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Matrix representation of $\lambda-$oscillator algebras and their corresponding Leibniz algebras

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Abstract: In this paper, we systematically characterize the types of oscillator algebras, ranging from the simplest case $\mathfrak{D}_4$ to the most general $\mathfrak{D}^{\lambda}_{2m+2}$. Subsequently, we construct multiple faithful matrix representations and vector field representations for $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$, and give a non-go theorem for a certain representation of $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{C})$. Furthermore, leveraging the faithful representation of $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$ via the special symplectic Lie algebra $\mathfrak{sp}_{2m+2}(\mathbb{R})$, we establish a Leibniz algebra associated with $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$.

Keywords: oscillator algebra; Lie algebra; matrix representation; algebra representation; Leibniz algebra

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1. Introduction

Real oscillator algebras were first introduced by Hilgert in [1] as double extensions of Hilbert spaces, and were subsequently renamed standard solvable Lorentzian Lie algebras in [2] to more accurately convey their intrinsic algebraic and geometric attributes. As a pivotal class of real solvable non-abelian Lie algebras, they are characterized by the existence of an invariant inner product with metric signature $(2m + 1, 1)$—a key feature that distinguishes them from other Lie algebra families and underpins their wide-ranging applications. The word oscillator originates from quantum mechanics, primarily because these algebras provide a rigorous mathematical framework for describing the dynamical behavior of harmonic oscillator systems in an m-dimensional Euclidean space. A classic illustrative case is the four-dimensional standard Lorentzian algebra, alternatively denoted as the harmonic oscillator algebra $\mathfrak{D}_4$, which serves as the Lie algebra corresponding to the harmonic oscillator group [3]—a group that has long been a cornerstone in the theoretical modeling of fundamental quantum mechanical phenomena. Notably, both oscillator groups and their associated algebras exhibit remarkable extensibility, as they can be naturally generalized to arbitrary even dimensions $2m \geq 4$. This generality has spurred extensive and in-depth research across multiple disciplines, encompassing both mathematics and physics. In the mathematical context, scholars have focused on exploring various algebraic structures associated with these algebras. For example, non-associative Leibniz algebra structures induced by oscillator algebras (wherein the latter are also referred to as diamond Lie algebras) were systematically investigated by Uguz and Camacho in [4,5]; Poisson algebras constructed on the basis of oscillator algebras were analyzed in detail by Albuerquerque et al. in [6]; invariant quantities and metric structures of these algebras were comprehensively studied by Benito and Roldán-López in [7]; and the derivations and biderivations of λ-generalized oscillator algebras, which are crucial for understanding algebraic deformations and rigidity, were thoroughly discussed by Chen and Li in [8].

In the field of physics, building on the well-established harmonic oscillator theory, oscillator groups and oscillator algebras have proven to be indispensable tools for addressing a series of frontier issues. Specifically, they have been employed in the study of Yang-Baxter equations—core equations in integrable systems by Boucetta and Medina in [9]; utilized to analyze Einstein-Yang-Mills equations, which play a vital role in classical field theory and gravitational physics by Diaz et al. in [10]; applied to explore the chronogeometric properties of electromagnetic waves by Levichev in [11]; and more recently, leveraged to investigate chiral oscillators by Bittencourt et al. in [12], among other significant research endeavors. These studies collectively highlight the profound practical value and broad application prospects of oscillator algebras in modern physics.

Against this backdrop of extensive research, the subsequent section will present a systematic investigation of three distinct types of oscillator algebras. To facilitate a clear and gradual understanding, our discussion will proceed in a hierarchical manner, starting from the simplest low-dimensional cases and progressively extending to more complex generalized structures. This organizational approach is intended to lay a solid foundation for the subsequent exploration of their deeper properties and potential applications, which constitutes the core focus of this work.

In this paper, we use $\mathbb{R}$ to denote real number field, $\mathbb{C}$ to denote complex number field, and $i = \sqrt{-1}$ to denote the imaginary unit whenever an algebra in $\mathbb{C}$ is mentioned.

2. Preliminaries

It should be noted that all algebras discussed in this paper are defined over a field of characteristic zero, specifically the real or complex number field.

2.1. Types of oscillator algebras

There mainly exist three types of oscillator algebras, from the most well-studied 4-dimen-sional case to more general $(2m+2)$-dimensional ones. In this paper, we name them harmonic oscillator algebra, general oscillator algebra, and λ-oscillator algebra.

Definition 2.1. *The harmonic oscillator algebra, also known as the four-dimensional oscillator algebra or diamond Lie algebra, contains the basis* $\{J, P, Q, T\}$ *and the following non-zero Lie brackets in the real number field, called* $\mathfrak{D}_4(\mathbb{R})$*:*

$$[J,P] = Q, [J,Q] = -P, [P,Q] = T.$$

While in the complex number field, its complexification $\mathfrak{D}_4(\mathbb{C})$ *has the complex basis* $\{J, P_+ = P - iQ,\ Q_- = P + iQ,\ T\}$*, and non-zero relations:*

$$[J,P_+] = iP_+, [J,Q_-] = -iQ_-, [P_+,Q_-] = 2iT.$$

Definition 2.2. *The real general oscillator algebra, also called the real general diamond algebra* $\mathfrak{D}_{2m+2}(\mathbb{R})$ *, is defined as having basis* $\{J, P_j, Q_j, T\}(1 \leq j \leq m)$ *with non-zero commutators:*

$$[J, P_j] = Q_j, [J, Q_j] = -P_j, [P_j, Q_j] = T.$$

The complexification of this algebra $\mathfrak{D}_{2m+2}(\mathbb{C})$ *reveals the following basis:* $\{J,\ P_k^+ = P_k - iQ_k,\ Q_k^- = P_k + iQ_k,\ T\}$ *with non-zero commutators:*

$$[J, P_k^+] = iP_k^+, [J, Q_k^-] = -iQ_k^-, [P_k^+, Q_k^-] = 2iT.$$

We can extend the real Lie algebra in Definition 2.2 by using a fixed real m-dimension-al vector λ with all positive entries, and it becomes the λ-oscillator algebra.

Definition 2.3. *In the real number field, the* λ*-oscillator Lie algebra* $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$ *is a* $(2m+2)$*-dimensional Lie algebra which has basis* $\{J, P_j, Q_j, T\}(1 \leq j \leq m)$*, and the non-zero Lie brackets:*

$$[J, P_j] = \lambda_j Q_j,\ [J, Q_j] = -\lambda_j P_j,\ [P_j, Q_j] = \lambda_j T,$$

where $\lambda = (\lambda_1, ..., \lambda_m)$*, with* $\lambda_k > 0,\ k = 1, 2, ..., m$*.*

The complexification of $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$ *is* $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{C})$*, with complex basis* $\{J,\ P_k^+ = P_k - iQ_k,$ $Q_k^- = P_k + iQ_k,\ T\}$*, and the following Lie commutators:*

$$[J, P_k^+] = i\lambda_k P_k^+, [J, Q_k^-] = -i\lambda_k Q_k^-, [P_k^+, Q_k^-] = 2i\lambda_k T.$$

It can be proved that all these types of oscillator Lie algebras are solvable, but not semisimple. In fact, oscillator Lie algebras are the only non-commutative solvable Lie algebras that carry a bi-invariant Lorentzian metric. The nilpotent radical of the $(2m+2)$-dimensional λ-oscillator algebra, is the $(2m+1)$-dimensional λ-Heisenberg algebra, which consists of the basis $\{P_k, Q_k, T\}$ $(k = 1, 2, ..., m)$, and the center is spanned by T.

2.2. Triangular matrix representations of complex general oscillator algebras

Ado's theorem in Lie theory states that every finite-dimensional complex Lie algebra can be represented as a matrix Lie algebra, formed by matrices. However, that result does not specify the minimal order of the matrices involved in such representations. Here, in the case of general oscillator algebras, we give a minimal-order faithful representation.

Lemma 2.4. *[13] For the* $(2m+1)$*-dimensional Heisenberg Lie algebra, the minimal order of its faithful representation is equal to* $(m+2)$*.*

Based on this lemma, the following matrix representations can be found in previous studies.

Proposition 2.5. *The complex harmonic oscillator algebra* $\mathfrak{D}^{\lambda}_4(\mathbb{C})$ *can be realized as a subalgebra of* $\mathfrak{sl}_3(\mathbb{C})$ *through the following map :*

$$aJ + bP_+ + cQ_- + dT \mapsto \begin{pmatrix} \frac{i\lambda}{3}a & b & -\frac{i}{2\lambda}d \\ 0 & -\frac{2i\lambda}{3}a & c \\ 0 & 0 & \frac{i\lambda}{3}a \end{pmatrix}$$

Proposition 2.6. *Let $\mathfrak{D}_{2m+2}(\mathbb{C})$ be a (2m+2)-dimensional general oscillator algebra, its minimal order faithful representation is constructed by*

$$aJ + \sum_{j=1}^{m} b_j P_j^+ + \sum_{k=1}^{m} c_k Q_k^- + dT \mapsto$$

$$\begin{pmatrix} \frac{mi}{m+2}a & b_m & b_{m-1} & \cdots & b_2 & b_1 & -\frac{i}{2}d \\ 0 & -\frac{2i}{m+2}a & 0 & \cdots & 0 & 0 & c_m \\ 0 & 0 & -\frac{2i}{m+2}a & \cdots & 0 & 0 & c_{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{2i}{m+2}a & 0 & c_2 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{2i}{m+2}a & c_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{mi}{m+2}a \end{pmatrix}$$

It can be revealed that these kinds of (2m+2)-dimensional complex general oscillator algebra representations have a minimal matrix order (m+2). However, such kinds of triangular matrix representation cannot be extended to generalized situations $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{C})$, as all entries of the vector λ in $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{C})$ cannot always be the same. We will prove this later in Section 3.4.

2.3. Leibniz algebra

Definition 2.7. *A Leibniz algebra L over a field $\mathbb{K}$ satisfies the Leibniz identity:*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \quad x, y, z \in L.$$

The Leibniz identity of x, y, z can be written as $\{x, y, z\}$ for short.

It can be seen that all Lie algebras are Leibniz algebras. But in general cases, an arbitrary Leibniz algebra does not have the anti-symmetric property, that is, $[x, x] = 0$ fails.

In the following sections, we mainly talk about the representations of λ-oscillator algebra on the real number field, i.e., $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$.

3. Representations

3.1. $\mathfrak{sp}_{2m+2}(\mathbb{R})$-modules as $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$-modules

In previous studies, the (2m+2)-dimensional general oscillator algebra $\mathfrak{D}_{2m+2}(\mathbb{R})$ can be represented through $\mathfrak{sp}_{2m+2}(\mathbb{R})$ as a module. Using the similar construction method, in this subsection, we give two faithful representations of $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$ through $\mathfrak{sp}_{2m+2}(\mathbb{R})$.

Theorem 3.1. *Suppose that $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$ is a $(2m + 2)$-dimensional real oscillator algebra with a basis $\{J, P_j, Q_j, T, 1 \leq j \leq m)\}$. Then there exists a faithful representation ϕ given by*

$$aJ + \sum_{k=1}^{m} b_k P_k + \sum_{k=1}^{m} c_k Q_k + dT \mapsto$$

$$\begin{pmatrix}
0 & b_1 & \cdots & b_{m-1} & b_m & c_m\lambda_m & c_{m-1}\lambda_{m-1} & \cdots & c_1\lambda_1 & 2d \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -a\lambda_m^2 & c_1\lambda_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & -a\lambda_2^2 & \cdots & 0 & c_{m-1}\lambda_{m-1} \\
0 & 0 & \cdots & 0 & 0 & -a\lambda_1^2 & 0 & \cdots & 0 & c_m\lambda_m \\
0 & 0 & \cdots & 0 & a & 0 & 0 & \cdots & 0 & -b_m \\
0 & 0 & \cdots & a & 0 & 0 & 0 & \cdots & 0 & -b_{m-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & a & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -b_1 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}.$$

Proof. Consider the bilinear map given by

$$\begin{aligned}
&\phi(J) = -\sum_{k=2}^{m+1} \lambda_{m+2-k}^2 e_{k,2m+3-k} + \sum_{k=m+2}^{2m+1} e_{k,2m+3-k}, \\
&\phi(P_k) = e_{1,1+k} - e_{2m+2-k,2m+2}, \\
&\phi(Q_k) = \lambda_k e_{1,2m+2-k} + \lambda_k e_{k+1,2m+2}, \\
&\phi(T) = 2e_{1,2m+2}, \quad 1 \le k \le m.
\end{aligned}$$

For any $x = aJ + \sum_{i=1}^m b_iP_i + \sum_{j=1}^m c_jQ_j + dT$, $y = \alpha J + \sum_{i=1}^m \beta_iP_i + \sum_{j=1}^m \gamma_jQ_j + \delta T$, then

$$[x, y] = \sum_{i=1}^m [(\alpha c_i - a\gamma_i)\lambda_iP_i] + \sum_{j=1}^m [(a\beta_j - \alpha b_j)\lambda_jQ_j] + \sum_{k=1}^m [(b_k\gamma_k - \beta_kc_k)\lambda_kT],$$

$$\phi[x, y] = \begin{pmatrix}
0 & B_1 & \cdots & B_m & C_m & \cdots & C_1 & D \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & C_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & C_m \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -B_m \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -B_1 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0
\end{pmatrix},$$

where $B_k = (\alpha c_k - a\gamma_k)\lambda_k$, $C_k = (a\beta_k - \alpha b_k)\lambda_k^2$, $D = 2\sum_{i=1}^m (b_i\gamma_i - \beta_ic_i)\lambda_i$. We have

$$\phi(x)\phi(y) =$$

$$\begin{pmatrix} 0 & \alpha c_1\lambda_1 & \cdots & \alpha c_m\lambda_m & -\alpha b_m\lambda_m^2 & \cdots & -\alpha b_1\lambda_1^2 & \sum_{i=1}^m(b_i\gamma_i-\beta_i c_i)\lambda_i \\ 0 & -a\alpha\lambda_1^2 & \cdots & 0 & 0 & \cdots & 0 & a\beta_1\lambda_1^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -a\alpha\lambda_m^2 & 0 & \cdots & 0 & a\beta_m\lambda_m^2 \\ 0 & 0 & \cdots & 0 & -a\alpha\lambda_1^2 & \cdots & 0 & a\gamma_m\lambda_m \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -a\alpha\lambda_m^2 & a\gamma_1\lambda_1 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

By checking $\phi[x,y]=\phi(x)\phi(y)-\phi(y)\phi(x)$ for all $x,y\in\mathfrak{D}_{2m+2}^{\lambda}(\mathbb{R})$, we can get that ϕ is a representation of algebras. Also, using the fact that all λ_i are positive real numbers, the mapping is injective. ϕ is a faithful representation. □

Theorem 3.2. *Based on the algebra in Theorem 3.1, there also exists a faithful representation ψ given by*

$$aJ+\sum_{j=1}^m b_jP_j+\sum_{k=1}^m c_kQ_k+dT\mapsto$$

$$\begin{pmatrix} 0 & b_1\lambda_1 & \cdots & b_{m-1}\lambda_{m-1} & b_m\lambda_m & c_m & c_{m-1} & \cdots & c_1 & 2d \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & -a & c_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & -a & \cdots & 0 & c_{m-1} \\ 0 & 0 & \cdots & 0 & 0 & -a & 0 & \cdots & 0 & c_m \\ 0 & 0 & \cdots & 0 & a\lambda_m^2 & 0 & 0 & \cdots & 0 & -b_m\lambda_m \\ 0 & 0 & \cdots & a\lambda_{m-1}^2 & 0 & 0 & 0 & \cdots & 0 & -b_{m-1}\lambda_{m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a\lambda_1^2 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -b_1\lambda_1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Proof. Notice that the bilinear map is given by

$$\begin{aligned} \psi(J)&=-\sum_{k=2}^{m+1}e_{k,2m+3-k}+\sum_{k=m+2}^{2m+1}\lambda_{2m+2-k}^2e_{k,2m+3-k}, \\ \psi(P_k)&=\lambda_ke_{1,1+k}-\lambda_ke_{2m+2-k,2m+2}, \\ \psi(Q_k)&=e_{1,2m+2-k}+e_{k+1,2m+2}, \\ \psi(T)&=2e_{1,2m+2},\quad 1\le k\le m. \end{aligned}$$

The proof process is similar to that in Theorem 3.1, by using the definition of a faithful representation. □

3.2. $\mathfrak{sl}_{2m+2}(\mathbb{R})$-*modules as* $\mathfrak{D}_{2m+2}^{\lambda}(\mathbb{R})$-*modules*

In [14], Thompson et al. described the matrix and vector field representations of all six-dimensional indecomposable solvable Lie algebras, or Mubarakazyanov algebras. In their table, $\mathfrak{D}_6^{\lambda}(\mathbb{R})$ corresponds to the algebra $g_{6.92}$ by changing the basis appropriately.

First, we find a formula of changing the basis of $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$, in order to match up the corresponding Lie algebra in Thompson's paper.

Change the basis of $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$ as follows:

$$T = T,\ \ X_1 = \lambda_1^{-\frac{1}{2}} P_1, X_2 = \lambda_2^{-\frac{1}{2}} P_2, ..., X_m = \lambda_m^{-\frac{1}{2}} P_m,$$

$$Y_1 = \lambda_1^{-\frac{1}{2}} Q_1, Y_2 = \lambda_2^{-\frac{1}{2}} Q_2, ..., Y_m = \lambda_m^{-\frac{1}{2}} Q_m, K = -\lambda_1^{-1} J.$$

Then we get a $(2m+2)$-dimensional Lie algebra $\mathfrak{M}^{\lambda}_{2m+2}(\mathbb{R})$ with the basis $\{T, X_j, Y_j, K\}$ $(j = 1, 2, ..., m)$. It satisfies the Lie brackets:

$$\begin{aligned}
&[X_1, Y_1] = [X_2, Y_2] = ... = [X_n, Y_n] = T,\ \ [T, K] = 0,\\
&[X_1, K] = \mu_1 Y_1,\ \ [X_2, K] = \mu_2 Y_2,\ \ ...,\ \ [X_m, K] = \mu_m Y_m,\\
&[Y_1, K] = -\mu_1 X_1,\ \ [Y_2, K] = -\mu_2 X_2,\ \ ...,\ \ [Y_m, K] = -\mu_m X_m,
\end{aligned}$$

where $\mu_1 = \frac{\lambda_1}{\lambda_1} = 1,\ \ \mu_2 = \frac{\lambda_2}{\lambda_1}, \mu_3 = \frac{\lambda_3}{\lambda_1}, ..., \mu_m = \frac{\lambda_m}{\lambda_1}$.

Using this method, one can extend the representation of $\mathfrak{D}^{\lambda}_{6}(\mathbb{R})$ to the general cases $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$.

Theorem 3.3. *Suppose* $\mathfrak{M}^{\lambda}_{2m+2}(\mathbb{R})$ *is a (2m+2)-dimensional Lie algebra with the basis* $\{T, X_j, Y_j, K\}$ $(j = 1, 2, ..., m)$. *Then its representation* η *is given by*

$$aT + \sum_{j=1}^{m} b_j X_j + \sum_{k=1}^{m} c_k Y_k + dK \mapsto$$

$$\begin{pmatrix}
0 & -c_m & b_m & \cdots & -c_2 & b_2 & -c_1 & b_1 & 2a\\
0 & 0 & \mu_m d & \cdots & 0 & 0 & 0 & 0 & b_m\\
0 & -\mu_m d & 0 & \cdots & 0 & 0 & 0 & 0 & c_m\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 0 & \mu_2 d & 0 & 0 & b_2\\
0 & 0 & 0 & \cdots & -\mu_2 d & 0 & 0 & 0 & c_2\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \mu_1 d & b_1\\
0 & 0 & 0 & \cdots & 0 & 0 & -\mu_1 d & 0 & c_1\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{3.1}$$

Proof. Consider the bilinear map $\eta : \mathfrak{M}^{\lambda}_{2m+2}(\mathbb{R}) \to \mathfrak{sl}_{2m+2}(\mathbb{R})$:

$$\eta(T) = 2e_{1,2m+2},\ \ \eta(K) = \sum_{k=1}^{m} \mu_k (e_{2m-2k+2,2m-2k+3} - e_{2m-2k+3,2m-2k+2}),$$

$$\eta(X_i) = e_{1,2m+3-2i} + e_{2m+2-2i,2m+2},\ \ \eta(Y_j) = -e_{1,2m+2-2j} + e_{2m+3-2j,2m+2},\ \ 1 \le i, j \le m.$$

By checking $\eta([x, y]) = \eta(x)\eta(y) - \eta(y)\eta(x)$ for all x, y, and all $\mu_j (j = 1, 2, ..., m)$ are positive real numbers, η can be verified as a faithful representation. □

3.3. Vector field representation

Based on the $\mathfrak{M}^{\lambda}_{2m+2}(\mathbb{R})$-modules in Theorem (3.3) and using the DNA vector method of vector field representation of Mubarakazyanov algebra, we can give a (2m+2)-dimensional vector field on $\mathbb{R}^{2m+2}$.

In the vector field of $\mathbb{R}^{2m+2}$, using coordinates $(p_1, p_2, ..., p_{2m+2})$, set the derivatives of coordinate p_k as D_{p_k}, or D_k for short, and the vector field representation corresponding to Eq (3.1) can be realised as:

$$-2D_{2m+2},\ \ -D_{2k} + p_{2k+1}D_{2m+2},\ \ D_{2k+1} + p_{2k}D_{2m+2},$$
$$D_1 + \sum_{i=1}^{m} \mu_{m+1-i}(p_{2i+1}D_{2i} - p_{2i}D_{2i+1}),\ \ k = 1, 2, ..., m.$$

3.4. Non-existence of a faithful representation in upper triangular matrices

In this part, we will prove that $(m + 2) \times (m + 2)$ upper triangular matrices with trace zero cannot be faithful representations of $(2m + 2)$-dimensional complex λ-oscillator algebras. We use the six-dimensional $\mathfrak{D}^{\lambda}_{6}(\mathbb{C})$ as an example. For the case of higher dimensions, a similar proof can be provided.

We first give the following lemmas:

Lemma 3.4. *Let* $\mathfrak{t}_4(\mathbb{C})_0$ *be the Lie algebra of* 4×4 *upper triangular matrices with zero trace,* $X = \mathrm{diag}(\mu_1, \ldots, \mu_4) \in \mathfrak{t}_4(\mathbb{C})_0$*, and* E_{mn} *($m \leq n$) be the matrix unit. Then*

$$[X, E_{mn}] = (\mu_m - \mu_n)E_{mn}.$$

Proof. Immediate from $XE_{mn} = \mu_m E_{mn}$, $E_{mn}X = \mu_n E_{mn}$. □

Lemma 3.5. *If* $Z \in \mathfrak{t}_4(\mathbb{C})_0$ *commutes with a diagonal matrix* $D = \mathrm{diag}(d_1, d_2, d_3, d_4)$*, then* $Z_{uv} = 0$ *whenever* $d_u \neq d_v$.

Proof. Immediate from $[D, Z]_{uv} = (d_u - d_v)Z_{uv} = 0$. □

Theorem 3.6. *Let* $\mathfrak{g}$ *be a six-dimensional λ-oscillator Lie algebra* $\mathfrak{D}^{\lambda}_{6}(\mathbb{C})$*. For* $\lambda_1 \neq \lambda_2 > 0$*, there exists no faithful Lie algebra representation* $\rho : \mathfrak{g} \to \mathfrak{t}_4(\mathbb{C})_0$.

Proof. Suppose a faithful representation $\rho : \mathfrak{g} \to \mathfrak{t}_4(\mathbb{C})_0$ exists. Since $\mathbb{C}$ is algebraically closed, we may conjugate ρ so that $\rho(J)$ is diagonal (because $\rho(J)$ is upper triangular and can be triangularized; the conjugation preserves the Lie bracket structure). Write

$$\rho(J) = \mathrm{diag}(\mu_1, \mu_2, \mu_3, \mu_4), \qquad \sum_{k=1}^{4} \mu_k = 0.$$

Then the conclusion will be drawn in the following steps:

Step 1: Preliminary analysis of the multiplicities of eigenvalues of $\rho(J)$.

Since T is central in $\mathfrak{g}$, $\rho(T)$ commutes with $\rho(J)$. By Lemma 3.5, $\rho(T)_{uv} \neq 0$ implies $\mu_u = \mu_v$. Since ρ is faithful, $\rho(T) \neq 0$, so there exists at least one pair of indices (u, v) with $u < v$ such that $\mu_u = \mu_v$. Consequently, $\rho(J)$ has at least one repeated eigenvalue. Let c be an eigenvalue of $\rho(J)$ with the most multiplicity r, where $2 \leq r \leq 4$. Denote the remaining eigenvalues (if any) by other complex numbers.

Step 2: Eigenvalue difference conditions.

From the Lie bracket relations in $\mathfrak{g}$,

$$[J, P_1^+] = i\lambda_1 P_1^+, \quad [J, Q_1^-] = -i\lambda_1 Q_1^-,$$

$$[J, P_2^+] = i\lambda_2 P_2^+, \quad [J, Q_2^-] = -i\lambda_2 Q_2^-.$$

Under the representation, we have

$$[\rho(J), \rho(P_1^+)] = i\lambda_1 \rho(P_1^+), \quad [\rho(J), \rho(Q_1^-)] = -i\lambda_1 \rho(Q_1^-),$$

$$[\rho(J), \rho(P_2^+)] = i\lambda_2 \rho(P_2^+), \quad [\rho(J), \rho(Q_2^-)] = -i\lambda_2 \rho(Q_2^-).$$

Let $\rho(P_1^+) = \sum_{m\leq n} p_{mn} E_{mn}$. By Lemma 3.4, if $p_{mn} \neq 0$, then

$$\mu_m - \mu_n = i\lambda_1. \tag{3.2}$$

Similarly, if $\rho(Q_1^-) = \sum_{m\leq n} q_{mn} E_{mn}$ with $q_{mn} \neq 0$, then

$$\mu_m - \mu_n = -i\lambda_1. \tag{3.3}$$

Analogous conditions hold for P_2^+ and Q_2^- with λ_1 replaced by λ_2.

Since $\rho(P_1^+) \neq 0$ and $\rho(Q_1^-) \neq 0$ (by faithfulness), there exist indices $m < n$ satisfying (3.2) and $m' < n'$ satisfying (3.3). This implies that the set of eigenvalues must contain two distinct differences $i\lambda_1$ and $-i\lambda_1$.

Step 3: Determination of the multiplicity r and the structure of eigenvalues.

We successively exclude the cases $r = 4$ and $r = 3$ and the case $r = 2$ with $d = e$.

3.1 Exclusion of $r = 4$. If $r = 4$, then all eigenvalues are equal: $\mu_1 = \mu_2 = \mu_3 = \mu_4 = c$. Then for all $m < n$, $\mu_m - \mu_n = 0$, but (3.2) requires a difference equal to $i\lambda_1 \neq 0$. Hence, $r \neq 4$.

3.2 Exclusion of $r = 3$. If $r = 3$, then the eigenvalues are $\{c, c, c, d\}$ with $d \neq c$. The trace condition gives $3c + d = 0$, i.e., $d = -3c$.

From (3.2), there exist $m < n$ such that $\mu_m - \mu_n = i\lambda_1$. The possible nonzero differences among the eigenvalues are:

- $c - d = c - (-3c) = 4c$,
- $d - c = -3c - c = -4c$.

Thus, we must have $4c = i\lambda_1$ or $-4c = i\lambda_1$. Similarly, from the condition for P_2, we must have $4c = i\lambda_2$ or $-4c = i\lambda_2$.

Consider the four combinations:

- If $4c = i\lambda_1$ and $4c = i\lambda_2$, or $4c = -i\lambda_1$ and $4c = -i\lambda_2$, then $\lambda_1 = \lambda_2$, contradicting the hypothesis.
- If $4c = i\lambda_1$ and $-4c = i\lambda_2$, or $-4c = i\lambda_1$ and $4c = i\lambda_2$, then $\lambda_1 = -\lambda_2$, which is impossible since $\lambda_1, \lambda_2 > 0$.

Therefore, the case $r = 3$ cannot simultaneously satisfy the conditions for P_1 and P_2. Hence, $r \neq 3$.

3.3 Analysis for $r = 2$. From Step 1 and the above exclusions, we must have $r = 2$. Write the eigenvalues of $\rho(J)$ as c, c, d, e, where c has multiplicity 2, and d, e are the remaining eigenvalues (they may be equal). The trace condition gives

$$2c + d + e = 0. \tag{3.4}$$

From (3.2), there exist $m < n$ such that $\mu_m - \mu_n = i\lambda_1$. The possible nonzero differences from the set $\{c, c, d, e\}$ are $c - d$, $c - e$, $d - c$, and $e - c$. Since $i\lambda_1$ is a positive imaginary number, we must have either $c - d = i\lambda_1$ or $c - e = i\lambda_1$ (the other two are their negatives).

We first consider the possibility that $d = e$.

3.3.1 Exclusion of $d = e$. If $d = e$, then the eigenvalues are $\{c, c, d, d\}$. The trace condition (3.4) becomes $2c + 2d = 0$, i.e., $d = -c$. The possible nonzero differences are $c - d = 2c$ and $d - c = -2c$. Then (3.2) requires $2c = i\lambda_1$ or $-2c = i\lambda_1$, so $c = i\lambda_1/2$ or $c = -i\lambda_1/2$. Similarly, the condition for P_2 requires $2c = i\lambda_2$ or $-2c = i\lambda_2$. As in the $r = 3$ case, this yields either $\lambda_1 = \lambda_2$ or $\lambda_1 = -\lambda_2$, contradicting the hypothesis. Hence, $d \neq e$.

3.3.2 Determination of the explicit eigenvalues. Since $d \neq e$ and both $i\lambda_1$ and $i\lambda_2$ are positive imaginary numbers, we consider the following generic case (other cases are symmetric and lead to the same conclusion):

$$c - d = i\lambda_1, \qquad c - e = i\lambda_2. \tag{3.5}$$

Then $d = c - i\lambda_1$ and $e = c - i\lambda_2$. Substituting into the trace condition (3.4),

$$2c + (c - i\lambda_1) + (c - i\lambda_2) = 4c - i(\lambda_1 + \lambda_2) = 0,$$

which yields

$$c = \frac{i(\lambda_1 + \lambda_2)}{4}.$$

Consequently,

$$d = \frac{i(\lambda_1 + \lambda_2)}{4} - i\lambda_1 = \frac{i(-3\lambda_1 + \lambda_2)}{4},$$
$$e = \frac{i(\lambda_1 + \lambda_2)}{4} - i\lambda_2 = \frac{i(\lambda_1 - 3\lambda_2)}{4}.$$

Note that $d \neq e$ because $\lambda_1 \neq \lambda_2$ ensures $-3\lambda_1 + \lambda_2 \neq \lambda_1 - 3\lambda_2$.

Thus, the four eigenvalues are c, c, d, e with $d \neq e$ as given above.

Step 4: Triangularity obstruction for Q_1^-.

We now verify the condition in Eq (3.3) for Q_1^-.

We nevertheless require indices $m < n$ such that $\mu_m - \mu_n = -\lambda_1 i$. The set of eigenvalues is $S = \{c, c, d, e\}$ with $d = c - \lambda_1 i$, $e = c - \lambda_2 i$. Compute the differences:

$$\begin{aligned}
c - d &= \lambda_1 i,\\
c - e &= \lambda_2 i,\\
d - c &= -\lambda_1 i,\\
e - c &= -\lambda_2 i,\\
d - e &= \frac{(-3\lambda_1 + \lambda_2)i}{4} - \frac{(\lambda_1 - 3\lambda_2)i}{4} = \frac{-4\lambda_1 + 4\lambda_2}{4} i = (\lambda_2 - \lambda_1)i,\\
e - d &= (\lambda_1 - \lambda_2)i.
\end{aligned}$$

The only negative pure imaginary $-\lambda_1 i$ in this list is $d - c$. However, we need $m < n$ with $\mu_m - \mu_n = -\lambda_1 i$.

However, the eigenvalue d has multiplicity 1, while c has multiplicity 2. In the ordering of rows for a 4×4 upper triangular matrix, the two positions corresponding to c must appear to the left of the position corresponding to d; otherwise, the condition $m < n$ cannot be satisfied. More concretely, we can label the rows so that rows 1 and 2 correspond to c, and row 3 corresponds to d. Then $m = 3$ and $n \in \{1, 2\}$, but $3 < 1$ and $3 < 2$ are both false. Hence, no pair (m, n) with $m < n$ satisfies $\mu_m = d$ and $\mu_n = c$.

Consequently, there exists no pair (m, n) with $m < n$ such that $\mu_m - \mu_n = -i\lambda_1$. Thus, any nonzero upper-triangular matrix satisfying $[\rho(J), \rho(Q_1^-)] = -i\lambda_1\rho(Q_1^-)$ must be zero. This forces $\rho(Q_1^-) = 0$, contradicting the faithfulness of ρ (since $Q_1 \neq 0$ in $\mathfrak{g}$).

Therefore, the triangularity requirement for $\rho(Q_1^-)$ is incompatible with the eigenvalue structure enforced by the existence of a nonzero $\rho(T)$ and the conditions on P_1^+ and P_2^+.

In conclusion, a faithful representation $\rho : \mathfrak{g} \to \mathfrak{t}_4(\mathbb{C})_0$ cannot exist when $\lambda_1 \neq \lambda_2$.

□

4. Leibniz algebras corresponding to $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$-modules

In fact, any non-Lie Leibniz algebra L has a non-trivial ideal I, which is the subspace spanned by the squares of the elements of the algebra L, and has the property of $[L, I] = 0$.

Using this construction method, in this section, we will construct a Leibniz algebra that satisfies the property mentioned above.

Consider the bilinear map of Theorem 3.1. Then we can have a natural $\phi(\mathfrak{D}^{\lambda}_{2m+2})$-module $V = \mathbb{R}^{2m+2}$ and endow it with a $\mathfrak{D}^{\lambda}_{2m+2}$-module structure, $V \times \mathfrak{D}^{\lambda}_{2m+2} \to V$, given by $[x, e] := x\phi(e)$ for any $x \in V, e \in \mathfrak{D}^{\lambda}_{2m+2}$.

Then we obtain the following non-zero commutation relations:

$$
\begin{aligned}
&[X_k, J] = -\lambda^2_{m+2-k} X_{2m+3-k}, \qquad 2 \leq k \leq m+1,\\
&[X_k, J] = X_{2m+3-k}, \qquad m+2 \leq k \leq 2m+1,\\
&[X_1, P_k] = X_{k+1}, \qquad 1 \leq k \leq m,\\
&[X_{2m+2-k}, P_k] = -X_{2m+2}, \qquad 1 \leq k \leq m,\\
&[X_1, Q_k] = \lambda_k X_{2m+2-k}, \qquad 1 \leq k \leq m,\\
&[X_{k+1}, Q_k] = \lambda_k X_{2m+2}, \qquad 1 \leq k \leq m,\\
&[X_1, T] = 2X_{2m+2}.
\end{aligned}
$$

We can construct a Leibniz algebra as follows.

Theorem 4.1. *Let L be a Leibniz algebra with a basis of*

$$\{J, P_1, ..., P_m, Q_1, ..., Q_m, T, X_1, ..., X_{2m+2}\},$$

and the multiplication table has the following form (products not mentioned are zero):

$$
\begin{aligned}
&[J, J] = \alpha X_{2m+2}, \qquad \alpha \in \mathbb{R} - \{0\}\\
&[J, P_k] = -[P_k, J] = \lambda_k Q_k, \qquad 1 \le k \le m,\\
&[J, Q_k] = -[Q_k, J] = -\lambda_k P_k, \qquad 1 \le k \le m,\\
&[P_k, Q_k] = -[Q_k, P_k] = \lambda_k T, \qquad 1 \le k \le m,\\
&[X_k, J] = -\lambda^2_{m+2-k} X_{2m+3-k}, \qquad 2 \le k \le m+1,\\
&[X_k, J] = X_{2m+3-k}, \qquad m+2 \le k \le 2m+1,\\
&[X_1, P_k] = X_{k+1}, \qquad 1 \le k \le m,\\
&[X_{2m+2-k}, P_k] = -X_{2m+2}, \qquad 1 \le k \le m,\\
&[X_1, Q_k] = \lambda_k X_{2m+2-k}, \qquad 1 \le k \le m,\\
&[X_{k+1}, Q_k] = \lambda_k X_{2m+2}, \qquad 1 \le k \le m,\\
&[X_1, T] = 2X_{2m+2}.
\end{aligned}
$$

Then L is a non-Lie Leibniz algebra, and it has an ideal I generated by squares

$$I = \mathrm{span}\{[x, x] \mid x \in L\}$$

with $[L, I] = 0$.

Proof. Set V as a $(2m+2)$-dimensional vector space spanned by $\{X_1, X_2, ..., X_{2m+2}\}$.

First, we compute all squares. For any $x \in L$:

- if $x = J$, then $[J, J] = \alpha X_{2m+2}$;
- if $x \in \mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$ and $x \neq J$, then $[x, x] = 0$ (since $[P_k, P_k] = [Q_k, Q_k] = [T, T] = 0$);
- if $x \in V$, then $[x, x] = 0$.

Thus, the subspace spanned by all squares is

$$I = \mathbb{R}X_{2m+2}, \alpha \neq 0.$$

Since $\alpha \neq 0$, then $[J, J] \neq 0$, hence L is not a Lie algebra.

We show that I is an ideal with $[L, I] = 0$. From the module action, X_{2m+2} commutes with every $g \in \mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$: $[g, X_{2m+2}] = 0$, and $[X_{2m+2}, V] = 0$, so $[L, X_{2m+2}] = 0$. Consequently $I = \mathbb{R}X_{2m+2}$ satisfies $[L, I] = 0$; in particular, I is a two-sided ideal.

□

Theorem 4.2. *Let L be the Leibniz algebra defined by the multiplication table in Theorem* (4.1). *Set*

$$V = \mathrm{span}\{X_1, X_2, \ldots, X_{2m+2}\}.$$

Then V is a minimum ideal of L consisting of $\{X_1, X_2, ..., X_{2m+2}\}$, *and the quotient L/V is isomorphic to the λ-oscillator Lie algebra* $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$.

Proof. We first prove that V is an ideal. From the multiplication table:

- For every $g \in \mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$ and $v \in V$, the module action gives $[g, v] \in V$.

- By definition $[v, g] = -[g, v]$, hence $[v, g] \in V$ as well.
- Moreover, $[V, V] = 0 \subseteq V$.

Thus, $[L, V] \subseteq V$ and $[V, L] \subseteq V$, so V is a two-sided ideal of L.

Consider the quotient algebra L/V. Denote the coset of an element $x \in L$ by $\overline{x}$. In L/V, we have $\overline{X_i} = 0$ for all i, hence $\overline{V} = 0$. The induced bracket is given by $[\overline{x}, \overline{y}] = \overline{[x, y]}$.

For the basis elements of $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$:

$$\begin{aligned}
\overline{[J, J]} &= \overline{\alpha X_{2m+2}} = 0,\\
\overline{[J, P_k]} &= \overline{\lambda_k Q_k} = \lambda_k \overline{Q_k},\\
\overline{[J, Q_k]} &= \overline{-\lambda_k P_k} = -\lambda_k \overline{P_k},\\
\overline{[P_k, Q_k]} &= \overline{\lambda_k T} = \lambda_k \overline{T},
\end{aligned}$$

and all other brackets among $\overline{J}, \overline{P_k}, \overline{Q_k}, \overline{T}$ are zero. The brackets involving any X_i vanish because $X_i \in V$ maps to zero. Consequently, the set $\{\overline{J}, \overline{P_1}, \ldots, \overline{P_m}, \overline{Q_1}, \ldots, \overline{Q_m}, \overline{T}\}$ is a basis of L/V, and the only non-zero brackets are

$$[\overline{J}, \overline{P_k}] = \lambda_k \overline{Q_k}, \qquad [\overline{J}, \overline{Q_k}] = -\lambda_k \overline{P_k}, \qquad [\overline{P_k}, \overline{Q_k}] = \lambda_k \overline{T}, \qquad (1 \le k \le m).$$

These are precisely the defining relations of the λ-oscillator Lie algebra $\mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$. Therefore, $L/V \cong \mathfrak{D}^{\lambda}_{2m+2}(\mathbb{R})$. □

Use of AI tools declaration

No Artificial Intelligence (AI) tools were used in the preparation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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