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Research article

$\phi-$Laplacian second-order boundary value problems with parameters

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Abstract: This paper deals with a second-order fully differential equation involving a $\phi-$Laplacian operator, which generalizes the more common p-Laplacian, with a parameter, applied to a nonlinearity depending on the unknown function and its first derivative. The main result states the problem's solvability for parameter values for which there are lower and upper solutions for the problem. Singular $\phi-$Laplacians considered with an application to the special relativity field.

Keywords: lower and upper solutions; Nagumo condition; regular and singular $\phi-$Laplacian equations; fixed-point theory

1. Introduction

This work studies the existence of solutions for the one-dimensional ϕ-Laplacian problem composed by the differential equation

$$(\phi(u'(t)))' = \lambda g\left(t, u\left(t\right), u'\left(t\right)\right) + f\left(t, u\left(t\right), u'(t)\right),\ \lambda \in \mathbb{R}, t \in (0,1), \tag{1}$$

together with the Sturm-Liouville boundary conditions

$$\begin{aligned} au(0) + bu'(0) &= A, \\ cu(1) + du'(1) &= B, \end{aligned} \tag{2}$$

where ϕ is an increasing homeomorphism with $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$, $g : [0,1] \times (0, +\infty)^2 \to \mathbb{R}^+$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are L^1-Carathéodory functions, λ is a parameter, $a, b, c, d, A, B \in \mathbb{R}$ with $a^2 + b^2 > 0$ and $c^2 + d^2 > 0$.

The so-called one-dimensional $\phi-$Laplacian equations generalize the classical Laplacian and p-Laplacian operators on strongly nonlinear problems and may be applied to several real phenomena. As examples, we refer to: [1], for the existence of a positive periodic solution for a singular φ-Laplacian Liénard equation; [2], proving a multiplicity result to a p-Laplacian Dirichlet problem with discontinuities; [3], applied to the one-dimensional superlinear semipositone p-Laplacian problem; [4], to obtain the existence of three positive solutions of the one-dimensional p-Laplacian problem; [5, 6], showing positive solutions for a superlinear p-Laplacian problem; [7], which used Krasnosel'skii expansion and compression cone theory to nonnegative nonlinearities; [8], related to the multiplicity of solutions for finite graphs; and [9], with impulsive coupled systems with two types of Laplacians operators.

Recently, several authors studied the singular ϕ-Laplacians, that is, homeomorphisms $\phi : (-r, r) \to \mathbb{R}$ with $0 < r < +\infty$. See: [10, 11] in reference to the p-Laplacian operator; [12–14], proving the existence and multiplicity of solutions; [15–17], which obtained the existence of heteroclinic solutions; [18], which defined an unbounded domain with functional boundary conditions; and [19], related to decay results in finite dimension. To the best of our knowledge, this is the first paper where the method of lower and upper solutions is applied to a $\phi-$Laplacian equation with parameters to discuss the solvability of the problem for some values of the parameters. In both cases, we have regular and singular problems involving $\phi-$Laplacian fully differential equations. The paper is organized as follows: In Section 2, we present some preliminary results such as definitions, the functional framework, a Nagumo-type growth condition and *a priori* bound for the first derivative, and the explicit solution for the associated linear problem. Section 3 contains an existence and localization for the values of the parameters where there are lower and upper solutions of the problem. Section 4 presents an example to underline the technique used to estimate the parameter values for which there is, at least, a solution. The last section applies our method to a singular $\phi-$Laplacian equation with an application to the theory of special relativity.

2. Preliminary results

In this section, we present some of the definitions and auxiliary results needed for the proof of our main result. Consider the space $X = C^1([0, 1])$ with the norm

$$\|u\|_X = \max\{\|u\|_\infty, \|u'\|_\infty\}$$

and with the norm

$$\|w\|_\infty = \max_{t\in[0,1]} |w(t)|.$$

The nonlinearities may have discontinuities but must have controlled growth via a Nagumo-type condition.

Definition 1. *A function $h : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a L^1-Carathéodory function if*

i. for each $(y_0, y_1) \in \mathbb{R}^2$, $t \mapsto h(t, y_0, y_1)$ is measurable on $[0, 1]$;
ii. for almost every $t \in [0, 1]$, $(y_0, y_1) \mapsto h(t, y_0, y)$ is continuous on $\mathbb{R}^2$;
iii. for each $L > 0$, there is a positive function $\rho_L \in L^1([0, 1])$ such that, for a.e., $t \in [0, 1]$, and $(y_0, y_1) \in \mathbb{R}^2$ with

$$\max\{|y_0|, |y_1|\} < L,$$

we have

$$|h(t, y_0, y_1)| \le \rho_L(t). \tag{3}$$

Definition 2. *Let $\alpha(t)$ and $\beta(t)$ be continuous functions such that*

$$\alpha(t) \le \beta(t), \forall t \in [0,1],$$

and consider the set

$$E = \left\{(t,x,y) \in [0,1] \times \mathbb{R}^2 : \alpha(t) \le x \le \beta(t)\right\}. \tag{4}$$

The L^1-Carathéodory function $h : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies a Nagumo-type condition in E if there is a continuous positive function $w : [0,+\infty[\to [0,+\infty[$, verifying

$$|h(t,x,y)| \le w(|y|), \forall (t,x,y) \in E, \tag{5}$$

such that

$$\int_{\phi(\eta)}^{+\infty} \frac{\tau}{w(\tau)} d\tau = +\infty, \tag{6}$$

for

$$\eta := \max\{\beta(t) - \alpha(\tau), t, \tau \in [0,1]\}. \tag{7}$$

Remark 3. *For convenience, define*

$$h_\lambda(t, u(t), u'(t)) := \lambda g(t, u(t), u'(t)) + f(t, u(t), u'(t)). \tag{8}$$

Since the functions f and g are L^1-Carathéodory functions, then h_λ is also an L^1-Carathéodory function, as long as λ belongs to a bounded set. Moreover, if f and g verify a Nagumo-type condition, then h_λ verifies a Nagumo-type condition, too, for finite λ.

This growth condition allows us to obtain an *a priori* estimation on the derivative.

Lemma 4. *Let $h_\lambda : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be an L^1-Carathéodory function satisfying a Nagumo-type condition, according to Definition 2 in E.*
Then there exists $N > 0$ (depending only on α, β, and w), such that every solution of

$$(\phi(u'(t)))' = h_\lambda(t, u(t), u'(t)) \tag{9}$$

in E satisfies

$$\|u'\|_\infty < N.$$

Proof. Let $(t, u, u') \in E$ and u be the solution of (9). Consider $N > 0$, sufficiently large, such that $N > \eta$ and

$$\int_{\phi(\eta)}^{\phi(N)} \frac{\left|\phi^{-1}(\tau)\right|}{w\left(\left|\phi^{-1}(\tau)\right|\right)} d\tau > \max_{t \in [0,1]} \beta(t) - \min_{t \in [0,1]} \alpha(t).$$

If $|u'(t)| \le \eta$, $\forall t \in [0,1]$, then it is enough to define $\eta := N$ and the proof is complete. By the mean value theorem, there are $t_0 \in]0,1[$ such that

$$u'(t_0) = u(1) - u(0). \tag{10}$$

The case $|u'(t)| > \eta$, $\forall t \in [0,1]$, is not possible. In fact, if $u'(t) > \eta$, $\forall t \in [0,1]$, we obtain, by (10), (4), (7), and the contradiction

$$u'(t_0) = u(1) - u(0) \le \beta(1) - \alpha(0) \le \eta.$$

If $u'(t) < -\eta, \forall t \in [0,1]$, the contradiction is similar. We assume that are $t_1, t_2 \in]0,1[$ with $t_1 < t_2$, such that

$$u'(t_1) \leq \eta \text{ and } u'(t_2) > \eta .$$

By continuity of u', there is $t_3 \in [t_1, t_2[$ such that $u'(t_3) = \eta$.

Consider $N > \eta$ such that

$$\int_{\phi(\eta)}^{\phi(N)} \frac{\left|\phi^{-1}(\tau)\right|}{w\left(\left|\phi^{-1}(\tau)\right|\right)} d\tau > \eta. \tag{11}$$

Making a convenient change of variable, using (5), (4), and (7),

$$\begin{aligned}
\int_{\phi(u'(t_3))}^{\phi(u'(t_2))} \frac{\left|\phi^{-1}(\tau)\right|}{w\left(\left|\phi^{-1}(\tau)\right|\right)} d\tau &= \int_{t_3}^{t_2} \frac{\left|\phi^{-1}(\phi(u'(s)))\right|}{w\left(\left|\phi^{-1}(\phi(u'(s)))\right|\right)} (\phi(u'(s)))' \, ds \\
&= \int_{t_3}^{t_2} \frac{|u'(s)|}{w(|u'(s)|)} (\phi(u'(s)))' \, ds \\
&= \int_{t_3}^{t_2} \frac{|u'(s)|}{w(|u'(s)|)} |h_\lambda(s, u(s), u'(s))| \, ds \\
&\leq \int_{t_3}^{t_2} |u'(s)| \, ds = \int_{t_3}^{t_2} u'(s) \, ds \\
&= u(t_2) - u(t_3) \leq \beta(t_2) - \alpha(t_3) \leq \eta,
\end{aligned}$$

and by (11),

$$\int_{\phi(u'(t_3))}^{\phi(u'(t_2))} \frac{\left|\phi^{-1}(\tau)\right|}{w\left(\left|\phi^{-1}(\tau)\right|\right)} d\tau \leq \eta < \int_{\phi(\eta)}^{\phi(N)} \frac{\left|\phi^{-1}(\tau)\right|}{w\left(\left|\phi^{-1}(\tau)\right|\right)} d\tau.$$

As $\phi(u'(t_3)) = \phi(\eta)$, then $\phi(u'(t_2)) < \phi(N)$, and $u'(t_2) < N$. Moreover, t_2 is taken arbitrarily, and then $u'(t) < N$ for the values whenever $u'(t) > \eta$.

The case for $t_1 > t_2$ follows similar arguments.

The other possible case where

$$u'(t_1) > -\eta \text{ and } u'(t_2) < -\eta$$

can be proved by previous techniques. Therefore, $\|u'\|_\infty < N$.

For the corresponding linear problem, we can evaluate the explicit solution.

Lemma 5. *Let $h_\lambda^* : [0,1] \mapsto \mathbb{R}$ be an L^1-Carathéodory function, with λ a fixed real parameter and $C, D \in \mathbb{R}$.*
The problem composed by the differential equation

$$(\phi(u'(t)))' = h_\lambda^*(t) \tag{12}$$

and the boundary conditions

$$\begin{aligned} u(0) &= C, \\ u'(1) &= D, \end{aligned} \tag{13}$$

has a unique solution, for every fixed $\lambda \in \mathbb{R}$, given by

$$u(t) = C + \int_0^t \phi^{-1}\left(\phi(D) - \int_s^1 h_\lambda^*(\xi)d\xi\right)ds.$$

Proof. Integrating (12), in $[s, 1]$, we have

$$u'(t) = \phi^{-1}\left(\phi(D) - \int_s^1 h_\lambda^*(s)ds\right),$$

and, by a new integration from 0 to t,

$$u(t) = C + \int_0^t \phi^{-1}\left(\phi(D) - \int_s^1 h_\lambda^*(\xi)d\xi\right)ds.$$

In this work, the lower and upper functions are defined as follows:

Definition 6. *Given $A, B, a, b, c, d \in \mathbb{R}$, a function $\alpha \in C^2[0, 1]$ with $\phi(\alpha') \in AC[0, 1]$ is a lower solution of problem (1) and (2), if*

$$\begin{cases} (\phi(\alpha'(t)))' \geq \lambda g(t, \alpha(t), \alpha'(t)) + f(t, \alpha(t), \alpha'(t)), \ t \in (0, 1), \\ a\alpha(0) + b\alpha'(0) \geq A, \\ c\alpha(1) + d\alpha'(1) > B. \end{cases} \tag{14}$$

A function $\beta \in C^2[0, 1]$ with $\phi(\beta') \in AC[0, 1]$ is an upper solution of problem (1) and (2) if the inverse inequalities hold.

Going forward, we will use the following lemma, given in [20]:

Lemma 7. *For $v, w \in C(I)$ such that $v(x) \leq w(x)$, for every $x \in I$, define*

$$q(x, u) = max\{v, min\{u, w\}\}.$$

Then, for each $u \in C^1(I)$, the next two properties hold:

(a) $\frac{d}{dx}q(x, u(x))$ exists for a.e., $x \in I$.
(b) If $u, u_m \in C^1(I)$ and $u_m \to u$ in $C^1(I)$, then

$$\frac{d}{dx}q(x, u_m(x)) \to \frac{d}{dx}q(x, u(x)) \textit{ for a.e., } x \in I.$$

The main argument for the existence part will be Schauder's fixed point theorem:

Theorem 8. *([21]) Let Y be a nonempty, closed, bounded, and convex subset of a Banach space, and suppose that $P : Y \to Y$ is a compact operator. Then P has at least one fixed point in Y.*

3. Existence and localization result

Going forward, we consider the following assumptions:

(H$_1$) $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$, and

$$\left|\phi^{-1}(x)\right| \leq \phi^{-1}(|x|).$$

The main result will be an existence and localization theorem:

Theorem 9. *Let $A, B, a, b, c, d \in \mathbb{R}$, and a homeomorphism ϕ verifies (H1). Assume that there are lower and upper solutions of (1) and (2), α and β, respectively, such that*

$$\alpha(t) \leq \beta(t), \forall t \in [0, 1],$$

and the L^1−Carathéodory functions $f, g : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ satisfy Nagumo conditions as in Definition 2, in the set

$$S_* = \{(t, y_0, y_1) \in [0, 1] \times \mathbb{R}^2 : \alpha(t) \leq y_0 \leq \beta(t)\}.$$

Then, there is at least a solution $u \in C^1([0, 1])$ of (1) and (2) and, moreover,

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \forall t \in [0, 1],$$

and

$$\|u'\|_\infty \leq N,$$

with N given by Lemma 4.

Proof. Define the truncation function $\delta : [0, 1] \times \mathbb{R} \to \mathbb{R}$, given by

$$\delta(t, w) = \begin{cases} \beta(t) & \text{if} \quad w > \beta(t), \\ w & \text{if} \quad \alpha(t) \leq w \leq \beta(t), \\ \alpha(t) & \text{if} \quad w < \alpha(t). \end{cases} \tag{15}$$

Consider the following modified problem composed by the truncated and perturbed differential equation

$$(\phi(u'(t)))' = \lambda g(t, \delta(t, u(t)), u'(t)) + f(t, \delta(t, u(t)), u'(t)) - \frac{\delta(t, u(t)) - u(t)}{1 + |\delta(t, u(t))) - u(t)|}, \tag{16}$$

with the truncated boundary conditions

$$\begin{cases} u(0) = a\, \delta(0, u(0)) + b\, \frac{d}{dt}\delta(t, w)\Big|_{(0,u(0))} - A + \delta(0, u(0)), \\ u'(1) = c\, \delta(1, u(1)) + d\, \frac{d}{dt}\delta(t, w)\Big|_{(1,u(1))} - B + \frac{d}{dt}\delta(t, w)\Big|_{(1,u(1))}. \end{cases} \tag{17}$$

Define the operator $T : C^1([0,1]) \to C^1([0,1])$ given by

$$(Tu)(t) = a\,\delta(0,u(0)) + b\,\frac{d}{dt}\delta(0,u(0)) - A + \delta(0,u(0))$$
$$+\int_0^t \phi^{-1}\left(\phi(c\,\delta(1,u(1)) + d\,\frac{d}{dt}\delta(1,u(1)) - B + \frac{d}{dt}\delta(1,u(1))) - \int_s^1 h_\lambda^*(\xi)d\xi\right)ds,$$

where
$$h_\lambda^*(t) := \lambda g(t,\delta(t,u(t)),u'(t)) + f(t,\delta(t,u(t)),u'(t)) - \frac{\delta(t,u(t)) - u(t)}{1 + |\delta(t,u(t))) - u(t)|}.$$

Note that, with this definition, the fixed points of T are solutions of the modified problem.

Let $(t,u,u') \in S_*$.

Therefore, by Lemma 4, there is $K_1 > 0$ such that

$$\|u\|_X \le K_1. \tag{18}$$

Notice that, as there are lower and upper solutions of (1), then λ is finite, and since f and g are L^1-Carathéodory functions then h_λ^* is also L^1-Carathéodory, that is, there is a non-negative function $\rho_{K_1} \in L^1([0,1])$, such that

$$\left|h_\lambda^*(t)\right| \le \rho_{K_1}(t),\ a.e.,\ t \in [0,1]. \tag{19}$$

Step 1: *T is well defined and uniformly bounded.*

For simplification, we denote

$$C := a\,\delta(0,u(0)) + b\,\frac{d}{dt}\delta(t,w)_{(0,u(0))} - A + \delta(0,u(0)),$$
$$D := c\,\delta(1,u(1)) + d\,\frac{d}{dt}\delta(t,w)_{(1,u(1))} - B + \frac{d}{dt}\delta(t,w)_{(1,u(1))}.$$

In fact, from the Lebesgue dominated convergence theorem, (H_1), and (19),

$$\begin{aligned}
|(Tu)(t)| &\le |C| + \int_0^t \left|\phi^{-1}\left(\phi(D) - \int_s^1 h_\lambda^*(\xi)d\xi\right)\right| ds \\
&\le |C| + \int_0^t \phi^{-1}\left(\left|\phi(D) - \int_s^1 h_\lambda^*(\xi)d\xi\right|\right) ds \\
&\le |C| + \int_0^t \phi^{-1}\left(|\phi(D)| + \int_s^1 \left|h_\lambda^*(\xi)\right| d\xi\right) ds \\
&\le |C| + \int_0^1 \phi^{-1}\left(|\phi(D)| + \int_s^1 \rho_{K_1}(\xi)\,d\xi\right) ds < +\infty,
\end{aligned}$$

and

$$\begin{aligned}
\left|(Tu)'(t)\right| &= \left|\phi^{-1}\left(\phi(D) - \int_t^1 h_\lambda^*(\xi)d\xi\right)\right| \\
&\le \phi^{-1}\left(|\phi(D)| + \int_0^1 \rho_{K_1}(\xi)\,d\xi\right) < +\infty.
\end{aligned}$$

Therefore, $Tu \in C^1([0,1])$. Define the set $M \subseteq C^1([0,1])$ as

$$M := \left\{u \in C^1([0,1]) : \|u\|_X \le K_2\right\},$$

with $K_2 > 0$ such that

$$K_2 := \max\left\{\begin{array}{c} K_1, \\ |C| + \int_0^1 \phi^{-1}\left(|\phi(D)| + \int_s^1 \rho_{K_1}(\xi)\, d\xi\right) ds, \\ \phi^{-1}\left(|\phi(D)| + \int_0^1 \rho_{K_1}(\xi)\, d\xi\right) \end{array}\right\},$$

where K_1 is given by (18). Therefore, from above, TM is uniformly bounded and $TM \subset M$.

Step 2: *T is equicontinuous.*

Consider $t_1, t_2 \in [0,1]$ such that, without loss of generality, $t_1 \le t_2$. For $u \in M$, we get the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{t_1 \to t_2} |(Tu)(t_1) - (Tu)(t_2)| &\le \lim_{t_1 \to t_2} \left| \int_{t_1}^{t_2} \phi^{-1}\left(\phi(D) + \int_s^1 \left|h_\lambda^*(\xi)\right| d\xi\right) ds \right| \\ &\le \lim_{t_1 \to t_2} \left| \int_{t_1}^{t_2} \phi^{-1}\left(\phi(D) + \int_s^1 \rho_{K_1}(\xi) d\xi\right) \right| = 0, \end{aligned}$$

and

$$\begin{aligned} & \lim_{t_1 \to t_2} \left|(Tu)'(t_1) - (Tu)'(t_2)\right| \\ = \ & \lim_{t_1 \to t_2} \left| \phi^{-1}\left(\phi(D) + \int_{t_1}^1 h_\lambda^*(\xi) d\xi\right) - \phi^{-1}\left(\phi(D) - \int_{t_2}^1 h_\lambda^*(\xi) d\xi\right) \right| \\ = \ & 0. \end{aligned}$$

Therefore, TM is equicontinuous on $C^1([0,1])$, and, by the Arzèla-Ascoli theorem, T is a compact operator. From Schauder's fixed point theorem, the operator T has a fixed point, u_*, which is a solution of the modified problem (16) and (17).

Step 3: *This fixed point, u_*, is a solution of (1) and (2).*

By (15), to prove this claim, it is enough to show that

$$\alpha(t) \le u_*(t) \le \beta(t), \quad \forall t \in [0,1],$$

and in this case, the modified problem (16) and (17) is equal to the initial problem (1) and (2).

For the second inequality, assume, by contradiction, that there is $t \in [0,1]$ such that $u_*(t) > \beta(t)$, and define

$$\max_{t \in [0,1]} (u_*(t) - \beta(t)) := u_*(t_0) - \beta(t_0) > 0.$$

If $t_0 \in]0,1[$, then

$$u'_*(t_0) - \beta'(t_0) = 0. \tag{20}$$

By Definition 6,

$$\begin{aligned}
0 &= (\phi(u'_*(t_0)))' - (\phi(\beta'(t_0)))' = \lambda g(t_0, \delta(t_0, u_*(t_0)), u'_*(t_0)) + f(t_0, \delta(t_0, u_*(t_0)), u'_*(t_0)) \\
&\quad - \frac{\delta(t_0, u_*(t_0)) - u_*(t_0)}{1 + |\delta(t_0, u_*(t_0))) - u_*(t_0)|} - (\phi(\beta'(t_0)))' \\
&= \lambda g(t_0, \beta(t_0), \beta'(t_0)) + f(t_0, \beta(t_0), \beta'(t_0)) - \frac{\beta(t_0) - u_*(t_0)}{1 + [\beta(t_0) - u_*(t_0)]} - (\phi(\beta'(t_0)))' \\
&\geq \frac{u_*(t_0) - \beta(t_0)}{1 + [\beta(t_0) - u_*(t_0)]} > 0.
\end{aligned}$$

If $t_0 = 0$, then

$$\max_{t\in[0,1]} \{u_*(t) - \beta(t)\} := u_*(0) - \beta(0) > 0.$$

By (17) and (15), and Definition 6, we obtain the contradiction

$$\begin{aligned}
\beta(0) &< u_*(0) = a\,\delta(0, u(0)) + b\,\frac{d}{dt}\delta(t, w)_{(0,u(0))} - A + \delta(0, u(0)) \\
&= a\,\beta(0) + b\,\beta'(0) - A + \beta(0) \\
&\leq \beta(0).
\end{aligned}$$

If the maximum is attained at $t = 1$, then

$$\begin{aligned}
u_*(1) - \beta(1) &> 0, \\
u'_*(1) - \beta'(1) &\geq 0,
\end{aligned}$$

and, by Definition 6, this contradiction holds:

$$\begin{aligned}
\beta'(1) &\leq u'_*(1) = c\,\delta(1, u(1)) + d\frac{d}{dt}\delta(t, w)_{(1,u(1))} - B + \frac{d}{dt}\delta(t, w)_{(1,u(1))} \\
&= c\,\beta(1) + d\,\beta'(1) - B + \beta'(1) \\
&< \beta'(1).
\end{aligned}$$

Therefore, $u_*(t) \leq \beta(t), \forall t \in [0,1]$. The inequality $\alpha(t) \leq u_*(t), \forall t \in [0,1]$, follows similar arguments, and u_* is a solution of the initial problem (1) and (2).

4. Example

Consider the following example:

$$\begin{aligned}
3\,(u'(t))^2\,u''(t) &= \lambda\left(u'(t) - u^3(t)\right) - (u'(t))^2 - 5u(t), \; t \in (0,1), \\
u(0) - u'(0) &= 0, \\
-2u(1) + u'(1) &= -1.
\end{aligned} \tag{21}$$

The previous problem is a particular case of (16) and (2), with

$$\begin{aligned} a &= 1,\ b=-1,\ A=0, c=-2,\ d=1,\ B=-1, \\ g(t,y,z) &= z-y^3,\ f(t,y,z)=-5y-z^2. \end{aligned}$$

The homeomorphism $\phi:\mathbb{R}\mapsto\mathbb{R}$ is defined by $\phi(v)=v^3$, which trivially verifies the condition (H_1). The functions f and g are continuous and L^1−Carathéodory, as for $L>0$ such that

$$\max\{|y|,|z|\}<L,$$

we have

$$|g(t,y,z)|\le L+L^3, |f(t,y,z)|\le 5L+L^2, \forall t\in[0,1].$$

The functions $\alpha(t)=-2t^2-\frac{t}{2}-\frac{1}{2}$ and $\beta(t)=2t^2+2t+1$, respectively, lower, and upper solutions of problem (21) for $\lambda\in[-57;5.25]$. Moreover, the functions f and g satisfy a Nagumo condition relative to the set

$$S_*=\left\{(t,y,z)\in[0,1]\times\mathbb{R}^2: -2t^2-\frac{t}{2}-\frac{1}{2}\le y\le 2t^2+2t+1\right\}.$$

In fact, in S_*,

$$\begin{aligned} |g(t,y,z)| &\le 125+|z| := w_1(|z|), \\ |f(t,y,z)| &\le 25+z^2 := w_2(|z|), \end{aligned}$$

and the integral condition (6) holds, as

$$\int_0^{+\infty}\frac{\sqrt[3]{|\tau|}}{w_1\left(\sqrt[3]{|\tau|}\right)}d\tau=\int_0^{+\infty}\frac{\sqrt[3]{|\tau|}}{125+\sqrt[3]{|\tau|}}d\tau=+\infty,$$

and

$$\int_0^{+\infty}\frac{\sqrt[3]{|\tau|}}{w_2\left(\sqrt[3]{|\tau|}\right)}d\tau=\int_0^{+\infty}\frac{\sqrt[3]{|\tau|}}{25+\left(\sqrt[3]{|\tau|}\right)^2}d\tau=+\infty.$$

By Theorem 9, there is a solution u^* of problem (21), for every $\lambda\in[-57;5.25]$. Moreover,

$$-2t^2-\frac{t}{2}-\frac{1}{2}\le u^*(t)\le 2t^2+2t+1, \forall t\in[0,1],$$

and there exists $N\ge 1$, such that

$$\|u^{*\prime}\|_\infty<N.$$

Note that the trivial null function is not a solution to problem (21).

5. Singular ϕ-Laplacian equations and applications

The study carried out in this paper can be applied to singular ϕ-Laplacian equations with two restrictions:

- Assumption (H_1) must be replaced by:

(Hs) $\phi : (-\eta, \eta) \to \mathbb{R}$, for some $0 < \eta < +\infty$, is an increasing homeomorphism with $\phi(0) = 0$, $\phi(-\eta, \eta) = \mathbb{R}$, such that

$$\left|\phi^{-1}(w)\right| \leq \phi^{-1}(|w|).$$

- In Lemma 4, the constant N must be chosen such that

$$0 < N < \eta.$$

Therefore, a solution to the problem (1) and (2) is a function $u(x) \in X = C^1([a, b])$ such that $u'(x) \in (-\eta, \eta)$ for all $x \in [a, b]$, satisfying (1) and (2). These singular ϕ-Laplacian operators are useful in the theory of special relativity, which has been crucial in the development of the modern theory of classical electrodynamics. The fact that an electric charge q generates an electric field $\mathbf{E}$ and in motion generates a magnetic field $\mathbf{B}$ is intuitively compatible with the statement that the electric and magnetic fields are covariant under a Lorentz transformation from one inertial system to another. For more details, see [22]. Relativity implies that physical laws do not depend on the chosen reference frame. In special relativity, the speed of light c is recognized as the maximum speed with which information can travel in free space from one frame of reference to another, [23]. Consider two frames of reference $\mathcal{P}_0$ and $\mathcal{P}$ in uniform relative motion to each other, that is, moving with relative speed v. Taking into account the upper limit c of the speed of information propagation, the space-time coordinates of the frames $\mathcal{P}_0$ and $\mathcal{P}$ must be related by *Lorentz transformations*, [24]. The Lorentz factor depends nonlinearly on the relative velocity v and is defined by

$$\Gamma \equiv \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}.$$

Motivated by this physical frame, we consider the problem

$$\begin{cases} \left(\dfrac{u'(x)}{\sqrt{1 - (u'(x))^2}}\right)' = \lambda\left(-2u(x) + \dfrac{5}{22}(u'(x))^2\right) + 2\,(u\,(x))^3 - 2u'(x), \\ x \in (-1, 1)\,, \end{cases} \tag{22}$$

with the boundary conditions

$$u(1) = 0, \; u'(-1) = \frac{1}{3}. \tag{23}$$

Problem (1) and (2) is defined in the normalized interval $[0, 1]$, but a similar result can be derived for a generic interval $[a, b]$. So, problem (22) and (23) can be considered as a particular case of problem (1) and (2) in $[-1, 1]$:

$$\phi(z) = \frac{z}{\sqrt{1 - z^2}}, \tag{24}$$

$$g(x, y, z) = -2y + \frac{5}{22}z^2, \text{ and } f(x, y, z) = 2y^3 - z.$$

As the function $\phi : (-1, 1) \to \mathbb{R}$, given in (24), is an increasing homeomorphism such that $\phi(0) = 0$, $\phi(-1, 1) = \mathbb{R}$,

$$\phi^{-1}(w) = \frac{w}{\sqrt{1 + w^2}},$$

then assumption (Hs) holds.

Note that, we consider the particular case of boundary conditions (2) with $a = d = B = 0$, $A = \frac{1}{3}$, $b > 0$, and $c < 0$ arbitrary. In this case, the functions $\alpha : [-1, 1] \to \mathbb{R}$, given by

$$\alpha(x) = \frac{9}{10}x - 1,$$

$\beta : [-1, 1] \to \mathbb{R}$,

$$\beta(x) = -\frac{1}{3}x + \frac{2}{3},$$

are, respectively, lower and upper solutions of problem (22) and (23), according to Definition 6, for $\lambda \in [-1.28; 1.74]$.

The functions f and g satisfy Nagumo conditions relative to the set

$$S = \left\{(x, y, z) \in [-1, 1] \times \mathbb{R}^2 : \frac{9}{10}x - 1 \le y \le -\frac{1}{3}x + \frac{2}{3}\right\},$$

as, in S,

$$\begin{aligned} |g(x, y, z)| &= \left|-2y + \frac{5}{22}z^2\right| \le 4 + z^2 := w_1(|z|), \\ |f(x, y, z)| &= \left|2y^3 - z\right| \le 16 + |z| := w_2(|z|), \end{aligned}$$

$$\int_0^{+\infty} \frac{|\phi^{-1}(s)|}{w_1\left(|\phi^{-1}(s)|\right)}ds = \int_0^{+\infty} \frac{|s|}{4\sqrt{1+s^2} + |s|}ds = +\infty,$$

and

$$\int_0^{+\infty} \frac{|\phi^{-1}(s)|}{w_2\left(|\phi^{-1}(s)|\right)}ds = \int_0^{+\infty} \frac{\left(1 + s^2\right)|s|}{(4 + 5s^2)\sqrt{1+s^2}}ds = +\infty.$$

So, by Theorem 9, there is at least a solution $u(x) \in C^1([-1, 1])$ of problem (22) and (23) for $\lambda \in [-1.28; 1.74]$, and, moreover,

$$\frac{9}{10}x - 1 \le u(x) \le -\frac{1}{3}x + \frac{2}{3}, \quad \forall x \in [-1, 1].$$

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Author contributions

All authors contributed to the study conception and design, as well as read and approved the final manuscript.

Conflict of interest

The authors have no relevant financial or non-financial interests to disclose. All authors declare no conflicts of interest in this paper.

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