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Research article

Periodic dynamics of a switching model with multiple stable states and linear harvest

Qing Zhu, Yu Li and Huaqin Peng*

School of Mathematics and Statistics, Guangxi Normal University, Guilin 541004, China

* **Correspondence:** Email: huaqinpeng@126.com.

Abstract: Fishing moratoriums are vital for fishery management, restoring fish stocks, safeguarding ecosystems, and ensuring sustainable development. In this study, we explored and studied a canonical model with a Holling type-III functional response consisting of two sub-equations switching each other. If the length of the closed season $\bar{T} \leq T^*$, the trivial steady state is globally asymptotically stable. For $\bar{T} > T^*$, we provided a complete analysis of the existence of a unique periodic solution and three exact periodic solutions. We also provided sufficient conditions for the existence of a globally asymptotically stable periodic solution. Finally, numerical examples were provided to illustrate the theoretical results.

Keywords: periodic solutions; switching dynamical systems; Holling type-III functional response; global asymptotic stability

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1. Introduction

In natural environments, the growth rates of biological populations, resource availability, and environmental conditions (temperature, precipitation, and seasonal variations) typically fluctuate over time. Traditional autonomous models (e.g., logistic models) assume constant parameters and cannot reflect these dynamics. In contrast, non-autonomous models introduce time-varying parameters, such as the growth rate $r(t)$, environmental carrying capacity $K(t)$, making them more consistent with the complexity of real-world ecosystems. Seasonality is a fundamental characteristic of the Earth's environment and is influenced by the tilt of the Earth's axis and its revolution around the sun. Seasonal changes lead to periodic fluctuations in environmental conditions, including the temperature and daylight duration, precipitation, and resource availability, also called seasonal succession in the literature [1–7].

The growth of organisms requires specific environmental conditions and is not always suitable for the growth of certain species. For instance, arctic plants grow only briefly during the summer, whereas low winter temperatures force their metabolism to stagnate and enter a dormant state. In tropical rainforests, tall trees dominate the upper canopy for sunlight, leaving under story herbaceous plants to rely on fleeting sun flecks during the rainy season for rapid growth. During the dry season, because insufficient light, they enter dormancy. It can be modeled by a periodic switching system [8–10]. Time periodic mathematical models have been developed to incorporate periodic environmental factors into the population growth, disease transmission, and other life systems. Existing extensive studies [11–13] suggest that seasonality plays a crucial role in the outbreak and evolution of epidemics and the transmission of diseases. Diseases transmitted by mosquitoes, such as malaria, Zika virus, and dengue fever, severely impact human health, the pulsed and periodic release of sterile mosquitoes carrying *Wolbachia* bacteria is an effective method for controlling these mosquito-borne diseases. In 2018, Yu [14] proposed a novel modeling framework that treats the release quantity of sterile mosquitoes as a predetermined function and subsequently establishes a time-delay differential equation system based on this formulation. Further research on this release strategy can be found in [15–19].

Fish populations can often exist in multiple stable states due to complex ecological interactions, such as predation, resource availability, competition, and environmental disturbances. This phenomenon not only highlights the intricacy of aquatic ecosystems but also offers important implications for their management and conservation. A classical model [20–22] based on a Holling type III functional response is commonly used to describe the dynamics of fish populations under predation pressure. This type of functional response accounts for nonlinear changes in predation rate, providing insight into its role in population stability. The governing equations are as follows:

$$\frac{dw}{dt} = rw(1 - \frac{w}{k}) - \frac{apw^2}{b^2 + w^2}, \tag{1.1}$$

where $w(t)$ is the population density at time t, r is the intrinsic growth rate, k is the carrying capacity, a is the predator consumption rate, p is the constant predator density and b is the half-saturation level of predation. Various differential equations with harvesting strategies have been proposed to describe phenomena arising in population ecology [23–30]. Based on (1.1), Takashina and Mougi [31] introduced a proportional harvesting term for model (1.1) as follows:

$$\frac{dw}{dt} = rw(1 - \frac{w}{k}) - \frac{apw^2}{b^2 + w^2} - qEw, \tag{1.2}$$

where q is the catch ability coefficient and E is the fishing effort.

Fishery management strategies can be broadly categorized into input control and output control mechanisms. Input control involves regulating the level of fishing effort, such as by imposing restrictions on the number, size, and horsepower of vessels, as well as specifications for fishing gear dimensions and mesh sizes. Output control, entails direct management of harvest quantities through measures like seasonal closures, spatially designated no-take zones, and total allowable catch limits. A substantial body of modeling work has been developed to analyze the effects of seasonal fishing closures [32–37].

Most harvesting models operate under the assumption of constant environmental condition. However, the periodic alternation between the open and closed seasons [32–35] leads to a living environment for populations that vary cyclically. This periodic variation has a profound influence on the dynamics of population growth and likelihood of species invasion. In this paper, we consider the following system, Our main objective is to understand the effect of seasonal harvesting on the survival of the population. Therefore, we formulated the single species model with seasonal linear type sawtooth wave periodic harvesting as follows:

$$\begin{cases} \frac{dw}{dt} = rw(1 - \frac{w}{k}) - \frac{apw^2}{b^2+w^2}, & t \in (nT, nT + \bar{T}], \\ \frac{dw}{dt} = rw(1 - \frac{w}{k}) - \frac{apw^2}{b^2+w^2} - qEw, & t \in (nT + \bar{T}, (n+1)T], \end{cases} \tag{1.3}$$

where $n = 0, 1, 2, \cdots$, T is the period of the seasonal fluctuation environment where the single species population lives in. $\bar{T}$ measures the length of the closed season (non-harvesting season); namely, $T - \bar{T}$ is the length of the open season (harvesting season). For simplicity, let $\bar{w} = \frac{w}{b}$, $\bar{t} = rt$, $Q = \frac{k}{b}$, $R = \frac{rb}{ap}$, $h = \frac{qE}{r}$, by dropping the bars, we have

$$\frac{dw}{dt} = w(1 - \frac{w}{Q}) - \frac{1}{R}\frac{w^2}{1 + w^2}, \quad t \in (nT, nT + \bar{T}], \tag{1.4}$$

and

$$\frac{dw}{dt} = w(1 - \frac{w}{Q}) - \frac{1}{R}\frac{w^2}{1 + w^2} - hw, \quad t \in (nT + \bar{T}, (n+1)T], \tag{1.5}$$

where h is a composite parameter reflecting the relative strength of fishing mortality versus natural recruitment/growth (a measure of fishing intensity). Biologically, $h > 1$ indicates high fishing intensity (fishing mortality dominates natural replenishment), threatening population survival.

The remainder of this paper is organized as follows. In Section 2, we present a stability analysis of Eq (1.4) including the determination of equilibria and their stability. In Section 3, we provide a detailed dynamic analysis of the existence and stability of periodic solutions. Several numerical examples illustrating our theoretical results are provided in Section 4. We finally conclude our paper with a brief discussion in Section 5.

2. Preliminaries

In this section, we first present the stability analysis of Eq (1.4), including the determination of the equilibria and their stability. We then define a threshold value T^* for the seasonal fluctuation environment, where some important lemmas play a crucial role in understanding the global asymptotically stability of the system.

2.1. Stability analysis of Eq (1.4)

We rewrite Eq (1.4) as

$$\frac{dw}{dt} = wf(w), \tag{2.1}$$

where $f(w) = (1 - \frac{w}{Q}) - \frac{1}{R}\frac{w}{1+w^2}$. In addition, the trivial equilibrium $w_0 = 0$, which is always unstable. The remaining zeros of (2.1) satisfy $f(w) = 0$, which yields

$$R(1 - \frac{w}{Q}) = \frac{w}{1 + w^2}. \tag{2.2}$$

Let $g_1(w) = R(1 - \frac{w}{Q})$ and $g_2(w) = \frac{w}{1+w^2}$. Then, the zeros of $f(w)$ are the points of $g_1(w) = g_2(w)$. Thus, a graphical solution to Eq (2.2) can be constructed following the approach established by [21]. As illustrated in Figure 1, although $g_1(w)$ also depends on Q, we have suppressed this dependence in the notation, since our primary interest lies in analyzing the behavior of the roots when Q is held fixed (Q represents a time-invariant or slowly varying ecological characteristic of the target fish population, such as carrying capacity of the habitat and baseline survival rate independent of fishing pressure). In real fisheries, such parameters are determined by long-term environmental conditions (e.g., water volume, food availability, habitat quality) and do not fluctuate with the periodic fishing/closed season regime, which is the core focus of this study and R is varied.

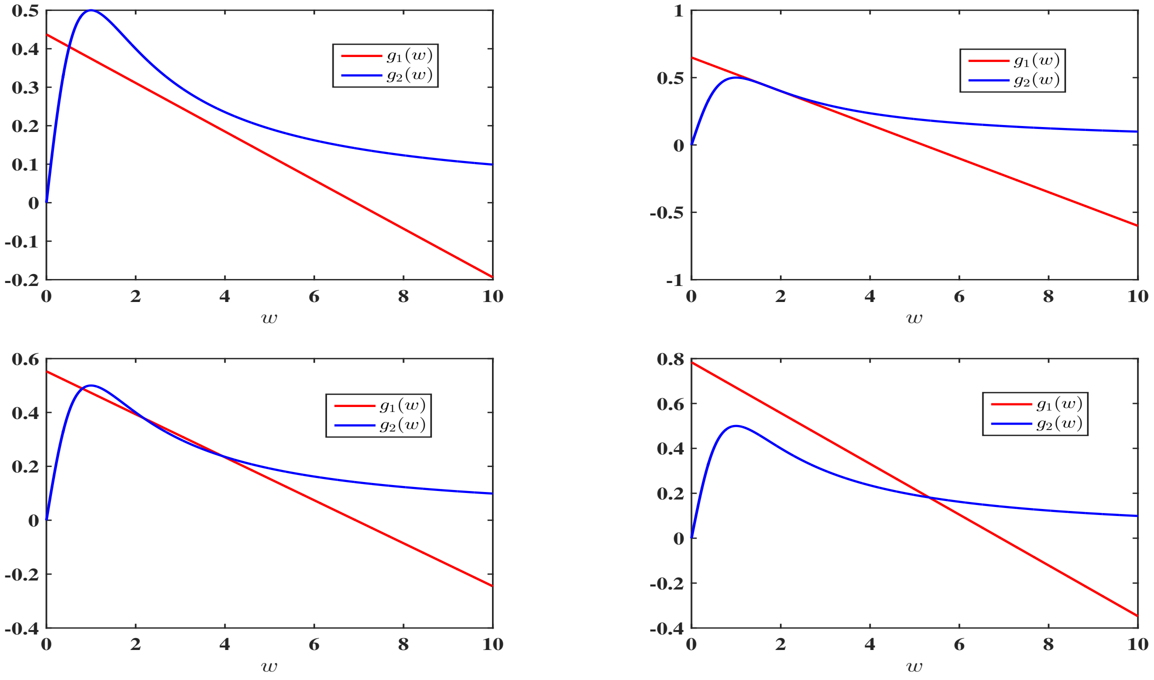

Figure 1. The zeros of $f(w)$ are the points of intersection of $g_1(w)$ and $g_2(w)$.

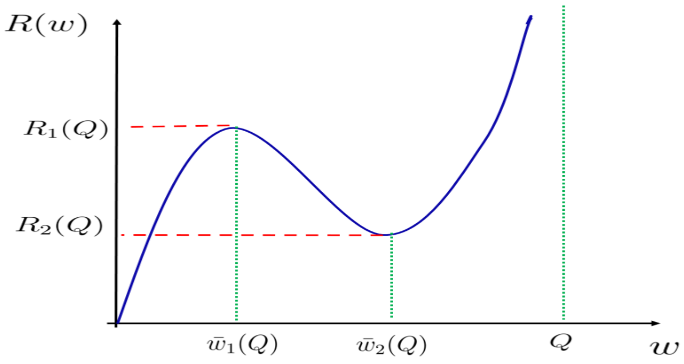

Figure 2. The graph of $R(w)$ versus w for a sufficiently large value of Q.

For sufficiently large Q, there are two distinct values of R for the curve $g_2(w)$ admits two distinct tangent lines of the form $g_1(w)$, and we call them $R_1(Q)$ and $R_2(Q)$ with $R_1(Q) > R_2(Q)$. Denote the w-coordinates of the respective tangency points by $\bar{w}_1(Q)$ and $\bar{w}_2(Q)$. For each $w \in (0, Q)$, there exists a unique line $g_1(w)$ passing through the point $(w, g_2(w))$ on $g_2(w)$, with associated parameter $R(w)$. As illustrated in Figure 2, the function $R(w)$ exhibits the following monotonic behavior:

- $R(w)$ increases from 0 to $R_1(Q)$ as w increases from 0 to $\bar{w}_1(Q)$.
- $R(w)$ decreases from $R_1(Q)$ to $R_2(Q)$ as w increases from $\bar{w}_1(Q)$ to $\bar{w}_2(Q)$.
- $R(w)$ increases from $R_2(Q)$ to $+\infty$ as w increases from $\bar{w}_2(Q)$ to Q.

When Q is sufficiently small, the line $g_1(w)$ does not become tangent to the curve $g_2(w)$, and R increases monotonically with w. To determine the critical value of Q that distinguishes these two regimes, we observe that at $w = \bar{w}_1(Q)$, the curve $g_2(w)$ must be concave downward, whereas at $w = \bar{w}_2(Q)$, it must be concave upward. By shifting Q to the left until the two tangents coincide, that is, considering the value of Q such that $\bar{w}_1(Q) = \bar{w}_2(Q)$. We note that $\bar{w}_1(Q)$ corresponds to the limit point where the second derivative $g_2''(w)$ is non-positive, and $\bar{w}_2(Q)$ corresponds to the limit point where $g_2''(w)$ is non-negative. Therefore, at $\bar{w} = \bar{w}_1(Q) = \bar{w}_2(Q)$, we must have $g_2''(w) = 0$. A brief calculation yields $\bar{w}_1(Q) = \bar{w}_2(Q) = \sqrt{3}$. Constructing the tangent at this point, we obtain $Q = 3\sqrt{3}$, with $R_1(Q) = R_2(Q) = \frac{3\sqrt{3}}{8}$. Hence, for $Q \leq 3\sqrt{3}$, the function $R(w)$ is monotonic in w. This leads to the characterization of the behavior of $f(w) = 0$, a conclusion that is confirmed in Figure 3.

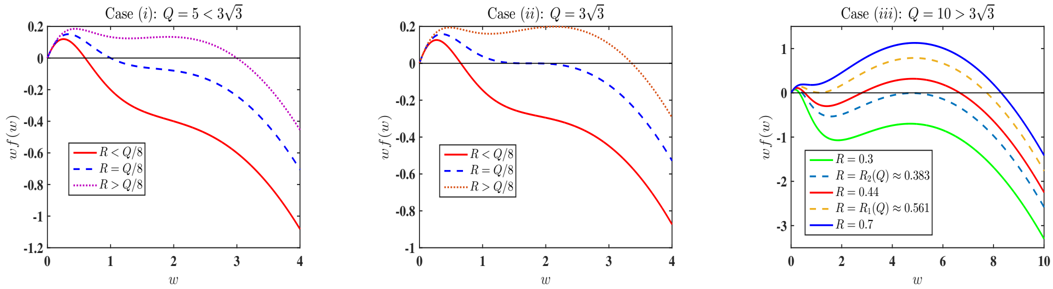

Figure 3. For a fixed value of Q, the behavior of $f(w) = 0$ for different cases of $R < \frac{Q}{8}$, $R = \frac{Q}{8}$ and $R > \frac{Q}{8}$.

In summary, we have the following results for the existence and stability of the equilibria of Eq (2.1).

Theorem 2.1. *Equation* (2.1) *has* $w_0 = 0$ *as a trivial equilibrium, that is unstable. In addition,*

(1) *If* $Q \leq 3\sqrt{3}$, *for each* $0 < R < \infty$, *Eq* (2.1) *has one positive equilibrium,* $w_1 < Q$, *which is locally asymptotically stable and corresponds to a low equilibrium density.*

(2) (i) *If* $Q > 3\sqrt{3}$, $0 < R < R_2(Q)$, *Eq* (2.1) *has one positive equilibrium,* $w_1 < Q$, *which is locally asymptotically stable.*

(ii) *If* $Q > 3\sqrt{3}$, $R = R_2(Q)$, *Eq* (2.1) *has two positive equilibria,* w_1 *and* $\bar{w}_2(Q)$, *where* w_1 *is locally asymptotically and* $\bar{w}_2(Q)$ *is unstable.*

(iii) *If* $Q > 3\sqrt{3}$, $R_2(Q) < R < R_1(Q)$, *Eq* (2.1) *has three positive equilibria,* w_1, w_2, *and* w_3, *where* w_1 *is locally asymptotically stable,* w_2 *is unstable and* w_3 *is locally asymptotically stable.*

(iv) *If* $Q > 3\sqrt{3}$, $R = R_1(Q)$, *Eq* (2.1) *has two positive equilibria,* $\bar{w}_1(Q)$ *and* w_3, *where* $\bar{w}_1(Q)$ *is unstable and* w_3 *is locally asymptotically stable.*

(v) *If* $Q > 3\sqrt{3}$, $R > R_1(Q)$, *Eq* (2.1) *has one positive equilibrium,* w_3, *where* w_3 *is locally asymptotically stable.*

Proof. We only provide a stability analysis of case (1), and similar discussions can be applied to other cases. When $Q \leq 3\sqrt{3}$, for each $0 < R < \infty$, it is easy to obtain $wf(w) > 0$ if $0 < w < w_1$ and $wf(w) < 0$ if $w_1 < w$. This implies that w_1 is asymptotically localized. The proof is complete. □

Remark 2.1. *Our graphical construction shows that as Q increases from* $3\sqrt{3}$ *to* $+\infty$, $R_1(Q)$ *decreases from* $\frac{3\sqrt{3}}{8}$ *to* $\frac{1}{2}$, *whereas* $R_2(Q)$ *decreases from* $\frac{3\sqrt{3}}{8}$ *to* 0. *We can also see that as Q increases from* $3\sqrt{3}$ *to* $+\infty$, $\bar{w}_1(Q)$ *decreases from* $\sqrt{3}$ *to* 1, *whereas* $\bar{w}_2(Q)$ *increases from* $\sqrt{3}$ *to* $+\infty$.

Remark 2.2. *From Figure 2, we observe that* $w_1 < \bar{w}_1(Q) < \sqrt{3}$. *Therefore,* w_1 *can be regarded as a low-density persistent state. When* $Q > 3\sqrt{3}$ *and* $R_2(Q) < R < R_1(Q)$, *we have* $0 < w_1 < \bar{w}_1(Q) < w_2 < \bar{w}_2(Q) < w_3 < Q$. *We may interpret* w_1 *as a low-density persistent state,* w_3 *as a high-density persistent state, and* w_2 *as a threshold density. Thus, if* $0 < w < w_2$ *at the initial time* $t = 0$, *then* $w \to w_1$ *as* $t \to \infty$. *However, if* $w > w_2$ *at* $t = 0$, *then* $w \to w_3$ *as* $t \to \infty$.

2.2. Basic lemmas for (1.4) and (1.5)

Let $w(t) = w(t; 0, u)$ denote the solution of models (1.4) and (1.5), with $w(0) = u > 0$. The solution $w(t)$ satisfying Eq (1.4) is determined with the initial value $w(0) = u$ for $t \in [0, \bar{T})$, and satisfying Eq (1.5) is determined with the initial value $w(\bar{T}) = \lim\limits_{t \to \bar{T}^-}$ for $t \in [\bar{T}, T)$. The solution can be defined in the same manner at other intervals. Because the right side of models (1.4) and (1.5) are T-periodic, the solution $w(t)$ for $t \in [0, T]$ can be determined for all $t > 0$. To describe the periodic dynamics of the solution $w(t)$, we define:

$$F(u) = w(T; 0, u), \quad \bar{F}(u) = w(\bar{T}; 0, u),$$

where F is the Poincaré map of models (1.4) and (1.5). The solution $w(t; 0, u)$ of (1.4) and (1.5) is T-periodic if and only if $F(u) = u$. For convenience, we set

$$H(u) = F(u) - u, \quad \bar{H}(u) = \bar{F}(u) - u.$$

Therefore, the existence of the T-periodic solutions of models (1.4) and (1.5) is equivalent to the existence of the positive roots of $H(u) = 0$. Furthermore, we denote sequences $\{H_n(u)\}$ and $\{\bar{H}_n(u)\}$ as follows:

$$H_n(u) = w(nT; 0, u) - u, \quad \bar{H}_n(u) = w(nT + \bar{T}; 0, u) - u, \quad n = 0, 1, 2, \cdots.$$

By induction, we obtain

$$H_0(u) = 0, \; H_{n+1}(u) = H_n(u) + H(H_n(u) + u), \quad n = 0, 1, 2, \cdots.$$

$$\bar{H}_0(u) = w(\bar{T}; 0, u) - u, \; \bar{H}_{n+1}(u) = \bar{H}_n(u) + \bar{H}(\bar{H}_n(u) + u), \quad n = 0, 1, 2, \cdots.$$

Lemma 2.1. *The following results hold for any given initial value* $u > 0$ *and* $n = 0, 1, 2, \cdots$.

(i) *Sequences* $\{H_n(u)\}$ *and* $\left\{\bar{H}_n(u)\right\}$ *are both strictly increasing if* $H(u) > 0$.
(ii) $H_n(u) \equiv 0$ *if* $H(u) = 0$.
(iii) *Sequences* $\{H_n(u)\}$ *and* $\left\{\bar{H}_n(u)\right\}$ *are both strictly decreasing if* $H(u) < 0$.

We rewrite (1.4) and (1.5) as

$$\begin{cases} \frac{dw}{dt} = wf(w) = F(w), & t \in (nT, nT + \bar{T}], \\ \frac{dw}{dt} = wg(w) = G(w), & t \in (nT + \bar{T}, (n+1)T], \end{cases} \tag{2.3}$$

where $f(w) = 1 - \frac{w}{Q} - \frac{1}{R}\frac{w}{1+w^2}$ and $g(w) = 1 - h - \frac{w}{Q} - \frac{1}{R}\frac{w}{1+w^2}$.

To further determine the signs of $H(u)$, we must analyze the qualitative properties of $H(u)$. We solved the initial value problem (2.3) with $w(0) = u$ for $t \in [0, \bar{T})$ and $w(\bar{T}) = \bar{F}(u)$ for $t \in [\bar{T}, T)$. For convenience, let us assume $F(w) \neq 0$ and $G(w) \neq 0$. Thus

$$\frac{dw}{wf(w)} = \left(\frac{\alpha_1}{w} + \frac{J_1(w)}{f(w)}\right)dw = dt, \quad t \in (0, \bar{T}] \tag{2.4}$$

$$\frac{dw}{wg(w)} = \left(\frac{\alpha_2}{w} + \frac{J_2(w)}{g(w)}\right)dw = dt, \quad t \in (\bar{T}, T] \tag{2.5}$$

where $J_1(w)$ and $J_2(w)$ are polynomial.

Set

$$Y_1(w) = w^{\alpha_1}F_1(w), \quad Y_2(w) = w^{\alpha_2}G_1(w), \tag{2.6}$$

where

$$F_1(w) = \exp\left(\int \frac{J_1(w)}{f(w)}dw\right), \quad G_1(w) = \exp\left(\int \frac{J_2(w)}{g(w)}dw\right).$$

Thus,

$$\frac{Y_1'(w)}{Y_1(w)} = \frac{\alpha_1}{w} + \frac{J_1(w)}{f(w)} = \frac{1}{wf(w)}, \tag{2.7}$$

and

$$\frac{Y_2'(w)}{Y_2(w)} = \frac{\alpha_2}{w} + \frac{J_2(w)}{g(w)} = \frac{1}{wg(w)}, \tag{2.8}$$

we have $\ln Y_1(w) = \int \frac{1}{wf(w)}dw$, $\ln Y_2(w) = \int \frac{1}{wg(w)}dw$.

Integrating (2.4) from 0 to $\bar{T}$ yields

$$Y_1(w(\bar{T})) = e^{\bar{T}}Y_1(w(0)),$$

and we obtain

$$Y_1(\bar{F}(u)) = e^{\bar{T}}Y_1(u). \tag{2.9}$$

Taking the derivative on both sides of Eq (2.9) with respect to u yields

$$Y_1'(\bar{F}(u))\bar{F}'(u) = e^{\bar{T}}Y_1'(u). \tag{2.10}$$

From (2.7) and (2.9), we have

$$\bar{F}'(u) = \frac{f(\bar{F}(u))}{f(u)} \times \frac{\bar{F}(u)}{u} = \frac{F(\bar{F}(u))}{F(u)}. \tag{2.11}$$

Similarly, integrating (2.5) from $\bar{T}$ to T, we get

$$Y_2(F(u)) = e^{(T-\bar{T})}Y_2(\bar{F}(u)). \tag{2.12}$$

In analogy with (2.10), we arrive at

$$Y_2'(F(u))F'(u) = e^{(T-\bar{T})}Y_2'(\bar{F}(u))\bar{F}'(u)$$

In fact, from Eqs (2.6), (2.8), and (2.12),

$$\frac{F'(u)}{\bar{F}'(u)} = \frac{F(u)}{\bar{F}(u)} \times \frac{g(F(u))}{g(\bar{F}(u))}. \tag{2.13}$$

Therefore,

$$\begin{aligned} F'(u) &= \frac{G(F(u))}{G(\bar{F}(u))} \times \frac{F(\bar{F}(u))}{F(u)} \\ &= \frac{F(\bar{F}(u))/G(\bar{F}(u))}{F(u)/G(F(u))} \end{aligned} \tag{2.14}$$

Because $u > 0$, we cannot derive the expression for $F'(0)$. Substituting (2.6) into (2.9) and (2.12) yields

$$\frac{\bar{F}(u)}{u} = \left[\frac{F_1(u)}{F_1(\bar{F}(u))}\right]^{\frac{1}{\alpha_1}} e^{\frac{\bar{T}}{\alpha_1}}$$

and

$$\frac{F(u)}{\bar{F}(u)} = \left[\frac{G_1(\bar{F}(u))}{G_1(F(u))}\right]^{\frac{1}{\alpha_2}} e^{\frac{(T-\bar{T})}{\alpha_2}},$$

which implies that

$$\frac{F(u)}{u} = \frac{F(u)}{\bar{F}(u)} \times \frac{\bar{F}(u)}{u} = \left[\frac{G_1(\bar{F}(u))}{G_1(F(u))}\right]^{\frac{1}{\alpha_2}} \left[\frac{F_1(u)}{F_1(\bar{F}(u))}\right]^{\frac{1}{\alpha_1}} e^{(\frac{\bar{T}}{\alpha_1}+\frac{(T-\bar{T})}{\alpha_2})}. \tag{2.15}$$

It follows from $F(u) \to 0, \bar{F}(u) \to 0$ as $u \to 0$ that

$$F'(0) = e^{(\frac{\bar{T}}{\alpha_1}+\frac{(T-\bar{T})}{\alpha_2})}.$$

Starting from the identity

$$\frac{\alpha_1}{w} + \frac{J_1(w)}{f(w)} = \frac{1}{wf(w)},$$

multiplying both sides by $wf(w)$ yields

$$\alpha_1 f(w) + wJ_1(w) = 1.$$

Taking the limit as $w \to 0$, and noting that $wJ_1(w) \to 0$ (since $J_1(w)$ is bounded near zero), we obtain

$$\alpha_1 f(0) = 1.$$

Similarly, for α_2, we derive $\alpha_2 = \frac{1}{g(0)}$ from the corresponding identity. Substituting these results into the expression for $F'(0)$, we arrive at

$$F'(0) = e^{\left(f(0)\bar{T}+g(0)(T-\bar{T})\right)}. \tag{2.16}$$

By direct computations, $f(0) = 1$ and $g(0) = 1 - h$, we have

$$F'(0) = e^{h(\bar{T}-T^*)}, \tag{2.17}$$

where $T^* = \frac{h-1}{h}T$, which is the critical minimum closed season duration required to ensure population persistence. If the actual closed season duration $\bar{T} > T^*$, we have $F'(0) > 1$, the population can persist. If $\bar{T} \leq T^*$, then the closed season is too short to allow sufficient recovery, leading to population decline or collapse. Specifically, $\frac{h-1}{h} = 1 - \frac{1}{h}$ shows that as fishing intensity h increases (approaching infinity), $T^* \to T$ meaning the closed season must cover nearly the entire year to sustain the fish population under extreme overfishing. This reveals a clear trade-off: higher fishing intensity demands a longer closed season to maintain the population's viability.

Next, we focus on the qualitative properties of $H(u)$ at u satisfying $H(u) = 0$, so

$$H'(u) = \begin{cases} \frac{B(\bar{F}(u))}{B(u)} - 1, & u > 0, \\ e^{h(\bar{T}-T^*)} - 1, & u = 0, \end{cases} \tag{2.18}$$

where

$$B(u) = \frac{F(u)}{G(u)}.$$

Taking the derivative of $B(u)$ yields

$$B'(u) = \frac{R(1+u^2)^2 + Q(1-u^2)}{QR(1+u^2)^2 g^2(u)} h = \frac{p(u)}{QR(1+u^2)^2 g^2(u)} h, \tag{2.19}$$

where $p(u) = Ru^4 + (2R - Q)u^2 + R + Q$.

Set $\Delta = Q(Q - 8R)$, if $Q \geq 8R$, we have $u_1^2 = \frac{(Q-2R)-\sqrt{Q^2-8QR}}{2R} > 1$ and $u_2^2 = \frac{(Q-2R)+\sqrt{Q^2-8QR}}{2R}$.

Remark 2.3. *After a simple calculation, we have* $\bar{w}_1(Q) = u_1$, $\bar{w}_2(Q) = u_2$.

Lemma 2.2. *The periodic solution* $\bar{w}(t)$ *is asymptotically stable if and only if there is* $\delta > 0$ *such that* $H(u)(u - \bar{u}) < 0$ *for* $0 < |u - \bar{u}| < \delta$.

Proof. Clearly the presence of δ was sufficient. To prove the necessity, we start by proving $\bar{w}(t)$ is stable.

By contradiction, assume that there exists $\varepsilon > 0$ and $t_0 \geq 0$ such that for any $0 < \delta_1 < \min\{\delta, \varepsilon\}$, there exists a $u \in (\bar{u} - \delta_1, \bar{u}) \bigcup (\bar{u}, \bar{u} + \delta_1)$ and a $t_1 > t_0$ such that

$$|w(t_1) - \bar{w}(t_1)| = \varepsilon, \text{ and } |w(t) - \bar{w}(t)| < \varepsilon, \text{for } t \in [t_0, t_1).$$

Without loss of generality, we assume that $\bar{w}(t_1) - w(t_1) = \varepsilon$, we have $H(u) > 0$ for $0 < \bar{u} - u < \delta_1$, implying that $\{H_n(u)\}$ and $\{\bar{H}_n(u)\}$ are both strictly increasing. There must be a nonnegative integer p such that either $t_1 = pT$, or $t_1 = pT + T_1$, or $t_1 \in (pT, pT + T_1) \bigcup (pT + T_1, (p+1)T)$. We assume that $t_0 \leq (p-1)T$; otherwise, we may reverse the solutions $w(t)$ and $\bar{w}(t)$ to point $t = (p-1)T$.

Suppose $t_1 = pT$, then we have

$$w(pT) + \varepsilon = \bar{w}(pT) \text{ and } \bar{w}(t) - w(t) < \varepsilon \text{ for } t \in [t_0, pT).$$

Which implies that

$$w(pT) + \varepsilon = \bar{w}(pT) = \bar{w}((p-1)T) < w((p-1)T) + \varepsilon.$$

Hence, from Lemma 2.1, $\{H_n(u)\}$ is decreases strictly, which is a contradiction.

Suppose $t_1 = pT + T_1$. Then, we have

$$\bar{w}(pT + T_1) = w(pT + T_1) + \varepsilon \text{ and } \bar{w}(t) < w(t) + \varepsilon \text{ for } t \in [t_0, pT + T_1),$$

we reach

$$\begin{aligned} w(pT + T_1) + \varepsilon &= \bar{w}(pT + T_1) \\ &= \bar{w}((p-1)T + T_1) \\ &< w((p-1)T + T_1) + \varepsilon, \end{aligned}$$

which means by Lemma 2.1 that series $\{\bar{H}_n(u)\}$ is strictly decreasing, again a contradiction.

If $t_1 \in (pT, pT + T_1)$, integrating (2.4) from pT to t_1, we have

$$Y_1(w(t_1)) = Y_1(w(pT))e^{(t_1 - pT)},$$

or, equivalently,

$$Y_1(\bar{w}(t_1 - T) - \varepsilon) = Y_1(w(pT))e^{(t_1 - pT)}. \tag{2.20}$$

Similarly, integrating (2.4) from $(p-1)T$ to $t_1 - T$, we obtain

$$Y_1(w(t_1 - T)) = Y_1(w((p-1)T))e^{(t_1 - pT)}. \tag{2.21}$$

However, $Y_1'(w) = \frac{Y_1(w)}{F(w)}$ indicates that $Y_1(w)$ is a locally monotonous function if $F(w)$ does not change its signs. Thus, by combining with $\bar{w}(t_1 - T) - \varepsilon < w(t_1 - T)$, (2.20) and (2.21) imply that $w(pT) < w((p-1)T)$, which leads to the same contradiction.

If $t_2 \in (pT + T_1, (p+1)T)$, integrating (2.4) from $pT + T_1$ to t_1, we have

$$Y_2(w(t_1)) = Y_2(w(pT + T_1))e^{(t_1 - pT - T_1)},$$

or, equivalently,

$$Y_2(\bar{w}(t_1 - T) - \varepsilon) = Y_2(w(pT + T_1))e^{(t_1 - pT - T_1)}. \tag{2.22}$$

Similarly, integrating (2.4) from $(p-1)T + T_1$ to $t_1 - T$, we obtain

$$Y_2(w(t_1 - T)) = Y_2(w((p-1)T + T_1))e^{(t_1 - pT - T_1)}. \tag{2.23}$$

Because $\bar{w}(t_1 - T) - \varepsilon < w(t_1 - T)$ with (2.22) and (2.23) imply that $w(pT + T_1) < w((p-1)T + T_1)$. This also means that series $\{\bar{H}_n(u)\}$ is strictly decreasing, again a contradiction. Therefore, $\bar{w}(t)$ was stable.

We then give the proof of attractive for $\bar{w}(t)$. We claim

$$\lim_{t\to\infty}(w(t) - \bar{w}(t)) = 0, \text{ for } 0 < |u - \bar{u}| < \delta.$$

Otherwise, we have $\lim\limits_{t\to\infty} w(t) \neq \bar{u}$, which contradicts to $H(u)(u - \bar{u}) < 0$ for $0 < |u - \bar{u}| < \delta$. Thus, periodic solution $\bar{w}(t)$ is locally asymptotically stable. The proof is complete. □

Corollary 2.1. *The periodic solution $\bar{w}(t)$ is asymptotically stable if and only if there is $\delta > 0$ such that $H(u)(u - \bar{u}) < 0$ for $u \in (0, \bar{u}) \bigcup (\bar{u}, \infty)$.*

3. Periodic dynamics analysis of (1.4) and (1.5)

In this section, we establish the main results concerning the persistence properties of the solutions to system (1.4) and (1.5). Because the model of (1.4) and (1.5) exhibits different dynamic properties in response to varying fishing intensities, we consider the following two cases.

Case 1: $h > 1$. In this case, we have

Theorem 3.1. *If $\bar{T} > T^*$ holds. Thus, the system in (2.3) has at least one positive T-periodic solution. Furthermore, (2.3) has a unique T-periodic solution if either $Q \leq 3\sqrt{3}$ or $Q > 3\sqrt{3}$ together with one of the following conditions:*

1) $R \leq R_2(Q)$,

2) $R > \frac{Q}{8}$,

and the unique T-periodic solution is globally asymptotically stable.

Proof. (a) When $Q \leq 3\sqrt{3}$, for any $R \geq 0$, $F(w)$ has only one small positive root $w_1 < u_1$ and $B'(u) > 0$ for $u \in [0, \infty)$. If $u \in [w_1, \infty)$, we have $\frac{dw}{dt} \leq 0$ for $t \in (0, \bar{T}]$, which leads to $u \geq \bar{F}(u)$. Since $\frac{dw}{dt} < 0$ for $t \in (\bar{T}, T]$, then $F(u) < \bar{F}(u) \leq u$. By Lemma 2.1, we have $H(u) < 0$ for $u \in [w_1, \infty)$.

If $\bar{T} > T^*$, it follows from (2.18) that $H'(0) > 0$. $H(0) = 0$ implies that there is a sufficiently small $\varepsilon > 0$ such that $H(u) > 0$ for $u \in [0, \varepsilon)$. Therefore, there must be $\bar{u} \in (\varepsilon, w_1)$ such that $H(\bar{u}) = 0$.

When $Q > 3\sqrt{3}$, for different values of R, there are five cases, and due to similarity, we only prove $R = R_2(Q)$. In this case, $F(w)$ has two positive root $w_1 < u_1$ and $\bar{w}_2(Q)$.

When $u \in [\bar{w}_2(Q), \infty)$, we have $\frac{dw}{dt} \leq 0$ for $t \in [0, \bar{T})$. When $u \in [w_1, \bar{w}_2(Q))$, we have $\frac{dw}{dt} \leq 0$ for $t \in [0, \bar{T})$, which leads to $u \geq \bar{F}(u)$. Since $\frac{dw}{dt} < 0$ for $t \in [\bar{T}, T)$, then $F(u) < \bar{F}(u) \leq u$. By Lemma 2.1, we have $H(u) < 0$ for $u \in [w_1, \infty)$.

If $\bar{T} > T^*$, it follows from (2.18) that $H'(0) > 0$. Since $H(0) = 0$ implies that there is sufficiently small $\varepsilon > 0$ such that $H(u) > 0$ for $u \in [0, \varepsilon)$. Therefore, there must be $\bar{u} \in (\varepsilon, w_1)$ such that $H(\bar{u}) = 0$.

From the above, it can be concluded that, when $\bar{T} > T^*$, the system in (2.3) has at least one positive T-periodic solution.

(b) We now prove that $H(u)$ has a unique positive zero point $\bar{u} \in (\varepsilon, w_1)$.

When $Q \leq 3\sqrt{3}$, it is obvious that $u < \bar{F}(u)$, $B'(u) > 0$, and $B(u) < 0$ for $u \in (0, w_1)$, and thus we obtain

$$H'(\bar{u}) = \frac{B(\bar{F}(\bar{u}))}{B(\bar{u})} - 1 < 0,$$

which suffices to show the uniqueness of $\bar{u}$. Furthermore, we have $H(u) > 0$ for $u \in [0, \bar{u})$ and $H(u) < 0$ for $u \in [\bar{u}, \infty)$, by Corollay 2.1, the T-periodic solution is globally asymptotically.

When $Q > 3\sqrt{3}$, because of the similarity, we only prove $R = R_2(Q)$. It is obvious that $u < \bar{F}(u)$, $B'(u) > 0$, and $B(u) < 0$ for $u \in (0, w_1)$, and we obtain

$$H'(\bar{u}) = \frac{B(\bar{F}(\bar{u}))}{B(\bar{u})} - 1 < 0,$$

which suffices to show the uniqueness of $\bar{u}$. Furthermore, we have $H(u) > 0$ for $u \in [0, \bar{u})$ and $H(u) < 0$ for $u \in [\bar{u}, \infty)$, and by Corollay 2.1, the T-periodic solution is globally asymptotically stable. □

Theorem 3.2. *If $\bar{T} > T^*$, $Q > 3\sqrt{3}$ and $R_2(Q) < R \leq \frac{Q}{8}$ hold. Thus, model (2.3) has at most three periodic solutions.*

Proof. Because of this similarity, we only prove $R_2(Q) < R < R_1(Q)$, where $F(w)$ has three positive roots $0 < w_1 < \bar{w}_1(Q) < w_2 < \bar{w}_2(Q) < w_3 < Q$.

When $u \in [w_1, w_2]$, we have $\frac{dw}{dt} \leq 0$ for $t \in (0, \bar{T}]$, which leads to $u \geq \bar{F}(u)$. Since $\frac{dw}{dt} < 0$ for $t \in (\bar{T}, T]$, then $F(u) < \bar{F}(u) \leq u$. By Lemma 2.1, we have $H(u) < 0$ for $u \in [w_1, w_2]$.

When $u \in [w_3, \infty)$, we have $\frac{dw}{dt} \leq 0$ for $t \in (0, \bar{T}]$, which leads to $u \geq \bar{F}(u)$. Since $\frac{dw}{dt} < 0$ for $t \in (\bar{T}, T]$, then $F(u) < \bar{F}(u) \leq u$. By Lemma 2.1, we have $H(u) < 0$ for $u \in [w_3, \infty)$.

If $w_2 \leq u_1^2$, we have $B'(u) > 0$ for $u \in (w_2, u_1^2)$, $B'(u) < 0$ for $u \in (u_1^2, w_3)$ and $B'(u) = 0$ for $u = u_1^2$. Thus, $H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 < 0$ for $u \in (w_2, u_1^2)$, $H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 > 0$ for $u \in (u_1^2, u_2^2)$ and $H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 < 0$ for $u \in (u_2^2, w_3)$. If $\bar{T} > T^*$, it follows from (2.18) that $H'(0) > 0$. Since $H(0) = 0$ implies that there is a sufficiently small $\varepsilon > 0$ such that $H(u) > 0$ for $u \in [0, \varepsilon)$. Therefore, there must be $\bar{u} \in (\varepsilon, w_1)$ must exist, such that $H(\bar{u}) = 0$. Thus, system (2.3) has a maximum of three T-periodic solutions.

If $w_2 > u_1^2$, we have $B'(u) < 0$ for $u \in (w_2, u_2^2)$, which implies that $H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 > 0$. We have $B'(u) > 0$ for $u \in (u_2^2, w_3)$, which implies that $H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 < 0$. If $\bar{T} > T^*$, it follows from (2.18) that $H'(0) > 0$. $H(0) = 0$ implies that there is a sufficiently small $\varepsilon > 0$ such that $H(u) > 0$ for $u \in [0, \varepsilon)$. Therefore, there must be $\bar{u} \in (\varepsilon, w_1)$ must exist, such that $H(\bar{u}) = 0$. Thus, system (2.3) has a maximum of three T-periodic solutions. □

Theorem 3.3. *If $\bar{T} \leq T^*$ holds, and then the origin w_0 of (2.3) is globally asymptotically stable.*

Proof. The linearization of system (2.3) at 0 is

$$\begin{cases} \frac{dw}{dt} = wf(0), & t \in (0, \bar{T}], \\ \frac{dw}{dt} = wg(0), & t \in (\bar{T}, T]. \end{cases}$$

Thus, any fundamental solution of Eq (2.3) admits the Floquet representation $w(t) = e^{-\lambda t}\psi(t)$, where $\lambda \in \mathbb{C}$ is the characteristic exponent and $\psi(t + T) = \psi(t)$ for all $t \in \mathbb{R}$, which leads to the following eigenvalue problem:

$$\begin{cases} \frac{d\psi}{dt} = \psi f(0) + \lambda\psi, & t \in (0, \bar{T}], \\ \frac{d\psi}{dt} = \psi g(0) + \lambda\psi, & t \in (\bar{T}, T], \\ \psi(0) = \psi(T). \end{cases}$$

Integrating the first equation with respect to t from 0 to $\bar{T}$, and the second equation from $\bar{T}$ to T, we obtain

$$\lambda T + \int_0^{\bar{T}} f(0)dt + \int_{\bar{T}}^{T} g(0)dt = 0.$$

We first show that there is no positive T-periodic solution to system (2.3), and by contradiction, that there exists $\bar{w}(t) > 0$ satisfying

$$\begin{cases} \frac{d\bar{w}}{dt} = \bar{w}f(\bar{w}), & t \in (0, \bar{T}], \\ \frac{d\bar{w}}{dt} = \bar{w}g(\bar{w}), & t \in (\bar{T}, T], \\ \bar{w}(0) = \bar{w}(T). \end{cases}$$

or, equivalently

$$\begin{cases} \frac{d\bar{w}}{\bar{w}} = f(\bar{w})dt, & t \in (0, \bar{T}], \\ \frac{d\bar{w}}{\bar{w}} = g(\bar{w})dt, & t \in (\bar{T}, T], \\ \bar{w}(0) = \bar{w}(T). \end{cases}$$

It follows that

$$\int_0^{\bar{T}} f(\bar{w})dt + \int_0^{\bar{T}} g(\bar{w})dt = 0,$$

and thus we have

$$\lambda T = \int_0^{\bar{T}} (f(\bar{w}) - f(0))dt + \int_0^{\bar{T}} (g(\bar{w}) - g(0))dt = 0,$$

since $f(\bar{w}) < f(0)$, $g(\bar{w}) < g(0)$, we obtain $\lambda < 0$. However, since $T \leq T^*$, we have $\lambda \geq 0$. This leads to a contradiction; hence, Eq (2.3) has no T-periodic solution. Using a standard monotone iteration scheme and the nonexistence of a positive T- periodic solution of system (2.3), the trivial equilibrium point 0 of system (2.3) is globally asymptotically stable. This completes this proof. □

Case 2: $h \leq 1$. In this case, we have the following:

Theorem 3.4. *If one of the following conditions holds:*
1) $Q \leq 3\sqrt{3}$,
2) $Q > 3\sqrt{3}$, *and* $R \leq R_2(Q)$,
3) $Q > 3\sqrt{3}$ *and* $R > \frac{Q}{8}$,
then system (2.3) has a unique T-periodic solution, which is globally asymptotically stable.

Proof. Due to similarity, we only prove the case $Q < 3\sqrt{3}$. When $Q < 3\sqrt{3}$, for any $R \leq \frac{Q}{8}$, $F(w)$ has only one small positive root $w_1 < u_1^2$. If $R > \frac{Q}{8}$, $F(w)$ has only one small positive root w_1 and $B'(u) > 0$ for $u \in [0, \infty)$. $G(w)$ has only one small positive root $\widetilde{w}_1 < w_1$. When $u \in (0, \widetilde{w}_1]$, we have $\frac{dw}{dt} > 0$ for $t \in [0, \bar{T})$, which leads to $u < \bar{F}(u)$. Since $\frac{dw}{dt} \geq 0$ for $t \in [\bar{T}, T)$, then $F(u) \geq \bar{F}(u) > u$. By Lemma 2.1, we have $H(u) > 0$ for $u \in (0, \widetilde{w}_1]$. When $u \in [w_1, \infty)$, we have $\frac{dw}{dt} \leq 0$ for $t \in [0, \bar{T})$, which leads to $u \geq \bar{F}(u)$. Since $\frac{dw}{dt} < 0$ for $t \in [\bar{T}, T)$, then $F(u) < \bar{F}(u) \leq u$. By Lemma 2.1, we have $H(u) < 0$ for $u \in [w_1, \infty)$. Thus, there must exist a point $u_0 \in (\widetilde{w}_1, w_1)$ such that $H(u_0) = 0$.

We now prove that $H(u)$ has a unique positive zero point $\bar{u} \in (\widetilde{w}_1, w_1)$. It is obvious that $u < \bar{F}(u)$, $B'(u) > 0$ and $B(u) < 0$ for $u \in (\widetilde{w}_1, w_1)$, and we obtain

$$H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 < 0,$$

which suffices to show the uniqueness of $\bar{u}$. Furthermore, we have $H(u) > 0$ for $u \in [0, \bar{u})$ and $H(u) < 0$ for $u \in [\bar{u}, \infty)$, and by Corollay 2.1, the T-periodic solution is globally asymptotically. The proof is complete. □

Theorem 3.5. *If* $Q > 3\sqrt{3}$, $R_2(Q) < R \leq R_1(Q)$ *and* $R(1-h) \leq R_2(Q)$ *hold. System (2.3) has at most three periodic solutions.*

Proof. It is easy to see that $F(w)$ has three positive roots; if $R(1-h) < R_2(Q)$, $G(w)$ has one positive root $\widetilde{w}_1$; and if $R(1-h) = R_2(Q)$, $G(w)$ has two positive roots $\widetilde{w}_1$ and $\widetilde{w}_2$. Now, we discuss the case $G(w)$ has one positive root.

When $u \in (0, \widetilde{w}_1)$, we have $\frac{dw}{dt} > 0$ for $t \in [0, \bar{T})$, which leads to $u \leq \bar{F}(u)$. Since $\frac{dw}{dt} > 0$ for $t \in [\bar{T}, T)$, then $F(u) > \bar{F}(u) \geq u$. By Lemma 2.1, we have $H(u) > 0$ for $u \in (0, \widetilde{w}_1)$.

When $u \in [w_1, w_2)$, we have $\frac{dw}{dt} \leq 0$ for $t \in [0, \bar{T})$, which leads to $u \geq \bar{F}(u)$. Since $\frac{dw}{dt} < 0$ for $t \in [\bar{T}, T)$, then $F(u) < \bar{F}(u) \leq u$. By Lemma 2.1, we have $H(u) < 0$ for $u \in [w_1, w_2)$.

When $u \in [w_3, \infty)$, we have $\frac{dw}{dt} \leq 0$ for $t \in [0, \bar{T})$, which leads to $u \geq \bar{F}(u)$. Since $\frac{dw}{dt} < 0$ for $t \in [\bar{T}, T)$, then $F(u) < \bar{F}(u) \leq u$. By Lemma 2.1, we have $H(u) < 0$ for $u \in [w_3, \infty)$.

If $w_2 \leq u_1^2$, we have $B'(u) > 0$ for $u \in (w_2, u_1^2)$, thus, $H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 < 0$. $B'(u) < 0$, $H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 > 0$ for $u \in (u_1^2, u_2^2)$. $B'(u) > 0$, $H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 < 0$ for $u \in (u_2^2, w_3)$.

If $w_2 > u_1^2$, we have $B'(u) < 0$, $H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 > 0$ for $u \in (w_2, u_2^2)$. $B'(u) > 0$, $H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 < 0$ for $u \in (u_2^2, w_3)$.

As another case is similar, we omit it here, and the proof is complete. □

Theorem 3.6. *If $Q > 3\sqrt{3}$, $R_2(Q) < R < R_1(Q)$, $F(w)$ has three positive real roots w_1,w_2,w_3, $G(w)$ has three positive roots $\widetilde{w}_1$, $\widetilde{w}_2$, $\widetilde{w}_3$ satisfying $\widetilde{w}_1 < w_1 < w_2 < \widetilde{w}_2 < \widetilde{w}_3 < w_3$, if one of the following conditions holds:*

(a) $u_1^2 < w_2 < u_2^2 < \widetilde{w}_2$,
(b) $u_1^2 < w_2 < \widetilde{w}_2 < u_2^2 < \widetilde{w}_3$
(c) $u_1^2 < w_2 < \widetilde{w}_2 < \widetilde{w}_3 < u_2^2 < w_3$
(d) $w_2 < u_1^2 < u_2^2 < \widetilde{w}_2$
(e) $w_2 < u_1^2 < \widetilde{w}_2 < u_2^2 < \widetilde{w}_3$
(h) $w_2 < u_1^2 < \widetilde{w}_2 < \widetilde{w}_3 < u_2^2 < w_3$

then system (2.3) contains three periodic solutions.

Proof. We only provide the details for case (a), as the other cases can be dealt with in a similar manner.

When $u \in (0, \widetilde{w}_1]$, we have $\frac{dw}{dt} > 0$ for $t \in [0, \bar{T})$, which leads to $u < \bar{F}(u)$. Since $\frac{dw}{dt} \geq 0$ for $t \in [\bar{T}, T)$, then $h(u) \geq \bar{F}(u) > u$. By Lemma 2.1, we have $H(u) > 0$ for $u \in (0, \widetilde{w}_1)$.

When $u \in [w_1, w_2]$, we have $\frac{dw}{dt} \leq 0$ for $t \in [0, \bar{T})$, which leads to $u \geq \bar{F}(u)$. Since $\frac{dw}{dt} < 0$ for $t \in [\bar{T}, T)$, then $F(u) < \bar{F}(u) \leq u$. By Lemma 2.1, we have $H(u) < 0$ for $u \in [w_1, w_2]$.

When $u \in [\widetilde{w}_1, \widetilde{w}_2]$, we have $\frac{dw}{dt}$ 0 for $t \in [0, \bar{T})$, which leads to $u < \bar{F}(u)$. Since $\frac{dw}{dt} \geq 0$ for $t \in [\bar{T}, T)$, then $F(u) \geq \bar{F}(u) > u$. By Lemma 2.1, we have $H(u) > 0$ for $u \in [\widetilde{w}_1, \widetilde{w}_2]$.

When $u \in [w_3, \infty)$, we have $\frac{dw}{dt} \leq 0$ for $t \in [0, \bar{T})$, which leads to $u \geq \bar{F}(u)$. Since $\frac{dw}{dt} < 0$ for $t \in [\bar{T}, T)$, then $F(u) < \bar{F}(u) \leq u$. By Lemma 2.1, we have $H(u) < 0$ for $u \in [w_3, \infty)$.

Now, we prove that $H(u)$ has a unique positive zero point $\bar{u}_1 \in (\widetilde{w}_1, w_1)$. It is obvious that $u < \bar{F}(u)$, $B'(u) > 0$ and $B(u) < 0$ for $u \in (\widetilde{w}_1, w_1)$, and we obtain

$$H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 < 0,$$

which suffices to show the uniqueness of $\bar{u}_1$.

When $u \in (w_2, u_2^2)$, we have $B'(u) < 0$, $B(u) < 0$, $u < \bar{F}(u)$, $H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 > 0$. Thus, $H(u)$ has a unique positive zero point $\bar{u}_2 \in (w_2, u_2^2)$.

When $u \in (\widetilde{w}_3, w_3)$, we have $B'(u) > 0$, $B(u) < 0$, $u < \bar{F}(u)$, $H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 < 0$. Thus, $H(u)$ has a unique positive zero point $\bar{u}_3 \in (\widetilde{w}_3, w_3)$. The proof is complete. □

Theorem 3.7. *If $Q > 3\sqrt{3}$, $R = R_1(Q)$, the equation $F(w) = 0$ has positive real roots w_1,w_3, $G(w)$ has three positive roots $\widetilde{w}_1$, $\widetilde{w}_2$, $\widetilde{w}_3$ satisfying $\widetilde{w}_1 < w_1 < \widetilde{w}_2 < \widetilde{w}_3 < w_3$, if one of the following conditions holds:*

(a) $w_1 < u_1^2 < u_2^2 < \widetilde{w}_2$
(b) $w_1 < u_1^2 < \widetilde{w}_2 < u_2^2 < \widetilde{w}_3$
(c) $w_1 < u_1^2 < \widetilde{w}_2 < \widetilde{w}_3 < u_2^2 < w_3$

then system (2.3) has exactly three distinct periodic solutions.

Proof. We only provide the details for case (a), as the other cases can be dealt with in a similar manner.

When $u \in (0, \widetilde{w}_1]$, we have $\frac{dw}{dt} > 0$ for $t \in [0, \bar{T})$, which leads to $u < \bar{F}(u)$. Since $\frac{dw}{dt} \geq 0$ for $t \in [\bar{T}, T)$, then $F(u) \geq \bar{F}(u) > u$. By Lemma 2.1, we have $H(u) > 0$ for $u \in (0, \widetilde{w}_1)$.

When $u \in [\widetilde{w}_1, \widetilde{w}_2]$, we have $\frac{dw}{dt} > 0$ for $t \in [0, \bar{T})$, which leads to $u < \bar{F}(u)$. Since $\frac{dw}{dt} \geq 0$ for $t \in [\bar{T}, T)$, then $F(u) \geq \bar{F}(u) > u$. By Lemma 2.1, we have $H(u) > 0$ for $u \in [\widetilde{w}_1, \widetilde{w}_2]$.

When $u \in [w_3, \infty)$, we have $\frac{dw}{dt} \leq 0$ for $t \in [0, \bar{T})$, which leads to $u \geq \bar{F}(u)$. Since $\frac{dw}{dt} < 0$ for $t \in [\bar{T}, T)$, then $F(u) < \bar{F}(u) \leq u$. By Lemma 2.1, we have $H(u) < 0$ for $u \in [w_3, \infty)$.

Now, we prove that $H(u)$ has a unique positive zero point $\bar{u}_1 \in (\widetilde{w}_1, w_1)$. It is obvious that $u < \bar{F}(u)$, $B'(u) > 0$ and $B(u) < 0$ for $u \in (\widetilde{w}_1, w_1)$,$H(w_1) < 0$ we obtain

$$H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 < 0,$$

which sufficient to demonstrate the uniqueness of $\bar{u}_1$.

When $u \in (u_1^2, u_2^2)$, we have $B'(u) < 0$, $B(u) < 0$, $u < \bar{F}(u)$, $H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 > 0$. Thus, $H(u)$ has a unique positive zero point $\bar{u}_2 \in (u_1^2, u_2^2)$.

When $u \in (\widetilde{w}_3, w_3)$, we have $B'(u) > 0$, $B(u) < 0$, $u < \bar{F}(u)$, $H'(u) = \frac{B(\bar{F}(u))}{B(u)} - 1 < 0$. Thus, $H(u)$ has a unique positive zero point $\bar{u}_3 \in (\widetilde{w}_3, w_3)$. The proof is complete. □

4. Numerical examples

We now present several numerical simulations to validate the analytical findings established above.

Examlpe 4.1. *Given parameters*

$$Q = 4, \quad h = 1.5, \quad \bar{T} = 3, \quad T = 6, \tag{4.1}$$

we have $T^* = \frac{h-1}{h} T = 2$.

Since $Q = 4 < 3\sqrt{3}$ and $\frac{Q}{8} = 0.5$, we chose $R = 0.4$, 0.5, and 0.6, respectively. According to Theorem 3.1, there exists a unique globally asymptotically stable T-periodic solution as shown in Figure 4(a)–(c).

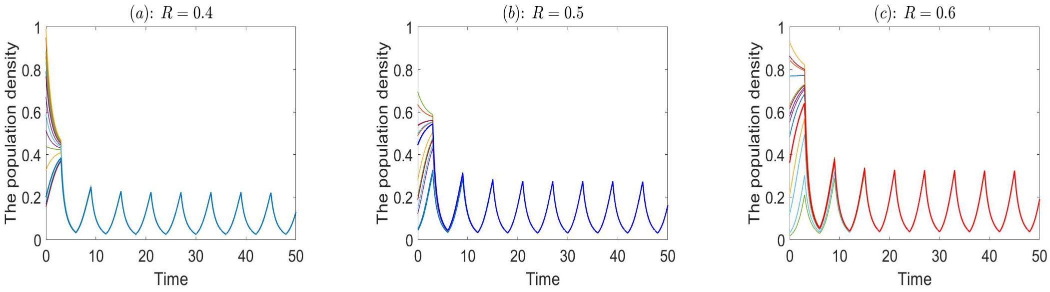

Figure 4. The solution graphs of system of (2.3) for different values of R satisfying conditions in Theorem 3.1. It shows that there exists a unique globally asymptotically stable T-periodic solution.

Examlpe 4.2. *Given parameters*

$$Q = 9, \quad h = 1.5, \quad \bar{T} = 5, \quad T = 6, \tag{4.2}$$

we have $T^* = \frac{h-1}{h}T = 2.$

Since $Q = 9 > 3\sqrt{3}$ and $\frac{Q}{8} = 1.125$, we choose $R = 0.2$, 0.43, and 1.13, respectively, and there exists a unique globally asymptotically stable T-periodic solution as shown in Figure 5(a),(b).

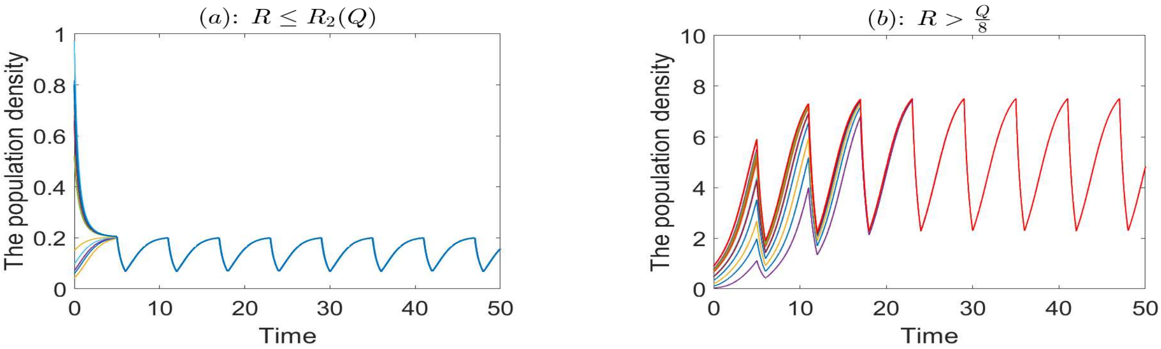

Figure 5. The solution graphs of system of (2.3) for different values of R satisfying conditions in Theorem 3.1. It shows that there exists a unique globally asymptotically stable T-periodic solution.

Examlpe 4.3. *Given parameters*

$$R = 0.44, \quad h = 1.5, \quad \bar{T} = 5, \quad T = 6, \tag{4.3}$$

we have $T^* = \frac{h-1}{h}T = 2.$

We set $Q = 9$. According to Theorem 3.2, there exists a unique T-Periodic solution as shown in Figure 6(a). We set $Q = 20$, and there exists three T-Periodic solution as shown in Figure 6(b).

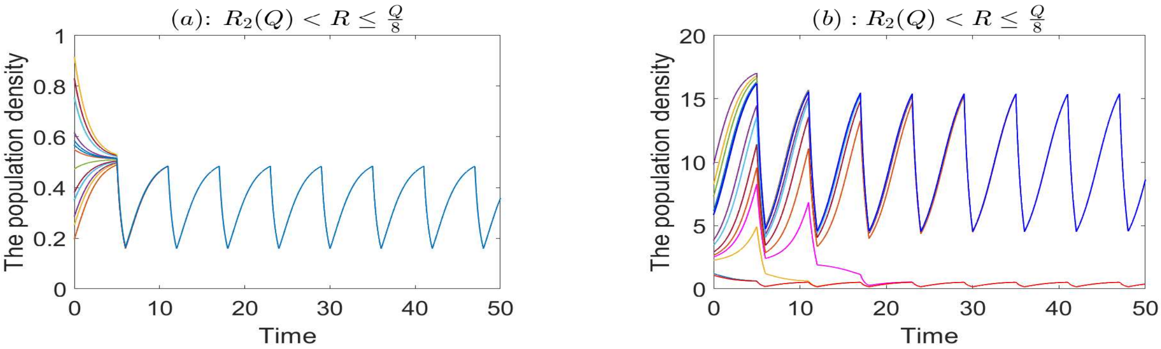

Figure 6. The solution graphs of system of (2.3) for different values of Q satisfying conditions in Theorem 3.2. It shows that there are three T-periodic solutions, the largest and smallest periodic solutions are stable, and the intermediate periodic solution is unstable.

Examlpe 4.4. *Given parameters*

$$(a)\ \ Q = 4,\ \ R = 0.4,\ \ h = 1.5,\ \ \bar{T} = 1.5,\ \ T = 6,$$

$$(b)\ \ Q = 9,\ \ R = 0.43,\ \ h = 1.5,\ \ \bar{T} = 1.5,\ \ T = 6,$$

$$(c)\ \ Q = 20,\ \ R = 0.44,\ \ h = 1.5,\ \ \bar{T} = 2,\ \ T = 6,$$

we have $T^* = \frac{h-1}{h}T = 2.$

For the case of $\bar{T} \leq T^*$, by Theorem 3.3, w_0 is globally asymptotically stable, which is show in Figure 7(a)–(c).

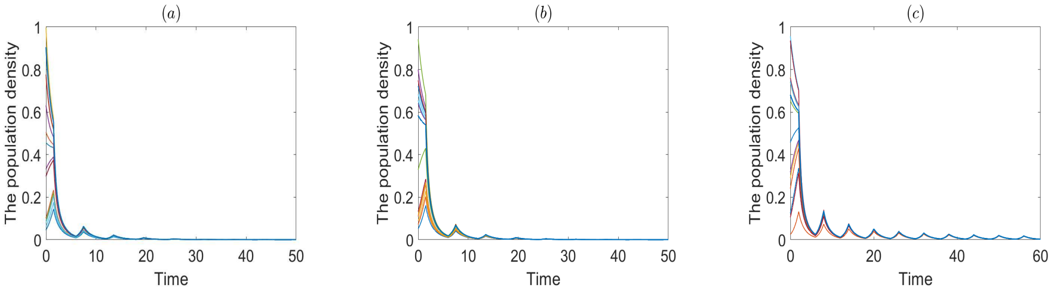

Figure 7. The solution graphs of system of (2.3) for the values of R satisfying conditions in Theorem 3.3. All panels show that origin w_0 is globally asymptotically stable.

Examlpe 4.5. *Given parameters*

$$(a)\ \ Q = 4,\ \ R = 0.4,\ \ h = 0.5,\ \ \bar{T} = 3,\ \ T = 6,$$

$$(b)\ \ Q = 9,\ \ R = 0.2,\ \ h = 0.5,\ \ \bar{T} = 3,\ \ T = 6,$$

$$(d)\ \ Q = 9,\ \ R = 1.13,\ \ h = 0.5,\ \ \bar{T} = 3,\ \ T = 6.$$

when $Q = 4$, we have $Q < 3\sqrt{3}$; when $Q = 9$, $R = 0.2$, we have $Q > 3\sqrt{3}$ and $R < R_2(Q)$; when $Q = 9$, $R = 1.13$ we have $R > \frac{Q}{8}$. According to Theorem 3.4, there exists a unique globally asymptotically stable T-periodic solution, which is show in Figure 8(a)–(c).

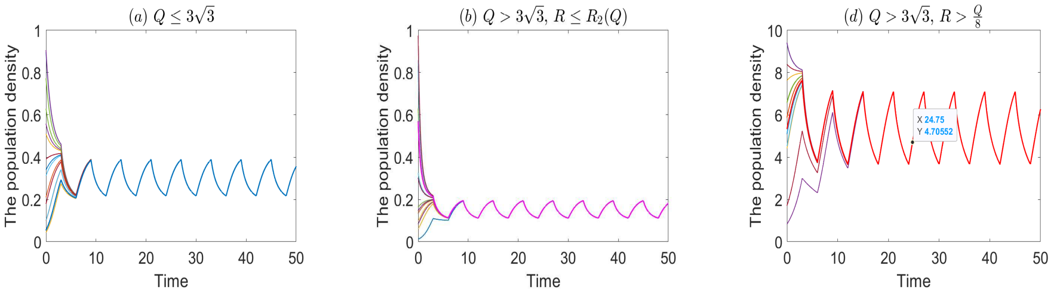

Figure 8. The solution graphs of system of (2.3) for different values satisfying conditions in Theorem 3.4. It shows that there exists a unique globally asymptotically stable T-periodic solution.

Examlpe 4.6. *Given parameters* $h = 0.5$, $\bar{T} = 3$, $T = 6$.

We set $Q = 9 > 3\sqrt{3}$, $R = 0.4$, then $R_2(Q) < R \leq R_1(Q)$ holds. According to Theorem 3.5, there exists a unique T-Periodic solution as shown the left part of Figure 9. We set $Q = 20$, $R = 0.44$, and there exists three T-Periodic solution, as shown in the right part of Figure 9.

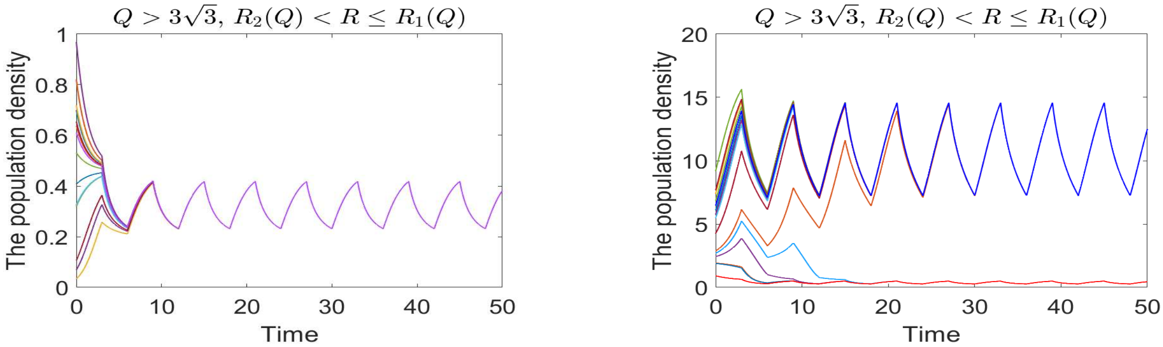

Figure 9. The solution graphs of system of (2.3) for the values of R satisfying conditions in Theorem 3.5. It shows that there are three T-periodic solutions, the largest and smallest periodic solutions are stable, and the intermediate periodic solution is unstable.

5. Conclusions and discussion

Under natural conditions, the combined effects of various ecological mechanisms, such as predation, resource limitation, competition, environmental disturbances, and feedback mechanisms, can lead to multiple stable states in a population.

In this paper, we propose and analyze a fish population model, that consists of two subsystem equations describing the alternating dynamics of the closed season and the fishing period. Specifically, the classical model with multiple stable characteristics was used during the closed season phase and a linear fishing term was introduced during the fishing period to reflect the impact of fishery production activities. Under high fishing pressure ($h > 1$), from Theorems 3.1 and 3.2, we find that when the duration of the closed fishing season exceeds a certain threshold ($\bar{T} > T^*$), we can prove that system (2.3) has at least one positive T-periodic solution. Furthermore, system (2.3) has a unique globally asymptotically stable positive T-periodic solution under additional conditions. When $\bar{T} \leq T^*$, we conclude that fish populations will die out, and the trivial equilibrium point 0 of system (2.3) is globally asymptotically stable. Under weak fishing intensity conditions ($h \leq 1$), from Theorems 3.4–3.7, we find that system (2.3) always admits periodic solutions. Under certain conditions, there exists a unique asymptotically stable periodic solution, whereas under other conditions, the system may exhibit up to three periodic solutions or exactly three periodic solutions.

Due to the complexity of the equations, we are unable to derive explicit analytical expressions for their roots. Therefore, when $Q > 3\sqrt{3}$, $R_2(Q) < R \leq \frac{Q}{8}$, we cannot definitively determine whether the condition $w_3 > u_2^2$ is satisfied, and under this condition, we conjecture that the system may exhibit multiple stable states and display multiple periodic solutions. System (2.3) may have three periodic solutions . The largest and smallest periodic solutions were stable, whereas the intermediate periodic solution appeared to be unstable.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. U. Sommer, Z. Gliwicz, W. Lampert, A. Duncan, The PEG-model of seasonal succession of planktonic events in fresh waters, *Arch. Hydrobiol.*, **106** (1986), 433–471. https://doi.org/10.1127/archiv-hydrobiol/106/1986/433
2. C. A. Klausmeier, Successional state dynamics: a novel approach to modeling nonequilibrium foodweb dynamics, *J. Theor. Biol.*, **262** (2010), 584–595. https://doi.org/10.1016/j.jtbi.2009.10.018
3. S. B. Hsu, X. Q. Zhao, A Lotka-Volterra competition model with seasonal succession, *J. Math. Biol.*, **64** (2012), 109–130. https://doi.org/10.1007/s00285-011-0408-6
4. R. Peng, X. Q. Zhao, The diffusive logistic model with a free boundary and seasonal succession, *Discrete Contin. Dyn. Syst.*, **33** (2013), 2007–2031. https://doi.org/10.3934/dcds.2013.33.2007
5. L. Niu, Y. Wang, X. Xie, Carrying simplex in the Lotka-Volterra competition model with seasonal succession with applications, *Discrete Contin. Dyn. Syst. B*, **26** (2021), 2161–2172. https://doi.org/10.3934/dcdsb.2021014
6. M. Wang, Q. Zhang, X.Q. Zhao, Dynamics for a diffusive competition model with seasonal succession and different free boundaries, *J. Differ. Equations*, **285** (2021), 536–582. https://doi.org/10.1016/j.jde.2021.03.006
7. L. Pu, Z. Lin, Y. Lou, A West Nile virus nonlocal model with free boundaries and seasonal succession, *J. Math. Biol.*, **86** (2023), 25. https://doi.org/10.1007/s00285-022-01860-x
8. Y. Lou, K. Liu, D. He, D. Gao, S. Ruan, Modelling diapause in mosquito population growth, *J. Math. Biol.*, **78** (2019), 2259–2288. https://doi.org/10.1007/s00285-019-01343-6
9. Z. Bai, Y. Lou, X.Q. Zhao, A delayed succession model with diffusion for the impact of diapause on population growth, *SIAM J. Appl. Math.*, **80** (2020), 1493–1519. https://doi.org/10.1137/19M1236448
10. X. Pan, H. Shu, L. Wang, X. Wang, J. Yu, On the periodic solutions of switching scalar dynamical systems, *J. Differ. Equations*, **415** (2025), 365–382. https://doi.org/10.1016/j.jde.2024.09.032

11. E. Augeraud-Véron, N. Sari, Seasonal dynamics in an SIR epidemic system, *J. Math. Biol.*, **68** (2014), 701–725. https://doi.org/10.1007/s00285-013-0645-y

12. Y. Tang, D. Xiao, W. Zhang, D. Zhu, Dynamics of epidemic models with asymptomatic infection and seasonal succession, *Math. Biosci. Eng.*, **14** (2017), 1407–1424. https://doi.org/10.3934/mbe.2017073

13. J. Duarte, C. Januario, N. Martins, S. Rogovchenko, Y. Rogovchenko, Chaos analysis and explicit series solutions to the seasonally forced SIR epidemic model, *J. Math. Biol.*, **578** (2019), 2235–2258. https://doi.org/10.1007/s00285-019-01342-7

14. J. Yu, Modeling mosquito population suppression based on delay differential equations, *SIAM J. Appl. Math.*, **78** (2018), 3168–3187. https://doi.org/10.1137/18M1204917

15. J. Yu, J. Li, Global asymptotic stability in an interactive wild and sterile mosquito model, *J. Differ. Equations*, **269** (2020), 6193–6215. https:/doi.org/10.1016/j.jde.2020.04.036

16. B. Zheng, J. Yu, J. Li, Modeling and analysis of the implementation of the Wolbachia incompatible and sterile insect technique for mosquito population suppressions, *SIAM J. Appl. Math.*, **81** (2021), 718–740. https://doi.org/10.1137/20M1368367

17. Z. Zhu, B. Zheng, Y. Shi, R. Yan, J. Yu, Stability and periodicity in a mosquito population suppression model composed of two sub-models, *Nonlinear Dyn.*, **107** (2022), 1383–1395. https://doi.org/10.21203/rs.3.rs-666866/v1

18. S. Ai, J. Li, J. Yu, B. Zheng, Stage-structured models for interactive wild and periodically and impulsively released sterile mosquitoes, *Discrete Contin. Dyn. Syst. B*, **27** (2022), 3039–3052. https://doi.org/10.3934/dcdsb.2021172

19. Y. Wang, Y. Chen, B. Zheng, J. Yu, Periodic dynamics of a mosquito population suppression model based on Wolbachia-infected males, *Discrete Contin. Dyn. Syst.*, **44** (2024), 2043–2437. https://doi.org/10.3934/dcds.2024033

20. R. May, Thresholds and breakpoints in ecosystems with a multiplicity of stable states, *Nature*, **10** (1977), 471–477. https://doi.org/10.1038/269471a0

21. D. Ludwig, D. G. Aronson, H. F. Weinberger, Spatial patterning of the spruce budworm, *J. Math. Biol.*, **8** (1979), 217–258. https://doi.org/10.1007/bf00276310

22. J. H. Steele, E. W. Henderson, Modeling long-term uctuations in fish stocks, *Science*, **224** (1984), 985–987. https://doi.org/10.1126/science.224.4652.985

23. G. Dai, M. Tang, Coexistence region and global dynamics of a harvested predator-prey systems, *SIAM J. Appl. Math.*, **58** (1998), 193–210. https://doi.org/10.1137/S0036139994275799

24. J. Chen, J. Huang, S. Ruan, J. Wang, Bifurcations of invariant tori in predator-prey models with seasonal prey harvesting, *SIAM J. Appl. Math.*, **73** (2013), 1876–1905. https://doi.org/10.1137/120895858

25. R. Gupta, P. Chandra, Bifurcation analysis of modified Leslie-Gower predator-prey model with Michaelis-Menten type prey harvesting, *J. Math. Anal. Appl.*, **398** (2013), 278–295. https://doi.org/10.1016/j.jmaa.2012.08.057

26. Y. Li, M. Wang, Dynamics of a diffusive predator-prey model with modified Leslie-Gower term and Michaelis-Menten type prey harvesting, *Acta Appl. Math.*, **140** (2015), 147–172. https://doi.org/10.1016/j.camwa.2013.04.004
27. W. Liu, Y. Jiang, Bifurcation of a delayed Gause predator-prey model with Michaelis-Menten type harvesting, *J. Theor. Biol.*, **438** (2017), 116–132. https://doi.org/10.1016/j.jtbi.2017.11.007
28. I. Darabsah, Y. Yuan, A stage-structured mathematical model for fish stocks with harvesting, *SIAM J. Appl. Math.*, **78** (2018), 145–170. https://doi.org/10.1137/16M1097092
29. B. Liu, R. Wu, L. Chen, Patterns induced by super cross-diffusion in a predator-prey system with Michaelis-Menten type harvesting, *Math. Biosci.*, **298** (2018), 71–79. https://doi.org/10.1016/j.mbs.2018.02.002
30. W. Gan, Z. Lin, M. Pedersen, Delay-driven spatial patterns in a predator-prey model with constant prey harvesting, *Z. Angew. Math. Phys.*, **73** (2022), 120. https://doi.org/10.1007/s00033-022-01761-5
31. N. Takashina, A. Mougi, Effects of marine protected areas on overfished fishing stocks with multiple stable states, *J. Theor. Biol.*, **341** (2014), 64–70. https://doi.org/10.1016/j.jtbi.2013.09.027
32. D. Xiao, Dynamics and bifurcations on a class of population model with seasonal constant-yield harvesting, *Discrete Contin. Dyn. Syst. B*, **21** (2017), 699–719. https://doi.org/10.3934/dcdsb.2016.21.699
33. X. Feng, Y. Liu, S. Ruan, J. Yu, Periodic dynamics of a single species model with seasonal Michaelis-Menten type harvesting, *J. Differ. Equations*, **354** (2023), 237–263. https://doi.org/10.1016/j.jde.2023.01.014
34. Y. Liu, X. Feng, S. Ruan, J. Yu, Periodic dynamics of a single species model with seasonal Michaelis-Menten type harvesting, II: Existence of two periodic solutions, *J. Differ. Equations*, **388** (2024), 253–285. https://doi.org/10.1016/j.jde.2024.01.004
35. Y. Liu, J. Yu, J. Li, Global dynamics of a competitive system with seasonal succession and different harvesting strategies, *J. Differ. Equations*, **382** (2024), 211–245. https://doi.org/10.1016/j.jde.2023.11.024
36. Y. Liu, J. Yu, Y. Chen, Z. Guo, Periodic dynamics of a reaction-diffusion-advection model with Michaelis-Menten type harvesting in heterogeneous environments, *SIAM J. Appl. Math.*, **84** (2024), 1891–1909. https://doi.org/10.1137/23M1600852
37. Y. Liu, H. Peng, J. Yu, Y. Chen, Z. Guo, Global dynamics of a reaction-diffusion switching model with multiple stable states and linear harvesting rate, *J. Math. Biol.*, **91** (2025), 81. https://doi.org/10.1007/s00285-025-02312-y

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