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Research article

Cubic spline rule to compute hypersingular integral on a circle

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Abstract: A novel approach for the high-precision evaluation of hypersingular integrals on a circle by the spline approximation of the periodic density function is presented. A cubic spline interpolation function with periodic boundary conditions, enforced through a cyclic tridiagonal system, is constructed through the uniform partitioning of the periodic interval. Through the analytical properties of the Clausen functions, an explicit expression for the integral is derived, and a rigorous error analysis is conducted. Theoretical results demonstrate that a convergence rate of $O(h^3)$ at non-superconvergent points and $O(h^4)$ superconvergence at the zeros of the special function $\Phi(\tau)$ are attained. It is further demonstrated that the superconvergence phenomenon is uniformly discerned whenever the singular point coincides with the zeros of $\Phi(\tau)$, regardless of the singular point's relative position within the mesh. Finally, a numerical example is presented for illustrating the effectiveness of the proposed method. The computed errors across diverse mesh sizes and singular point locations are in remarkable agreement with theoretical predictions.

Keywords: hypersingular integral; cubic spline interpolation; Hadamard finite-part integral; superconvergence; Clausen functions

1. Introduction

Hypersingular integrals, especially those defined in the sense of Hadamard finite part, constitute a fundamental yet arduous subject in numerical computation and possess substantial engineering applications. These integrals emerge extensively in boundary element methods [1, 2], fracture mechanics [3–5], acoustic scattering [2], and the analysis of fractional partial differential equations [6, 7], among other domains. This paper focuses on the numerical evaluation of a hypersingular integral defined on a unit circle.

$$\text{f.p.} \int_c^{c+2\pi} \frac{f(x)}{\sin^2 \frac{x-s}{2}} \mathrm{d}x = g(s), \quad s \in (c, c + 2\pi). \tag{1.1}$$

In this formulation, f.p. $\int_c^{c+2\pi}$ represents the Hadamard finite-part integral over a single period. The function $f(x)$ is defined as the density function with a period of 2π, and $g(s)$ signifies the resultant integral value. This integral, which functions as the natural boundary integral equation for harmonic problems within unbounded domains [8], serves as a theoretical basis for the establishment of efficient domain decomposition algorithms and hybrid numerical schemes.

The integral is defined in the sense of the Hadamard finite part as

$$\text{f.p.} \int_c^{c+2\pi} \frac{f(x)}{\sin^2 \frac{x-s}{2}} \mathrm{d}x = \lim_{\varepsilon \to 0} \left\{ \left(\int_c^{s-\varepsilon} + \int_{s+\varepsilon}^{c+2\pi} \right) \frac{f(x)}{\sin^2 \frac{x-s}{2}} \mathrm{d}x - \frac{8 f(s)}{\varepsilon} \right\} \tag{1.2}$$

or

$$\text{f.p.} \int_c^{c+2\pi} \frac{f(x)}{\sin^2 \frac{x-s}{2}} \mathrm{d}x = \lim_{\varepsilon \to 0} \left\{ \left(\int_c^{s-\varepsilon} + \int_{s+\varepsilon}^{c+2\pi} \right) \frac{f(x)}{\sin^2 \frac{x-s}{2}} \mathrm{d}x - 4 f(s) \cot \frac{\varepsilon}{2} \right\}. \tag{1.3}$$

It can be proven that formulas (1.2) and (1.3) are equivalent.

In recent years, the development of efficient and high-precision numerical methods for singular and hypersingular integrals has emerged as an active research domain. The definitions and numerical approaches for this category of integrals were systematically expounded in the fundamental review by Monegato [9]. One of the primary research directions pertains to the construction of specialized quadrature formulas. For instance, Gaussian-type quadrature formulas have been put forward for diverse types of singularities [4, 10–14]. Newton-Cotes-type quadrature rules, particularly the trapezoidal rule [15–23], have been comprehensively investigated due to their outstanding performance in dealing with periodic problems. The Euler-Maclaurin expansion functions have been used as a potent tool for analyzing the error of such quadrature rules [21, 24–30]. Computational precision can be effectively improved by the Richardson extrapolation method, which is grounded in the asymptotic expansion of the error [6, 18, 21, 22]. Moreover, variable transformation techniques [24, 25, 30] and specialized methods for highly oscillatory singular integrals [31] have also been demonstrated to be effective.

A significant milestone in this domain has been the discovery and theoretical explication of the superconvergence phenomenon. Studies have indicated that for singular integrals on a circle, when the singularity is situated at particular positions within a grid cell, frequently associated with the zeros of special functions such as the Clausen function, the convergence order of composite quadrature rules can be notably improved. This phenomenon has been rigorously verified for the trapezoidal rule [16,17,23], the midpoint rule [32–34], and Newton-Cotes rules [35].

Concurrently with the advancement of quadrature methods, spline-based approximation techniques have emerged as an alternative effective means for dealing with singular integrals and associated differential equations, owing to their flexibility and robustness. Among these techniques, cubic splines have exhibited particularly remarkable performance. Their applications span a broad spectrum, ranging from the estimation of pit excavation volumes using non-uniform grids [36] and the evaluation of optical surface errors in Ronchi tests [37] to the solution of quasilinear parabolic equations [38] and convection-diffusion equations via cubic trigonometric B-splines [39]. Notably, Gu et al. [40] were the first to directly apply a cubic spline rule to Hadamard finite-part integrals on an interval, accompanied by corresponding error estimates and superconvergence results. Nevertheless, the application of this approach to the circular hypersingular integrals investigated in this paper remains unexplored. Moreover, the potential for attaining high accuracy near interval endpoints is also demonstrated in the work of Hasegawa [41] on numerically stable interpolation rules for approximating the finite Hilbert transform.

Other notable related studies encompass computational methodologies for multidimensional singular and hypersingular integrals [11–13, 33, 34], along with the numerical analysis of hypersingular integral equations [1, 5, 29, 32, 42]. Contemporary research trends also involve the uniqueness of positive solutions for nonlinear p-Laplacian Hadamard fractional boundary-value problems [43] and nontrivial solutions for Hadamard fractional integral boundary-value problems through topological degree methods [6].

During the analysis, the Clausen series $\mathrm{Cl}_n(\theta)$ [44] functions as a crucial special function, and a concise introduction to it will be provided hereinafter.

$$\mathrm{Cl}_n(\theta)=\begin{cases}\displaystyle\sum_{k=1}^{\infty}\frac{\sin(k\theta)}{k^n}, & n\in \text{even},\\ \displaystyle\sum_{k=1}^{\infty}\frac{\cos(k\theta)}{k^n}, & n\in \text{odd}.\end{cases}$$

If $n=1$, it takes the equivalent form

$$\mathrm{Cl}_1(\theta)=-\ln\left|2\sin\frac{\theta}{2}\right|, \tag{1.4}$$

and for $n=2$, it reduces to the well-known Clausen integral

$$\mathrm{Cl}_2(\theta)=-\int_0^{\theta}\ln\left|2\sin\frac{t}{2}\right|\mathrm{d}t. \tag{1.5}$$

The Clausen functions satisfy a number of useful relations.
(1) Recurrence relation. For any positive integer n , Clausen functions satisfy the differential recurrence relation

$$\mathrm{Cl}'_{n+1}(\theta)=(-1)^{n+1}\,\mathrm{Cl}_n(\theta).$$

(2) Periodicity. For any positive integer n and any integer m, the function is 2π-periodic.

$$\mathrm{Cl}_n(\theta)=\mathrm{Cl}_n(\theta+2m\pi).$$

(3) Parity. For any positive integer n, the Clausen function exhibits the symmetry property

$$\mathrm{Cl}_n(-\theta)=(-1)^{n+1}\,\mathrm{Cl}_n(\theta).$$

This study endeavors to address the research gap between cubic spline interpolation and the superconvergence analysis of hypersingular integrals on a circle. Leveraging cubic spline interpolation, a novel composite quadrature rule for integral (1.1) is constructed, integrating spline-based approaches with the special function framework typically employed in periodic integral analysis. The primary contribution of this paper lies in conducting a rigorous asymptotic error analysis. This analysis not only offers general error bounds but also precisely delineates the conditions for superconvergence. It is proven that superconvergence occurs at the zeros of a specific special function, $\Phi(\tau)$. Notably, since $f(x)$ is a periodic function, this property persists even at the nodal points of the periodic interval, presenting a substantial advantage over the finite interval scenario [15, 41, 45]. The theoretical results are verified through extensive numerical experiments, which illustrate the high precision and outstanding applicability of the proposed cubic spline method.

2. Main results

Consider a uniform partitioning of the periodic interval $[c, c+2\pi]$ on the unit circle, which is defined by nodes

$$c = x_0 < x_1 < \cdots < x_{n-1} < x_n = c + 2\pi,$$

where the mesh size is denoted as $h = 2\pi/n$.

Let $S_3(x) \in C^2[c, c + 2\pi]$ denote a cubic spline interpolant. On each sub-interval $[x_i, x_{i+1}]$, $S_3(x)$ represents a cubic polynomial, fulfilling the following conditions:

$$S_3(x_i) = f(x_i) = f_i, \quad i = 0, 1, \cdots, n,$$

where the values f_i are given at the nodal points x_i. Since $f(x)$ is a periodic function on the unit circle, the periodic boundary conditions are naturally imposed on the interval $[c, c + 2\pi]$ as follows:

$$S_3(x_0) = S_3(x_n), \quad S_3'(x_0) = S_3'(x_n), \quad S_3''(x_0) = S_3''(x_n).$$

Under these conditions, the nodal derivatives $m_i = S_3'(x_i)$ $(i = 0, 1, \ldots, n-1)$ are determined, with $m_n = m_0$. Subsequently, the cubic spline interpolation function $S_3(x)$ on each sub-interval $[x_i, x_{i+1}]$ is expressed as

$$S_3(x) = \sum_{j=i}^{i+1} \Big[\alpha_j(x) f_j + \beta_j(x) m_j\Big], \quad x \in [x_i, x_{i+1}], \tag{2.1}$$

where

$$\begin{aligned}
\alpha_i(x) &= \left(1 + 2\frac{x - x_i}{x_{i+1} - x_i}\right)\left(\frac{x - x_{i+1}}{x_i - x_{i+1}}\right)^2, \\
\alpha_{i+1}(x) &= \left(1 + 2\frac{x - x_{i+1}}{x_i - x_{i+1}}\right)\left(\frac{x - x_i}{x_{i+1} - x_i}\right)^2, \\
\beta_i(x) &= (x - x_i)\left(\frac{x - x_{i+1}}{x_i - x_{i+1}}\right)^2, \\
\beta_{i+1}(x) &= (x - x_{i+1})\left(\frac{x - x_i}{x_{i+1} - x_i}\right)^2,
\end{aligned} \tag{2.2}$$

$$f_i = f(x_i), \quad m_i = S_3'(x_i), \quad i = 0, 1, \cdots, n,$$

and

$$\begin{bmatrix} 4 & 1 & & & 1 \\ 1 & 4 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 4 & 1 \\ 1 & & & 1 & 4 \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ \vdots \\ m_{n-2} \\ m_{n-1} \end{bmatrix} = \frac{3}{h} \begin{bmatrix} f_1 - f_{n-1} \\ f_2 - f_0 \\ \vdots \\ f_{n-1} - f_{n-3} \\ f_n - f_{n-2} \end{bmatrix}. \tag{2.3}$$

Introduce a linear transformation

$$x = \hat{x}_i(\tau) := \frac{(\tau+1)h}{2} + x_i, \quad \tau \in [-1,1], \tag{2.4}$$

which transforms the reference element $[-1,1]$ into the sub-interval $[x_i, x_{i+1}]$. By substituting $f(x)$ in formula (1.1) with $S_3(x)$, we obtain

$$Q_n(f,s) = \text{f.p.} \int_c^{c+2\pi} \frac{S_3(x)}{\sin^2 \frac{x-s}{2}} \mathrm{d}x = \sum_{i=0}^{n-1} \text{f.p.} \int_{x_i}^{x_{i+1}} \frac{S_3(x)}{\sin^2 \frac{x-s}{2}} \mathrm{d}x = g(s) - \mathcal{E}_n(f,s). \tag{2.5}$$

Furthermore

$$\sum_{i=0}^{n-1} \text{f.p.} \int_{x_i}^{x_{i+1}} \frac{S_3(x)}{\sin^2 \frac{x-s}{2}} \mathrm{d}x = \sum_{i=0}^{n-1} \left[a_i(s) f_i + b_i(s) f_{i+1} + c_i(s) m_i + d_i(s) m_{i+1} \right], \tag{2.6}$$

where $a_i(x), b_i(x), c_i(x), d_i(x)$ are given by the following expressions:

$$a_i(s) = 2\cot\frac{x_i - s}{2} + \frac{24}{h^2}\left[\mathrm{Cl}_2(x_{i+1}-s) + \mathrm{Cl}_2(x_i - s)\right] + \frac{48}{h^3}\left[\mathrm{Cl}_3(x_{i+1}-s) - \mathrm{Cl}_3(x_i - s)\right], \tag{2.7}$$

$$b_i(s) = -2\cot\frac{x_{i+1} - s}{2} - \frac{24}{h^2}\left[\mathrm{Cl}_2(x_{i+1}-s) + \mathrm{Cl}_2(x_i - s)\right] - \frac{48}{h^3}\left[\mathrm{Cl}_3(x_{i+1}-s) - \mathrm{Cl}_3(x_i - s)\right], \tag{2.8}$$

$$c_i(s) = 4\,\mathrm{Cl}_1(x_i - s) + \frac{8}{h}\left[\mathrm{Cl}_2(x_{i+1}-s) + 2\,\mathrm{Cl}_2(x_i - s)\right] + \frac{24}{h^2}\left[\mathrm{Cl}_3(x_{i+1}-s) - \mathrm{Cl}_3(x_i - s)\right], \tag{2.9}$$

$$d_i(s) = -4\,\mathrm{Cl}_1(x_{i+1} - s) + \frac{8}{h}\left[2\,\mathrm{Cl}_2(x_{i+1}-s) + \mathrm{Cl}_2(x_i - s)\right] + \frac{24}{h^2}\left[\mathrm{Cl}_3(x_{i+1}-s) - \mathrm{Cl}_3(x_i - s)\right]. \tag{2.10}$$

The derivation of $a_i(s)$ is presented as an illustrative example.

First, we substitute the density function $f(x)$ with the interpolation basis function $\alpha_i(x)$. By expanding $\alpha_i(x)$ as a polynomial in x, we obtain

$$\begin{aligned} a_i(s) &= \text{f.p.} \int_{x_i}^{x_{i+1}} \left[\left(\frac{1}{h^2} - \frac{2x_i}{h^3} \right) \cdot \frac{(x - x_{i+1})^2}{\sin^2\left(\frac{x-s}{2}\right)} + \frac{2}{h^3} \cdot \frac{x(x-x_{i+1})^2}{\sin^2\left(\frac{x-s}{2}\right)} \right] dx \\ &= \frac{3(x_i + x_{i+1})}{h^3} \cdot \text{f.p.} \int_{x_i}^{x_{i+1}} \frac{x^2}{\sin^2\left(\frac{x-s}{2}\right)} dx + \frac{2}{h^3} \cdot \text{f.p.} \int_{x_i}^{x_{i+1}} \frac{x^3}{\sin^2\left(\frac{x-s}{2}\right)} dx \\ &\quad + \frac{x_{i+1}^3 - 3x_i x_{i+1}^2}{h^3} \cdot \text{f.p.} \int_{x_i}^{x_{i+1}} \frac{1}{\sin^2\left(\frac{x-s}{2}\right)} dx + \frac{6x_i x_{i+1}}{h^3} \cdot \text{f.p.} \int_{x_i}^{x_{i+1}} \frac{x}{\sin^2\left(\frac{x-s}{2}\right)} dx. \end{aligned}$$

As an example, according to the Hadamard finite-part definition in formula (1.3), we perform integration by parts on the term involving x^2 in the above formula.

$$\begin{aligned}
\text{f.p.}\int_{x_i}^{x_{i+1}} \frac{x^2}{\sin^2\frac{x-s}{2}}dx &= \lim_{\epsilon\to 0}\left\{\left(\int_{x_i}^{s-\epsilon}+\int_{s+\epsilon}^{x_{i+1}}\right)\frac{x^2}{\sin^2\frac{x-s}{2}}dx - 4s^2\cot\frac{\epsilon}{2}\right\} \\
&= \lim_{\varepsilon\to 0}\left\{-2x^2\cot\frac{x-s}{2}\bigg|_{x_i}^{s-\varepsilon} - 2x^2\cot\frac{x-s}{2}\bigg|_{s+\varepsilon}^{x_{i+1}}\right. \\
&\left. + 4\int_{x_i}^{s-\varepsilon} x\cot\frac{x-s}{2}dx + 4\int_{s+\varepsilon}^{x_{i+1}} x\cot\frac{x-s}{2}dx - 4s^2\cot\frac{\varepsilon}{2}\right\} \\
&= \left[-2x^2\cot\frac{x-s}{2}\right]_{x_i}^{x_{i+1}} + 4\text{p.v.}\int_{x_i}^{x_{i+1}} x\cot\frac{x-s}{2}dx.
\end{aligned} \tag{2.11}$$

The divergent term $4s^2\cot(\epsilon/2)$ is precisely neutralized by the subtraction term $-4s^2\cot(\epsilon/2)$ specified in the Hadamard definition. Consequently, as $\epsilon \to 0$, the hypersingular integral reduces to a Cauchy principal value (p.v.) integral. A second integration by parts is then performed on the resulting p.v. integral. We obtain

$$4\text{p.v.}\int_{x_i}^{x_{i+1}} x\cot\frac{x-s}{2}dx = \left[8x\ln\left|\sin\frac{x-s}{2}\right|\right]_{x_i}^{x_{i+1}} - 8\int_{x_i}^{x_{i+1}}\ln\left|\sin\frac{x-s}{2}\right|dx. \tag{2.12}$$

Finally, the Clausen functions are introduced by invoking their integral representations, specifically formula (1.5).

By substituting the definition of the Clausen function, we eventually arrive at the explicit formula for a_i(s) given in formula (2.7). The coefficients b_i(s), c_i(s), and d_i(s) are obtained similarly.

Let $H_3(x)$ denote the piecewise Hermite interpolating function of $f(x)$ defined on the interval $[c, c+2\pi]$, which has the same interval partition and linear transformation as $S_3(x)$. Regarding the error $\mathcal{E}_n(f, s)$ of the cubic spline quadrature, its expression can be decomposed and calculated according to the following procedure:

$$\begin{aligned}
\mathcal{E}_n(f,s) &= \text{f.p.}\int_c^{c+2\pi}\frac{f(x)-S_3(x)}{\sin^2\frac{x-s}{2}}\mathrm{d}x \\
&= \text{f.p.}\int_c^{c+2\pi}\frac{f(x)-H_3(x)}{\sin^2\frac{x-s}{2}}\mathrm{d}x + \text{f.p.}\int_c^{c+2\pi}\frac{H_3(x)-S_3(x)}{\sin^2\frac{x-s}{2}}\mathrm{d}x \\
&= {}^{\mathrm{H}}\mathcal{E}_n(f,s) + \mathcal{E}_n^{(a)}(f,s),
\end{aligned} \tag{2.13}$$

where

$${}^{\mathrm{H}}\mathcal{E}_n(f,s) = \text{f.p.}\int_c^{c+2\pi}\frac{f(x)-H_3(x)}{\sin^2\frac{x-s}{2}}\mathrm{d}x, \tag{2.14}$$

$$\mathcal{E}_n^{(a)}(f,s) = \text{f.p.}\int_c^{c+2\pi}\frac{H_3(x)-S_3(x)}{\sin^2\frac{x-s}{2}}\mathrm{d}x. \tag{2.15}$$

Define a functional $\kappa_s(x)$ as

$$\kappa_s(x) = \begin{cases} \dfrac{(x-s)^2}{\sin^2\frac{x-s}{2}}, & x \neq s, \\ 4, & x = s. \end{cases} \tag{2.16}$$

Let $x \in [c, c+2\pi]$, and the maximal value can be characterized as

$$\varrho(\tau) = \max_{c \le x \le c+2\pi} \{\kappa_s(x)\}.$$

Theorem 2.1. *Assume $f(x) \in C^4[c, c+2\pi]$, $s \neq x_i$, $i = 0, 1, 2, \ldots, n$. Employing the Hermite interpolation function, with $Q_n(f, s)$ defined by formula (2.5), the following estimate holds:*

$$|\mathcal{E}_n(f, s)| \le C \left\{ (\varrho(\tau) + 1)\gamma^{-1}(\tau) + 2|\ln \gamma(\tau)| + |\ln h| \right\} h^3,$$

where

$$\gamma(\tau) = \min_{0 \le i \le n} \frac{|s - x_i|}{h} = \frac{1 - |\tau|}{2}, \quad -1 < \tau < 1. \tag{2.17}$$

Theorem 2.2. *Assume $f(x) \in C^5[c, c+2\pi]$, $s \neq x_i$, $i = 0, 1, 2, \ldots, n$. Applying cubic spline quadrature with $Q_n(f, s)$ defined in formula (2.5), the following holds:*

$$\mathcal{E}_n(f, s) = \frac{f^{(4)}(s) h^3}{\pi^2} \Phi(\tau) + \mathcal{R}_n(f) + \mathcal{R}_n(f, s),$$

and

$$|\mathcal{R}_n(f) + \mathcal{R}_n(f, s)| \le C\{(\varrho(\tau) + 1)\gamma^{-1}(\tau) + (\varrho(\tau) + 1)|\ln \gamma(\tau)| + \varrho(\tau)|\ln h|\} h^4.$$

Define

$$\Phi(\tau) = \mathrm{Cl}_3[(1 + \tau)\pi]. \tag{2.18}$$

Under the assumptions of Theorem 2, when τ^* is a zero of $\Phi(\tau)$, the following estimate holds:

$$|\mathcal{E}_n(f, s)| \le C\{(\varrho(\tau^*) + 1)\gamma^{-1}(\tau^*) + (\varrho(\tau^*) + 1)|\ln \gamma(\tau^*)| + \varrho(\tau^*)|\ln h|\} h^4.$$

3. Proof of Theorems 1 and 2

Lemma 3.1 ([45])**.** *Suppose $f(x) \in C^4[c, c+2\pi]$, $s \neq x_i, i = 0, 1, 2, \cdots, n$. Using the Hermite integration formula, we have $R(x) = f(x) - H_3(x)$, $|R(x)| \le Ch^4$, then the corresponding error satisfies*

$$|{}^{\mathrm{H}}\mathcal{E}_n(f, s)| \le C \left(\gamma^{-1}(\tau) + |\ln \gamma(\tau)| + |\ln h| \right) h^3.$$

Lemma 3.2. *Suppose $f(x) \in C^4[c, c+2\pi]$, $s \neq x_i$, $i = 0, 1, 2, \ldots, n$. Under the periodic cubic spline construction, the following error estimate holds:*

$$|\mathcal{E}_n^{(a)}(f, s)| \le C \left(|\ln \gamma(\tau)| + \gamma^{-1}(\tau) + |\ln h| \right) h^3.$$

Proof. From the construction of the periodic cubic spline, the nodal derivatives m_i satisfy the cyclic tridiagonal system $Ae = \delta$. It can be derived from formula (2.3) that

$$\begin{bmatrix} 4 & 1 & & & & 1 \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ 1 & & & & 1 & 4 \end{bmatrix} \begin{bmatrix} m_0 - f_0' \\ m_1 - f_1' \\ \vdots \\ m_{n-1} - f_{n-1}' \end{bmatrix} = \begin{bmatrix} \delta_0 \\ \delta_1 \\ \vdots \\ \delta_{n-1} \end{bmatrix}.$$

By solving this cyclic tridiagonal system, we obtain the error vector $e = [m_0 - f_0', \ldots, m_{n-1} - f_{n-1}']^T$ and A is a cyclic tridiagonal matrix with diagonal elements 4 and off-diagonal elements 1. The components of the truncation error δ_i are given by

$$\delta_i = \frac{3}{h}(f_{i+1} - f_{i-1}) - (f_{i-1}' + 4f_i' + f_{i+1}'), \quad i = 0, 1, \ldots, n-1.$$

Here, f_{-1} denotes the result obtained by periodic extension over the aforementioned sub-intervals. Since f is a 2π-periodic function, it follows that $f_{-1} = f_{n-1}$. By performing Taylor expansion for $f \in C^4$, it can be shown that

$$|\delta_i| \leq Ch^3, \quad i = 0, 1, \ldots, n-1.$$

Since the matrix A is strictly diagonally dominant, its inverse is bounded by $\|A^{-1}\|_\infty \leq C$, independent of h. Taking the infinity norm on both sides of $Ae = \delta$,we obtain

$$\|e\|_\infty \leq \|A^{-1}\|_\infty \|\delta\|_\infty \leq C \cdot Ch^3 = O(h^3).$$

Thus,

$$|m_i - f_i'| \leq Ch^3, \quad i = 0, 1, \ldots, n-1.$$

Assume that $s \in (x_m, x_{m+1})$, $\mathcal{E}_n^{(a)}(f, s)$ can be estimated by the formula given below

$$\mathcal{E}_n^{(a)}(f, s) = \sum_{\substack{i=0\\ i\neq m}}^{n-1} \int_{x_i}^{x_{i+1}} \frac{H_3(x) - S_3(x)}{\sin^2 \frac{x-s}{2}} \mathrm{d}x + \text{f.p.} \int_{x_m}^{x_{m+1}} \frac{H_3(x) - S_3(x)}{\sin^2 \frac{x-s}{2}} \mathrm{d}x.$$

With respect to the first part of the above formula, the following expression holds:

$$\begin{aligned} &\left| \sum_{\substack{i=0\\ i\neq m}}^{n-1} \int_{x_i}^{x_{i+1}} \frac{H_3(x) - S_3(x)}{\sin^2 \frac{x-s}{2}} \mathrm{d}x \right| \\ &\leq \max_{0\leq i\leq n-1} |H_3(\zeta_i) - S_3(\zeta_i)| \sum_{\substack{i=0\\ i\neq m}}^{n-1} \int_{x_i}^{x_{i+1}} \frac{1}{\sin^2 \frac{x-s}{2}} \mathrm{d}x \\ &\leq \max_{0\leq i\leq n-1} |H_3(\zeta_i) - S_3(\zeta_i)| \sum_{\substack{i=0\\ i\neq m}}^{n-1} \int_{x_i}^{x_{i+1}} \frac{\kappa_s(x)}{(x-s)^2} \mathrm{d}x \\ &\leq Ch^4 \varrho(\tau) \sum_{\substack{i=0\\ i\neq m}}^{n-1} \int_{x_i}^{x_{i+1}} \frac{1}{(x-s)^2} \mathrm{d}x \\ &\leq C\varrho(\tau)\gamma^{-1}(\tau) h^3, \end{aligned} \tag{3.3}$$

where $\zeta_i \in (c, c + 2\pi)$ and $\varrho(\tau)$ have already been defined in formula (2.16). For the second part of the above formula, it holds

$$\begin{aligned}
&\left|\text{f.p.}\int_{x_m}^{x_{m+1}}\frac{H_3(x)-S_3(x)}{\sin^2\frac{x-s}{2}}\mathrm{d}x\right| \\
&\le \max_{0\le i\le n-1}|H_3(\zeta_i)-S_3(\zeta_i)|\left|\text{f.p.}\int_{x_m}^{x_{m+1}}\frac{1}{\sin^2\frac{x-s}{2}}\mathrm{d}x\right| \\
&\le 4\max_{0\le i\le n-1}|H_3(\zeta_i)-S_3(\zeta_i)|\left|\text{f.p.}\int_{x_m}^{x_{m+1}}\frac{1}{(x-s)^2}\mathrm{d}x\right| \\
&+\max_{0\le i\le n-1}|H_3(\zeta_i)-S_3(\zeta_i)|\left|\text{f.p.}\int_{x_m}^{x_{m+1}}\frac{\kappa_s(x)-4}{(x-s)^2}\mathrm{d}x\right|.
\end{aligned} \tag{3.4}$$

From the first part of formula (3.4), we obtain

$$4\max_{0\le i\le n-1}|H_3(\zeta_i)-S_3(\zeta_i)|\left|\text{f.p.}\int_{x_m}^{x_{m+1}}\frac{1}{(x-s)^2}\mathrm{d}x\right|\le C\left\{\gamma^{-1}(\tau)+|\ln\gamma(\tau)|\right\}h^3. \tag{3.5}$$

The second term holds

$$\begin{aligned}
&\max_{0\le i\le n-1}|H_3(\zeta_i)-S_3(\zeta_i)|\left|\text{f.p.}\int_{x_m}^{x_{m+1}}\frac{\kappa_s(x)-4}{(x-s)^2}\mathrm{d}x\right| \\
&\le Ch^4\left|\text{f.p.}\int_{x_m}^{x_{m+1}}\frac{\kappa_s(x)-4}{(x-s)^2}\mathrm{d}x\right| \\
&= Ch^4\left|\text{f.p.}\int_{x_m}^{x_{m+1}}\frac{1}{\sin^2\frac{x-s}{2}}\mathrm{d}x-4\text{f.p.}\int_{x_m}^{x_{m+1}}\frac{1}{(x-s)^2}\mathrm{d}x\right| \\
&= Ch^4\left[-2\cot\frac{s-x_m}{2}-2\cot\frac{x_{m+1}-s}{2}+\frac{4h}{(x_{m+1}-s)(s-x_m)}\right] \\
&\le C\gamma^{-1}(\tau)h^3,
\end{aligned} \tag{3.6}$$

where $\zeta_i\in(c,c+2\pi)$. By applying the triangle inequality to absolute values in the derivation above, we get

$$|H_3(x)-S_3(x)|\le h\left(|f'_m-m_m|+|f'_{m+1}-m_{m+1}|\right).$$

It follows that

$$\left|\text{f.p.}\int_{x_m}^{x_{m+1}}\frac{H_3(x)-S_3(x)}{\sin^2\frac{x-s}{2}}\mathrm{d}x\right|\le C\left\{\gamma^{-1}(\tau)+|\ln\gamma(\tau)|\right\}h^3. \tag{3.7}$$

Therefore, by combining this result with formulas (3.3) and (3.7), we conclude that

$$\left|\mathcal{E}_n^{(a)}(f,s)\right|\le C\left\{\gamma^{-1}(\tau)+|\ln\gamma(\tau)|\right\}h^3.$$

Thus, Theorem 2.1 is directly derived from Lemmas 3.1 and 3.2.

Lemma 3.3 ([40])**.** *Suppose $f(x)\in C^5[c,c+2\pi]$, m_0, m_n are defined as before. Then the following estimation holds:*

$$|m_i-f'_i|\le Ch^4,\quad i=0,1,\cdots,n.$$

Proof. It is noted that for a periodic function $f(x) \in C^4$, the general theory of periodic splines provides a nodal derivative error of $O(h^3)$. However, when the regularity of $f(x)$ is elevated to C^5, a higher-order truncation error can be attained through a more refined Taylor expansion. From formula (3.1), we derive

$$\begin{bmatrix} m_0 - f_0' \\ m_1 - f_1' \\ \vdots \\ m_{n-1} - f_{n-1}' \end{bmatrix} = A^{-1} \begin{bmatrix} \delta_0 \\ \delta_1 \\ \vdots \\ \delta_{n-1} \end{bmatrix}, \tag{3.8}$$

where A denotes a cyclic tridiagonal matrix.

By performing Taylor expansion on f_{i+1}, f_{i-1}, f_{i+1}' and f_{i-1}' at x_i and then performing operations, we can obtain

$$\delta_i = \frac{3}{h}(f_{i+1} - f_{i-1}) - (f_{i-1}' + 4f_i' + f_{i+1}') = -\frac{h^4}{30} f^{(5)}(\xi_i) + O(h^5), \quad i = 0, 1, \cdots, n-1. \tag{3.9}$$

When performing Taylor expansion on x_0 and x_n periodically, we expand f_1 and f_{n-1} separately,

$$m_0 - f_0' = \frac{h^4}{30} f^{(5)}(\xi_0) + O(h^5), \tag{3.10}$$

$$m_n - f_n' = \frac{h^4}{30} f^{(5)}(\xi_n) + O(h^5). \tag{3.11}$$

Moreover, given that the matrix A^{-1} exhibits strict diagonal dominance, the anticipated outcome can be achieved

$$|m_i - f_i'| \leq Ch^4, \quad i = 0, 1, \cdots, n.$$

Lemma 3.4 ([45])**.** *Assume $f(x) \in C^5[c, c+2\pi]$, $s \in (c, c+2\pi)$, using Hermite rule, we have*

$${}^{\mathrm{H}}\mathcal{E}_n(f, s) = \frac{f^{(4)}(s)h^3}{\pi^2}\Phi(\tau) + \mathcal{R}_n(f), \tag{3.12}$$

where

$$|\mathcal{R}_n(f)| \leq C\varrho(\tau)\left(\gamma^{-1}(\tau) + |\ln \gamma(\tau)| + |\ln h|\right) h^4.$$

Lemma 3.5. *Suppose $f(x) \in C^5[c, c+2\pi]$, $s \in (c, c+2\pi)$ and $s \neq x_i, i = 0, 1, \cdots, n-1$. Then the following conclusion holds:*

$$\int_c^{c+2\pi} \frac{H_3(x) - S_3(x)}{\sin^2 \frac{x-s}{2}} \mathrm{d}x = R_n(f, s),$$

where the remainder term satisfies the estimation

$$|R_n(f, s)| \leq C\left\{(\varrho(\tau) + 1)\gamma^{-1}(\tau) + |\ln \gamma(\tau)|\right\} h^4.$$

Proof. For any singular point $s \in (x_m, x_{m+1})$, one can derive

$$R_n(f,s) = \sum_{\substack{i=0\\ i\neq m}}^{n-1} \int_{x_i}^{x_{i+1}} \frac{H_3(x)-S_3(x)}{\sin^2\frac{x-s}{2}}\mathrm{d}x + \text{f.p}\int_{x_m}^{x_{m+1}} \frac{H_3(x)-S_3(x)}{\sin^2\frac{x-s}{2}}\mathrm{d}x.$$

With respect to the first part of the above formula, the following expression holds:

$$\begin{aligned}
&\left|\sum_{\substack{i=0\\ i\neq m}}^{n-1} \int_{x_i}^{x_{i+1}} \frac{H_3(x)-S_3(x)}{\sin^2\frac{x-s}{2}}\mathrm{d}x\right| \\
&\quad \le \max_{0\le i\le n-1} |H_3(\zeta_i)-S_3(\zeta_i)| \sum_{\substack{i=0\\ i\neq m}}^{n-1} \int_{x_i}^{x_{i+1}} \frac{1}{\sin^2\frac{x-s}{2}}\mathrm{d}x \\
&\quad \le \max_{0\le i\le n-1} |H_3(\zeta_i)-S_3(\zeta_i)| \sum_{\substack{i=0\\ i\neq m}}^{n-1} \int_{x_i}^{x_{i+1}} \frac{\kappa_s(x)}{(x-s)^2}\mathrm{d}x \\
&\quad \le Ch^5\varrho(\tau) \sum_{\substack{i=0\\ i\neq m}}^{n-1} \int_{x_i}^{x_{i+1}} \frac{1}{(x-s)^2}\mathrm{d}x \\
&\quad \le C\varrho(\tau)\gamma^{-1}(\tau)h^4.
\end{aligned}$$

For the second part of the above formula, it can be seen that

$$\begin{aligned}
&\left|\text{f.p.}\int_{x_m}^{x_{m+1}} \frac{H_3(x)-S_3(x)}{\sin^2\frac{x-s}{2}}\mathrm{d}x\right| \\
&\quad \le \max_{0\le i\le n-1} |H_3(\zeta_i)-S_3(\zeta_i)| \left|\text{f.p.}\int_{x_m}^{x_{m+1}} \frac{1}{\sin^2\frac{x-s}{2}}\mathrm{d}x\right| \\
&\quad \le 4\max_{0\le i\le n-1} |H_3(\zeta_i)-S_3(\zeta_i)| \left|\text{f.p.}\int_{x_m}^{x_{m+1}} \frac{1}{(x-s)^2}\mathrm{d}x\right| \\
&\quad + \max_{0\le i\le n-1} |H_3(\zeta_i)-S_3(\zeta_i)| \left|\text{f.p.}\int_{x_m}^{x_{m+1}} \frac{\kappa_s(x)-4}{(x-s)^2}\mathrm{d}x\right|.
\end{aligned} \tag{3.13}$$

The first item has the following estimation:

$$4\max_{0\le i\le n-1} |H_3(\zeta_i)-S_3(\zeta_i)| \left|\text{f.p.}\int_{x_m}^{x_{m+1}} \frac{1}{(x-s)^2}\mathrm{d}x\right| \le C\left\{\gamma^{-1}(\tau) + |\ln\gamma(\tau)|\right\}h^4. \tag{3.14}$$

The second term holds

$$\begin{aligned}
&\max_{0\le i\le n-1}|H_3(\zeta_i)-S_3(\zeta_i)|\left|\text{f.p.}\int_{x_m}^{x_{m+1}}\frac{\kappa_s(x)-4}{(x-s)^2}\mathrm{d}x\right|\\
&\le Ch^5\left|\text{f.p.}\int_{x_m}^{x_{m+1}}\frac{\kappa_s(x)-4}{(x-s)^2}\mathrm{d}x\right|\\
&=Ch^5\left|\text{f.p.}\int_{x_m}^{x_{m+1}}\frac{1}{\sin^2\frac{x-s}{2}}\mathrm{d}x-4\text{f.p.}\int_{x_m}^{x_{m+1}}\frac{1}{(x-s)^2}\mathrm{d}x\right|\\
&=Ch^5\left[-2\cot\frac{s-x_m}{2}-2\cot\frac{x_{m+1}-s}{2}+\frac{4h}{(x_{m+1}-s)(s-x_m)}\right]\\
&\le C\gamma^{-1}(\tau)h^4.
\end{aligned} \tag{3.15}$$

Based on the aforementioned estimation, the following result can be derived

$$|R_n(f,s)|\le C\left\{(\varrho(\tau)+1)\gamma^{-1}(\tau)+|\ln\gamma(\tau)|\right\}h^4.$$

Theorems 2.1 and 2.2 are proven by leveraging Lemmas 3.4 and 3.5.

4. Numerical examples

The errors arising from the approximation of the Hadamard finite-part integral on the circle by means of the cubic spline quadrature rule are presented in Tables 1 and 2 for two test cases.

In the example, the singular points are chosen as $s = x_{[n/4]} + (1+\tau)h/2$ and $s = x_n - (1+\tau)h/2$. Herein, h^α denotes the convergence order calculated from the error values corresponding to adjacent mesh sizes.

In the example, the subsequent hypersingular integral is taken into account

$$\text{f.p.}\int_0^{2\pi}\frac{1+2\cos x+2\cos 2x}{\sin^2\frac{x-s}{2}}\mathrm{d}x=-8\pi(\cos s+2\cos 2s).$$

It is indicated by the data presented in Table 1 that, under the configuration of singular points located within the grid cells, the order of the error corresponding to non-superconvergence parameters remains at $O(h^3)$. In contrast, the order of the error is elevated to $O(h^4)$ for superconvergence parameters, thereby re-validating Theorem 2.2.

It is further demonstrated by the results in Table 2 that even when the singular point is situated at the nodal points x_0 or x_n, the superconvergence phenomenon is still clearly discerned. Due to the cyclic tridiagonal system employed in the periodic cubic spline construction, these nodal points are theoretically equivalent to any other nodal points on the circle. As a result, the superconvergence behavior is preserved at the mesh closure, and the order of error is notably enhanced. This confirms that the proposed method is robustly applicable to periodic scenarios without any accuracy degradation at the nodal boundaries.

Table 1. Errors of the cubic spline rule with $s = x_{[n/4]} + (1 + \tau)h/2$.

n	$\tau = 0.2$		$\tau = -0.2$		$\tau = 0.8$		$\tau = -0.8$		$\tau = 0.53834070$		$\tau = -0.53834070$	
	Error	h^α	Error	h^α	Error	h^α	Error	h^α	Error	h^α	Error	h^α
16	1.9517e-01	–	1.7667e-01	–	8.1549e-02	–	1.4701e-01	–	6.5712e-02	–	1.6930e-02	–
32	2.1069e-02	3.21	2.0518e-02	3.11	1.6806e-02	2.28	1.8757e-02	2.97	2.8412e-03	4.53	1.3888e-03	3.61
64	2.4754e-03	3.09	2.4612e-03	3.06	2.3548e-03	2.84	2.4052e-03	2.96	1.3652e-04	4.38	9.9053e-05	3.81
128	3.0117e-04	3.04	3.0094e-04	3.03	3.0495e-04	2.95	3.0579e-04	2.98	7.2696e-06	4.23	6.6422e-06	3.90
256	3.7211e-05	3.02	3.7217e-05	3.01	3.8583e-05	2.98	3.8563e-05	2.99	4.4917e-07	4.02	4.6433e-07	3.84
$h^{\bar{\alpha}}$		3.09		3.05		2.76		2.97		4.29		3.79

Table 2. Errors of the cubic spline rule with $s = x_n - (1 + \tau)h/2$.

n	$\tau = 0.2$		$\tau = -0.2$		$\tau = 0.8$		$\tau = -0.8$		$\tau = 0.53834070$		$\tau = -0.53834070$	
	Error	h^α	Error	h^α	Error	h^α	Error	h^α	Error	h^α	Error	h^α
16	2.0444e-01	–	1.8396e-01	–	8.5117e-02	–	1.5759e-01	–	6.9167e-02	–	1.5164e-02	–
32	2.2236e-02	3.20	2.1569e-02	3.09	1.7669e-02	2.27	2.0034e-02	2.98	3.0481e-03	4.50	1.2879e-03	3.56
64	2.6219e-03	3.08	2.6006e-03	3.05	2.4873e-03	2.83	2.5627e-03	2.97	1.4910e-04	4.35	9.3044e-05	3.79
128	3.1951e-04	3.04	3.1883e-04	3.03	3.2300e-04	2.94	3.2538e-04	2.98	8.0417e-06	4.21	6.2736e-06	3.89
256	3.9502e-05	3.02	3.9481e-05	3.01	4.0933e-05	2.98	4.1009e-05	2.99	4.9475e-07	4.02	4.3913e-07	3.84
$h^{\bar{\alpha}}$		3.08		3.05		2.76		2.98		4.27		3.77

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no potential conflict of interests.

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