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Research article

Infinitely many solutions for a quasilinear Kirchhoff -Schrödinger-Poisson system with critical growth

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Abstract: This paper is concerned with the quasilinear Kirchhoff-Schrödinger-Poisson system with critical growth. By using the Krasnoselskii genus theory and variational methods, we prove the existence of infinitely many solutions for this system.

Keywords: Schrödinger-Poisson system; critical exponent; p-Laplacian; variational methods; infinitely many solutions

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1. Introduction and main result

This paper investigates infinitely many solutions for a quasilinear Schrödinger-Poisson system

$$\begin{cases} -M\left(\displaystyle\int_\Omega |\nabla u|^p dx\right)\Delta_p u + \phi|u|^{p-2}u = |u|^{p^*-2}u + \lambda|u|^{q-2}u, & \text{in } \Omega, \\ -\Delta\phi = |u|^p, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $\frac{3}{2} < q < p < 3$, $p^* = \frac{3p}{3-p}$ is the critical Sobolev exponent, and $\lambda > 0$ is a real parameter.

(M^*) Assume that $M(t) : [0, +\infty) \to (0, +\infty)$ is continuous and increasing and there exists $m_0 > 0$ such that $M(t) \geq m_0 = M(0)$ for all $t \in [0, +\infty)$.

The Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + \eta\phi f(u) = h(x, u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 2F(u), & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

a quantum mechanical model describing the interaction between an electron wave and its own electrostatic field, was first proposed by Benci and Fortunato in [1]. For further background information on (1.2), please refer to [2–4].

In [5, 6], the authors investigated the existence of ground state solutions for system (1.2). There has been much research regarding the positive solutions of (1.2); we can see [7–9]. Moreover, numerous researchers have investigated the existence of infinitely many solutions to system (1.2), for example, [10–14].

Recently, the quasilinear Schrödinger-Poisson system has been studied by numerous scholars

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \mu\phi|u|^{q-2}u = |u|^{r-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^q, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $1 < p < 3$, $p \leq r < p^* = \frac{3p}{3-p}$, and $\lambda > 0$ is a parameter. Through scaling transformation and ingenious methods, Du et al. [15, 16] demonstrated that (1.3) admits nontrivial solutions. In the case when $r = 2$, the existence of $\mu^* > 0$ was demonstrated by Xue and Wang [17], ensuring that system (1.3) admits two positive solutions for any $\mu \in (0, \mu^*)$. In particular, in [18], the following quasilinear Schrödinger-Poisson system with critical growth was investigated:

$$\begin{cases} -\Delta_p u + |u|^{p-2}u + \mu\phi|u|^{p-2}u = \lambda|u|^{q-2}u + |u|^{p^*-2}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi = |u|^p, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.4}$$

For system (1.4), different results are obtained under varying conditions for p, q, μ, and $\lambda > 0$; please refer to [19–22].

Moreover, some researchers have taken an interest in the following elliptic problems with critical Sobolev growth:

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda|u|^{q-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $1 < p < N$, and Δ_p is the p-Laplacian operator. By using critical point methods, the authors obtained the existence of multiple nontrivial solutions of (1.5) in [23]. The work of Azorero and Alonso [24] proved that Eq (1.5) has at least two positive solutions for the parameter $\frac{2N}{N+2} < p < 3$ with $1 < q < p$ or $p > 3$ with $p > q > \frac{p^*-2}{p-1}$. If $q = p$ and $N > p^2 + p$, Cao et al. in [25] verified the existence of an infinite number of solutions to (1.5) using variational techniques. In addition, in [26], the authors studied the existence of infinitely many solutions of nonlinear elliptic problems of p&q-Laplacian type involving the critical Sobolev exponent.

Based on the above results, we consider the existence of infinitely many solutions of the quasilinear Schrödinger-Poisson system with critical growth. One of the characteristics of this paper is that this class of problems lacks compactness created by the critical term. This undoubtedly makes it difficult to prove the existence of solutions to this system, so we overcome the compactness by using the concentration compactness principle. Moreover, the emergence of the nonlocal term renders the existence of infinitely many solutions considerably more challenging than is typically the case (1.1). Furthermore, in Reference [23], quasilinear equations with critical and subcritical types were considered. In our study, quasilinear equations with dual local types are examined, where the variational functional becomes more complex and overcoming compactness issues becomes more challenging.

Next, we define the truncation function M. Since $p < p^*$, there exists $\kappa \in (p, p^*)$. By (M^*), there exists $t_0 > 0$ such that $m_0 < M(t_0) < \frac{\kappa}{p} m_0$. That is,

$$M_0(t) := \begin{cases} M(t), & \text{if } 0 \leq t \leq t_0, \\ M(t_0), & \text{if } t \geq t_0, \end{cases} \tag{1.6}$$

and we have

$$M_0(t) \leq \frac{\kappa}{p} m_0. \tag{1.7}$$

Now, consider the following auxiliary problem:

$$\begin{cases} -M_0\left(\displaystyle\int_\Omega |\nabla u|^p dx\right) \Delta_p u + \phi |u|^{p-2} u = |u|^{p^*-2} u + \lambda |u|^{q-2} u, & \text{in } \Omega, \\ -\Delta \phi = |u|^p, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Then, we have the main result.

Theorem 1.1. *If* (M^*) *and* $\frac{3}{2} < q < p < 3$ *hold, then, there exists* $\lambda^* > 0$*, such that for all* $0 < \lambda < \lambda^*$*, system* (1.1) *has infinitely many nontrivial solutions.*

From now on, we use $C_i (i = 1, 2, \cdots)$ to denote (possibly different) positive constants. We denote by S_r (respectively, B_r) the sphere (respectively, the closed ball) of center zero and radius r, i.e., $S_r = \{u \in W_0^{1,p}(\Omega) : \|u\| = r\}, B_r = \{u \in W_0^{1,p}(\Omega) : \|u\| \leq r\}$. In addition, Du [27] proved the best Sobolev constant

$$S := \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p dx}{\left(\displaystyle\int_\Omega |u|^{p^*} dx\right)^{p/p^*}}.$$

We always use the notations $\|\cdot\|$ and $|\cdot|_p$ to represent the norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$ and $L^p(\Omega)$ norm. Besides, let $D^{1,2}(\Omega)$ be equipped with the norm

$$\|u\|^2_{D^{1,2}(\Omega)} = \int_\Omega |\nabla u|^2 dx.$$

2. Some preliminary results

In this section, we shall introduce some preparatory knowledge.

By the Lax-Milgram theorem, for all $u \in W_0^{1,p}(\Omega)$, the equation

$$\begin{cases} -\Delta \phi = |u|^p, & \text{in } \Omega, \\ \phi = 0, & \text{on } \partial\Omega \end{cases}$$

has a unique solution $\phi_u \in D^{1,2}(\Omega)$, and by [15], we have $\phi_u \geq 0$ for $x \in \Omega$ and for each $t > 0$, $\phi_{tu} = t^p \phi_u$, and

$$\|\phi_u\|^2_{D^{1,2}(\Omega)} = \int_\Omega \phi_u |u|^p dx \leq \|u\|^p_{L^{\frac{6p}{5}}(\Omega)} \|\phi_u\|_{L^6(\Omega)} \leq C\|u\|^p \|\phi_u\|_{D^{1,2}(\Omega)},$$

for all $u, v \in W_0^{1,p}(\Omega)$,

$$\int_\Omega \phi_u |v|^p dx = \int_\Omega \phi_v |u|^p dx.$$

Furthermore, we can reduce the system (1.8) to the following problem:

$$\begin{cases} -M_0\left(\int_\Omega |\nabla u|^p dx\right)\Delta_p u + \phi_u |u|^{p-2}u = |u|^{p^*-2}u + \lambda |u|^{q-2}u, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

For (2.1), define the functional

$$I_\lambda(u) = \frac{1}{p}\widehat{M}_0(\|u\|^p) + \frac{1}{2p}\int_\Omega \phi_u |u|^p dx - \frac{1}{p^*}\int_\Omega |u|^{p^*} dx - \frac{\lambda}{q}\int_\Omega |u|^q dx,$$

where $\widehat{M}_0(t) = \int_0^t M_0(s)ds$. If $u \in W_0^{1,p}(\Omega)$ is a weak solution of Eq (2.1), then u satisfies

$$\begin{aligned} M_0(\|u\|^p)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx + \int_\Omega \phi_u |u|^{p-2}u\varphi dx \\ = \int_\Omega |u|^{p^*-2}u\varphi dx + \lambda \int_\Omega |u|^{q-2}u\varphi dx, \end{aligned}$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

Now, we define the Krasnoselskii genus in [28]:

$$\gamma(A) = \min\{k \in \mathbb{N} | \exists \phi \in C(A, \mathbb{R}^k \setminus \{0\}), \phi(-x) = \phi(x)\}.$$

If such a mapping does not exist for any $k > 0$, we set $\gamma(A) = \infty$. Moreover, by definition, $\gamma(\emptyset) = 0$.

Let $\mathbb{S}^{k-1}$ be a $k-1$ dimensional sphere in $\mathbb{R}^k$, then $\gamma(\mathbb{S}^{k-1}) = k$. We give the result due to Clarke [29].

Lemma 2.1. *Let* $X = \mathbb{R}^k$ and $\partial\Omega$ be the boundary of an open, symmetric, and bounded subset $\Omega \subset \mathbb{R}^k$ with $0 \in \Omega$. Then, $\gamma(\partial\Omega) = k$.

Theorem 2.2. *Suppose that the functional* $I_\lambda \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ *satisfies the* $(PS)_c$ *condition and the following conditions:*

(i) I_λ *is bounded from below and even;*

(ii) *There exists a compact, symmetric set* $K \subset Y$ *such that* $\gamma(K) = k$ *and*

$$\sup_{u \in K} I_\lambda(u) < I_\lambda(0).$$

Then, I_λ *possesses at least* k *distinct pairs of critical points with corresponding critical values less than* $I_\lambda(0)$.

According to the Sobolev and Hölder inequalities, for $u \in W_0^{1,p}(\Omega)$, $\frac{3}{2} < q < p < 3$, and $q < p^*$, one has

$$\int_\Omega |u|^q dx \le S^{-\frac{q}{p}} |\Omega|^{\frac{p^*-q}{p^*}} \|u\|^q. \tag{2.2}$$

By (2.2), we get

$$I_\lambda(u) \geq \frac{m_0}{p}\|u\|^p - \frac{1}{p^*}S^{-\frac{p^*}{p}}\|u\|^{p^*} - \frac{\lambda}{q}S^{-\frac{q}{p}}|\Omega|^{\frac{p^*-q}{p^*}}\|u\|^q = g(\|u\|^p).$$

Let

$$g(t) = \frac{m_0}{p}t - \frac{1}{p^*}S^{-\frac{p^*}{p}}t^{\frac{p^*}{p}} - \frac{\lambda}{q}S^{-\frac{q}{p}}|\Omega|^{\frac{p^*-q}{p^*}}t^{\frac{q}{p}}.$$

Then, there exists a sufficiently small $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$, g attains a positive maximum. Let $T_0(\lambda) < T_1(\lambda)$ be the only roots of g. Then, $g(t) < 0$ for $0 < t < T_0(\lambda)$, $g(t) > 0$ for $T_0(\lambda) < t < T_1(\lambda)$, and $g(t) < 0$ for $T_1(\lambda) < t$.

From [23], we set $\psi \in C_0^\infty([0, +\infty))$ such that $\psi(t) \in [0, 1]$ for any $t \in [0, +\infty)$ and

$$\psi(t) := \begin{cases} 1, & t \in [0, T_0], \\ 0, & t \in [T_1, +\infty). \end{cases}$$

Then, we consider the following truncated functional:

$$J_\lambda(u) = \frac{1}{p}\widehat{M}_0(\|u\|^p) + \frac{1}{2p}\int_\Omega \phi_u|u|^p dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}\psi(\|u\|^p)dx - \frac{\lambda}{q}\int_\Omega |u|^q dx.$$

Note that $J_\lambda \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$, $J_\lambda(u) \geq \bar{g}(\|u\|^p)$, where

$$\bar{g}(t) = \frac{m_0}{p}t - \frac{1}{p^*}S^{-\frac{p^*}{p}}t^{\frac{p^*}{p}}\psi(t) - \frac{\lambda}{q}S^{-\frac{q}{p}}|\Omega|^{\frac{p^*-q}{p^*}}t^{\frac{q}{p}}.$$

Therefore, assume that $\|u\|^p \leq T_0$, one has $J_\lambda(u) = I_\lambda(u)$, and if $\|u\|^p \geq T_1$, we obtain

$$J_\lambda(u) = \frac{1}{p}\widehat{M}_0(\|u\|^p) + \frac{1}{2p}\int_\Omega \phi_u|u|^p dx - \frac{\lambda}{q}\int_\Omega |u|^q dx.$$

Then, we deduce J_λ is coercive and bounded below.

Lemma 2.3. *If $\frac{3}{2} < q < p < 3$ holds, the functional I_λ satisfies the $(PS)_c$ condition for*

$$c < c^* = \frac{1}{p}\left(1 - \frac{\kappa}{p^*}\right)(m_0 S)^{\frac{3}{p}} - D\lambda^{\frac{p}{p-q}},$$

where $D = \frac{1}{pqp^*}\left[\frac{(p^*-q)|\Omega|^{\frac{p^*-q}{p^*}}}{m_0^{\frac{q}{p}}S^{\frac{q}{p}}(p^*-\kappa)^{\frac{q}{p}}}\right]^{\frac{p}{p-q}}$.

Proof. Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a $(PS)_c$ sequence such that

$$I_\lambda(u_n) \to c \text{ and } I_\lambda'(u_n) \to 0, \text{ as } n \to \infty. \tag{2.3}$$

From (2.2) and (2.3), we have

$$\begin{aligned} |c| + 1 + o(\|u_n\|) &\geq I_\lambda(u_n) - \frac{1}{p^*}\langle I_\lambda'(u_n), u_n\rangle \\ &\geq \frac{m_0}{p}\left(1 - \frac{\kappa}{p^*}\right)\|u_n\|^p - \lambda\left(\frac{1}{q} - \frac{1}{p^*}\right)S^{-\frac{q}{p}}|\Omega|^{\frac{p^*-q}{p^*}}\|u_n\|^q, \end{aligned}$$

which implies that $\{u_n\} \subset W_0^{1,p}(\Omega)$ is a bounded sequence for all $q < p < p^*$. Hence, there exist a subsequence (by denoted itself), and $u \in W_0^{1,p}(\Omega)$ as $n \to \infty$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } W_0^{1,p}(\Omega), \\ u_n \to u, & \text{strongly in } L^s(\Omega)\ (p \le s < p^*), \\ u_n(x) \to u(x), & \text{a.e. in } \Omega. \end{cases} \tag{2.4}$$

We proceed to confirm the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega)$. By the concentration compactness principle (see [30]), we have

$$\begin{aligned} |u_n|^{p^*}dx &\rightharpoonup d\nu = |u|^{p^*}dx + \sum_{j\in J} \nu_j \delta_{x_j}, \\ |\nabla u_n|^p dx &\rightharpoonup d\mu \ge |\nabla u|^p dx + \sum_{j\in J} \mu_j \delta_{x_j},\ S\nu_j^{\frac{p}{p^*}} \le \mu_j. \end{aligned} \tag{2.5}$$

Now, define a smooth cut-off function $\varphi_{\varepsilon,j}(x)$, for $\varepsilon > 0$, such that $0 \le \varphi_{\varepsilon,j} \le 1$, $|\nabla\varphi_{\varepsilon,j}| \le \frac{2}{\varepsilon}$,

$$\varphi_{\varepsilon,j}(x) = \begin{cases} 1, & \text{in } B(x_j,\varepsilon), \\ 0, & \text{in } \Omega \setminus B(x_j, 2\varepsilon). \end{cases}$$

Notice that $\{\varphi_{\varepsilon,j}u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ by $\langle I'_\lambda(u_n), \varphi_{\varepsilon,j}u_n\rangle \to 0$ as $n \to \infty$.

$$\begin{aligned} &M_0(\|u_n\|^p)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(\varphi_{\varepsilon,j}u_n)dx + \int_\Omega \phi_{u_n}|u_n|^p\varphi_{\varepsilon,j}dx \\ &= \int_\Omega |u_n|^{p^*}\varphi_{\varepsilon,j}dx + \lambda\int_\Omega |u_n|^q\varphi_{\varepsilon,j}dx + o(1). \end{aligned} \tag{2.6}$$

With the help of (2.4), we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \phi_{u_n}|u_n|^p\varphi_{\varepsilon,j}dx \le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{B(x_j,2\varepsilon)} \phi_{u_n}|u_n|^p dx = 0,$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega |u_n|^q\varphi_{\varepsilon,j}dx \le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{B(x_j,2\varepsilon)} |u_n|^q dx = 0.$$

Since $|\nabla\phi_{\varepsilon,j}| \le \frac{2}{\varepsilon}$, by using the Hölder inequality and (2.4), we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi_{\varepsilon,j}u_n dx\right| \le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p-1}|\nabla\varphi_{\varepsilon,j}||u_n|dx \\ &\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(\int_\Omega |\nabla u_n|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |u_n|^p|\nabla\varphi_{\varepsilon,j}|^p dx\right)^{\frac{1}{p}} \\ &\le C\lim_{\varepsilon\to 0}\left(\int_{B(x_j,2\varepsilon)} |u|^{p^*}dx\right)^{\frac{1}{p^*}}\left(\int_{B(x_j,2\varepsilon)} |\nabla\varphi_{\varepsilon,j}|^{\frac{pp^*}{p^*-p}}dx\right)^{\frac{p^*-p}{pp^*}} \\ &\le C\lim_{\varepsilon\to 0}\left(\int_{B(x_j,2\varepsilon)} |u|^{p^*}dx\right)^{\frac{1}{p^*}}\left(\int_{B(x_j,2\varepsilon)} \left(\frac{2}{\varepsilon}\right)^{\frac{pp^*}{p^*-p}}dx\right)^{\frac{p^*-p}{pp^*}} \\ &\le C_1\lim_{\varepsilon\to 0}\left(\int_{B(x_j,2\varepsilon)} |u|^{p^*}dx\right)^{\frac{1}{p^*}} = 0, \end{aligned}$$

where $C_1 > 0$, and we also derive that

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_\Omega |\nabla u_n|^p\varphi_{\varepsilon,j}dx \geq \lim_{\varepsilon\to0}\int_\Omega |\nabla u|^p\varphi_{\varepsilon,j}dx + \mu_j = \mu_j,$$

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_\Omega |u_n|^{p^*}\varphi_{\varepsilon,j}dx = \lim_{\varepsilon\to0}\int_\Omega |u|^{p^*}\varphi_{\varepsilon,j}dx + \nu_j = \nu_j.$$

Consequently, taking function $u_n\varphi_{\varepsilon,j}$ in (2.3), from the above information, we have

$$\begin{aligned}
0 &= \lim_{\varepsilon\to0}\lim_{n\to\infty}\langle I'_\lambda(u_n), u_n\varphi_{\varepsilon,j}\rangle \\
&= \lim_{\varepsilon\to0}\lim_{n\to\infty}\Big\{M_0(\|u_n\|^p)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla(\varphi_{\varepsilon,j}u_n)dx + \int_\Omega \phi_{u_n}|u_n|^p\varphi_{\varepsilon,j}dx \\
&\quad - \int_\Omega |u_n|^{p^*}\varphi_{\varepsilon,j}dx - \lambda\int_\Omega |u_n|^q\varphi_{\varepsilon,j}dx\Big\} \\
&= \lim_{\varepsilon\to0}\lim_{n\to\infty}\Big\{m_0\int_\Omega\Big(|\nabla u_n|^p\varphi_{\varepsilon,j} + |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi_{\varepsilon,j}u_n\Big)dx - \int_\Omega |u_n|^{p^*}\varphi_{\varepsilon,j}dx\Big\} \\
&\geq \lim_{\varepsilon\to0}\Big\{m_0\Big(\int_\Omega |\nabla u|^p\varphi_{\varepsilon,j}dx + \mu_j\Big) - \int_\Omega |u|^{p^*}\varphi_{\varepsilon,j}dx - \nu_j\Big\} \\
&\geq m_0\mu_j - \nu_j.
\end{aligned}$$

Thus, we have $\nu_j \geq m_0\mu_j$. This together with (2.5) implies that either $\mu_j = 0$ or

$$\mu_j \geq S^{\frac{p^*}{p^*-p}} m_0^{\frac{p}{p^*-p}}. \tag{2.7}$$

To proceed further, we prove that (2.7) is impossible. If there exists $j_0 \in J$ such that $\mu_{j_0} \geq S^{\frac{p^*}{p^*-p}} m_0^{\frac{p}{p^*-p}}$ and $x_{j_0} \in \Omega$, by combining with (2.3) and (2.5), there holds

$$\begin{aligned}
c &= \lim_{n\to\infty}\Big\{I_\lambda(u_n) - \frac{1}{p^*}\langle I'_\lambda(u_n), u_n\rangle\Big\} \\
&\geq \frac{m_0}{p}\Big(1 - \frac{\kappa}{p^*}\Big)(\|u\|^p + \mu_{j_0}) - \lambda\Big(\frac{1}{q} - \frac{1}{p^*}\Big)S^{-\frac{q}{p}}|\Omega|^{\frac{p^*-q}{p^*}}\|u\|^q \\
&\geq \frac{1}{p}\Big(1 - \frac{\kappa}{p^*}\Big)(m_0S)^{\frac{N}{p}} + \frac{m_0}{p}\Big(1 - \frac{\kappa}{p^*}\Big)\|u\|^p \\
&\quad - \lambda\Big(\frac{1}{q} - \frac{1}{p^*}\Big)S^{-\frac{q}{p}}|\Omega|^{\frac{p^*-q}{p^*}}\|u\|^q.
\end{aligned}$$

Let

$$h(t) = \frac{m_0}{p}\Big(1 - \frac{\kappa}{p^*}\Big)t^p - \lambda\Big(\frac{1}{q} - \frac{1}{p^*}\Big)S^{-\frac{q}{p}}|\Omega|^{\frac{p^*-q}{p^*}}t^q, \quad t > 0,$$

then, we can deduce that $\min_{t>0} h(t)$ attains at $t_0 > 0$ and

$$t_0 = \left(\frac{\lambda(p^* - q)|\Omega|^{\frac{p^*-q}{p^*}}}{m_0(p^* - \kappa)S^{\frac{q}{p}}}\right)^{\frac{1}{p-q}}.$$

Consequently, we obtain

$$c \geq \frac{1}{p}\left(1 - \frac{\kappa}{p^*}\right)(m_0 S)^{\frac{3}{p}} - D\lambda^{\frac{p}{p-q}},$$

where $D = \frac{1}{pqp^*}[\frac{(p^*-q)|\Omega|^{\frac{p^*-q}{p^*}}}{m_0^{\frac{q}{p}} S^{\frac{q}{p}} (p^*-\kappa)^{\frac{q}{p}}}]^{\frac{p}{p-q}}$. Hence, we deduce $c \geq c^*$. This is a contradiction, that is to say, $\nu_j = \mu_j = 0$ for all $j \in J$, which implies that

$$\int_\Omega |u_n|^{p^*} dx \to \int_\Omega |u|^{p^*} dx \text{ as } n \to \infty. \tag{2.8}$$

Let $\varphi \in W_0^{1,p}(\Omega)$ be fixed, and let us define linear functional $\mathcal{B}_\varphi$:

$$\mathcal{B}_\varphi(v) = \int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\nabla v dx$$

for all $v \in W_0^{1,p}(\Omega)$. Using the Hölder inequality, one has

$$|\mathcal{B}_\varphi(v)| \leq \|\varphi\|^{p-1}\|v\|.$$

By $I'_\lambda(u_n) \to 0$ in $(W_0^{1,p}(\Omega))^*$, $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, that is,

$$\begin{aligned} o(1) &= \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle \\ &= M_0(\|u_n\|^p)\left[\mathcal{B}_{u_n}(u_n - u) - \mathcal{B}_u(u_n - u)\right] + \left[M_0(\|u_n\|^p) - M_0(\|u\|^p)\right]\mathcal{B}_u(u_n - u) \\ &\quad + \int_\Omega (\phi_{u_n}|u_n|^{p-2}u_n - \phi_u|u|^{p-2}u)(u_n - u)dx - \int_\Omega (|u_n|^{p^*-2}u_n - |u|^{p^*-2}u)(u_n - u)dx \\ &\quad - \lambda\int_\Omega (|u_n|^{q-2}u_n - |u|^{q-2}u)|u_n|^{q-2}u_n(u_n - u)dx. \end{aligned} \tag{2.9}$$

According to $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ and (2.4), we know that

$$\begin{aligned} &\lim_{n\to\infty} M_0(\|u_n\|^p)\mathcal{B}_{u_n}(u_n - u) = 0 \text{ and } \lim_{n\to\infty} M_0(\|u_n\|^p)\mathcal{B}_u(u_n - u) = 0, \\ &\lim_{n\to\infty}\int_\Omega |u_n|^{q-2}u_n(u_n - u)dx = 0 \text{ and } \lim_{n\to\infty}\int_\Omega |u|^{q-2}u(u_n - u)dx = 0. \end{aligned} \tag{2.10}$$

By [15] Proposition 1, the Hölder and Sobolev inequalities, there holds

$$\begin{aligned} &\left|\int_\Omega (\phi_{u_n}|u_n|^{p-2}u_n - \phi_u|u|^{p-2}u)(u_n - u)dx\right| \\ &\leq \int_\Omega |\phi_{u_n}|u_n|^{p-2}u_n(u_n - u)|dx + \int_\Omega |\phi_u|u|^{p-2}u(u_n - u)|dx \\ &\leq \left(\int_\Omega |\phi_{u_n}|u_n|^{p-2}u_n|^{\frac{2p}{2p-1}}dx\right)^{\frac{2p-1}{2p}}\left(\int_\Omega |u_n - u|^{2p}\right)^{\frac{1}{2p}} \\ &\quad \times\left(\int_\Omega |\phi_u|u|^{p-2}u|^{\frac{2p}{2p-1}}dx\right)^{\frac{2p-1}{2p}}\left(\int_\Omega |u_n - u|^{2p}\right)^{\frac{1}{2p}} \\ &\leq C\left[\|u_n\|^p\left(\int_\Omega |u_n|^{2p}dx\right)^{\frac{p-1}{2p}} + \|u\|^p\left(\int_\Omega |u|^{2p}dx\right)^{\frac{p-1}{2p}}\right]\left(\int_\Omega |u_n - u|^{2p}\right)^{\frac{1}{2p}} \\ &\leq C\left(\int_\Omega |u_n - u|^{2p}\right)^{\frac{1}{2p}} \to 0, \text{ as } n \to \infty. \end{aligned} \tag{2.11}$$

Moreover, with the help of (2.8) and the Brezis-Lieb lemma, one obtains that

$$\int_\Omega |u_n-u|^{p^*}dx=\int_\Omega |u_n|^{p^*}dx-\int_\Omega |u|^{p^*}dx+o(1)\to 0,\ \text{as } n\to\infty. \tag{2.12}$$

Combining with (2.12) and the Hölder inequality, we see that

$$\int_\Omega (|u_n|^{p^*-2}u_n-|u|^{p^*-2}u)(u_n-u)dx\to 0,\ \text{as } n\to\infty.$$

Consequently, by (2.9), we can get that

$$\lim_{n\to\infty} M_0(\|u_n\|^p)\,[\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u)]=0.$$

Notice that $0<m_0\le M_0(\|u_n\|^p)$, and we have

$$\lim_{n\to\infty}[\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u)]=0. \tag{2.13}$$

For $a,b\in\mathbb{R}$, we have Simon inequalities

$$|a-b|^p\le\begin{cases} c_p(|a|^{p-2}a-|b|^{p-2}b)(a-b), & \text{for } p\ge 2,\\ C_p[(|a|^{p-2}a-|b|^{p-2}b)(a-b)]^{\frac{p}{2}}(|a|^p+|b|^p)^{\frac{2-p}{2}}, & \text{for } 1<p<2.\end{cases} \tag{2.14}$$

Clearly, we have two cases:
Case (*i*): When $2\le p<3$, by (2.13) and (2.14) as $n\to\infty$, we get

$$\begin{aligned}\|u_n-u\|^p&=\int_\Omega|\nabla(u_n-u)|^pdx\\&\le c_p\int_\Omega(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)(\nabla u_n-\nabla u)dx\\&=c_p[\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u)]\to 0.\end{aligned}$$

Case (*ii*): When $\frac{3}{2}<p<2$, since $\|u_n\|^p$ and $\|u\|^p$ are bounded in $W_0^{1,p}(\Omega)$, for all $a,b\ge 0$ and $1<p<2$, we have the subadditivity inequality

$$(a+b)^{\frac{2-p}{2}}\le a^{\frac{2-p}{2}}+b^{\frac{2-p}{2}}.$$

Letting $a=\nabla u_n$ and $b=\nabla u$ in (2.14) as $n\to\infty$, we have

$$\begin{aligned}\|u_n-u\|^p&\le C_p[\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u)]^{\frac{p}{2}}(\|u_n\|^p+\|u\|^p)^{\frac{2-p}{2}}\\&\le C_p[\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u)]^{\frac{p}{2}}(\|u_n\|^{\frac{p(2-p)}{2}}+\|u\|^{\frac{p(2-p)}{2}})\\&\le C_p[\mathcal{B}_{u_n}(u_n-u)-\mathcal{B}_u(u_n-u)]^{\frac{p}{2}}\to 0.\end{aligned}$$

Hence, we deduce that $u_n\to u$ in $W_0^{1,p}(\Omega)$.

Lemma 2.4. *If $J_\lambda(u) < 0$, then $\|u\|^p < T_0$ and $J_\lambda(u) = I_\lambda(u)$. Moreover, there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$, J_λ satisfies the $(PS)c$ condition for $c < 0$.*

Proof. From the definition of $\bar{g}$ and $J_\lambda(u) < 0$, that is,

$$\bar{g}(\|u\|^p) \leq J_\lambda(u) < 0$$

for all $\lambda \in (0, \lambda_0)$, we have

$$\|u\|^p < T_0 \text{ and } J_\lambda(u) = I_\lambda(u).$$

Therefore, if $\{u_n\} \subset W_0^{1,p}(\Omega)$ is a sequence such that $I_\lambda(u_n) \to c < 0$ and $I'_\lambda(u_n) \to 0$, we get

$$I_\lambda(u_n) = J_\lambda(u_n) \to c < 0 \text{ and } I'_\lambda(u_n) = J'_\lambda(u_n) \to 0, \text{ as } n \to \infty.$$

Since J_λ is coercive, there holds that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Moreover, there exists $\lambda_1 > 0$ such that for every $0 < \lambda < \lambda_1$, it holds that $c^* > 0$. Hence, by Lemma 2.3, J_λ satisfies the $(PS)c$ condition for $c < 0$. Choosing $\lambda^* = \min\{\lambda_0, \lambda_1\}$, we conclude.

3. Proof of main results

Proof of Theorem 1.1. We prove Theorem 1.1 in three steps.

Step 1. Assume that $k \in \mathbb{N}$ is given, and there exists $\varepsilon = \varepsilon(k) > 0$ such that $\gamma(J_\lambda^{-\varepsilon}) \geq k$, where $J_\lambda^{-\varepsilon} = \{u \in W_0^{1,p}(\Omega) : J_\lambda(u) \leq -\varepsilon\}$.

In fact, let $k \in \mathbb{N}$ and $E_k \subset W_0^{1,p}(\Omega)$ be a k-dimensional subspace. Since all norms are equivalent on a finite dimensional Banach space, thus, for all $u \in E_k$, we have

$$C(k)\|u\|^q \leq \int_\Omega |u|^q dx, \tag{3.1}$$

where $C(k) > 0$ depending on k. By [15] Proposition 1 and (3.1), we get

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\widehat{M_0}(\|u\|^p) + \frac{1}{2p}\int_\Omega \phi_u |u|^p dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}\psi(\|u\|^p)dx - \frac{\lambda}{q}\int_\Omega |u|^q dx \\ &\leq \frac{\kappa m_0}{p^2}\|u\|^p + C\|u\|^p - C(k)\|u\|^q = \|u\|^q\left(C\|u\|^{p-q} - C(k)\right). \end{aligned}$$

Let $R > 0$ be small enough. Hence, there exists $\varepsilon = \varepsilon(R) > 0$ for all $u \in \mathcal{S}_R = \{u \in E_k : \|u\| = R\}$, and one has

$$J_\lambda(u) < -\varepsilon < 0.$$

Since E_k and $\mathbb{R}^k$ are isomorphic and $\mathcal{S}_R$ and $\mathbb{S}^{k-1}$ are homeomorphic, therefore, by Lemma 2.1, we can see that $\gamma(\mathcal{S}_R) = \gamma(\mathbb{S}^{k-1}) = k$. $\mathcal{S}_R \subset J_\lambda^{-\varepsilon}$, $J_\lambda^{-\varepsilon}$ is symmetric and closed, that is,

$$\gamma(J_\lambda^{-\varepsilon}) \geq \gamma(\mathcal{S}_R) = k.$$

Now, we define

$$\Gamma_k = \{A \subset W_0^{1,p}(\Omega) \setminus \{0\} : A \text{ is closed, } A = -A \text{ and } \gamma(A) \geq k\},$$

$$K_c = \{u \in W_0^{1,p}(\Omega) \setminus \{0\} : J_\lambda(u) = c \text{ and } J'_\lambda(u) = 0\},$$

and

$$c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} J_\lambda(u).$$

Step 2. Assume that $c = c_k = c_{k+1} = \cdots = c_{k+\tau}$ for some $\tau \in \mathbb{N}$, then there exists $\lambda^* > 0$ such that $\gamma(K_c) \geq \tau + 1$ for all $\lambda \in (0, \lambda^*)$.

In fact, by step 1, that is $\gamma(J_\lambda^{-\varepsilon}) \geq k$. Moreover, $J_\lambda^{-\varepsilon} \in \Gamma_k$ and $\sup_{u \in J_\lambda^{-\varepsilon}} J_\lambda(u) \leq -\varepsilon$, so we obtain

$$-\infty < c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} J_\lambda(u) \leq \sup_{u \in J_\lambda^{-\varepsilon}} J_\lambda(u) \leq -\varepsilon < 0.$$

Since $c_k < 0$ for all $\lambda \in (0, \lambda^*)$, by step 1, we know that K_c is a compact set. If $\gamma(K_c) \leq \tau$, then there exists a closed and symmetric set U verifying $K_c \subset U$ such that $\gamma(U) \leq \tau$. It follows from the deformation lemma [31] that the homeomorphism $\eta : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$, then one has

$$\eta(J_\lambda^{c+\delta} - U) \subset J_\lambda^{c-\delta}$$

for some $\delta > 0$ with $0 < \delta < -c$. Thus, we have $J_\lambda^{c+\delta} \subset J_\lambda^0$. By the definition of $c = c_{k+\tau} = \inf_{A \in \Gamma_k} \sup_{u \in A} J_\lambda(u)$, there exists $B \in \Gamma_{k+\tau}$ such that $\sup_{u \in A} J_\lambda(u) < c + \delta$, that is, $B \in J_\lambda^{c+\delta}$ and

$$\eta(B - U) \subset \eta(J_\lambda^{c+\delta} - U) \subset J_\lambda^{c-\delta}. \tag{3.2}$$

Moreover, $\gamma(\overline{B - U}) \geq \gamma(B) - \gamma(U) \geq k$ and $\gamma(\eta(\overline{B - U})) \geq \gamma(\overline{B - U}) \geq k$. Then, we conclude that $\gamma(\overline{B - U}) \in \Gamma_k$. This contradicts with (3.2).

Step 3. By the above arguments and Theorem 2.2, we have the following two cases:
Case (*i*): If $-\infty < c_1 < c_2 < \cdots < c_k < \cdots < 0$, since every c_k is a critical value of J_λ, then we have infinitely many critical points of J_λ. Therefore, Eq (2.1) has infinitely many nontrivial solutions.
Case (*ii*): If there are two constants $c_k = c_{k+\tau}$, then $c = c_k = c_{k+1} = \cdots = c_{k+\tau}$. From Theorem 2.2, we obtain that K_c has infinitely many points. Consequently, Eq (2.1) has infinitely many nontrivial solutions. In addition, if u_λ is the nontrivial solution of Eq (2.1), then $J_\lambda(u_\lambda) = I_\lambda(u_\lambda) < 0$. Thus,

$$||u_\lambda||^p \leq T_0 \leq t_0. \tag{3.3}$$

From (1.6), we have that

$$M_0(||u_\lambda||^p) = M(||u_\lambda||^p),$$

that is to say, u_λ is a solution to system (1.1).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

References

1. V. Benci, D. Fortunato, Solitons in Schrödinger-Maxwell equations, *J. Fixed Point Theory Appl.*, **15** (2014), 101–132. https://doi.org/10.1007/s11784-014-0184-1
2. V. Benci, D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, *Topol. Methods Nonlinear Anal.*, **11** (1998), 283–293.
3. V. Benci, D. Fortunato, Solitary waves of the nonlinear Klein-Gordon equation coupled with the Maxwell equations, *Rev. Math. Phys.*, **14** (2002), 409–420. https://doi.org/10.1142/S0129055X02001168
4. T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, *Proc. Roy. Soc. Edinburgh Sect. A. Math.*, **134** (2004), 893–906. https://doi.org/10.1017/S030821050000353X
5. A. Azzollini, A. Pomponio, Ground state solutions for the nonlinear Schrödinger-Maxwell equations, *J. Math. Anal. Appl.*, **345** (2008), 90–108. https://doi.org/10.1016/j.jmaa.2008.03.057
6. L. Zhao, F. Zhao, On the existence of solutions for the Schrödinger-Poisson equations, *J. Math. Anal. Appl.*, **346** (2008), 155–169. https://doi.org/10.1016/j.jmaa.2008.04.053
7. L. Zhao, F. Zhao, Positive solutions for Schrödinger-Poisson equations with a critical exponent, *Nonlinear Anal. Theory Methods Appl.*, **70** (2009), 2150–2164. https://doi.org/10.1016/j.na.2008.02.116
8. F. Y. Li, Y. H. Li, J. P. Shi, Existence of positive solutions to Schrödinger-Poisson type systems with critical exponent, *Commun. Contemp. Math.*, **16** (2014), 1450036. https://doi.org/10.1142/S0219199714500369
9. D. Ruiz, The Schrödinger–Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.*, **237** (2006), 655–674. https://doi.org/10.1016/j.jfa.2006.04.005
10. A. Ambrosetti, On Schrödinger-Poisson systems, *Milan J. Math.*, **76** (2008), 257–274. https://doi.org/10.1007/s00032-008-0094-z
11. A. Ambrosetti, D. Ruiz, Multiple bound states for the Schrödinger-Poisson problem, *Commun. Contemp. Math.*, **10** (2008), 391–404. https://doi.org/10.1142/S021919970800282X
12. P. D'Avenia, A. Pomponio, G. Vaira, Infinitely many positive solutions for a Schrödinger-Poisson system, *Appl. Math. Lett.*, **24** (2011), 661–664. https://doi.org/10.1016/j.aml.2010.12.002

13. G. B. Li, S. J. Peng, S. S. Yan, Infinitely many positive solutions for the nonlinear Schrödinger-Poisson system, *Commun. Contemp. Math.*, **12** (2010), 1069–1092. https://doi.org/10.1142/S0219199710004068
14. S. Liu, Infinitely many solutions for Schrödinger-Poisson systems and Schrödinger-Kirchhoff equations, *Mathematics*, **12** (2024), 2233. https://doi.org/10.3390/math12142233
15. Y. Du, J. Su, C. Wang, On a quasilinear Schrödinger-Poisson system, *J. Math. Anal. Appl.*, **505** (2022), 125446. https://doi.org/10.1016/j.jmaa.2021.125446
16. Y. Du, J. Su, C. Wang, The Schrödinger-Poisson system with p-Laplacian, *Appl. Math. Lett.*, **120** (2021), 107286. https://doi.org/10.1016/j.aml.2021.107286
17. J. Xue, L. Wang, The fibering method approach for a Schrödinger-Poisson system with p-Laplacian in bounded domains, *Open Math.*, **22** (2024), 20240015. https://doi.org/10.1515/math-2024-0015
18. Y. Du, J. Su, C. Wang, On the critical Schrödinger-Poisson system with p-Laplacian, *Commun. Pure Appl. Anal.*, **21** (2022), 1329–1342. https://doi.org/10.3934/cpaa.2022020
19. L. Huang, J. Su, Multiple solutions for nonhomogeneous quasilinear Schrödinger-Poisson system, *J. Appl. Anal. Comput.*, **13** (2023), 1597–1612. https://doi.org/10.11948/20220404
20. L. Huang, J. Su, Multiple positive solutions of the quasilinear Schrödinger-Poisson system with critical exponent in $D^{1,p}(\mathbb{R}^3)$, *J. Math. Phys.*, **65** (2024), 051512. https://doi.org/10.1063/5.0202378
21. D. Xiao, T. V. Nguyen, S. Liang, Normalized solutions for critical Schrödinger-Poisson system involving p-Laplacian in $\mathbb{R}^3$, *Z. Angew. Math. Phys.*, **76** (2025), 1–25. https://doi.org/10.1007/s00033-024-02408-3
22. B. Yang, D. Zhang, S. Liang, On Kirchhoff-Schrödinger-Poisson-type systems with singular and critical nonlinearity, *Adv. Nonlinear Anal.*, **13** (2024), 20240050. https://doi.org/10.1515/anona-2024-0050
23. J. Azorero, I. Alonso, Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term, *Trans. Am. Math. Soc.*, **323** (1991), 877–895. https://doi.org/10.1090/S0002-9947-1991-1083144-2
24. J. Azorero, I. Alonso, Some results about the existence of a second positive solution in a quasilinear critical problem, *Indiana Univ. Math. J.*, **43** (1994), 941–957. https://doi.org/10.1512/iumj.1994.43.43041
25. D. Cao, S. Peng, S. Yan, Infinitely many solutions for p-Laplacian equation involving critical Sobolev growth, *J. Funct. Anal.*, **262** (2012), 2861–2902. https://doi.org/10.1016/j.jfa.2012.01.006
26. G. B. Li, Z. Guo, Multiple solutions for the p&q-Laplacian problem with critical exponent, *Acta Math. Sci.*, **29** (2009), 903–918. https://doi.org/10.1016/S0252-9602(09)60077-1
27. L. Du, Bounds for subcritical best Sobolev constants in $W^{1,p}$, *Commun. Pure Appl. Anal.*, **20** (2021), 3871–3886. https://doi.org/10.3934/cpaa.2021135
28. P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Dierential Equations*, AMS, 1986. https://doi.org/10.1090/cbms/065

29. D. C. Clarke, A variant of the Lusternik-Schnirelman theory, *Indiana Univ. Math. J.*, **22** (1973), 65–74. https://doi.org/10.1512/iumj.1973.22.22008
30. P. L. Lions, The concentration-compactness principle in the calculus of variations, the limit case, part 1, *Rev. Mat. Iberoam.*, **1** (1985),145–201. https://doi.org/10.4171/RMI/6
31. V. Benci, On critical points theory for indefinite functionals in the presence of symmetric, *Trans. Am. Math. Soc.*, **274** (1982), 533–572. https://doi.org/10.1090/S0002-9947-1982-0675067-X