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Research article

A blow-up criterion for a p-Laplacian-type pseudo-parabolic equation involving singular potential and logarithmic source

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Abstract: The main goal of this paper was to develop new skills to investigate the blow-up properties of solutions to an initial-boundary value problem for a p-Laplacian-type pseudo-parabolic equation with singular potential and logarithmic source. By establishing a crucial reverse Sobolev inequality, we derived that the solutions blow up in finite time under a new blow-up criterion and we estimated the lifespan of the weak solutions from both above and below. It is worthy to point out that our blow-up criterion implies that this problem admits finite time blow-up solutions at an arbitrarily high initial energy level. From methods to results, we partially extended some results obtained in recent literatures.

Keywords: pseudo-parabolic equation; p-Laplacian; singular potential; logarithmic nonlinearity; arbitrarily high initial energy; blow-up; lifespan

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1. Introduction

This work is concerned with the following initial-boundary value problem for a p-Laplacian-type pseudo-parabolic equation involving singular potential and logarithmic nonlinearity:

$$\begin{cases} \frac{u_t}{|x|^s} - \Delta_p u - \Delta u_t = |u|^{q-2}u \ln |u|, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n > 2)$ is a bounded open domain with smooth boundary $\partial\Omega$, for $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $|x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$, $u_0 \in W_0^{1,p}(\Omega) \setminus \{0\}$, the p-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and the parameter s, p, q satisfy the following assumptions:

$$0 \le s < \min\left\{\frac{n(p-2)}{p}, 2\right\}, \quad 2 < p < q < p^* = \begin{cases} +\infty, & n \le p, \\ \frac{np}{n-p}, & n > p. \end{cases} \tag{1.2}$$

Due to

$$\lim_{u\to 0}|u|^{q-2}u\ln|u| = 0, \tag{1.3}$$

when $u = 0$, we let $|u|^{q-2}u\ln|u| = 0$.

When the compressible fluid flows in a homogeneous isotropic and rigid porous medium, then the relationship between the volumetric moisture content $\theta(x)$, the fluid density ρ, and the macroscopic velocity $\vec{V}$ can be characterized by the following model [1]

$$\theta(x)\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\vec{V}) = f(\rho), \tag{1.4}$$

where $f(\rho)$ is the source, and the momentum velocity $\rho\vec{V}$ satisfies

$$\rho\vec{V} = -\lambda|\nabla\rho|^{\alpha-2}\nabla\rho.$$

According to parameter α, the media can be classified. If $1 < \alpha < 2$, the media are called pseudo-plastics; if $\alpha = 2$, they are called Newtonian fluids; and those with $\alpha > 2$ are called dilatant fluids.

In recent years, a huge amount of literature has been devoted to the investigation of qualitative properties of solutions to problem (1.4) with $\theta(x) = \frac{1}{|x|^s}$, called a singular potential. For instance, Hao and Zhou [2] considered the following parabolic equation with a singular potential

$$\frac{u_t}{|x|^s} - \Delta u = |u|^{p-2}u. \tag{1.5}$$

They derived the threshold results for the solutions to exist globally or to blow up in finite time by using the potential well method [3], when the initial energy is subcritical. With the help of a new functional, they obtained some finite-time blow-up sufficient conditions that include the possibility of the critical and supercritical initial energy. Deng and Zhou [4] studied a semilinear heat equation with singular potential and logarithmic nonlinearity that can be used to describe many phenomena in the viscoelastic mechanics, and quantum mechanics theory [5–7]:

$$\frac{u_t}{|x|^s} - \Delta u = u\log(|u|). \tag{1.6}$$

With the help of the modified potential well method [8,9], they obtained, among many other interesting results, some sufficient conditions on the initial data with lower energy such that the weak solutions blow up at ∞. Deng and Zhou [10] also researched a fast diffusion p-Laplace evolution equation with singular potential

$$\frac{u_t}{|x|^s} - \Delta_p u = |u|^{q-1}u. \tag{1.7}$$

They proved that the solution of (1.7) exists globally and obtained the conditions of extinction and non-extinction. Lian et al. [11] investigated the following pseudo-parabolic equation with strong damped and singular potential

$$\frac{u_t}{|x|^s} - \Delta u - \Delta u_t = |u|^{p-2}u. \tag{1.8}$$

They obtained global existence, asymptotic behavior, and blow-up of the solutions for subcritical and critical initial energy, and proved the finite-time blow-up for the high energy level by introducing an invariant manifold. Xie et al. [12] dealt with a fractional pseudo-parabolic equation with singular potential:

$$\frac{u_t}{|x|^{2s}} + (-\Delta)^s u_t + (-\Delta)^s u = |u|^{p-2}u. \tag{1.9}$$

They established the local existence and uniqueness of the weak solutions to (1.9) by means of the Galerkin method and contraction mapping principle. Yuan et al. [13] studied local and global well-posedness to problem (1.1). Under appropriate conditions, they revealed the existence and decay estimate of the global solutions and the blow-up phenomena of solutions to problem (1.1) at subcritical or critical initial energy level.

Motivated by the results in [11, 13], it is natural to ask whether problem (1.1) admits finite-time blow-up solutions with arbitrarily high initial energy. However, for supercritical initial energy, the potential well method and the invariance of the unstable set may be no longer valid, and some new techniques and methods need to be invented to overcome this difficulty. For this, by constructing an appropriate auxiliary functional, we first obtain that the unstable set $\mathcal{N}_-$ (defined in (2.3)) is invariant under the semiflow of problem (1.1). With the help of the invariance of the set $\mathcal{N}_-$ and Gagliardo-Nirenberg's interpolation inequality, we derive a reverse Sobolev inequality, i.e., $\|u\|_q^q$ can be bounded from below by $\|\nabla u\|_2^2$ up to a multiplicative constant (see (3.12)). Based on this relationship between $\|u\|_q^q$ and $\|\nabla u\|_2^2$, we provide a concavity inequality that guarantees finite-time blow-up for problem (1.1) and gives an upper bound for the blow-up time. Further, by employing the Fountain Theorem, we show that for some initial data, finite-time blow-up of solutions to problem (1.1) will occur with high initial energy. Finally, the blow-up time is estimated from below. These results extend and improve the blow-up results obtained in [13].

The rest of this paper is organized as follows: in Section 2, we present some notations, definitions and lemmas that will be used in the sequel. The main results will be stated and proved in Section 3.

2. Preliminaries

We shall present some notations, definitions and lemmas, in order to state our main results more clearly. Throughout this paper, we use $\|\cdot\|_r$ $(r \geq 1)$ and $(\cdot,\cdot)$ to denote the norm in $L^r(\Omega)$ and the $L^2(\Omega)$-inner product, respectively. When $q > 1$, the Sobolev space is denoted by $W_0^{1,q}(\Omega)$, which means both u and $|\nabla u|$ belong to $L^q(\Omega)$ for any $u \in W_0^{1,q}(\Omega)$. The space $W_0^{1,q}(\Omega)$ is endowed with the norm

$$\|u\|_{W_0^{1,q}(\Omega)} = \|\nabla u\|_q,$$

which is equivalent to the standard norm by Poincaré's inequality, in particular, when $q = 2$, $W_0^{1,q}(\Omega)$ is written as $H_0^1(\Omega)$. In addition, we define

$$(u,v)_* = \int_\Omega \frac{uv}{|x|^s}\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x,\ \|u\|_*^2 = (u,u)_* = \int_\Omega \frac{u^2}{|x|^s}\mathrm{d}x + \|\nabla u\|_2^2.$$

The solution $u(x,t)$ to problem (1.1) is studied in the weak sense as follows:

Definition 2.1. *(Weak Solution [13]) Function $u = u(x,t)$ is called a weak solution to problem* (1.1) *on $\Omega \times [0,T)$, if $u \in L^\infty(0,T; W_0^{1,p}(\Omega))$ with $u_t \in L^2(0,T; H_0^1(\Omega))$ satisfies*

(1)

$$\left(\frac{u_t}{|x|^s},\phi\right)+(\nabla u_t,\nabla\phi)+(|\nabla u|^{p-2}\nabla u,\nabla\phi)=(|u|^{q-2}u\ln|u|,\phi),$$

for any $\phi\in W_0^{1,p}(\Omega)$ *and a.e.* $t\in[0,T]$.

(2) $u(0)=u_0\in W_0^{1,p}(\Omega)\setminus\{0\}$.

Local existence and uniqueness of the weak solution can be derived via Galerkin's method and the method of a priori estimates (see [13] for the details), i.e., let $u_0\in W_0^{1,p}(\Omega)\setminus\{0\}$, p,q,s satisfy (1.2), then there exists a $T^*>0$, such that problem (1.1) admits a unique weak solution u on $[0,T^*]$.

Definition 2.2. *(Maximal existence time) Let* $u=u(x,t)$ *be a weak solution to problem* (1.1). *We define the maximal existence time* $T_{\max}$ *of* u *as follows:*

(1) If u *exists for all* $0\le t<\infty$, *then* $T_{\max}=\infty$;

(2) If there exists a $t_0\in(0,\infty)$ *such that* u *exists for* $0\le t<t_0$, *but does not exist at* $t=t_0$, *then* $T_{\max}=t_0$.

Definition 2.3. *(Finite-time blow-up) Let* $u(x,t)$ *be a weak solution to problem* (1.1). *We call* $u(x,t)$ *blowing up in finite time if the maximal existence time* $T_{\max}<\infty$ *and*

$$\lim_{t\to T_{\max}}\|u(\cdot,t)\|_*^2=\infty.$$

In order to investigate the blow-up properties of solutions, we introduce the energy functional $J(u)$, its Fréchet derivative $J'(u)$, and the Nehari's functional $I(u)$ associated with problem (1.1) as follow: For $u\in W_0^{1,p}(\Omega)$, we set

$$J(u)=\frac{1}{p}\|\nabla u\|_p^p-\frac{1}{q}\int_\Omega|u|^q\ln|u|\mathrm{d}x+\frac{1}{q^2}\|u\|_q^q. \tag{2.1}$$

Furthermore, for any $u,v\in W_0^{1,p}(\Omega)$,

$$\langle J'(u),v\rangle=\int_\Omega|\nabla u|^{p-2}\nabla u\cdot\nabla v\mathrm{d}x-\int_\Omega|u|^{q-2}u\ln|u|v\mathrm{d}x.$$

$$I(u)=\langle J'(u),u\rangle=\|\nabla u\|_p^p-\int_\Omega|u|^q\ln|u|\mathrm{d}x. \tag{2.2}$$

Using Gagliardo-Nirenberg multiplicative embedding inequality (see [14,15]), it is easily verified that $J(u)$ and $I(u)$ are C^1 functionals in $W_0^{1,p}(\Omega)$. Besides, we define the unstable set

$$\mathcal{N}_-=\{u\in W_0^{1,p}(\Omega)\mid I(u)<0\}. \tag{2.3}$$

The following lemmas are essential tools to establish our main results. The first one is the energy inequality for problem (1.1), which has been given in [13]. The second is the Hardy-Sobolev inequality, the third is an inequality involving the logarithmic function, and the fourth is a special form of Gagliardo-Nirenberg's interpolation inequality, which, combined with the third lemma, will be utilized to deal with the logarithmic source. The final one is Levine's concavity lemma.

Lemma 2.1. *([13]) Let* $u_0 \in W_0^{1,p}(\Omega)$*,* T_{max} *be the maximal existence time of the solution* $u = u(x,t)$ *to problem* (1.1)*. Then we have*

$$J(u) + \int_0^t \|u_\tau\|_*^2 \mathrm{d}\tau \le J(u_0), \quad a.e.\ t \in [0, T_{max}).$$

Lemma 2.2. *([13, 16]) Let* $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$*, with* $2 \le k \le n$*, and* $x = (x', z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$*, for given real number* q, s *satisfying* $1 < q < n, 0 \le s \le q$ *and* $s < k$*, set* $q_*(s,n,q) = \frac{q(n-s)}{n-q}$*, there exists a positive constant* $C_H = C(s,q,n,k)$ *such that*

$$\int_{\mathbb{R}^n} \frac{|u|^{q_*(s)}}{|x'|^s} \mathrm{d}x \le C_H \left(\int_{\mathbb{R}^n} |\nabla u|^q \mathrm{d}x \right)^{\frac{n-s}{n-q}}, \ \forall u \in W_0^{1,q}(\mathbb{R}^n). \tag{2.4}$$

Remark 2.1. *In* (2.4)*, when* $q_*(s) = 2$*, then* $q = \frac{2n}{n-s+2}$ *and* $\frac{n-s}{n-q} = \frac{n-s+2}{n}$*. Since* $0 \le s < 2, n > 2$*, we have* $1 < \frac{2n}{n-s+2} < 2$*. In accordance with Hölder's inequality and* (2.4)*, we derive*

$$\int_\Omega \frac{u^2}{|x|^s} \mathrm{d}x \le C_H \left(\int_\Omega |\nabla u|^{\frac{2n}{n-s+2}} \mathrm{d}x \right)^{\frac{n-s+2}{n}} \le C_H |\Omega|^{\frac{2-s}{n}} \|\nabla u\|_2^2.$$

Lemma 2.3. *For any* $\rho > 0, x > 0$*, there holds*

$$\ln x \le \frac{1}{e\rho} x^\rho.$$

Proof. For any $\rho > 0$, set $f(x) = \frac{\ln x}{x^\rho}, \forall x > 0$. By direct differentiation, we obtain $f'(x) > 0$ for $x \in (0, e^{\frac{1}{\rho}})$; $f'(x) < 0$ for $x \in (e^{\frac{1}{\rho}}, +\infty)$. Then, $f(x)$ attains its maximum at $x = e^{\frac{1}{\rho}}$, which means that $f(x) \le f(e^{\frac{1}{\rho}}) = \frac{1}{e\rho}, \forall x > 0$. □

Lemma 2.4. *([17]) Assume that* $\gamma < q + \sigma < p^*$*. Then for any* $u \in W_0^{1,p}(\Omega)$*, it holds that*

$$\|u\|_{q+\sigma}^{q+\sigma} \le C_{G,\gamma} \|\nabla u\|_p^{\alpha(q+\sigma)} \|u\|_\gamma^{(1-\alpha)(q+\sigma)},$$

where $\alpha \in (0,1)$ *is determined by* $\alpha = \frac{q+\sigma-\gamma}{\gamma(q+\sigma)}(\frac{1}{\gamma} + \frac{1}{n} - \frac{1}{p})^{-1}$ *and* $C_{G,\gamma} > 0$ *is a constant depending on* n, p, q, σ *and* γ*.*

Lemma 2.5. *([18, 19]) Assume that a positive, twice-differentiable function* $\psi(t)$ *satisfies the inequality*

$$\psi''(t)\psi(t) - (1+\theta)(\psi'(t))^2 \ge 0,$$

where $\theta > 0$*. If* $\psi(0) > 0$ *and* $\psi'(0) > 0$*, then* $\psi(t) \to \infty$ *as*

$$t \to t_* \le t^* = \frac{\psi(0)}{\theta \psi'(0)}.$$

3. Main results

In this section, we focus on the blow-up properties of solutions to problem (1.1) with supercritical initial energy. Prior to proving the main theorem, we establish two crucial lemmas, which aim to prove that the unstable set $\mathcal{N}_-$ is invariant under the semi-flow of problem (1.1). By virtue of them, a new finite-time blow-up criterion for problem (1.1) can be established. For simplicity, $u(x,t)$ will be written as $u(t)$ unless confusion arises.

Lemma 3.1. *Let* $u_0 \in W_0^{1,p}(\Omega)$ *and* $u = u(t)$ *be a weak solution to problem* (1.1) *such that* $u(t) \in \mathcal{N}_-$ *on* $[0, T_{max})$. *Then*

$$\{t \mapsto \|u(t)\|_*^2\}$$

is strictly increasing on $(0, T_{max})$.

Proof. Define

$$U(t) = \|u(t)\|_*^2, \qquad t \in [0, T_{max}). \tag{3.1}$$

By computing its derivative and recalling (1.1) and (2.3), this yields

$$\begin{aligned} U'(t) =& 2(\frac{u}{|x|^s}, u_t) + 2(\nabla u, \nabla u_t) \\ =& 2(-\|\nabla u\|_p^p + \int_\Omega |u|^q \ln |u| \mathrm{d}x) \\ =& -2I(u) > 0, \end{aligned} \tag{3.2}$$

and it is clear to see that $U(t)$ is strictly increasing on $(0, T_{max})$. The proof is complete. □

Lemma 3.2. *(Invariance of* $\mathcal{N}_-$*) Assume that* $u_0 \in \mathcal{N}_-$ *satisfies*

$$0 < J(u_0) < \frac{C^*}{q^2(C_H|\Omega|^{\frac{2-s}{n}} + 1)}\|u_0\|_*^2 \triangleq C_0\|u_0\|_*^2, \tag{3.3}$$

where $C^* = \left(\frac{e(p-2)(np+pq-nq)}{C_{G,q}(np+2p-2n)}\right)^{\frac{np+2p-2n}{p^2}} |\Omega|^{\frac{2}{p}-1}$. *Then the solution* $u(t)$ *to problem* (1.1) *belongs to* $\mathcal{N}_-$ *for all* $t \in [0, T_{max})$.

Proof. We claim that $u(t) \in \mathcal{N}_-$ for all $t \in [0, T_{max})$. Otherwise, by the continuity of $I(u)$, there would exist a $t_0 \in (0, T_{max})$ such that

$$I(u(t)) < 0, \quad t \in [0, t_0), \tag{3.4}$$

and

$$I(u(t_0)) = 0. \tag{3.5}$$

On one hand, according to (3.4) and Lemma 3.1, we know that $U(t) = \|u(t)\|_*^2$ is strictly increasing on $[0, t_0)$, which, combined with the assumption (3.3), leads to

$$0 < J(u_0) < C_0\|u_0\|_*^2 < C_0\|u(t)\|_*^2, \quad t \in (0, t_0).$$

Due to the continuity of $\|u(t)\|_*^2$, we arrive at

$$0 < J(u_0) < C_0\|u_0\|_*^2 < C_0\|u(t_0)\|_*^2. \tag{3.6}$$

On the other hand, from (3.5), Lemma 2.3, and Lemma 2.4, we obtain

$$\begin{aligned}\|\nabla u(t_0)\|_p^p &= \int_\Omega |u(t_0)|^q \ln |u(t_0)| \mathrm{d}x \\ &\leq \frac{1}{e\sigma} \|u(t_0)\|_{q+\sigma}^{q+\sigma} \\ &\leq \frac{C_{G,q}}{e\sigma} \|\nabla u(t_0)\|_p^{\alpha(q+\sigma)} \|u(t_0)\|_q^{(1-\alpha)(q+\sigma)}.\end{aligned} \tag{3.7}$$

Here,

$$\alpha = \frac{\sigma}{q(q+\sigma)} \left(\frac{1}{q} + \frac{1}{n} - \frac{1}{p} \right)^{-1}, \tag{3.8}$$

and we choose $\sigma = \frac{(p-2)(np+pq-nq)}{np+2p-2n}$ (which ensures that $q + \sigma < p^*$). In accordance with (3.6), it is seen that $u(t_0) \neq 0$, i.e., $\|\nabla u(t_0)\|_p \neq 0$. Then, dividing both sides of (3.7) by $\|\nabla u(t_0)\|_p^{\alpha(q+\sigma)}$ yields that

$$\|\nabla u(t_0)\|_p^{p-\alpha(q+\sigma)} \leq \frac{C_{G,q}}{e\sigma} \|u(t_0)\|_q^{(1-\alpha)(q+\sigma)},$$

and taking both sides to the power of $\frac{q}{(1-\alpha)(q+\sigma)}$ and letting

$$\frac{p - \alpha(q+\sigma)}{(1-\alpha)(q+\sigma)} q = 2 \tag{3.9}$$

(in fact, substituting α into this formula yields σ), it shows that

$$\|\nabla u(t_0)\|_p^{\frac{p-\alpha(q+\sigma)}{(1-\alpha)(q+\sigma)} q} = \|\nabla u(t_0)\|_p^2 \leq \left(\frac{C_{G,q}}{e\sigma} \right)^{\frac{q}{(1-\alpha)(q+\sigma)}} \|u(t_0)\|_q^q. \tag{3.10}$$

In view of Hölder's inequality, we know that

$$\|\nabla u(t_0)\|_2^2 \leq |\Omega|^{1-\frac{2}{p}} \|\nabla u(t_0)\|_p^2. \tag{3.11}$$

Therefore, by (3.10) and (3.11), we have

$$|\Omega|^{\frac{2}{p}-1} \|\nabla u(t_0)\|_2^2 \leq \|\nabla u(t_0)\|_p^2 \leq \left(\frac{C_{G,q}}{e\sigma} \right)^{\frac{q}{(1-\alpha)(q+\sigma)}} \|u(t_0)\|_q^q,$$

and then, substituting α and σ, we give the following crucial reverse Sobolev inequality:

$$\|u(t_0)\|_q^q \geq \left(\frac{e(p-2)(np+pq-nq)}{C_{G,q}(np+2p-2n)} \right)^{\frac{np+2p-2n}{p^2}} |\Omega|^{\frac{2}{p}-1} \|\nabla u(t_0)\|_2^2 \triangleq C^* \|\nabla u(t_0)\|_2^2, \tag{3.12}$$

where the exponent is determined as follows:

$$\begin{aligned}\frac{q}{(1-\alpha)(q+\sigma)} = \frac{2}{p - \alpha(q+\sigma)} &= \frac{2}{p - \frac{\sigma}{q} \left(\frac{1}{q} + \frac{1}{n} - \frac{1}{p} \right)^{-1}} \\ &= \frac{2}{p - \frac{(p-2)(np+pq-nq)}{q(np+2p-2n)} \frac{npq}{np+pq-nq}} = \frac{np+2p-2n}{p^2}.\end{aligned}$$

It follows from Remark 2.1 that

$$\|\nabla u\|_2^2 \geq \frac{1}{C_H|\Omega|^{\frac{2-s}{n}}}\left\|\frac{u}{|x|^{\frac{s}{2}}}\right\|_2^2. \tag{3.13}$$

With the help of Lemma 2.1, (3.5), (3.12), and (3.13), we get

$$\begin{aligned}
J(u_0) \geq J(u(t_0)) &\geq \frac{1}{q^2}\|u(t_0)\|_q^q \\
&\geq \frac{C^*}{q^2}\|\nabla u(t_0)\|_2^2 \\
&= \frac{C^* C_H|\Omega|^{\frac{2-s}{n}}}{q^2(C_H|\Omega|^{\frac{2-s}{n}}+1)}\|\nabla u(t_0)\|_2^2 + \left(\frac{C^*}{q^2} - \frac{C^* C_H|\Omega|^{\frac{2-s}{n}}}{q^2(C_H|\Omega|^{\frac{2-s}{n}}+1)}\right)\|\nabla u(t_0)\|_2^2 \\
&\geq \frac{C^*}{q^2(C_H|\Omega|^{\frac{2-s}{n}}+1)}\|u(t_0)\|_*^2 = C_0\|u(t_0)\|_*^2,
\end{aligned}$$

which contradicts (3.6). The proof is complete. □

In what follows, we shall demonstrate that the solutions to problem (1.1) blow up in finite time, when positive initial energy is bounded from above by $C_0\|u_0\|_*^2$ for some $C_0 > 0$. During this process, an upper bound for the blow-up time is estimated.

Theorem 3.1. *Let both assumptions in Lemma 3.2 hold. Then the solution* $u(t)$ *to problem* (1.1) *blows up in finite time. Moreover,* T_{max} *can be estimated from above as follows:*

$$T_{max} \leq \frac{8\|u_0\|_*^2}{(q-2)^2\left(C_0\|u_0\|_*^2 - J(u_0)\right)}, \tag{3.14}$$

where $C_0 > 0$ *is the constant given in Lemma 3.2.*

Proof. The proof employs Levine's concavity argument with a specific choice of parameters. We argue by contradiction: assume that $u(t)$ is a global weak solution to problem (1.1), which implies $T_{max} = \infty$. For any $T > 0$, $b > 0$ and, $\eta > 0$, define a nonnegative auxiliary function

$$F(t) = \int_0^t \|u(\tau)\|_*^2 \mathrm{d}\tau + (T-t)\|u_0\|_*^2 + \beta(t+\xi)^2, \quad t \in [0, T]. \tag{3.15}$$

Direct computation gives

$$\begin{aligned}
F'(t) =&\|u(t)\|_*^2 - \|u_0\|_*^2 + 2\beta(t+\xi) \\
=&2\int_0^t (u, u_\tau)_* \mathrm{d}\tau + 2\beta(t+\xi), \quad t \in [0, T],
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
F''(t) =&2(u, u_t)_* + 2\beta \\
=&2(u, \Delta_p u + |u|^{q-2}u\ln|u|) + 2\beta \\
=&-2I(u) + 2\beta \\
\geq&-2qJ(u_0) + 2q\int_0^t \|u_\tau\|_*^2 \mathrm{d}\tau + \frac{2(q-p)}{p}\|\nabla u\|_p^p + \frac{2}{q}\|u\|_q^q + 2\beta \quad t \in [0, T].
\end{aligned} \tag{3.17}$$

From the Cauchy-Schwarz and Hölder's inequalities, it follows that

$$\int_0^t (u, u_\tau)_* \mathrm{d}\tau \leq \int_0^t \|u\|_* \|u_\tau\|_* \mathrm{d}\tau \leq \Big(\int_0^t \|u\|_*^2 \mathrm{d}\tau \Big)^{\frac{1}{2}} \Big(\int_0^t \|u_\tau\|_*^2 \mathrm{d}\tau \Big)^{\frac{1}{2}},$$

which together with (3.16) and Cauchy's inequality, yields

$$\begin{aligned}
&(F'(t))^2 \\
&\leq 4 \left[\int_0^t \|u\|_*^2 \mathrm{d}\tau \int_0^t \|u_\tau\|_*^2 \mathrm{d}\tau + 2\beta(t+\xi) \Big(\int_0^t \|u\|_*^2 \mathrm{d}\tau \Big)^{\frac{1}{2}} \Big(\int_0^t \|u_\tau\|_*^2 \mathrm{d}\tau \Big)^{\frac{1}{2}} + \beta^2 (t+\xi)^2 \right] \\
&\leq 4 \left[\int_0^t \|u\|_*^2 \mathrm{d}\tau \int_0^t \|u_\tau\|_*^2 \mathrm{d}\tau + \beta \int_0^t \|u\|_*^2 \mathrm{d}\tau + \beta(t+\xi)^2 \int_0^t \|u_\tau\|_*^2 \mathrm{d}\tau + \beta^2 (t+\xi)^2 \right] \qquad (3.18) \\
&= 4 \left(\int_0^t \|u\|_*^2 \mathrm{d}\tau + \beta(t+\xi)^2 \right) \left(\int_0^t \|u_\tau\|_*^2 \mathrm{d}\tau + \beta \right).
\end{aligned}$$

By similar calculations to (3.7)–(3.12), the reverse Sobolev inequality remains valid, namely,

$$\|u\|_q^q > \left(\frac{e(p-2)(np+pq-nq)}{C_{G,q}(np+2p-2n)} \right)^{\frac{np+2p-2n}{p^2}} |\Omega|^{\frac{2}{p}-1} \|\nabla u\|_2^2 = C^* \|\nabla u\|_2^2. \qquad (3.19)$$

By (3.15), (3.17)–(3.19), and the fact that $q > 2$ we see that

$$\begin{aligned}
&F(t)F''(t) - \frac{q+2}{4}(F'(t))^2 \\
&\geq F(t) \left[-2qJ(u_0) + (q-2) \int_0^t \|u_\tau\|_*^2 \mathrm{d}\tau + \frac{2(q-p)}{p} \|\nabla u\|_p^p + \frac{2}{q} \|u\|_q^q - q\beta \right] \\
&\geq F(t) \left[\frac{2C^*}{q} \|\nabla u\|_2^2 - 2qJ(u_0) - q\beta \right] \qquad (3.20) \\
&\geq 2qF(t) \left[\frac{C^*}{q^2 (C_H |\Omega|^{\frac{2-s}{n}} + 1)} \|u\|_*^2 - J(u_0) - \frac{\beta}{2} \right] \\
&\geq 2qF(t) \left[\frac{C^*}{q^2 (C_H |\Omega|^{\frac{2-s}{n}} + 1)} \|u_0\|_*^2 - J(u_0) - \frac{\beta}{2} \right].
\end{aligned}$$

Since $F(t) \geq 0$ on $[0, T]$, we deduce from (3.20) that

$$F(t)F''(t) - \frac{q+2}{4}(F'(t))^2 \geq 0, \qquad (3.21)$$

for any $t \in [0, T]$ and $\beta \in \left(0, 2\left(\frac{C^*}{q^2(C_H|\Omega|^{\frac{2-s}{n}}+1)} \|u_0\|_*^2 - J(u_0) \right)\right]$. Choose

$$\xi > \frac{2\|u_0\|_*^2}{\beta(q-2)}, \qquad (3.22)$$

which is independent of T, then

$$F(0) = T\|u_0\|_*^2 + \beta\xi^2 > 0,$$

$$F'(0) = 2\beta\xi > 0,$$

and

$$\frac{4F(0)}{(q-2)F'(0)} = \frac{2(T\|u_0\|_*^2 + \beta\xi^2)}{\beta\xi(q-2)} < T, \tag{3.23}$$

for T large enough. Recalling Lemma 2.5, there exists a $t_* > 0$ satisfying

$$t_* \le \frac{4F(0)}{(q-2)F'(0)} (< T) \tag{3.24}$$

such that

$$F(t) \to \infty \text{ as } t \to t_*^-.$$

This is a contradiction to the assumption of $u(t)$ being global. At this point, we have proved that $T_{max} < \infty$.

Note that the above argument itself can not provide an upper bound estimation for blow-up time, since it is obtained under the false assumption that the solution $u(t)$ is global. To estimate T_{max} from above, we define $\bar{F}(t)$ similarly to (3.15), for any $T \in (0, T_{max})$,

$$\bar{F}(t) = \int_0^t \|u(\tau)\|_*^2 \mathrm{d}\tau + (T_{max} - t)\|u_0\|_*^2 + \beta(t+\xi)^2, \quad t \in [0, T].$$

Based on the foregoing arguments, we can still derive

$$T \le \frac{2(T_{max}\|u_0\|_*^2 + \beta\xi^2)}{\beta\xi(q-2)},$$

where $\beta \in \left(0, 2\left(\frac{C^*}{q^2(C_H|\Omega|^{\frac{2-s}{n}}+1)}\|u_0\|_*^2 - J(u_0)\right)\right]$ and ξ is still required to satisfy (3.22). By the arbitrariness of $T < T_{max}$, it follows that

$$T_{max} \le \frac{2(T_{max}\|u_0\|_*^2 + \beta\xi^2)}{(q-2)\beta\xi}, \tag{3.25}$$

which is equivalent to

$$T_{max} \le T(\beta, \xi) \triangleq \frac{2\beta\xi^2}{\beta\xi(q-2) - 2\|u_0\|_*^2}. \tag{3.26}$$

Fix a $\beta \in \left(0, 2\left(\frac{C^*}{q^2(C_H|\Omega|^{\frac{2-s}{n}}+1)}\|u_0\|_*^2 - J(u_0)\right)\right]$. Then, minimizing $T(\xi,\beta)$ on $(\frac{2\|u_0\|_*^2}{\beta(q-2)}, +\infty)$, we have

$$T_{min}(\xi,\beta) = T(\xi_0,\beta) = \frac{16\|u_0\|_*^2}{\beta(q-2)^2},$$

where $\xi_0 = \frac{4\|u_0\|_*^2}{\beta(q-2)}$.

Minimizing $T(\xi_0,\beta)$ on $\left(0, 2\left(\frac{C^*}{q^2(C_H|\Omega|^{\frac{2-s}{n}}+1)}\|u_0\|_*^2 - J(u_0)\right)\right]$, we further obtain

$$T_{min}(\xi_0,\beta) = T(\xi_0,\beta_0) = \frac{8\|u_0\|_*^2}{(q-2)^2\left(\frac{C^*}{q^2(C_H|\Omega|^{\frac{2-s}{n}}+1)}\|u_0\|_*^2 - J(u_0)\right)},$$

where $\beta_0 = 2\left(\frac{C^*}{q^2(C_H|\Omega|^{\frac{2-s}{n}}+1)}\|u_0\|_*^2 - J(u_0)\right)$.

In conclusion,

$$T_{max} \le \frac{8\|u_0\|_*^2}{(q-2)^2\left(\frac{C^*}{q^2(C_H|\Omega|^{\frac{2-s}{n}}+1)}\|u_0\|_*^2 - J(u_0)\right)} = \frac{8\|u_0\|_*^2}{(q-2)^2\left(C_0\|u_0\|_*^2 - J(u_0)\right)}.$$

The proof of Theorem 3.1 is complete. □

It is noteworthy that Theorem 3.1 implies that the solution to problem (1.1) blows up in finite time at arbitrarily high initial energy level.

Remark 3.1. *For any positive initial energy, there always exists an appropriate function u_0 such that the solution $u(x,t)$ to problem* (1.1) *with initial datum u_0 will blow up in finite time.*

More concretely, for any $M > 0$, we can construct a function u_0 such that $J(u_0) = M$, while also satisfying $u_0 \in \mathcal{N}_-$ and $J(u_0) < C_0\|u_0\|_*^2$, which meets both the assumptions in Theorem 3.1. Therefore, the corresponding solution $u(x,t)$ to problem (1.1) *with such u_0 as initial datum blows up in finite time.*

To explain this, we recall a well-known result that can be proved by using the Fountain Theorem [20, 21], i.e., for any bounded smooth domain Ω in $R^n (n > 2)$ and $p < q < p^*$, the energy functional $J(u)$ defined on $W_0^{1,p}(\Omega)$ has a sequence of critical points $\{\varpi_k\}_{k=1}^{\infty} \subset W_0^{1,p}(\Omega)$ such that

$$J(\varpi_k) = \frac{1}{p}\|\nabla\varpi_k\|_p^p - \frac{1}{q}\int_{\Omega}|\varpi_k|^q \ln|\varpi_k| dx + \frac{1}{q^2}\|\varpi_k\|_q^q \to +\infty, \ \ k \to \infty. \tag{3.27}$$

Let Ω_1 and Ω_2 be two arbitrary disjoint smooth subdomains of Ω. From (3.27)*, there exists a sequence $\{\varpi_k\}_{k=1}^{\infty} \subset W_0^{1,p}(\Omega_1)$ such that*

$$\frac{1}{p}\int_{\Omega_1}|\nabla\varpi_k|^p dx - \frac{1}{q}\int_{\Omega_1}|\varpi_k|^q \ln|\varpi_k| dx + \frac{1}{q^2}\int_{\Omega_1}|\varpi_k|^q dx \to +\infty, \quad k \to \infty. \tag{3.28}$$

On the other hand, choose υ to be any nontrivial function in $W_0^{1,p}(\Omega_2) \subset H_0^1(\Omega_2)$. Then, for any $M > 0$, there exists a $\gamma_1 > 0$ such that

$$\int_{\Omega_2}\left|\frac{\gamma_1\upsilon}{|x|^{\frac{s}{2}}}\right|^2 + |\gamma_1\nabla\upsilon|^2 dx = \gamma_1^2\int_{\Omega_2}\frac{|\upsilon|^2}{|x|^s} + |\nabla\upsilon|^2 dx > \frac{M}{C_0}. \tag{3.29}$$

Combining $q > p$ with monotonicity theory, it is not difficult to verify that

$$\begin{aligned} Z(\gamma) = M - \frac{\gamma^p}{p}\int_{\Omega_2}|\nabla\upsilon|^p dx + \frac{\gamma^q \ln\gamma}{q}\int_{\Omega_2}|\upsilon|^q dx \\ + \frac{\gamma^q}{q}\int_{\Omega_2}|\upsilon|^q \ln|\upsilon| dx - \frac{\gamma^q}{q^2}\int_{\Omega_2}|\upsilon|^q dx \to +\infty, \end{aligned} \tag{3.30}$$

as $\gamma \to +\infty$. By virtue of (3.28) *and* (3.30)*, there exists $k_0 \in \mathbb{N}$ and $\gamma_0 > \gamma_1$, both sufficiently large,* *such that*

$$
\begin{aligned}
&\frac{1}{p}\int_{\Omega_1}|\nabla\varpi_{k_0}|^p\mathrm{d}x-\frac{1}{q}\int_{\Omega_1}|\varpi_{k_0}|^q\ln|\varpi_{k_0}|\mathrm{d}x+\frac{1}{q^2}\int_{\Omega_1}|\varpi_{k_0}|^q\mathrm{d}x\\
=&M-\frac{{\gamma_0}^p}{p}\int_{\Omega_2}|\nabla\upsilon|^p\mathrm{d}x+\frac{{\gamma_0}^q\ln\gamma_0}{q}\int_{\Omega_2}|\upsilon|^q\mathrm{d}x\\
&+\frac{{\gamma_0}^q}{q}\int_{\Omega_2}|\upsilon|^q\ln|\upsilon|\mathrm{d}x-\frac{{\gamma_0}^q}{q^2}\int_{\Omega_2}|\upsilon|^q\mathrm{d}x,
\end{aligned}
\tag{3.31}
$$

and

$$
\begin{aligned}
&\int_{\Omega_1}|\nabla\varpi_{k_0}|^p\mathrm{d}x-\int_{\Omega_1}|\varpi_{k_0}|^q\ln|\varpi_{k_0}|\mathrm{d}x\\
<&-{\gamma_0}^p\int_{\Omega_2}|\nabla\upsilon|^p\mathrm{d}x+{\gamma_0}^q\ln\gamma_0\int_{\Omega_2}|\upsilon|^q\mathrm{d}x+{\gamma_0}^q\int_{\Omega_2}|\upsilon|^q\ln|\upsilon|\mathrm{d}x.
\end{aligned}
\tag{3.32}
$$

Extend ϖ_{k_0} *and* υ *to be* 0 *in* $\Omega\setminus\Omega_1$ *and* $\Omega\setminus\Omega_2$*, respectively, and denote them, respectively, by* $\widetilde{\varpi}$ *and* $\widetilde{\upsilon}$*. Thus,* $\widetilde{\varpi},\widetilde{\upsilon}\in W_0^{1,p}(\Omega)$*. Let* $u_0=\widetilde{\varpi}+\gamma_0\widetilde{\upsilon}$*. By* (3.31) *and* (3.32)*, we obtain* $J(u_0)=J(\widetilde{\varpi})+J(\gamma_0\widetilde{\upsilon})=M$ *and* $u_0\in\mathcal{N}_-$*. Besides,* (3.29) *shows that*

$$
\begin{aligned}
\|u_0\|_*^2=\|\widetilde{\varpi}+\gamma_0\widetilde{\upsilon}\|_*^2=&\int_{\Omega}\left|\frac{\widetilde{\varpi}+\gamma_0\widetilde{\upsilon}}{|x|^{\frac{s}{2}}}\right|^2+|\widetilde{\varpi}+\gamma_0\widetilde{\upsilon}|^2\mathrm{d}x\\
\geq&\int_{\Omega_1}\left|\frac{\widetilde{\varpi}}{|x|^{\frac{s}{2}}}\right|^2+|\widetilde{\varpi}|^2\mathrm{d}x+\int_{\Omega_2}\left|\frac{\gamma_0\widetilde{\upsilon}}{|x|^{\frac{s}{2}}}\right|^2+|\gamma_0\widetilde{\upsilon}|^2\mathrm{d}x\\
\geq&\int_{\Omega_2}\left|\frac{\gamma_1\upsilon}{|x|^{\frac{s}{2}}}\right|^2+|\gamma_1\nabla\upsilon|^2\mathrm{d}x\\
>&\frac{M}{C_0}=\frac{J(u_0)}{C_0}.
\end{aligned}
$$

In accordance with Theorem 3.1, it is seen that, when the positive initial energy is arbitrary, the solution $u(x,t)$ *to problem* (1.1) *with such* $u_0=\widetilde{\varpi}+\gamma_0\widetilde{\upsilon}$ *as initial datum blows up in finite time.*

In general, it is seldom possible to obtain the explicit blow-up time when blow-up occurs. Therefore, estimating the blow-up time from both above and below is of considerable importance. In practice, a lower bound is often more useful, as it provides a safe time interval for the system under consideration. At the end of this section, we will derive a lower bound for the lifespan of solutions to problem (1.1), provided that blow-up does occur.

Theorem 3.2. *Let all the assumptions in Theorem 3.1 hold and* $q<p(1+\frac{2}{n})$*. Then the maximal existence time satisfies*

$$
T_{max}\geq\frac{\|u_0\|_*^{2(1-\kappa)}}{\widetilde{C}'(\kappa-1)}.
$$

where $\widetilde{C}'$ *and* κ *are two positive constants that will be determined in the proof.*

Proof. From the assumptions in Theorem 3.1, we know that $I(u(t)) < 0$ for all $t \in [0, T_{max})$. Then, we have

$$\|\nabla u(t)\|_p^p < \int_\Omega |u(t)|^q \ln |u(t)| \mathrm{d}x \leq \frac{1}{e\sigma}\|u(t)\|_{q+\sigma}^{q+\sigma}, \ t \in [0, T_{max}). \tag{3.33}$$

By using Lemma 2.4 and (3.33), we obtain

$$\begin{aligned}\|u(t)\|_{q+\sigma}^{q+\sigma} &\leq C_{G,2}\|\nabla u(t)\|_p^{\alpha(q+\sigma)}\|u(t)\|_2^{(1-\alpha)(q+\sigma)} \\ &< C_{G,2}\left(\frac{\|u(t)\|_{q+\sigma}^{q+\sigma}}{e\sigma}\right)^{\frac{\alpha(q+\sigma)}{p}}\left(\|u(t)\|_2^2\right)^{\frac{(1-\alpha)(q+\sigma)}{2}} \\ &< C_{G,2}\left(\frac{\|u(t)\|_{q+\sigma}^{q+\sigma}}{e\sigma}\right)^{\frac{\alpha(q+\sigma)}{p}}\left[(\mathrm{diam}(\Omega))^s\|u(t)\|_*^2\right]^{\frac{(1-\alpha)(q+\sigma)}{2}} \\ &\triangleq \widetilde{C}\left(\|u(t)\|_{q+\sigma}^{q+\sigma}\right)^{\frac{\alpha(q+\sigma)}{p}}\left(\|u(t)\|_*^2\right)^{\frac{(1-\alpha)(q+\sigma)}{2}}.\end{aligned}$$

This means that

$$\left(\|u(t)\|_{q+\sigma}^{q+\sigma}\right)^{1-\frac{\alpha(q+\sigma)}{p}} < \widetilde{C}\left(\|u(t)\|_*^2\right)^{\frac{(1-\alpha)(q+\sigma)}{2}}, \tag{3.34}$$

where $\widetilde{C} = C_{G,2}(\frac{1}{e\sigma})^{\frac{\alpha(q+\sigma)}{p}}(\mathrm{diam}(\Omega))^{\frac{s(1-\alpha)(q+\sigma)}{2}}$, $\alpha = (\frac{1}{2} - \frac{1}{q+\sigma})(\frac{1}{2} + \frac{1}{n} - \frac{1}{p})^{-1}$, $\mathrm{diam}(\Omega) > 0$ is the diameter of Ω. Since $q < p(1 + \frac{2}{n})$, it is clear that there is a sufficiently small $\sigma > 0$ such that

$$1 - \frac{\alpha(q+\sigma)}{p} > 0,$$

$$\kappa \triangleq \frac{(1-\alpha)(q+\sigma)/2}{1-\alpha(q+\sigma)/p} > 1.$$

Thus,

$$\begin{aligned}\frac{d}{dt}\|u(t)\|_*^2 = -2I(u(t)) &< \frac{2}{e\sigma}\|u(t)\|_{q+\sigma}^{q+\sigma} \\ &< \frac{2}{e\sigma}\widetilde{C}^{\frac{1}{1-\alpha(q+\sigma)/p}}\left(\|u(t)\|_*^2\right)^\kappa \\ &\triangleq \widetilde{C}'\left(\|u(t)\|_*^2\right)^\kappa, \ t \in [0, T_{max}),\end{aligned} \tag{3.35}$$

where $\widetilde{C}' = \frac{2}{e\sigma}\widetilde{C}^{\frac{1}{1-\alpha(q+\sigma)/p}}$. According to the negativity of $I(u(t))$ on $[0, T_{max})$, we can conclude that $\|u(t)\|_*^2 > 0$ for $t \in [0, T_{max})$. Then, dividing both sides of (3.35) by $\left(\|u(t)\|_*^2\right)^\kappa$ and integrating the resulting inequality over $[0, t)$, we derive

$$\frac{1}{1-\kappa}\left[\left(\|u(t)\|_*^2\right)^{1-\kappa} - \left(\|u_0\|_*^2\right)^{1-\kappa}\right] \leq \widetilde{C}'t. \tag{3.36}$$

Theorem 3.1 and Definition 2.3 imply that $\lim\limits_{t\to T_{\max}} \|u(t)\|_*^2 = \infty$. Letting $t \to T_{\max}$ in inequality (3.36) and recalling that $\kappa > 1$, we have

$$T_{max} \geq \frac{\|u_0\|_*^{2(1-\kappa)}}{\widetilde{C}'(\kappa - 1)}.$$

The proof is complete. □

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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