3^*	1 or 3	1 or 3
G_4	1^*	1^*	1, 2, or 3^*
G_5	3 (if $\lambda = 0$)	1 or 3	1, 2, or 3
G_6	1, 3, or 3^*	1 or 3	3 (if $\lambda = 0$)
G_7	1^*	0 or 1^*	1, 2, or 3

Note:"0" means only the trivial solution $X = 0$ exists; "1", "2", "3" denote the dimension of the solution space; "$*$" indicates that non-trivial solutions exist only under specific parameter constraints.

From the table we can see that the distribution D_3 generally admits more CRCs than D_1 and D_2. For several groups, the space of CRCs for D_3 has dimension at least 1 for generic parameters, while for D_1 and D_2, non-trivial CRCs often require specific parameter conditions. This indicates that the choice of distribution significantly affects the symmetry properties of the Bott connection.

An important observation is that for the Bott connection, we find non-trivial CRCs with non-zero λ (e.g., in Theorem 2, case (2) for G_1 under D_1). In contrast, for the Levi-Civita connection, the existing results (see [14]) show that λ is always zero in the CRC equations, i.e., the CRCs reduce to Ricci collineations. Therefore, the Bott connection exhibits a richer variety of conformal symmetries, including those with non-zero conformal factor, which highlights its distinctive geometric features.

Our classification of conformal Ricci collineations associated with the Bott connection opens several avenues for future work. First, it would be interesting to extend this study to higher-dimensional Lorentzian manifolds or more general sub-Riemannian geometries. Second, the physical implications of these symmetries, particularly in gravitational theories, warrant further investigation. Finally, a comparative analysis with other affine connections, such as the Kobayashi-Nomizu connection, or Yano connection, could reveal deeper geometric insights.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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