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A global stability result for incompressible magnetohydrodynamics

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Abstract: We propose a result of global stability for the equations of homogeneous, incompressible magnetohydrodynamics (MHD) on a torus of any dimension $d \in \{2, 3, ...\}$, with positive viscosity and resistivity. This result applies to the C^∞ global solutions, with a conveniently defined decay property for large times; it is expressed by fully explicit estimates, formulated via H^p-type Sobolev norms of arbitrarily high order p. The present stability result is similar to that proposed by one of us for the Navier-Stokes (NS) equation (L. P., Appl. Math. Lett., 2021); it is derived from a suitable formulation of the MHD equations proposed in a previous work of ours (L. P. and E. T., Nonlinear Anal., 2020), emphasizing strong structural analogies with the NS case. A basic tool in the proof of the present stability result is a general theory of approximate solutions of the MHD Cauchy problem, which we developed in the above mentioned work on the grounds of previous results on the NS equation (C. Morosi and L. P., Nonlinear Anal., 2015) and of the above structural similarities. We also introduce a class of Beltrami-type initial data for the MHD equations; although being arbitrarily large, these data produce global and decaying MHD solutions, fitting the framework of the present stability result. Comparisons with the previous literature on these subjects are performed.

Keywords: NS and MHD equations; existence and regularity theory; global stability; Beltrami vector fields

1. Introduction

State of the art on global stability of the homogeneous, incompressible (NS and) MHD equations. The equations of magnetohydrodynamics (MHD) are deeply related to the Navier-Stokes (NS) equation. The homogeneous, incompressible NS and MHD equations in $d \geqslant 2$ space dimensions read,

respectively:

$$\frac{\partial u}{\partial t} = \nu\Delta u - (u\bullet\nabla)u - \nabla p\ ; \tag{1.1}$$

$$\frac{\partial u}{\partial t} = \nu\Delta u - (u\bullet\nabla)u + (b\bullet\nabla)b - \nabla(p + \frac{1}{2}|b|^2)\ , \qquad \frac{\partial b}{\partial t} = \eta\Delta b - (u\bullet\nabla)b + (b\bullet\nabla)u\ . \tag{1.2}$$

Both in (1.1) and in (1.2), one considers a d-dimensional spatial domain Ω, which can be either $\mathbf{R}^d$, or a domain in $\mathbf{R}^d$, or the torus $\mathbf{T}^d$ (see the forthcoming Eq (2.1)). Moreover: u and b are, respectively, the velocity and magnetic fields, two *divergence-free* vector fields depending on time t and on the space variables $x = (x_1, ..., x_d) \in \Omega$; $\nabla = (\partial_1, ..., \partial_d)$ is the gradient and, for each vector field $v : \Omega \to \mathbf{R}^d$, $v\bullet\nabla$ is the differential operator $\sum_{s=1}^d v_s\partial_s$; $p = p(x,t)$ is the pressure; Δ is the Laplacian; the constants $\nu, \eta \geqslant 0$ are the viscosity and the resistivity, respectively; the above equations assume the use of units in which the constant density of the fluid equals unity (*). Needless to say, for $b := 0$ the second equation in (1.2) is fulfilled, and the first equation in (1.2) takes the form (1.1).

In the sequel of this paper the adjectives "homogeneous" and "incompressible" will always be omitted, and we will simply refer to the NS and MHD equations.

As well known, after fixing a precise functional setting for the vector fields on Ω one can introduce the Leray projection $\mathfrak{L}$ onto the space of divergence free vector fields, which annihilates gradients; after application of this projection the NS and MHD equations (1.1) and (1.2) become, respectively,

$$\frac{\partial u}{\partial t} = \nu\Delta u - \mathfrak{L}((u\bullet\nabla)u)\,; \tag{1.3}$$

$$\frac{\partial u}{\partial t} = \nu\Delta u - \mathfrak{L}((u\bullet\nabla)u) + \mathfrak{L}((b\bullet\nabla)b)\ , \qquad \frac{\partial b}{\partial t} = \eta\Delta b - \mathfrak{L}((u\bullet\nabla)b) + \mathfrak{L}((b\bullet\nabla)u)\ . \tag{1.4}$$

The main advantage of using $\mathfrak{L}$ is the elimination of the pressure term p that was present in Eq (1.1) and in the first equation (1.2). The application of $\mathfrak{L}$ to the second equation (1.2) may appear to be pleonastic, since the right hand side of this second equation is automatically divergence free (†) ; however $\mathfrak{L}$ is useful even in this case, for reasons on which we return later.

Equation (1.3) is an evolution equation for the velocity field u, while Eq (1.4) gives a pair of evolution equations for the velocity and magnetic fields $\mathbf{u} = (u, b)$. It has been known for a long time that, sticking again to a suitable functional setting, one can prove the existence of a unique solution of maximal time domain for the Cauchy problems associated to (1.3) or to (1.4), for each specification of sufficiently regular initial data u_0 or $\mathbf{u}_0 = (u_0, b_0)$ (concerning the Cauchy problems and some related results, see [1–7] and the books [8, 9] as for the NS case, and [10–13] as for the MHD case; some of these references are reviewed or commented in our works [14, 15]). Here and in the sequel, the term "solution" is always intended in the strong, or even in the classical sense; the time domain of any solution is an interval $[0, T)$ with $T \in (0, +\infty]$, and the solution is called global if $T = +\infty$.

For $d = 2$ it is known that, in appropriate functional settings, each initial datum gives rise to a global (strong) solution, both in the case of the NS equation (1.3) with $\nu \geqslant 0$ [16], and in the case of the MHD

*For $d = 3$, one can write Eqs (1.1) and (1.2) in a more familiar form using the identities $(u\bullet\nabla)u = \nabla(|u|^2/2) + (\mathrm{rot}\, u) \wedge u$, $(b\bullet\nabla)b = \nabla(|b|^2/2) + (\mathrm{rot}\, b) \wedge b$ and $-(u\bullet\nabla)b + (b\bullet\nabla)u = \mathrm{rot}(u \wedge b)$ (with $\wedge$ the usual vector product; the first two identities hold for all vector fields u, b on Ω, the third one holds for all divergence free vector fields u, b). Since Eqs (1.2) have the usual form for $d = 3$ but make sense for any d, we keep the names "MHD equations", "velocity field", "magnetic field" for d arbitrary; we are aware that, according to first physical principles, a d-dimensional magnetic field should be represented as a 2-form, which is geometrically equivalent to a vector field only for $d = 3$.

†The proof is as in the subsequent footnote ††, with the vector fields v_0 and c_0 considered therein replaced, respectively, by u and b.

equations (1.4) with $\nu, \eta > 0$ [13]. For $d \geqslant 3$ and $\nu, \eta > 0$, the existence of global strong solutions is known for both the NS and MHD equations in the case of small initial data, and is an open problem for arbitrary data (it is almost unnecessary to recall the Millennium problem concerning global existence for the NS equation on $\mathbf{R}^3$ or $\mathbf{T}^3$ with arbitrary initial data, in an appropriate C^∞ functional setting [17]).

Let us come to the issue of global stability. If v_0 is an initial datum for the NS equation (1.3) giving rise to a global solution v, a global stability result for this datum is a statement ensuring that, for any initial datum u_0 sufficiently close to v_0, the corresponding solution u is global as well; a statement of this kind is typically accompanied by an estimate, indicating that the difference $u(t) - v(t)$ at any time $t \in [0, +\infty)$ is somehow controlled by the data difference $u_0 - v_0$. Global stability results for the MHD equations (1.4) have a similar structure, but of course the initial data and the corresponding solutions are pairs $\mathbf{v}_0 = (v_0, c_0)$, $\mathbf{v} = (v, c)$, $\mathbf{u}_0 = (u_0, b_0)$ and $\mathbf{u} = (u, b)$.

Global stability results have been known for a long time for both Eqs (1.3) and (1.4); these typically hold for all initial data with suitable regularity features, including large data.

In 1994, Ponce, Racke, Sideris and Titi [18] proved the global stability of the NS equation (1.3), on $\mathbf{R}^3$ or on a domain therein, for all H^1-type initial data v_0 yielding a solution v such that $\int_0^{+\infty} dt \, \|\nabla v(t)\|_{L^2}^4 < +\infty$; the basic estimates on the instantaneous separation between v and a solution u with initial datum close to v_0 are given via H^1-type norms (‡). Global stability of the NS equation was discussed (in different functional settings) until recent times, see e.g., [20–22]; the last one of these papers is a contribution by one of us, which inspired the present work on the MHD equations.

The issue of global stability for the MHD equations can be introduced recalling that Zhao and Li [19] proved in 2001 the global stability of Eqs (1.4) on $\mathbf{R}^3$ (or on domains therein) with $\nu, \eta > 0$, for all H^1-type initial data producing solutions with suitable decay properties. The functional setting of [19] is similar to that introduced in [18] for the NS equation, and the results of [19] can be described as follows when the spatial domain is the whole $\mathbf{R}^3$. Consider an initial datum $\mathbf{v}_0 = (v_0, c_0)$ (formed by two H^1-type vector fields on $\mathbf{R}^3$, representing the initial velocity and magnetic field); assume this datum produces for the MHD equations a global strong solution $\mathbf{v} = (v, c)$ which is decaying in time, in the sense that $\int_0^{+\infty} \|\nabla \mathbf{v}(t)\|_{L^2}^4 < +\infty$ (with $\|\nabla \mathbf{v}(t)\|_{L^2} := \sqrt{\|\nabla v(t)\|_{L^2}^2 + \|\nabla c(t)\|_{L^2}^2}$). Then, there exists a real $\delta > 0$ such that, for all initial data $\mathbf{u}_0 = (u_0, b_0)$ fulfilling $\|\mathbf{u}_0 - \mathbf{v}_0\|_{H^1} < \delta$, the MHD equations with datum $\mathbf{u}_0$ have a global strong solution $\mathbf{u} = (u, b)$, and for all $t \geqslant 0$ one has $\|\mathbf{u}(t) - \mathbf{v}(t)\|_{H^1} \leqslant M(\delta)$, where $M(\delta) \to 0$ as $\delta \to 0$.

The global stability of the MHD equations in space dimension $d = 3$ was the subject of subsequent investigations, falling outside the setting of [19]. A special attention was devoted to the MHD equations on $\mathbf{R}^3$ with an initial datum of the specific form $\mathbf{v}_0 = (0, B_0)$, where $B_0 \in \mathbf{R}^3$ is a constant vector (zero velocity and constant magnetic field); this case was analysed for $\nu = \eta \geqslant 0$ [23] and for $\nu > 0$, $\eta = 0$ [24, 25]. The cited works used for their estimates Sobolev norms of the types $W^{N,1}$, $W^{N,\infty}$ or H^N (including weighted variants), with integer N.

A recent paper [26] discusses (amongst else) the global stability of the zero mean solutions of the MHD equations on the torus $\mathbf{T}^3$ with $\nu, \eta > 0$, using H^k type norms of integer order k. In the panorama of the previous literature on global MHD stability, the cited reference is the one to which our present contribution is closest; let us sketch the stability result presented therein. Theorem 3.1 in [26] considers a global, smooth, zero mean solution $\mathbf{v}$ of the MHD equations on $\mathbf{T}^3$, with initial datum $\mathbf{v}_0$, and proves

‡Indeed, the results of [18] are more general and also apply to the NS case with external forces, not considered in the present work. The possibility of external forces is considered as well in paper [19] about MHD equations, cited shortly afterwards.

the following: if $r \in \mathbf{N} \setminus \{0\}$, $\mathbf{v}$ fulfills a suitable decay condition (depending on r), and $\mathbf{u}_0$ is an initial datum with $\|\mathbf{u}_0 - \mathbf{v}_0\|_{H^r}$ sufficiently small, then the corresponding MHD solution $\mathbf{u}$ is global as well and $\mathbf{u} - \mathbf{v}$ fulfills bounds, having essentially the form

$$\|\mathbf{u}(t) - \mathbf{v}(t)\|_{H^k} \leqslant \text{const.} \times \|\mathbf{u}_0 - \mathbf{v}_0\|_{H^k}\, e^{-\theta\mu t} \quad \text{for } t \geqslant 0,\ k = 0, 1, ..., r \text{ and any } \theta \in (0, 1); \tag{1.5}$$

here $\mu := \min(\nu, \eta)$, and the above constants depend on $\mathbf{v}$ and θ. We return later on [26] and on the previous literature for a comparison with our present contribution (see the comments after Eq (1.7), and Remarks 4.4 (iii) (iv) referring to the decay conditions in the present work and in [26]).

For completeness, let us mention that global stability results are available for some variants of the MHD equations (1.4) on $\mathbf{R}^3$, accounting for the Hall effect [27] or assuming partial dissipation and resistivity [28]; these variants will not be considered in the present work. We acknowledge one of the referees of the present paper for the indication of another variant of the MHD equations, proposed recently, where the viscosity and resistivity are not constant and depend on a real valued function ϕ (the "phase field"), that in turn evolves via a Cahn-Hilliard type equation; this model gives rise to the so-called Cahn-Hilliard MHD (CH-MHD) equations for the velocity field u, the magnetic field b and ϕ. To our knowledge, this model has been introduced in [29]. At present there are theoretical results on weak solutions of the CH-MHD Cauchy problem and on the convergence of certain fully discrete numerical schemes, which have also been employed in numerical experiments [29, 30]; we are not aware of global stability results. A hypothetical extension to the CH-MHD model of the setting of the present paper would require to generalize some fundamental inequalities of Section 2; however, this task seems to be nontrivial and we leave it as an open problem for future investigations.

Aims and contents of the present paper. The aims of the present work are:

(i) To present a global stability result for the MHD equations (1.4) on the torus $\mathbf{T}^d$ of any dimension $d \geqslant 2$, with $\nu, \eta > 0$; this result refers to a C^∞ setting for the MHD equations, and is expressed by fully quantitative estimates via H^p-type norms of arbitrarily high order p.

(ii) To introduce a class of *arbitrarily large* MHD initial data, yielding global solutions to which one can apply the result in (i). These data will be called *generalized Beltrami pairs*, by analogy with the generalized Beltrami flows often considered in the NS framework.

The C^∞ setting employed here for the MHD equations was proposed in our work [15], and is similar to the one considered previously for the NS equation in some papers by one of us and coworkers (see in particular [14, 22] and their bibliographies). So, the velocity field and the magnetic field at any time are C^∞ vector fields on $\mathbf{T}^d$; their time dependence is as well C^∞. Besides prescribing the velocity and magnetic fields to be divergence free, for technical reasons we require the vanishing of their means over $\mathbf{T}^d$.

The above functional setting is described in Section 2. Here we introduce the Fréchet space $\mathbb{H}^\infty_{\Sigma 0}$, formed by the C^∞ vector fields on $\mathbf{T}^d$ with vanishing divergence and mean; we characterize this space as the intersection of suitably defined, H^p-type Sobolev spaces $\mathbb{H}^p_{\Sigma 0}$ ($p \in \mathbf{R}$), whose norms are indicated with $\| \|_p$ (see Eqs (2.2), (2.3), (2.5), and (2.6)). To deal with the MHD equations, which involve pairs of vector fields, it is useful to consider the product spaces $\mathbf{H}^p_{\Sigma 0} := \mathbb{H}^p_{\Sigma 0} \times \mathbb{H}^p_{\Sigma 0}$ ($p \in \mathbf{R}$), whose norms are again denoted with $\| \|_p$; the intersection of these spaces, denoted with $\mathbf{H}^\infty_{\Sigma 0}$, coincides in fact with $\mathbb{H}^\infty_{\Sigma 0} \times \mathbb{H}^\infty_{\Sigma 0}$ (see Eqs (2.19) and (2.20)). Again in Section 2, we present formally the NS and MHD equations and their Cauchy problems, with $\nu, \eta > 0$; the solutions are C^∞ maps from a time interval to $\mathbb{H}^\infty_{\Sigma 0}$ or $\mathbf{H}^\infty_{\Sigma 0}$.

Following [15], *strong structural analogies are emphasized between the NS and the MHD case*; these concern, especially, the fundamental bilinear maps appearing in the right hand sides of Eqs (1.3) and (1.4). Certain inequalities about these bilinear maps, reviewed in the same section, contain constants $K_{np}, G_{np}, \hat{K}_{np}, \hat{G}_{np}$ (associated to pairs of Sobolev orders n, p), on which we have fully quantitative bounds from [15, 31–33]; these constants are essential for the purposes of the present work.

In Section 3 we review the basic tool yielding our stability result (i). This tool is a general theory for the *a posteriori* analysis of approximate solutions of the MHD Cauchy problem; we proposed such a theory in [15] on the grounds of the above mentioned analogies with the NS equation, and of previous works on approximants for the NS Cauchy problem (see [14] and other works by one of us and coworkers, cited therein; this approach was deeply inspired by [34]). In a few words, Section 3 and [15] consider the MHD Cauchy problem with an initial datum $\mathbf{u}_0$, and an approximate solution $\mathbf{u}_\mathrm{a}$ of this problem, fulfilling the evolution equations and the initial conditions up to certain errors. A general method allows to determine a lower bound on the interval of existence of the exact solution $\mathbf{u}$ and upper bounds on the Sobolev distances $\|\mathbf{u}(t) - \mathbf{u}_\mathrm{a}(t)\|_n$, $\|\mathbf{u}(t) - \mathbf{u}_\mathrm{a}(t)\|_p$ for some real $n > d/2 + 1$ and all real $p > n$, obtained by first solving a Riccati type, ordinary differential (in)equality corresponding to the Sobolev order n, and then some linear, ordinary differential (in)equalities corresponding to the Sobolev orders $p > n$; these are determined by the errors of $\mathbf{u}_\mathrm{a}$, and referred to as the "control (in)equalities" of orders n or p. These bounds on the interval of existence of $\mathbf{u}$ and on the Sobolev distances between $\mathbf{u}(t)$ and $\mathbf{u}_\mathrm{a}(t)$ form the content of Proposition 3.2. The completely quantitative nature of this proposition (with known contants, etc.) is very important in applications, especially in the numerical ones; in fact, given an approximant arising, say, from a numerical computation, its *a posteriori* analysis along the lines of Proposition 3.2 ultimately gives the *values* of the previously mentioned bounds on the time of existence of the exact solution, and on its Sobolev distances from the approximant (concerning numerical approximants, see also the comments at the end of Section 3).

The above framework has deep connections with the subsequent part of the paper, which we are going to describe hereafter.

Sections 4–6 of the paper contain the main results, connected with items (i) and (ii) at the beginning of the present paragraph.

In Section 4 we introduce the notion of global, decaying solution of the MHD equations (1.4); this is a global solution $\mathbf{v}$ such that

$$\|\mathbf{v}(t)\|_n \to 0 \qquad \text{as } t \to +\infty, \tag{1.6}$$

for some real $n > d/2 + 1$; this condition is in fact equivalent to the (seemingly stronger) requirement that, for each real p, $\|\mathbf{v}(t)\|_p$ decays like $e^{-\mu t}$, where again $\mu := \min(\nu, \eta) > 0$ (see Proposition 4.2). Any sufficiently small initial datum $\mathbf{v}_0$ produces a global, decaying solution; more precisely, such a solution occurs if $\|\mathbf{v}_0\|_n$ is below a known bound for some $n > d/2 + 1$ (see Lemma 4.1). This section is inspired by [22], where similar results are presented for the NS equation.

In Section 5, again inspired by reference [22] on the NS case, we derive our global stability result for the MHD equations; this is the subject of Theorem 5.1. Here we consider an initial datum $\mathbf{v}_0$ giving rise to a global, decaying MHD solution $\mathbf{v}$, and we show that any datum $\mathbf{u}_0$ sufficiently close to $\mathbf{v}_0$ produces as well a global, decaying solution $\mathbf{u}$. The closeness condition is a fully explicit upper bound on $\|\mathbf{u}_0 - \mathbf{v}_0\|_n$, for some real $n > d/2 + 1$, implying estimates on $\mathbf{u} - \mathbf{v}$; in the simplest version, such estimates have the form

$$\|\mathbf{u}(t) - \mathbf{v}(t)\|_p \leqslant \text{const.}\, \|\mathbf{u}_0 - \mathbf{v}_0\|_p\, e^{-\mu t} \quad \text{for } t \geqslant 0 \text{ and all real } p \geqslant n\,, \tag{1.7}$$

with explicit expressions for the above constants, that depend on $\mathbf{v}$ (see Eq (5.5)).

Some features of Theorem 5.1 somehow mark differences with respect to the previous literature. One such feature is the availability of estimates in the Sobolev norms of arbitrarily high Sobolev order p, in spite of a closeness condition of a fixed order n for the initial data; another feature we would mention is the availability of fully quantitative information.

These comments can be used, in particular, for a comparison with reference [26]; we recall our brief description of that reference, centered upon the previous Eq (1.5). In some sense, our result (1.7) (with $d = 3$) is complementary to the result (1.5) of [26]. In fact, with the assumption that $\mathbf{u}_0 - \mathbf{v}_0$ is small in the Sobolev norm of some (integer) order r, Eq (1.5) gives estimates on $\mathbf{u}(t) - \mathbf{v}(t)$ in the Sobolev norms of all (integer) orders *k less than or equal to r*. On the contrary, with the assumption that $\mathbf{u}_0 - \mathbf{v}_0$ is small in the Sobolev norm of some order n ($> d/2 + 1$), our Eq (1.7) gives estimates on $\mathbf{u}(t) - \mathbf{v}(t)$ in the Sobolev norms of all orders *p greater than or equal to n*. Let us add that the relations between our present work and [26] are conceptually similar to those between the already mentioned papers [22] and [21] on the global stability of the NS equation.

The proof that we propose for Theorem 5.1 has a feature that we also presume to be of some interest: it is a simple, almost automatic application of the general method proposed in our work [15], and summarized in Section 3, for the *a posteriori* analysis of approximants for the MHD Cauchy problem. The idea yielding the proof of Theorem 5.1 is that the global, decaying MHD solution $\mathbf{v}$ with initial datum $\mathbf{v}_0$ can be regarded as an approximate solution of the MHD Cauchy problem with datum $\mathbf{u}_0$ (to be precise, $\mathbf{v}$ satisfies exactly the MHD evolution equations, but fulfills the initial condition up to an error $\mathbf{u}_0 - \mathbf{v}_0$). So, Proposition 3.2 can be used to infer information about the exact solution $\mathbf{u}$ of this Cauchy problem and its separation from $\mathbf{v}$; the already emphasized, quantitative nature of Theorem 5.1 is a consequence of the quantitative features of Proposition 3.2.

To conclude, let us describe the subject of Section 6. Here we introduce the generalized Beltrami pairs mentioned in the previous item (ii), where they were indicated as possible, arbitrarily large initial data for the MHD equations; these data yield global, decaying solutions. Beltrami vector fields and their generalizations are well known characters in the framework of the NS equation (and are also discussed in [22]); the extension of these notions to the MHD equations has been considered to some extent in the literature, with motivations and approaches which match just partly the framework of Section 6 (see the references cited therein).

2. A smooth functional setting for the NS and MHD equations

Some function spaces ([§]). Here and in the rest of the present paper we work on a torus

$$\mathbf{T}^d := (\mathbf{R}/2\pi\mathbf{Z})^d \qquad (d = 2, 3, ...) \; ; \tag{2.1}$$

for $a, b \in \mathbf{C}^d$ we intend $a \bullet b := \sum_{r=1}^{d} a_r b_r$. We consider the space $D'(\mathbf{T}^d, \mathbf{R}^d) \equiv \mathbb{D}'$ of $\mathbf{R}^d$-valued distributions on $\mathbf{T}^d$ (and intend distributionally the differential operators mentioned in the sequel); each $v \in \mathbb{D}'$ has a weakly convergent Fourier expansion $v = \sum_{k \in \mathbf{Z}^d} v_k e_k$, where $e_k(x) := (2\pi)^{-d/2} e^{ik \bullet x}$

[§]The present paragraph on function spaces is similar to a paragraph in [22] with the same title; we think that reproducing some basic notations and facts can be useful. Other similarities between the present work and [15] or [22] are due to our aim to make the present paper self-contained, or are merely textual: in particular, the present work and [22] contain textually similar statements, which however concern different subjects (the MHD equations in the present work, and the NS equation in [22]).

and each coefficient $v_k = \overline{v_{-k}} \in \mathbf{C}^d$ is provided by the action of v on the test function e_{-k}. The mean value $\langle v \rangle$ is, by definition, the action of v on the constant test function $(2\pi)^{-d}$, and $\langle v \rangle = (2\pi)^{-d/2} v_0$. The Laplacian of $v \in \mathbb{D}'$ has Fourier coefficients $(\Delta v)_k = -|k|^2 v_k$; if $\langle v \rangle = 0$ and $p \in \mathbf{R}$, we define $(-\Delta)^{p/2} v$ to be the element of $\mathbb{D}'$ with mean zero and Fourier coefficients $((-\Delta)^{p/2} v)_k = |k|^p v_k$ for $k \in \mathbf{Z}^d \setminus \{0\}$. Let us consider $L^2(\mathbf{T}^d, \mathbf{R}^d) \equiv \mathbb{L}^2$, with the inner product $\langle v | w \rangle_{L^2} := \int_{\mathbf{T}^d} \overline{v} \bullet w \, dx = \sum_{k \in \mathbf{Z}^d} \overline{v_k} \bullet w_k$. For any $p \in \mathbf{R}$, we introduce the Sobolev-type space

$$\begin{aligned} \mathbb{H}^p_{\Sigma 0} &:= \{ v \in \mathbb{D}' \mid \operatorname{div} v = 0, \ \langle v \rangle = 0, \ (-\Delta)^{p/2} v \in \mathbb{L}^2 \} \\ &= \{ v \in \mathbb{D}' \mid \ k \bullet v_k = 0 \ \forall k \in \mathbf{Z}^d, \ \ v_0 = 0, \ \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2p} |v_k|^2 < +\infty \} \end{aligned} \tag{2.2}$$

(the subscripts Σ and 0 indicate the vanishing of the divergence and of the mean); this is a Hilbert space with the inner product and the norm

$$\langle v | w \rangle_p := \langle (-\Delta)^{p/2} v | (-\Delta)^{p/2} w \rangle_{L^2} = \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2p} \overline{v_k} \bullet w_k \,, \quad \|v\|_p := \sqrt{\langle v | v \rangle_p} = \|(-\Delta)^{p/2} v\|_{L^2} \,. \tag{2.3}$$

For real $p \geqslant \ell$, one has $\mathbb{H}^p_{\Sigma 0} \subset \mathbb{H}^\ell_{\Sigma 0}$ and $\| \ \|_p \geqslant \| \ \|_\ell$. For future reference, let us mention the following: for each real p and $v \in \mathbb{H}^{p+2}_{\Sigma 0}$, one has $\Delta v \in \mathbb{H}^p_{\Sigma 0}$ and

$$\|\Delta v\|_p = \|v\|_{p+2} \,, \qquad \langle \Delta v | v \rangle_p = -\|v\|^2_{p+1} \leqslant -\|v\|^2_p \,; \tag{2.4}$$

the first of these equalities indicates that Δ is continuous (indeed, isometric) from $\mathbb{H}^{p+2}_{\Sigma 0}$ to $\mathbb{H}^p_{\Sigma 0}$. The vector space

$$\mathbb{H}^\infty_{\Sigma 0} := \bigcap_{p \in \mathbf{R}} \mathbb{H}^p_{\Sigma 0} \tag{2.5}$$

can be equipped with the topology induced by the family of all Sobolev norms $\| \ \|_p$ ($p \in \mathbf{R}$), which coincides with that induced by the countable subfamily $\| \ \|_p$ ($p \in \mathbf{N}$); indeed, we have a Fréchet space. Due to the standard Sobolev imbeddings,

$$\mathbb{H}^\infty_{\Sigma 0} = \{ \, v \in C^\infty(\mathbf{T}^d, \mathbf{R}^d) \mid \operatorname{div} v = 0, \ \langle v \rangle = 0 \, \} \tag{2.6}$$

and the above mentioned topology on $\mathbb{H}^\infty_{\Sigma 0}$ coincides with that induced by the family of norms $\| \ \|_{C^r}$ ($r \in \mathbf{N}$), where $\| \ \|_{C^r}$ is the sup norm for the derivatives of all orders $\leqslant r$ (for $r \in \mathbf{N}$ and $p \in \mathbf{R}$ one has $\| \ \|_{C^r} \leqslant \text{const.} \| \ \|_p$ if $p > r + d/2$, and $\| \ \|_p \leqslant \text{const.} \| \ \|_{C^r}$ if $p \leqslant r$).

Obviously enough, Δ is continuous from $\mathbb{H}^\infty_{\Sigma 0}$ to itself. In the sequel we just occasionally consider for $p \in \mathbf{R}$ the spaces

$$\mathbb{H}^p_0 := \{ v \in \mathbb{D}' \mid \ \langle v \rangle = 0, \ (-\Delta)^{p/2} v \in \mathbb{L}^2 \} = \{ v \in \mathbb{D}' \mid \ v_0 = 0, \ \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2p} |v_k|^2 < +\infty \} \,, \tag{2.7}$$

which carry inner products and norms defined as in (2.3); their intersection

$$\mathbb{H}^\infty_0 := \bigcap_{p \in \mathbf{R}} \mathbb{H}^p_0 = \{ \, v \in C^\infty(\mathbf{T}^d, \mathbf{R}^d) \mid \ \langle v \rangle = 0 \, \} \tag{2.8}$$

is as well a Fréchet space with the norms $\| \ \|_p$ ($p \in \mathbf{R}$). $\mathbb{H}^p_{\Sigma 0}$ and $\mathbb{H}^\infty_{\Sigma 0}$ are closed subspaces of $\mathbb{H}^p_0$ and $\mathbb{H}^\infty_0$, respectively.

Leray projection. Working with the maximum generality, we can define this to be the map

$$\mathfrak{L} : \mathbb{D}' \to \mathbb{D}' , \quad v \mapsto \mathfrak{L}v \text{ such that } (\mathfrak{L}v)_k = \mathfrak{L}_k v_k \text{ for } k \in \mathbf{Z}^d ; \tag{2.9}$$

here $\mathfrak{L}_k$ is the orthogonal projection of $\mathbf{C}^d$ onto $k^\perp = \{a \in \mathbf{C}^d \mid k \bullet a = 0\}$ (so that, for $a \in \mathbf{C}^d$, $\mathfrak{L}_k a = a - (k \bullet a)k/|k|^2$ if $k \in \mathbf{Z}^d \setminus \{0\}$ and $\mathfrak{L}_0 a = a$). For each $v \in \mathbb{D}'$, we have: $\mathrm{div}(\mathfrak{L}v) = 0$; $\mathfrak{L}v = v$ if and only if $\mathrm{div}\, v = 0$; $\langle \mathfrak{L}v \rangle = \langle v \rangle$ and, in particular, $\langle \mathfrak{L}v \rangle = 0$ if $\langle v \rangle = 0$. If $p \in D'(\mathbf{T}^d, \mathbf{R})$ is any real valued distribution, we have $\mathfrak{L}(\nabla p) = 0$. Finally, $\mathfrak{L}\mathbb{L}^2 \subset \mathbb{L}^2$.

The fundamental bilinear map for the NS equation. Working with a good generality, let $v \in L^\infty(\mathbf{T}^d, \mathbf{R}^d) \equiv \mathbb{L}^\infty$, and $w \in \mathbb{D}'$ such that $\partial_s w \in \mathbb{L}^2$ for $s = 1, ...d$. Then the vector field $(v \bullet \nabla)w = \sum_{s=1}^d v_s \partial_s w$ is well defined and belongs to $\mathbb{L}^2$; we can form from it the vector field

$$\mathcal{P}(v, w) := -\mathfrak{L}((v \bullet \nabla)w) \in \mathbb{L}^2 . \tag{2.10}$$

By construction, $\mathrm{div}\mathcal{P}(v, w) = 0$; moreover, if $\mathrm{div}\, v = 0$ we have $\langle (v \bullet \nabla)w \rangle = 0$ (see, e.g., Lemma 2.1. in [32]) and, consequently, $\langle \mathcal{P}(v, w) \rangle = 0$. The bilinear map $\mathcal{P}$ sending v, w in the above mentioned spaces to $\mathcal{P}(v, w)$ is called in the sequel the *fundamental bilinear map* for the NS equation; $\mathcal{P}$ evidently appears in the NS equation (1.3) (as well as in the MHD equations (1.4), a fact on which we return later). Let $p, n \in \mathbf{R}$; it is known that $p > d/2$, $v \in \mathbb{H}^p_{\Sigma 0}$, $w \in \mathbb{H}^{p+1}_{\Sigma 0} \Rightarrow \mathcal{P}(v, w) \in \mathbb{H}^p_{\Sigma 0}$ and that, for p, n as below, there are constants $K_{pn}, G_{pn} \in (0, +\infty)$ such that the following holds:

$$\|\mathcal{P}(v, w)\|_p \leqslant \frac{1}{2} K_{pn}(\|v\|_p \|w\|_{n+1} + \|v\|_n \|w\|_{p+1}) \qquad \text{if } p \geqslant n > d/2, v \in \mathbb{H}^p_{\Sigma 0}, w \in \mathbb{H}^{p+1}_{\Sigma 0} , \tag{2.11}$$

$$|\langle \mathcal{P}(v, w) | w \rangle_p| \leqslant \frac{1}{2} G_{pn}(\|v\|_p \|w\|_n + \|v\|_n \|w\|_p) \|w\|_p \qquad \text{if } p \geqslant n > d/2 + 1, v \in \mathbb{H}^p_{\Sigma 0}, w \in \mathbb{H}^{p+1}_{\Sigma 0} . \tag{2.12}$$

The above results indicate, amongst else, that $\mathcal{P}$ maps continuously $\mathbb{H}^p_{\Sigma 0} \times \mathbb{H}^{p+1}_{\Sigma 0}$ to $\mathbb{H}^p_{\Sigma 0}$ for all $p > d/2$ (use (2.11) with $n = p$), and $\mathbb{H}^\infty_{\Sigma 0} \times \mathbb{H}^\infty_{\Sigma 0}$ to $\mathbb{H}^\infty_{\Sigma 0}$. Equation (2.11) is closely related to the basic norm inequalities about multiplication in Sobolev spaces [35]. Equation (2.12) was discovered in [3] for integer $p = n$, and extended in [36] to the case of noninteger $p = n$; inequalities very similar to (2.12) were proposed in [1, 7, 37]. In the sequel, we intend

$$K_q := K_{qq} , \qquad G_q := G_{qq} , \tag{2.13}$$

(typically, with $q = n$ or $q = p$). For our purposes, it is important to point out that fully quantitative, upper and lower bounds for the sharp constants K_{pn}, G_{pn} in Eqs (2.11) and (2.12)) are available (see [31] and the previous works [32,33] by one of us and coworkers). From here to the end of the paper K_{pn}, G_{pn} are (possibly nonsharp) constants fulfilling the above inequalities.

The NS and MHD equations in a smooth framework. Let us fix a viscosity and a resistivity

$$\nu, \eta \in (0, +\infty) , \tag{2.14}$$

and specify the notion of solution to be used from now on for the above equations and their Cauchy problems. To this purpose we consider the equalities

$$\text{(a)} \quad \frac{du}{dt} = \nu \Delta u + \mathcal{P}(u, u) , \qquad \text{(b)} \quad u(0) = u_0 . \tag{2.15}$$

From now to the end of the paper, a *solution of the NS equation* means a map $u \in C^\infty([0,T), \mathbb{H}^\infty_{\Sigma 0})$, with $T = T_u \in (0, +\infty]$, fulfilling Eq (2.15.a); a *solution of the NS Cauchy problem* with initial datum $u_0 \in \mathbb{H}^\infty_{\Sigma 0}$ is a map u as before, fulfilling Eqs (2.15.a) and (2.15.b).

We now consider the equalities

$$\begin{aligned} &\text{(a)} \quad \frac{du}{dt} = \nu\Delta u + \mathcal{P}(u,u) - \mathcal{P}(b,b)\,, \quad \frac{db}{dt} = \eta\Delta b + \mathcal{P}(u,b) - \mathcal{P}(b,u)\,; \\ &\text{(b)} \quad u(0) = u_0, \;\; b(0) = b_0\,. \end{aligned} \tag{2.16}$$

A *solution of the MHD equations* is a pair formed by two maps $u, b \in C^\infty([0,T), \mathbb{H}^\infty_{\Sigma 0})$, with $T = T_{u,b} \in (0, +\infty]$, fulfilling Eq (2.16.a); a solution of the *MHD Cauchy problem* with initial datum $(u_0, b_0) \in \mathbb{H}^\infty_{\Sigma 0} \times \mathbb{H}^\infty_{\Sigma 0}$ is a pair of maps (u, b) as before, fulfilling Eqs (2.16.a) and (2.16.b).

In the above, $\mathcal{P}$ is the fundamental bilinear map (2.10); one recognizes that the evolution equations in (2.15.a) and (2.16.a) correspond, respectively, to those appearing in Eqs (1.3) and (1.4). Let us repeat that a solution u or (u, b) with $T = +\infty$ is said to be *global*.

It is known that problem (2.15.a), (2.15.b) (resp., problem (2.16.a), (2.16.b)) has a unique maximal, i.e., not extendable solution. (¶)

Reformulation of the MHD equations; the fundamental MHD bilinear map. Maintaining the notations $D'(\mathbf{T}^d, \mathbf{R}^d) \equiv \mathbb{D}'$ and $L^2(\mathbf{T}^d, \mathbf{R}^d) \equiv \mathbb{L}^2$, let us define

$$\mathbf{D}' := \mathbb{D}' \times \mathbb{D}'\,; \tag{2.17}$$

$$\begin{gathered} \mathbf{L}^2 := \mathbb{L}^2 \times \mathbb{L}^2\,, \quad \langle \mathbf{v}|\mathbf{w}\rangle_{L^2} := \langle v|w\rangle_{L^2} + \langle c|g\rangle_{L^2}\,, \quad \|\mathbf{v}\|_{L^2} := \sqrt{\langle \mathbf{v}|\mathbf{v}\rangle_{L^2}} = \sqrt{\|v\|^2_{L^2} + \|c\|^2_{L^2}} \\ \text{for } \mathbf{v} = (v,c),\, \mathbf{w} = (w,g) \in \mathbf{L}^2\,; \end{gathered} \tag{2.18}$$

$$\begin{gathered} \mathbf{H}^p_{\Sigma 0} := \mathbb{H}^p_{\Sigma 0} \times \mathbb{H}^p_{\Sigma 0}\,, \quad \langle \mathbf{v}|\mathbf{w}\rangle_p := \langle v|w\rangle_p + \langle c|g\rangle_p, \quad \|\mathbf{v}\|_p := \sqrt{\langle \mathbf{v}|\mathbf{v}\rangle_p} = \sqrt{\|v\|^2_p + \|c\|^2_p} \\ \text{for } p \in \mathbf{R}\,, \mathbf{v} = (v,c), \mathbf{w} = (w,g) \in \mathbf{H}^p_{\Sigma 0}\,; \end{gathered} \tag{2.19}$$

of course, $\mathbf{L}^2$ and $\mathbf{H}^p_{\Sigma 0}$ are Hilbert spaces with the inner products $\langle\,|\,\rangle_{L^2}$ and $\langle\,|\,\rangle_p$, inducing the norms $\|\,\|_{L^2}$ and $\|\,\|_p$. We also set

$$\mathbf{H}^\infty_{\Sigma 0} := \bigcap_{p \in \mathbf{R}} \mathbf{H}^p_{\Sigma 0} = \mathbb{H}^\infty_{\Sigma 0} \times \mathbb{H}^\infty_{\Sigma 0}\,; \tag{2.20}$$

this is a Fréchet space with the infinitely many norms $\|\,\|_p$ ($p \in \mathbf{R}$ or, equivalently, $p = 0, 1, 2, ...$). In this subsection we will rephrase the MHD Cauchy problem (2.16.a), (2.16.b) using a linear operator $\mathcal{A}$ and a bilinear map $\mathcal{P}$ acting on the above product spaces. The first one of these characters is defined by

$$\mathcal{A} : \mathbf{D}' \to \mathbf{D}'\,, \qquad \mathbf{v} := (v,c) \mapsto \mathcal{A}\mathbf{v} := (\nu\Delta v, \eta\Delta c)\,. \tag{2.21}$$

¶See the classical references on the NS or MHD Cauchy problems cited in the Introduction. Most of these works present technical differences with respect to the setting considered here (e.g., use of the spatial domain $\mathbf{R}^d$ instead of $\mathbf{T}^d$, and use of Sobolev spaces of finite order in place of our infinite order space (2.5), (2.6)). However, one converts the results of the cited works to the present setting by rather standard arguments. We refer to paper [14] by one of us and coworkers for a detailed discussion of these issues in the NS case (relying basically on [7] for what concerns the C^∞ setting); our work [15] gives hints for the analogous discussion in the MHD case.

We note that Eq (2.4) and the related statements imply the following: for each real p and $\mathbf{v} = (v, c) \in \mathbf{H}^{p+2}_{\Sigma 0}$ one has $\mathcal{A}\mathbf{v} \in \mathbf{H}^{p}_{\Sigma 0}$, and

$$\begin{aligned} \|\mathcal{A}\mathbf{v}\|_p &= \sqrt{\nu^2\|v\|^2_{p+2} + \eta^2\|c\|^2_{p+2}} \leqslant M\|\mathbf{v}\|_{p+2}\,, \\ \langle \mathcal{A}\mathbf{v}|\mathbf{v}\rangle_p &= -\nu\|v\|^2_{p+1} - \eta\|c\|^2_{p+1} \leqslant -\mu\|\mathbf{v}\|^2_{p+1} \leqslant -\mu\|\mathbf{v}\|^2_p\,, \end{aligned} \tag{2.22}$$

$$M := \max(\nu, \eta)\,, \qquad \mu := \min(\nu, \eta)\,. \tag{2.23}$$

The first relation (2.22) implies that $\mathcal{A}$ is continuous from $\mathbf{H}^{p+2}_{\Sigma 0}$ to $\mathbf{H}^{p}_{\Sigma 0}$ for each real p, and from $\mathbf{H}^{\infty}_{\Sigma 0}$ to $\mathbf{H}^{\infty}_{\Sigma 0}$. To go on we put

$$\mathcal{P} : (\mathbf{v} = (v, c), \mathbf{w} = (w, g)) \mapsto \mathcal{P}(\mathbf{v}, \mathbf{w}) := (\mathcal{P}(v, w) - \mathcal{P}(c, g), \mathcal{P}(v, g) - \mathcal{P}(c, w))\,. \tag{2.24}$$

$\mathcal{P}$ is well defined and bilinear, say, on the domain formed by pairs $(\mathbf{v}, \mathbf{w})$ as above with $v, c \in \mathbb{L}^{\infty}$ and $\partial_s w, \partial_s g \in \mathbb{L}^2$ for $s \in \{1, ..., d\}$; this domain is mapped by $\mathcal{P}$ to $\mathbb{L}^2$. For reasons that will soon be clear, from now on $\mathcal{P}$ will be called the *fundamental MHD bilinear map*. Due to the properties of $\mathcal{P}$ reviewed previously, $\mathcal{P}$ maps continuously $\mathbf{H}^{p}_{\Sigma 0} \times \mathbf{H}^{p+1}_{\Sigma 0}$ to $\mathbf{H}^{p}_{\Sigma 0}$ for each $p > d/2$, and $\mathbf{H}^{\infty}_{\Sigma 0} \times \mathbf{H}^{\infty}_{\Sigma 0}$ to $\mathbf{H}^{\infty}_{\Sigma 0}$. It was shown in [15] (see the Appendix therein) that $\mathcal{P}$ fulfills inequalities very similar to (2.11), (2.12). More precisely, if $p, n \in \mathbf{R}$ are related as below, there are constants $\hat{K}_{pn}, \hat{G}_{pn} \in (0, +\infty)$ such that

$$\|\mathcal{P}(\mathbf{v}, \mathbf{w})\|_p \leqslant \frac{1}{2}\hat{K}_{pn}(\|\mathbf{v}\|_p\|\mathbf{w}\|_{n+1} + \|\mathbf{v}\|_n\|\mathbf{w}\|_{p+1}) \qquad \text{if } p \geqslant n > d/2,\ \mathbf{v} \in \mathbf{H}^{p}_{\Sigma 0}\,,\ \mathbf{w} \in \mathbf{H}^{p+1}_{\Sigma 0}\,, \tag{2.25}$$

$$|\langle \mathcal{P}(\mathbf{v}, \mathbf{w})|\mathbf{w}\rangle_p| \leqslant \frac{1}{2}\hat{G}_{pn}(\|\mathbf{v}\|_p\|\mathbf{w}\|_n + \|\mathbf{v}\|_n\|\mathbf{w}\|_p)\|\mathbf{w}\|_p \qquad \text{if } p \geqslant n > d/2 + 1,\ \mathbf{v} \in \mathbf{H}^{p}_{\Sigma 0}\,,\ \mathbf{w} \in \mathbf{H}^{p+1}_{\Sigma 0}\,; \tag{2.26}$$

as proved in [15], one can take

$$\hat{K}_{pn} := \sqrt{2}\, K_{pn}\,, \qquad \hat{G}_{pn} := \sqrt{2}\, G_{pn}\,, \tag{2.27}$$

where K_{pn}, G_{pn} are constants fulfilling (2.11), (2.12). In the sequel, we will use the notations

$$\hat{K}_q := \hat{K}_{qq}\,, \qquad \hat{G}_q := \hat{G}_{qq}\,, \tag{2.28}$$

(typically, with $q = n$ or $q = p$; needless to say, (2.13) and (2.27) ensure that we can take $\hat{K}_q := \sqrt{2}\, K_q$ and $\hat{G}_q := \sqrt{2}\, G_q$).

We are now ready to reconsider the solutions of the MHD equations and of the corresponding Cauchy problem. To this purpose, let us write the equalities

$$\text{(a)} \quad \frac{d\mathbf{u}}{dt} = \mathcal{A}\mathbf{u} + \mathcal{P}(\mathbf{u}, \mathbf{u})\,, \qquad\qquad \text{(b)} \quad \mathbf{u}(0) = \mathbf{u}_0\,. \tag{2.29}$$

As readily checked, *a solution of the MHD equations (2.16.a) is just a map* $\mathbf{u} = (u, b) \in C^{\infty}([0, T), \mathbf{H}^{\infty}_{\Sigma 0})$ *(with* $T = T_{\mathbf{u}} \in (0, +\infty]$*) fulfilling Eq (2.29.a)*; *a solution of the MHD Cauchy problem (2.16.a) (2.16.b) with initial datum* $\mathbf{u}_0 = (u_0, b_0) \in \mathbf{H}^{\infty}_{\Sigma 0}$ *is a map* $\mathbf{u}$ *as before, fulfilling Eqs (2.29.a) and (2.29.b).*

As emphasized in [15], *the formulation (2.29.a) of the MHD equations indicates strong similarities with the NS equation (2.15.a), on the grounds of the structural analogies between the linear maps* $\nu\Delta$ *and* $\mathcal{A}$ (compare Eqs (2.4) and (2.22) (2.23)), *and between the bilinear maps* $\mathcal{P}$ *and* $\mathcal{P}$ (see again Eqs (2.11) (2.12) and (2.25) (2.26)).

For convenience of the reader, in Table 1 we summarize the notations introduced in the present section.

Table 1. Symbols introduced in Section 2.

$\mathbb{D}'$	The space of $\mathbf{R}^d$-valued distributions (distributional vector fields) on $\mathbf{T}^d = (\mathbf{R}/2\pi\mathbf{Z})^d$
Δ	The Laplacian on $\mathbf{T}^d$
$\mathbb{H}^p_{\Sigma 0}$;	The space of divergence free, zero mean vector fields $v : \mathbf{T}^d \to \mathbf{R}^d$
	such that $(-\Delta)^p v$ is in L^2 (see Eq (2.2));
$\langle \mid \rangle_p, \Vert \Vert_p$ $(p \in \mathbf{R})$	the inner product and norm of this space (see Eq (2.3))
$\mathbb{H}^\infty_{\Sigma 0}$	The intersection $\cap_{p\in\mathbf{R}} \mathbb{H}^p_{\Sigma 0}$, made in fact of C^∞ vector fields (see Eqs (2.5) and (2.6))
$\mathfrak{L}$	The Leray projection onto the space of divergence free vector fields (see Eq (2.9))
$\mathcal{P}$	The fundamental bilinear map defined by Eq (2.10)
$du/dt = \nu\Delta u + \mathcal{P}(u,u)$	The NS equations in the present notations (see Eq (2.15))
K_{pn}, G_{pn}	The constants in inequalities (2.11), (2.12) about $\mathcal{P}$
K_q, G_q	The constants K_{qq} and G_{qq}, respectively (see Eq (2.13))
$\mathbf{D}'$	The product $\mathbb{D}' \times \mathbb{D}'$ (see Eq (2.17))
$\mathcal{A}$	The linear operator $\mathbf{D}' \to \mathbf{D}'$, $\mathbf{v} := (v, c) \mapsto \mathcal{A}\mathbf{v} := (\nu\Delta v, \eta\Delta c)$,
	with ν, η the viscosity and resistivity (see Eq (2.21))
$\mathbf{H}^p_{\Sigma 0}$;	The product $\mathbb{H}^p_{\Sigma 0} \times \mathbb{H}^p_{\Sigma 0}$;
$\langle \mid \rangle_p, \Vert \Vert_p$ $(p \in \mathbf{R})$	the inner product and norm of this space (see Eq (2.19))
$\mathbf{H}^\infty_{\Sigma 0}$	The intersection $\cap_{p\in\mathbf{R}} \mathbf{H}^p_{\Sigma 0}$, coinciding with the product $\mathbb{H}^\infty_{\Sigma 0} \times \mathbb{H}^\infty_{\Sigma 0}$ (see Eq (2.20))
$\boldsymbol{\mathcal{P}}$	The fundamental bilinear map defined by Eq (2.24)
$d\mathbf{u}/dt = \mathcal{A}\mathbf{u} + \boldsymbol{\mathcal{P}}(\mathbf{u},\mathbf{u})$	The MHD equations in the present notations (see Eq (2.29))
$\hat{K}_{pn}, \hat{G}_{pn}$	The constants in inequalities (2.25), (2.26) about $\boldsymbol{\mathcal{P}}$;
	one can take $\hat{K}_{pn} := \sqrt{2}\, K_{pn}$, $\hat{G}_{pn} := \sqrt{2}\, G_{pn}$ (see Eq (2.27))
$\hat{K}_q, \hat{G}_q$	The constants $\hat{K}_{qq}$ and $\hat{G}_{qq}$, respectively (see Eq (2.28))

3. Approximate solutions of the MHD Cauchy problem. Control inequalities

In [14], a general method was developed for the *a posteriori* analysis of the approximate solutions of the NS Cauchy problem (2.15.a), (2.15.b); this method, summarized in [22], allows to get estimates on the interval of existence of the exact solution, and on the Sobolev distances (of arbitrarily high order) between the exact and the approximate solution, using information provided exclusively by the approximate solution. In [15], an analogous method was constructed for the MHD Cauchy problem (2.29.a), (2.29.b), using the structural similarities with the NS case emphasized in the previous section.

Recalling that we have fixed $\nu, \eta > 0$, let $\mathbf{u}_0 = (u_0, b_0) \in \mathbf{H}^\infty_{\Sigma 0}$; the method of [15] can be summarized via the definition and the proposition that follow.

3.1. ***Definition.*** *An* approximate solution *of the MHD Cauchy problem (2.29.a), (2.29.b) is any map* $\mathbf{u}_\mathsf{a} = (u_\mathsf{a}, b_\mathsf{a}) \in C^1([0, T_\mathsf{a}), \mathbf{H}^\infty_{\Sigma 0})$*, with* $T_\mathsf{a} \in (0, +\infty]$*. Given such a function, (i) and (ii) are stipulated.*
i) The differential error *and the* datum error *of* $\mathbf{u}_\mathsf{a}$ *are, respectively:*

$$e(\mathbf{u}_\mathsf{a}) := \frac{d\mathbf{u}_\mathsf{a}}{dt} - \mathcal{A}\mathbf{u}_\mathsf{a} - \boldsymbol{\mathcal{P}}(\mathbf{u}_\mathsf{a}, \mathbf{u}_\mathsf{a}) \ \in C([0, T_\mathsf{a}), \mathbf{H}^\infty_{\Sigma 0}) \ ; \quad \mathbf{u}_0 - \mathbf{u}_\mathsf{a}(0) \in \mathbf{H}^\infty_{\Sigma 0} \ . \tag{3.1}$$

ii) Let $p \in \mathbf{R}$*. A* differential error estimator*, a* datum error estimator *and a* growth estimator *of order p for* u_a *are, respectively, a function* $\epsilon_p \in C([0, T_\mathsf{a}), [0, +\infty))$*, a number* $\delta_p \in [0, +\infty)$ *and a function*

$\mathcal{D}_p \in C([0,T_\mathsf{a}),[0,+\infty))$ *such that*

$$\|e(\mathbf{u}_\mathsf{a})(t)\|_p \leqslant \epsilon_p(t) \quad \textit{for } t\in[0,T_\mathsf{a})\,, \qquad \|\mathbf{u}_0-\mathbf{u}_\mathsf{a}(0)\|_p \leqslant \delta_p\,, \qquad \|\mathbf{u}_\mathsf{a}(t)\|_p \leqslant \mathcal{D}_p(t) \quad \textit{for } t\in[0,T_\mathsf{a})\,. \tag{3.2}$$

In particular the function $\epsilon_p(t) := \|e(\mathbf{u}_\mathsf{a})(t)\|_p$*, the number* $\delta_p := \|\mathbf{u}_0-\mathbf{u}_\mathsf{a}(0)\|_p$ *and the function* $\mathcal{D}_p(t) := \|\mathbf{u}_\mathsf{a}(t)\|_p$ *will be called the* tautological *estimators of order p for the differential error, the datum error and the growth of* $\mathbf{u}_\mathsf{a}$.

3.2. ***Proposition.*** *Let* $\mathbf{u}_\mathsf{a}\in C^1([0,T_\mathsf{a}),\mathbf{H}^\infty_{\Sigma 0})$ *be an approximate solution of the MHD Cauchy problem (2.29.a), (2.29.b). Assume that, for some* $n\in(d/2+1,+\infty)$*,* $\mathbf{u}_\mathsf{a}$ *has differential error, datum error and growth estimators of order n or n* + 1*, indicated with* $\epsilon_n,\delta_n,\mathcal{D}_n$ *and* $\mathcal{D}_{n+1}$*, and that there is a function* $\mathcal{R}_n\in C([0,T_\mathsf{c}),\mathbf{R})$*, with* $T_\mathsf{c}\in(0,T_\mathsf{a}]$*, fulfilling the following* control inequalities*:*

$$\frac{d^+\mathcal{R}_n}{dt} \geqslant -\mu\mathcal{R}_n + (\hat{G}_n\mathcal{D}_n + \hat{K}_n\mathcal{D}_{n+1})\mathcal{R}_n + \hat{G}_n\mathcal{R}_n^2 + \epsilon_n \textit{ everywhere on } [0,T_\mathsf{c}), \qquad \mathcal{R}_n(0)\geqslant\delta_n \tag{3.3}$$

(μ *as in (2.23),* $\hat{K}_n,\hat{G}_n$ *as in Eqs (2.25), (2.26), and (2.28); in the above we use the right, upper Dini derivative* $(d^+\mathcal{R}_n(t)/dt)(t) := \limsup_{h\to 0^+}(\mathcal{R}_n(t+h)-\mathcal{R}_n(t))/h$*). Consider the maximal solution* $\mathbf{u}\in C^\infty([0,T),\mathbf{H}^\infty_{\Sigma 0})$ *of problem (2.29.a), (2.29.b); then (i)(ii) hold.*
i) $\mathbf{u}$ *and its existence time T are such that*

$$T\geqslant T_\mathsf{c}\,, \qquad \|\mathbf{u}(t)-\mathbf{u}_\mathsf{a}(t)\|_n \leqslant \mathcal{R}_n(t) \qquad \textit{for } t\in[0,T_\mathsf{c})\,. \tag{3.4}$$

In particular, if $\mathcal{R}_n$ *is global (*$T_\mathsf{c}=+\infty$*), then* $\mathbf{u}$ *is global as well (*$T=+\infty$*).*
ii) Consider any $p\in(n,+\infty)$*, and let* ϵ_p*,* δ_p*,* $\mathcal{D}_p,\mathcal{D}_{p+1}$ *be differential error, datum error and growth estimators of order p or p* + 1 *for* $\mathbf{u}_\mathsf{a}$*. Let* $\mathcal{R}_p\in C([0,T_\mathsf{c}),\mathbf{R})$ *be a function fulfilling the* linear, order p control inequalities

$$\frac{d^+\mathcal{R}_p}{dt} \geqslant -\mu\mathcal{R}_p + (\hat{G}_p\mathcal{D}_p + \hat{K}_p\mathcal{D}_{p+1} + \hat{G}_{pn}\mathcal{R}_n)\mathcal{R}_p + \epsilon_p \textit{ everywhere on } [0,T_\mathsf{c})\,, \qquad \mathcal{R}_p(0)\geqslant\delta_p \tag{3.5}$$

($\hat{G}_{pn}$*,* $\hat{K}_p,\hat{G}_p$ *as in Eqs (2.25), (2.26), and (2.28); again,* d^+/dt *stands for the right, upper Dini derivative). Then*

$$\|\mathbf{u}(t)-\mathbf{u}_\mathsf{a}(t)\|_p \leqslant \mathcal{R}_p(t) \qquad \textit{for } t\in[0,T_\mathsf{c})\,. \tag{3.6}$$

The relations (3.5) are both fulfilled as equalities by a unique function $\mathcal{R}_p\in C^1([0,T_\mathsf{c}),\mathbf{R})$*, which is given explicitly by*

$$\begin{aligned}
&\mathcal{R}_p(t) = e^{-\mu t+\mathcal{A}_p(t)}\Big(\delta_p + \int_0^t ds\, e^{\mu s-\mathcal{A}_p(s)}\epsilon_p(s)\Big)\,,\\
&\mathcal{A}_p(t) := \int_0^t ds\,(\hat{G}_p\mathcal{D}_p(s)+\hat{K}_p\mathcal{D}_{p+1}(s)+\hat{G}_{pn}\mathcal{R}_n(s))\,.
\end{aligned} \tag{3.7}$$

Let us emphasize that all pieces of information provided by Proposition 3.2 about the exact solution $\mathbf{u}$ (lower bounds on the interval of existence, estimates on the Sobolev distances between $\mathbf{u}(t)$ and $\mathbf{u}_\mathsf{a}(t)$) arise solely from the analysis of the approximate solution $\mathbf{u}_\mathsf{a}$; so, it is correct to speak of an *a posteriori* approach.

In [15], after proving Proposition 3.2 we presented some applications. The most important application discussed therein is the *a posteriori* analysis of the approximate solution $\mathbf{u}_\mathsf{a}$ provided by a

projection method; $\mathbf{u}_a$ is the solution of the Cauchy problem obtained by projecting problem (2.29.a), (2.29.b) onto a finite-dimensional subspace of $\mathbf{H}^\infty_{\Sigma 0}$ (more precisely, onto the Galerkin subspace spanned by finitely many Fourier modes; the projected evolution equation is in fact equivalent to a system of finitely many ODEs for the corresponding Fourier components). This approach is semidiscrete, since $\mathbf{u}_a$ depends on a continuous time variable. A subsequent step would be the *a posteriori* analysis of a time-discretized projection method, where time derivatives are replaced by finite differences; this possibility was not considered in [15].

It should be mentioned that the *a posteriori* analysis of time-discretized projection methods have been discussed by several authors for evolutionary PDEs differing from those of MHD: a recent work by Brunk, Giesselmann and Tscherpel [38] treats the NS equation (on $\mathbf{T}^3$), Giesselmann and Kolbe have discussed the Keller-Segel model [39], while Blömker, Nolde and Robinson have previously analysed the evolution equation for a surface growth model [40–42].

In the case of MHD equations, we are not aware of an *a posteriori* analysis for a time-discretized projection method; the optimal error estimates for such a method, derived recently by Wang et al. [43], are *a priori* estimates, since they assume that the exact solution of the MHD Cauchy problem exists on a given interval $[0, T)$.

In the present paper we propose a novel application of Proposition 3.2, concerning the global stability of the decaying solutions of the MHD Cauchy problem; the notion of decay is defined hereafter in a formal way.

4. Global, decaying MHD solutions

We recall once more that $\nu, \eta > 0$ and μ, $\hat{G}_n$ are defined by Eqs (2.23), (2.26), and (2.28).

4.1. ***Lemma.*** *Let $\mathbf{w}_0 \in \mathbf{H}^\infty_{\Sigma 0}$ be such that $\|\mathbf{w}_0\|_n < \mu/\hat{G}_n$ for some $n \in (d/2 + 1, +\infty)$. Then, the maximal solution $\mathbf{w}$ of the MHD Cauchy problem (2.29.a), (2.29.b) with initial datum $\mathbf{w}_0$ is global and, for each $p \in \mathbf{R}$, there is a constant $C_p \in [0, +\infty)$ such that $\|\mathbf{w}(t)\|_p \leqslant C_p e^{-\mu t}$ for all $t \in [0, +\infty)$.*

Proof. This is just a simplified formulation of Proposition 5.2 in [15]. ([‖]) □

Using the previous Lemma, one can prove the equivalence of several decay conditions for a global MHD solution, as described hereafter:

4.2. ***Proposition.*** *Let $\mathbf{v} \in C^\infty([0, +\infty), \mathbf{H}^\infty_{\Sigma 0})$ be a global solution of the MHD equations (2.29.a). The following statements are equivalent:*

a) For some $n \in (d/2 + 1, +\infty)$ and $t_0 \in [0, +\infty)$, it is $\|\mathbf{v}(t_0)\|_n < \mu/\hat{G}_n$.

b) For some $n \in (d/2 + 1, +\infty)$, it is $\|\mathbf{v}(t)\|_n \to 0$ for $t \to +\infty$.

c) For all $p \in \mathbf{R}$, it is $\|\mathbf{v}(t)\|_p \to 0$ for $t \to +\infty$ (i.e., $\mathbf{v}(t) \to 0$ in the Fréchet space $\mathbf{H}^\infty_{\Sigma 0}$).

d) For some $n \in (d/2 + 1, +\infty)$ and $\gamma \in (0, +\infty)$, it is $\int_0^{+\infty} dt\, \|\mathbf{v}(t)\|_n^\gamma < +\infty$.

e) For all $p \in \mathbf{R}$ and $\gamma \in (0, +\infty)$, it is $\int_0^{+\infty} dt\, \|\mathbf{v}(t)\|_p^\gamma < +\infty$.

f) For some $n \in (d/2 + 1, +\infty)$, there is constant $C_n \in [0, +\infty)$ such that $\|\mathbf{v}(t)\|_n \leqslant C_n\, e^{-\mu t}$ for all $t \in [0, +\infty)$.

g) For each $p \in \mathbf{R}$, there is constant $C_p \in [0, +\infty)$ such that $\|\mathbf{v}(t)\|_p \leqslant C_p\, e^{-\mu t}$ for all $t \in [0, +\infty)$.

[‖]The cited Proposition from [15] gives slightly more refined estimates on the instantaneous Sobolev norms of the solution, which can be used to infer explicit expressions for the constants C_p. More precisely, with n as above, Eqs (5.9), (5.11), and (5.12) in [15] ensure that we can take $C_p = \|\mathbf{w}_0\|_p (1 - \hat{G}_n \|\mathbf{w}_0\|_n/\mu)^{-\hat{G}_{pn}/\hat{G}_n}$ for n as above and $p \geqslant n$; of course for $p < n$ we can take $C_p = C_n$, since $\|\mathbf{w}(t)\|_p \leqslant \|\mathbf{w}(t)\|_n$.

Proof. It is essentially identical to the proof of an analogous statement concerning the NS equation, namely, Proposition 4.2 in [22]. □

4.3. ***Definition.*** *i) A* global, decaying MHD solution *is a global solution* $\mathbf{v}$ *of the MHD equations (2.29.a) with the equivalent properties (a)–(g) of Proposition 4.2.*
ii) We say that $\mathbf{v}_0 \in \mathbf{H}^\infty_{\Sigma 0}$ gives rise to a global, decaying solution for the MHD Cauchy problem *if such features are possessed by the maximal solution* $\mathbf{v}$ *of problem (2.29.a), (2.29.b) with initial datum* $\mathbf{v}_0$.
iii) The subset of $\mathbf{H}^\infty_{\Sigma 0}$ *formed by the initial data* $\mathbf{v}_0$ *as in (ii) will be indicated with* $\mathbf{E}^\infty_{\Sigma 0,\, \nu\eta} \equiv \mathbf{E}^\infty_{\Sigma 0}$.

4.4. Remarks. i) Lemma 4.1 indicates that, for any $n \in (d/2+1, +\infty)$, $\mathbf{E}^\infty_{\Sigma 0}$ contains the ball $\{\mathbf{w}_0 \in \mathbf{H}^\infty_{\Sigma 0} \mid \|\mathbf{w}_0\|_n < \mu/\hat{G}_n\}$.
ii) In Section 6, we will give examples of arbitrarily large initial data $\mathbf{v}_0 \in \mathbf{E}^\infty_{\Sigma 0}$.
iii) We already mentioned that, due to the Sobolev inequalities, the family of norms $\|\,\|_p$ ($p \in \mathbf{R}$) on $\mathbb{H}^\infty_{\Sigma 0}$ is equivalent to the family of norms $\|\,\|_{C^r}$ ($r \in \mathbf{N}$): see the comments after Eq (2.6). Similarly, the family of norms $\|\,\|_p$ ($p \in \mathbf{R}$) on $\mathbf{H}^\infty_{\Sigma 0}$ is equivalent to the family of norms $\|\,\|_{C^r}$ ($r \in \mathbf{N}$), where $\|\mathbf{v}\|_{C^r} := \max(\|v\|_{C^r}, \|c\|_{C^r})$ for $\mathbf{v} = (v, c)$. That said, let us consider a global solution $\mathbf{v}$ of the MHD equations (2.29.a). Then, conditions (a)–(g) in Proposition 4.2, are equivalent to anyone of the following two conditions:
d') For some $r \in \mathbf{N}$ with $r > d/2 + 1$ and some $\gamma \in (0, +\infty)$, it is $\int_0^{+\infty} dt\, \|\mathbf{v}(t)\|^\gamma_{C^r} < +\infty$.
e') For all $r \in \mathbf{N}$ and $\gamma \in (0, +\infty)$, it is $\int_0^{+\infty} dt\, \|\mathbf{v}(t)\|^\gamma_{C^r} < +\infty$.
The claimed equivalences follow from the inequalities relating the norms $\|\,\|_p$ and $\|\,\|_{C^r}$: in any case, see Remark 4.4 (iii) in [22] for the verification of a very similar statement, concerning the global solutions of the NS equation (2.15.a).
iv) The previous Remark (iii) is useful for comparison with the very recent paper [26], mentioned in the Introduction as a reference of special interest on the global stability of the MHD equations. Here the authors consider the global, smooth, zero mean solutions $\mathbf{v}$ of the MHD equations on $\mathbf{T}^3$ with the property that, in our notations, reads $\int_0^{+\infty} dt\, \|\mathbf{v}(t)\|^2_{C^r} < +\infty$ for some $r \in \mathbf{N}$. On the grounds of Remark (iii), this requirement is equivalent to conditions (a)–(g) in Proposition 4.2 for $r \geqslant 3$, and is implied by such conditions for $r = 1, 2$.

5. A global stability result for the MHD Cauchy problem

We repeat again that $\nu, \eta > 0$ and $\mu, \hat{G}_{pn}, \hat{G}_n, \hat{G}_p, \hat{K}_n, \hat{K}_p$ are defined by Eqs (2.23), (2.25), (2.26), and (2.28). The forthcoming statement is the main result of the present paper.

5.1. ***Theorem.*** *Let* $\mathbf{v}_0 \in \mathbf{E}^\infty_{\Sigma 0}$ *(see Definition 4.3); denote with* $\mathbf{v} \in C^\infty([0, +\infty), \mathbf{H}^\infty_{\Sigma 0})$ *the global, decaying MHD solution with initial datum* $\mathbf{v}_0$. *For each* $p \in \mathbf{R}$, *let* J_p *denote any nonnegative real number such that*

$$\int_0^{+\infty} dt\, \|\mathbf{v}(t)\|_p \leqslant J_p\ ; \tag{5.1}$$

in addition, choose any $n \in (d/2+1, +\infty)$ *and define*

$$\rho_n := \frac{\mu}{\hat{G}_n} e^{-\hat{G}_n J_n - \hat{K}_n J_{n+1}}\ . \tag{5.2}$$

Then

$$\mathbf{u}_0 \in \mathbf{H}^\infty_{\Sigma 0}\,,\ \|\mathbf{u}_0 - \mathbf{v}_0\|_n < \rho_n \implies \mathbf{u}_0 \in \mathbf{E}^\infty_{\Sigma 0}\,. \tag{5.3}$$

If $\mathbf{u}_0 \in \mathbf{H}^\infty_{\Sigma 0}$, $\|\mathbf{u}_0 - \mathbf{v}_0\|_n < \rho_n$ *and* $\mathbf{u}$ *is the global, decaying MHD solution with datum* $\mathbf{u}_0$, *for all* $t \in [0, +\infty)$ *we have:*

$$\|\mathbf{u}(t) - \mathbf{v}(t)\|_n \leqslant \frac{e^{\hat{G}_n J_n + \hat{K}_n J_{n+1}}}{1 - \delta_n/\rho_n}\,\delta_n\, e^{-\mu t}, \qquad \delta_n := \|\mathbf{u}_0 - \mathbf{v}_0\|_n\,;$$

$$\|\mathbf{u}(t) - \mathbf{v}(t)\|_p \leqslant e^{\hat{G}_p J_p + \hat{K}_p J_{p+1} + \dfrac{\hat{G}_{pn}\,\delta_n/\rho_n}{\hat{G}_n(1 - \delta_n/\rho_n)}}\,\delta_p\, e^{-\mu t} \quad \text{for } p \in (n, +\infty), \qquad \delta_p := \|\mathbf{u}_0 - \mathbf{v}_0\|_p\,. \tag{5.4}$$

Under the stronger assumptions $\mathbf{u}_0 \in \mathbf{H}^\infty_{\Sigma 0}$ *and* $\delta_n \equiv \|\mathbf{u}_0 - \mathbf{v}_0\|_n \leqslant \rho_n/2$, *the bounds (5.4) imply these simpler bounds, with linear dependence on both variables* δ_n *and* δ_p*: for all* $t \in [0, +\infty)$,

$$\begin{aligned} \|\mathbf{u}(t) - \mathbf{v}(t)\|_n &\leqslant 2\, e^{\hat{G}_n J_n + \hat{K}_n J_{n+1}}\,\delta_n\, e^{-\mu t}\ , \\ \|\mathbf{u}(t) - \mathbf{v}(t)\|_p &\leqslant e^{\hat{G}_p J_p + \hat{K}_p J_{p+1} + \dfrac{\hat{G}_{pn}}{\hat{G}_n}}\,\delta_p\, e^{-\mu t} \quad \text{for } p \in (n, +\infty). \end{aligned} \tag{5.5}$$

Proof. It is very similar to that of Proposition 5.1 in [22], concerning an analogous statement about global, decaying solutions of the NS equation. In spite of this, we think it is the case to report explicitly the argument.

With $\mathbf{v}_0$ and $\mathbf{v}$ as in the above statement, let us consider the MHD Cauchy problem (2.29.a), (2.29.b), where the datum $\mathbf{u}_0 \in \mathbf{H}^\infty_{\Sigma 0}$ is for the moment arbitrary; we write $\mathbf{u}$ for the maximal exact solution. We apply to this Cauchy problem Proposition 3.2 on approximate solutions, with

$$\mathbf{u}_\mathsf{a} := \mathbf{v}\,. \tag{5.6}$$

Since $\mathbf{v}$ solves exactly the MHD equations with datum $\mathbf{v}_0$, the differential error of $\mathbf{v}$ is zero and the datum error (with respect to (2.29.b)) is $\mathbf{u}_0 - \mathbf{v}_0$; we will use the tautological error and growth estimators associated to $\mathbf{v}$ according to Definition 3.1, which are

$$\epsilon_p(t) := 0\,, \quad \delta_p := \|\mathbf{u}_0 - \mathbf{v}_0\|_p\,, \quad \mathcal{D}_p(t) := \|\mathbf{v}(t)\|_p \qquad \text{for } p \in \mathbf{R},\ t \in [0, +\infty)\,. \tag{5.7}$$

To discuss the control (in)equalities of Proposition 3.2 with these estimators, it is convenient to introduce the primitive functions

$$\mathcal{J}_p(t) := \int_0^t ds\, \|v(s)\|_p \ \leqslant J_p \qquad \text{for } p \in \mathbf{R},\ t \in [0, +\infty) \tag{5.8}$$

(the last inequality comes from comparison with (5.1)).

Let us choose $n \in (d/2 + 1, +\infty)$, and consider the control inequalities (3.3) with the above tautological estimators. We try to fulfill the relations (3.3) as equalities for an unknown C^1 function; this yields the Cauchy problem

$$\frac{d\mathcal{R}_n}{dt} = -\mu\mathcal{R}_n + (\hat{G}_n\|\mathbf{v}\|_n + \hat{K}_n\|\mathbf{v}\|_{n+1})\mathcal{R}_n + \hat{G}_n\mathcal{R}_n^2\,, \qquad \mathcal{R}_n(0) = \delta_n \tag{5.9}$$

for an unknown function $\mathcal{R}_n \in C^1([0, T_c), \mathbf{R})$. The maximal solution of (5.9) is

$$\mathcal{R}_n(t) := \delta_n \frac{e^{-\mu t} + \hat{G}_n \mathcal{J}_n(t) + \hat{K}_n \mathcal{J}_{n+1}(t)}{1 - \hat{G}_n \delta_n \mathcal{L}_n(t)} \quad \text{for } t \in [0, T_c) \, ; \tag{5.10}$$

in the above,

$$\mathcal{L}_n(t) := \int_0^t ds\, e^{-\mu s} + \hat{G}_n \mathcal{J}_n(s) + \hat{K}_n \mathcal{J}_{n+1}(s) \ \text{ for } t \in [0, +\infty),$$
$$T_c := \begin{cases} +\infty, \ \text{if } G_n \delta_n \mathcal{L}_n(t) \neq 1 \text{ for all } t \in (0, +\infty); \\ \text{the unique } t \in (0, +\infty) \text{ s.t. } \hat{G}_n \delta_n \mathcal{L}_n(t) = 1, \\ \text{if this exists} \end{cases} \tag{5.11}$$

(note that $\mathcal{L}_n$ is strictly increasing on $[0, +\infty)$). Due to (5.8) and to the definition (5.2) of ρ_n, we have

$$\begin{aligned} \mathcal{L}_n(t) &\leqslant e^{\hat{G}_n J_n + \hat{K}_n J_{n+1}} \int_0^t ds\, e^{-\mu s} = e^{\hat{G}_n J_n + \hat{K}_n J_{n+1}} \frac{1 - e^{-\mu t}}{\mu} \\ &\leqslant \frac{1}{\mu} e^{\hat{G}_n J_n + \hat{K}_n J_{n+1}} = \frac{1}{\hat{G}_n \rho_n} \qquad \text{for } t \in [0, +\infty) \, . \end{aligned} \tag{5.12}$$

From now on we assume, as in (5.3),

$$\delta_n \equiv \|\mathbf{u}_0 - \mathbf{v}_0\|_n < \rho_n \, . \tag{5.13}$$

Then, due to (5.12),

$$G_n \delta_n \mathcal{L}_n(t) \leqslant \frac{\delta_n}{\rho_n} < 1 \text{ for all } t \in [0, +\infty) \quad \text{whence, } T_c = +\infty \, ; \tag{5.14}$$

so, the solution $\mathcal{R}_n$ in Eqs (5.10) and (5.11) is globally defined. Due to Proposition 3.2, this implies that the solution $\mathbf{u}$ of the MHD Cauchy problem (2.29.a), (2.29.b) with datum $\mathbf{u}_0$ is global as well, and that

$$\|\mathbf{u}(t) - \mathbf{v}(t)\|_n \leqslant \mathcal{R}_n(t) \quad \text{for } t \in [0, +\infty) \, ; \tag{5.15}$$

$$\begin{aligned} &\|\mathbf{u}(t) - \mathbf{v}(t)\|_p \leqslant \mathcal{R}_p(t) \text{ for } p \in (n, +\infty), \, t \in [0, +\infty), \\ &\mathcal{R}_p(t) := \delta_p \, e^{-\mu t} + \mathcal{A}_p(t) \, , \quad \mathcal{A}_p(t) := \hat{G}_p \mathcal{J}_p(t) + \hat{K}_p \mathcal{J}_{p+1}(t) + \hat{G}_{pn} \int_0^t ds\, \mathcal{R}_n(s). \end{aligned} \tag{5.16}$$

(To derive Eq (5.16) one uses Eqs (3.6) and (3.7), recalling the form (5.7) of the error and growth estimators and Eq (5.8)). Now, let us return to the expression (5.10) for $\mathcal{R}_n$ in which we insert the inequalities (5.8) for $\mathcal{J}_n, \mathcal{J}_{n+1}$ and the inequality (5.14) for $\mathcal{L}_n$; this gives

$$\begin{aligned} \mathcal{R}_n(t) &\leqslant \frac{e^{\hat{G}_n J_n + \hat{K}_n J_{n+1}}}{1 - \delta_n/\rho_n} \delta_n \, e^{-\mu t} = \frac{\mu \, \delta_n/\rho_n}{\hat{G}_n (1 - \delta_n/\rho_n)} \, e^{-\mu t} \quad \text{for } t \in [0, +\infty) \ ; \\ &\int_0^{+\infty} dt\, \mathcal{R}_n(t) \leqslant \frac{\delta_n/\rho_n}{\hat{G}_n (1 - \delta_n/\rho_n)} \end{aligned} \tag{5.17}$$

(the above equality follows from (5.2); the bound on the integral is a consequence of the bound on $\mathcal{R}_n$). Now, $\|\mathbf{u}(t)\|_n \leqslant \|\mathbf{v}(t)\|_n + \|\mathbf{u}(t) - \mathbf{v}(t)\|_n \leqslant \|\mathbf{v}(t)\|_n + \mathcal{R}_n(t)$; from here we infer, using the inequality (5.1) for J_n and the inequality in (5.17) for the integral of $\mathcal{R}_n$,

$$\int_0^{+\infty} dt\, \|\mathbf{u}(t)\|_n \leqslant J_n + \frac{\delta_n/\rho_n}{\hat{G}_n (1 - \delta_n/\rho_n)} < +\infty \, . \tag{5.18}$$

Thus the solution $\mathbf{u}$ of the MHD Cauchy problem with datum $\mathbf{u}_0$, besides being global is decaying: this means that $\mathbf{u}_0 \in \mathbf{E}^\infty_{\Sigma 0}$, so statement (5.3) is proved.

To go on, let us insert the bound (5.17) for $\mathcal{R}_n$ in Eq (5.15); this yields the bound on $\|\mathbf{u}(t) - \mathbf{v}(t)\|_n$ in (5.4).

Now, let $p \in (n, +\infty)$ and let us consider the definition of $\mathcal{A}_p(t)$ in Eq (5.16); inserting therein the inequalities (5.8) about $\mathcal{J}_p$, $\mathcal{J}_{p+1}$, writing $\int_0^t ds\, \mathcal{R}_n(s) \leqslant \int_0^{+\infty} ds\, \mathcal{R}_n(s)$ and using for the last integral the bound (5.17), we obtain

$$\mathcal{A}_p(t) \leqslant \hat{G}_p J_p + \hat{K}_p J_{p+1} + \frac{\hat{G}_{pn}\, \delta_n/\rho_n}{\hat{G}_n(1 - \delta_n/\rho_n)} \quad \text{for } t \in [0, +\infty)\,. \tag{5.19}$$

Equations (5.16) and (5.19) yield the bound on $\|\mathbf{u}(t) - \mathbf{v}(t)\|_p$ in (5.4). To conclude, let us make the stronger assumption $\delta_n \leqslant \rho_n/2$; then the bounds (5.4) yield the simpler bounds (5.5), noting that $1/(1 - \delta_n/\rho_n) \leqslant 2$ and $(\delta_n/\rho_n)/(1 - \delta_n/\rho_n) \leqslant 1$. □

5.2. Remark. Equation (5.3) indicates that $\mathbf{E}^\infty_{\Sigma 0}$ is an *open* subset of $\mathbf{H}^\infty_{\Sigma 0}$ in the Fréchet topology.

6. Generalized Beltrami pairs and MHD

Throughout this section, following a tradition we often use the word "flow" as a synonym for "vector field". In the forthcoming subsection we review known facts about Beltrami-type flows and the NS equation. In the subsequent subsection we propose an extention of this setting to the MHD equations; this will be centered upon the notion of *generalized Beltrami pair*, introduced in Definition 6.10. We will show that the MHD Cauchy problem with initial datum a generalized Beltrami pair $\mathbf{v}_0 = (v_0, c_0) \in \mathbf{H}^\infty_{\Sigma 0}$ has a global, decaying solution in the sense of Definition 4.3; we will also show that, for any real p, there exists a Beltrami pair whose norm $\|\mathbf{v}_0\|_p$ is arbitrarily large.

Known facts about Beltrami-type flows and the NS equation. Our review of these facts begins as follows.

6.1. *Definition.* *For any $d \in \{2, 3, ...\}$, a* generalized Beltrami flow on $\mathbf{T}^d$ *is a vector field w_0 such that*

$$w_0 \in \mathbb{H}^\infty_{\Sigma 0}\,, \quad \Delta w_0 = -\kappa^2\, w_0 \ \textit{for some}\ \kappa \in [1, +\infty), \quad \mathcal{P}(w_0, w_0) = 0\,. \tag{6.1}$$

6.2. Remarks. i) The above notion was proposed in [22] for arbitrary d; for $d = 3$, the same notion (or its analog on $\mathbf{R}^3$) was previously considered in [44, 45].

ii) The condition $\kappa \geqslant 1$ in (6.1) is not at all restrictive, and reflects a general features of the eigenvalues of Δ as an operator mapping into itself the space $\mathbb{H}^\infty_{\Sigma 0}$ (or even, the space $\mathbb{H}^\infty_0$ of Eq (2.8)). By elementary considerations involving the Fourier basis (e_k) $(k \in \mathbf{Z}^d)$, we readily see that the eigenvalues of Δ in $\mathbb{H}^\infty_{\Sigma 0}$ (or in $\mathbb{H}^\infty_0$) coincide with the real numbers of the form $-\kappa^2$, where $\kappa = |k|$ for some $k \in \mathbf{Z}^d \setminus \{0\}$; this implies $\kappa \geqslant 1$. Any eigenvector w_0 of Δ associated to an eigenvalue $-\kappa^2$ is a sum $w_0 = \sum_{k \in \mathbf{Z}^d \setminus \{0\}, |k| = \kappa} w_{0,k} e_k$; this also implies

$$(-\Delta)^{p/2} w_0 = \kappa^p w_0\,, \quad \|w_0\|_p = \kappa^p \|w_0\|_{L^2} \quad \text{for } p \in \mathbf{R}\,. \tag{6.2}$$

(Needless to say, if $w_0 = 0$ the equality $\Delta w_0 = -\kappa^2\, w_0$ holds trivially for any real κ, so that we can assume again $\kappa \geqslant 1$; the relations (6.2) hold as well for trivial reasons.)

iii) For each $d \geqslant 2$, there exist nontrivial (i.e., non vanishing) Beltrami flows on $\mathbf{T}^d$: see, for instance, the subsequent Example (6.3).
iv) If w_0 is a generalized Beltrami flow and $\varrho \in \mathbf{R}$, then ϱw_0 is again a generalized Beltrami flow. If $w_0 \neq 0$, we can choose ϱ so as to make ϱw_0 arbitrarily large.

6.3. Example [22]. For any $d \in \{2, 3, ...\}$ let us define a vector field $w_0 : \mathbf{T}^d \to \mathbf{R}^d$ by

$$w_0(x) := \frac{\sqrt{2}}{(2\pi)^{d/2}} W \sin(k \bullet x + \psi) \quad \text{for } x \in \mathbf{T}^d \quad (W \in \mathbf{R}^d, k \in \mathbf{Z}^d \setminus \{0\}, \psi \in \mathbf{R}) . \tag{6.3}$$

Then w_0 is C^∞ and $\langle w_0 \rangle = 0$, so that $w_0 \in \mathbb{H}^\infty_0$ (see again (2.8)). Moreover

$$\Delta w_0 = -|k|^2 w_0 \tag{6.4}$$

and $\|w_0\|_{L^2} = |W|$, whence

$$\|w_0\|_p = |k|^p |W| \quad \text{for all } p \in \mathbf{R} \tag{6.5}$$

(concerning the last statement, recall Remark 6.2 (ii)). To go on, we note that $(\operatorname{div} w_0)(x) = \sqrt{2}(2\pi)^{-d/2}(W \bullet k) \cos(k \bullet x + \psi)$, and $((w_0 \bullet \nabla) w_0)(x) = (2\pi)^{-d}(W \bullet k) W \sin(2k \bullet x + 2\psi)$ (from here one could determine $\mathcal{P}(w_0, w_0) = -\mathfrak{L}((w_0 \bullet \nabla) w_0)$. Thus

$$\begin{aligned} & w_0 \in \mathbb{H}^\infty_{\Sigma 0} \quad \Longleftrightarrow \quad W \bullet k = 0 \\ \Longleftrightarrow \quad & w_0 \text{ is a generalized Beltrami flow, and (6.1) holds with } \kappa = |k| . \end{aligned} \tag{6.6}$$

□

In the case $d = 3$, the following notion is well known [21, 46, 47].

6.4. ***Definition.*** *A* Beltrami flow *(also called a Beltrami-Trkal flow) on* $\mathbf{T}^3$ *is a vector field* w_0 *such that*

$$w_0 \in C^\infty(\mathbf{T}^3, \mathbf{R}^3) \equiv \mathbb{C}^\infty , \qquad \textit{rot}\, w_0 = \pm \kappa\, w_0 \ \textit{for some } \kappa \in [1, +\infty) \tag{6.7}$$

(with any one of the above two signs).

It is hardly the case to mention that there exist nontrivial (i.e., non vanishing) Beltrami flows: see, for instance, Example 6.7. The forthcoming statement reflects known facts, and is reviewed just for completeness.

6.5. ***Proposition.*** *A Beltrami flow on* $\mathbf{T}^3$ *is a generalized Beltrami flow. In fact, if a vector field* w_0 *fulfills conditions (6.7), it fulfills as well conditions (6.1) (with the same* κ*).*

Proof. It is well known that, for each C^∞ vector field w on $\mathbf{T}^3$, one has: $\operatorname{div}(\operatorname{rot} w) = 0$; $\langle \operatorname{rot} w \rangle = 0$; $\Delta w = -\operatorname{rot} \operatorname{rot} w + \nabla(\operatorname{div} w)$; $(w \bullet \nabla) w = (\operatorname{rot} w) \wedge w + \nabla(|w|^2/2)$. That said, assume w_0 fulfills (6.7). Then $w_0 = (\pm\kappa)^{-1} \operatorname{rot} w_0$, whence $\operatorname{div} w_0 = (\pm\kappa)^{-1} \operatorname{div}(\operatorname{rot} w_0) = 0$ and $\langle w_0 \rangle = (\pm\kappa)^{-1} \langle \operatorname{rot} w_0 \rangle = 0$; thus, $w_0 \in \mathbb{H}^\infty_{\Sigma 0}$. Moreover $\Delta w_0 = -\operatorname{rot} \operatorname{rot} w_0 = -(\pm\kappa)^2 w_0 = -\kappa^2 w_0$. To go on, we note that $(w_0 \bullet \nabla) w_0 = (\operatorname{rot} w_0) \wedge w_0 + \nabla(|w_0|^2/2) = \pm\kappa\, w_0 \wedge w_0 + \nabla(|w_0|^2/2) = \nabla(|w_0|^2/2)$; this implies $\mathcal{P}(w_0, w_0) = -\mathfrak{L} \nabla(|w_0|^2/2) = 0$, since $\mathfrak{L}$ annihilates gradients. Summing up, (6.1) holds. □

6.6. Remark. The condition $\kappa \geqslant 1$ in (6.7) has the same degree of generality as the condition $\kappa > 0$. In fact, let $w_0 \in \mathbb{C}^\infty$ and $\operatorname{rot} w_0 = \pm\kappa\, w_0$ for some $\kappa > 0$. Then, repeating the argument of the previous proof, we infer that $w_0 \in \mathbb{H}^\infty_{\Sigma 0}$ and $\Delta w_0 = -\kappa^2 w_0$. From here, recalling Remark 6.2 (ii), we are led to $\kappa \geqslant 1$.

6.7. Example. Let $w_0 : \mathbf{T}^3 \to \mathbf{R}^3$ be the C^∞ vector field defined as follows:

$$\begin{aligned} w_0(x) &= \frac{1}{(2\pi)^{3/2}}\Big[\epsilon\,(\alpha,\beta,0)\sin(\kappa x_3) + (-\beta,\alpha,0)\cos(\kappa x_3)\Big] \\ &\text{for } x = (x_1,x_2,x_3) \in \mathbf{T}^3\,(\alpha,\beta \in \mathbf{R}, \epsilon \in \{\pm 1\}, \kappa \in \{1,2,3,...\})\,. \end{aligned} \tag{6.8}$$

One readily checks that

$$\operatorname{rot} w_0 = \epsilon\,\kappa\, w_0\,, \tag{6.9}$$

so we have a Beltrami flow. We note that

$$\|w_0\|_{L^2} = \sqrt{\alpha^2+\beta^2}\,, \qquad \|w_0\|_p = \kappa^p\,\sqrt{\alpha^2+\beta^2} \quad \text{for } p \in \mathbf{R} \tag{6.10}$$

(where the first statement is checked directly, and the second follows again from Remark (6.2) (ii)).

6.8. Remark. Let us reconsider the vector field w_0 of Eq (6.3), with $W \bullet k = 0$ as in (6.6). We already know w_0 to be a generalized Beltrami flow in any space dimension d; for $d = 3$, it should be noted that w_0 is *not* a Beltrami flow if $W \neq 0$. In fact $(\operatorname{rot} w_0)(x) = \sqrt{2}(2\pi)^{-3/2}(k \wedge W)\cos(k \bullet x + \psi)$; so, $\operatorname{rot} w_0$ is not proportional to w_0 if $W \neq 0$.

The relevance of generalized Beltrami flows for the NS equation is well known. In fact, from the very Definition 6.1 one immediately infers

6.9. *Proposition.* *Let $d \in \{2,3,...\}$, $\nu > 0$. If w_0 is a generalized Beltrami flow and κ is as in Eq (6.1), the NS Cauchy problem (2.15.a), (2.15.b) with initial datum w_0 has the global solution*

$$w(t) = e^{-\kappa^2\nu t} w_0 \qquad \textit{for } t \in [0,+\infty)\,. \tag{6.11}$$

Proof. The argument is well known, and reproduced here just for completeness. Let us look for a solution w of the Cauchy problem (2.15.a), (2.15.b) of the form $w(t) = f(t)\,w_0$, where $f \in C^\infty([0,+\infty), \mathbf{R})$. Then $(dw/dt)(t) = (df/dt)(t)\,w_0$ and $\nu\Delta w(t) + \mathcal{P}(w(t), w(t)) = \nu f(t)\Delta w_0 + f^2(t)\mathcal{P}(w_0, w_0) = -\kappa^2\nu f(t)\,w_0$ due to (6.1), so w fulfills the NS equation if $(df/dt)(t) = -\kappa^2\nu f(t)$; of course, the initial condition $w(0) = w_0$ holds if $f(0) = 1$. These requirements on f are satisfied if $f(t) := e^{-\kappa^2\nu t}$ for all $t \in [0,+\infty)$, which yields the thesis (6.11). (**) □

The function in Eq (6.11) fulfills the general definition of global, decaying NS solution given in [22]. Generalized Beltrami flows and the NS solutions with these data were employed in [22] to exemplify the global stability result derived therein for the NS equation.

Beltrami-like pairs and the MHD equations. Pairs of Beltrami-type flows have been considered for several purposes in the literature about MHD equations, on suitable spatial domains: see, for instance, [26,48–52] and references therein (some of these works consider the pairs formed by the zero velocity field and by a Beltrami-type magnetic field). Hereafter we propose a new, possibly more systematic approach to the subject, in the case where the space domain is $\mathbf{T}^d$; this will be connected with the stability result in Theorem 5.1 of the present work. First of all, we propose the following

6.10. *Definition.* *Let $d \in \{2,3,...\}$. A* generalized Beltrami pair *is a pair $\mathbf{v}_0 = (v_0, c_0)$ of vector fields on $\mathbf{T}^d$ with the following properties (i)(ii):*

(**) Note that, by the vanishing of the quadratic term $\mathcal{P}(w(t), w(t))$, the NS equation reduces to the heat equation along w.

(i) Both v_0 and c_0 are generalized Beltrami flows, in the sense of Definition 6.1. So,

$$\begin{aligned} &v_0 \in \mathbb{H}^\infty_{\Sigma 0}\,,\ \Delta v_0 = -\kappa^2\, v_0\ (\kappa \in [1,+\infty))\,,\ \mathcal{P}(v_0,v_0) = 0\,;\\ &c_0 \in \mathbb{H}^\infty_{\Sigma 0}\,,\ \Delta c_0 = -\lambda^2\, c_0\ (\lambda \in [1,+\infty))\,,\ \mathcal{P}(c_0,c_0) = 0\,. \end{aligned} \tag{6.12}$$

(ii) In addition,

$$(v_0 \bullet \nabla) c_0 = (c_0 \bullet \nabla) v_0\,. \tag{6.13}$$

6.11. Remarks. i) Concerning the conditions $\kappa, \lambda \geqslant 1$ in (6.12), let us recall Remark 6.2 (ii). According to the same remark, if $v_0, c_0 \in \mathbb{H}^\infty_{\Sigma 0}$ and $\Delta v_0, \Delta c_0$ are as in (6.12), we have

$$\|v_0\|_p = \kappa^p \|v_0\|_{L^2}\,, \quad \|c_0\|_p = \lambda^p \|c_0\|_{L^2} \quad \text{for } p \in \mathbf{R}\,. \tag{6.14}$$

ii) Let $v_0, c_0 \in \mathbb{H}^\infty_{\Sigma 0}$. The mere fact that v_0, c_0 are divergence free ensures $\mathrm{div}[(v_0 \bullet \nabla) c_0 - (c_0 \bullet \nabla) v_0] = 0$ ([††]). Thus $\mathcal{P}(v_0, c_0) - \mathcal{P}(c_0, v_0) = -\mathfrak{L}[(v_0 \bullet \nabla) c_0 - (c_0 \bullet \nabla) v_0] = -[(v_0 \bullet \nabla) c_0 - (c_0 \bullet \nabla) v_0]$, that implies the following:

$$(v_0 \bullet \nabla) c_0 = (c_0 \bullet \nabla) v_0 \quad \Longleftrightarrow \quad \mathcal{P}(v_0, c_0) = \mathcal{P}(c_0, v_0)\,. \tag{6.15}$$

iii) For each $d \geqslant 2$, there exist non trivial (i.e., non vanishing) Beltrami pairs on $\mathbf{T}^d$; see, for instance, the subsequent examples 6.13, 6.14.

iv) (Similar to the previous Remark 6.2 (iv)). If $\mathbf{v}_0 = (v_0, c_0)$ is a generalized Beltrami pair and $\varrho \in \mathbf{R}$, then $\varrho \mathbf{v}_0 = (\varrho v_0, \varrho c_0)$ is again a generalized Beltrami pair. If $\mathbf{v}_0 \neq 0$, we can choose ϱ so as to make $\varrho \mathbf{v}_0$ arbitrarily large.

6.12. *Proposition*. *Let $d \in \{2, 3, ...\}$ and ν, η, μ as in (2.14), (2.23). Assume $\mathbf{v}_0 = (v_0, c_0)$ is a generalized Beltrami pair for the MHD equations, and consider the MHD Cauchy problem (2.29.a), (2.29.b) (viz., (2.16.a), (2.16.b)) with this initial datum; let κ, λ be as in (6.12). Then the following holds:*
i)The maximal solution $\mathbf{v} = (v, c)$ of the Cauchy problem is global, and given by

$$v(t) = e^{-\kappa^2 \nu t} v_0\,, \qquad c(t) = e^{-\lambda^2 \eta t} c_0 \qquad \text{for all } t \in [0, +\infty)\,. \tag{6.16}$$

ii) For all $p \in \mathbf{R}$ and $t \in [0, +\infty)$, one has

$$\begin{aligned} \|\mathbf{v}(t)\|_p &= \sqrt{e^{-2\kappa^2 \nu t}\, \kappa^{2p} \|v_0\|^2_{L^2} + e^{-2\lambda^2 \eta t}\, \lambda^{2p} \|c_0\|^2_{L^2}} \leqslant e^{-\kappa^2 \nu t}\, \kappa^p \|v_0\|_{L^2} + e^{-\lambda^2 \eta t}\, \lambda^p \|c_0\|_{L^2} \\ &\leqslant e^{-\min(\kappa^2, \lambda^2) \mu t} \sqrt{\kappa^{2p} + \lambda^{2p}} \|\mathbf{v}_0\|_{L^2}\,; \end{aligned} \tag{6.17}$$

so, besides being global $\mathbf{v}$ is decaying in the sense of Definition 4.3, and $\mathbf{v}_0$ is in the space $\mathbf{E}^\infty_{\Sigma 0}$ introduced herein.
iii) For all $p \in \mathbf{R}$ one has

$$\int_0^{+\infty} dt\, \|\mathbf{v}(t)\|_p \leqslant J_p\,, \qquad J_p := \frac{\kappa^{p-2}}{\nu} \|v_0\|_{L^2} + \frac{\lambda^{p-2}}{\eta} \|c_0\|_{L^2}\,. \tag{6.18}$$

[††] In fact, $\mathrm{div}[(v_0 \bullet \nabla) c_0 - (c_0 \bullet \nabla) u_0] = \sum_{r,s=1}^d \partial_r (v_{0s} \partial_s c_{0r} - c_{0s} \partial_s v_{0r}) = \sum_{r,s=1}^d \partial_r v_{0s} \partial_s c_{0r} + \sum_{r,s=1}^d v_{0s} \partial_s \partial_r c_{0r} - \sum_{r,s=1}^d \partial_r c_{0s} \partial_s v_{0r} - \sum_{r,s=1}^d c_{0s} \partial_s \partial_r v_{0r} = \sum_{r,s=1}^d \partial_r v_{0s} \partial_s c_{0r} + (v_0 \bullet \nabla)(\mathrm{div} c_0) - \sum_{r,s=1}^d \partial_s c_{0r} \partial_r v_{0s} - (c_0 \bullet \nabla)(\mathrm{div} v_0) = 0$ (since the last two sums over r, s coincide, and $\mathrm{div} v_0 = \mathrm{div} c_0 = 0$).

Proof. i) Let us look for a solution $\mathbf{v} = (v, c)$ of the MHD Cauchy problem of the form $v(t) = f(t)\, v_0$ and $c(t) = g(t)\, c_0$, where $f, g \in C^\infty([0, +\infty), \mathbf{R})$. Then $(dv/dt)(t) = (df/dt)(t)\, v_0$ and $\nu\Delta v(t) + \mathcal{P}(v(t), v(t)) - \mathcal{P}(c(t), c(t)) = \nu f(t)\Delta v_0 + f^2(t)\mathcal{P}(v_0, v_0) - g^2(t)\mathcal{P}(c_0, c_0) = -\kappa^2 \nu f(t)\, v_0$ due to (6.12); moreover $(dc/dt)(t) = (dg/dt)(t)\, c_0$ and $\eta\Delta c(t) + \mathcal{P}(v(t), c(t)) - \mathcal{P}(c(t), v(t)) = \eta g(t)\Delta c_0 + f(t)g(t)[\mathcal{P}(v_0, c_0) - \mathcal{P}(c_0, v_0)] = -\lambda^2 \eta g(t) c_0$, due to (6.12), (6.13) and to the equivalence (6.15). Therefore, the MHD equations are fulfilled if $(df/dt)(t) = -\kappa^2 \nu f(t)$ and $(dg/dt)(t) = -\lambda^2 \eta g(t)$. Of course, the initial conditions $v(0) = v_0$, $c(0) = c_0$ are fulfilled if $f(0) = g(0) = 1$. These requirements about f, g hold if $f(t) := e^{-\kappa^2 \nu t}$, $g(t) := e^{-\lambda^2 \eta t}$ for all $t \in [0, +\infty)$, which yields the thesis (6.16).
ii) Equation (6.16) for $\mathbf{v}$ and Eq (2.19) for $\| \|_p$ give $\|\mathbf{v}(t)\|_p = \sqrt{e^{-2\kappa^2 \nu t}\, \|v_0\|_p^2 + e^{-2\lambda^2 \eta t}\, \|c_0\|_p^2}$; from here, representing $\|v_0\|_p$ and $\|c_0\|_p$ via (6.14), we obtain the exact expression for $\|\mathbf{v}(t)\|_p$ in (6.17). The first inequality in (6.17) follows noting that $\sqrt{\alpha^2 + \beta^2} \leqslant \alpha + \beta$ for all $\alpha, \beta \geqslant 0$; the second inequality in (6.17) follows from the first one writing $e^{-\kappa^2 \nu t}, e^{-\lambda^2 \eta t} \leqslant e^{-\min(\kappa^2, \lambda^2)\mu t}$, and applying the Schwarz inequality to the expression $\kappa^p \|v_0\|_{L^2} + \lambda^p \|c_0\|_{L^2}$.
iii) The first inequality in (6.17) implies $\int_0^{+\infty} dt\, \|\mathbf{v}(t)\|_p \leqslant \kappa^p \|v_0\|_{L^2} \int_0^{+\infty} dt e^{-\kappa^2 \nu t} + \lambda^p \|c_0\|_{L^2} \int_0^{+\infty} dt e^{-\lambda^2 \eta t}$, and computing the last two integrals we get the thesis (6.18). □

Of course, Theorem 5.1 can be applied to any generalized Beltrami pair $\mathbf{v}_0$, using for the related estimates the expressions (6.18) of the constants J_p. The rest of the present subsection will be devoted to giving examples of generalized Beltrami pairs. A first example is the following.

6.13. Example. Fix a dimension $d \geqslant 2$, and consider any generalized Beltrami flow w_0 on $\mathbf{T}^d$, in the sense of Definition 6.1 (this could be, e.g., the vector field described by Eqs (6.3) and (6.6), or, if $d = 3$, the Beltrami flow of Eq (6.8). Then, any pair of the form

$$\mathbf{v}_0 = (w_0, \alpha w_0) \quad \text{or} \quad \mathbf{v}_0 = (\alpha w_0, w_0) \qquad (\alpha \in \mathbf{R}) \tag{6.19}$$

is a generalized Beltrami pair; in fact, with the choices of v_0 and c_0 indicated by (6.19), conditions (6.12) follow immediately (with $\lambda = \kappa$) from the assumptions (6.1) on w_0, and (6.13) holds trivially.

The above example of a generalized Beltrami pair is rather trivial, but it has its own utility. Pairs of the form (6.19) have been considered in [48], and are related in [50] to the notion of *Alfvén solution*. An interesting result was obtained in [26] starting from the pair $\mathbf{v}_0 = (0, w_0)$, where w_0 is a Beltrami flow on $\mathbf{T}^3$; this has the second form in Eq (6.19), with $\alpha = 0$. Considering the MHD equations with initial datum $\mathbf{v}_0$, and applying their global stability result on these equations, the authors of [26] proved that certain MHD initial data, close to $\mathbf{v}_0$, display in their evolution the magnetic reconnection phenomenon. (‡‡) A second, certainly less trivial example of a Beltrami pair is as follows.

6.14. Example. For arbitrary $d \in \{2, 3, ...\}$, let $v_0, c_0 : \mathbf{T}^d \to \mathbf{R}^d$ be the vector fields defined by

$$v_0(x) := \frac{\sqrt{2}}{(2\pi)^{d/2}}\, V \sin(k \bullet x), \quad c_0(x) := \frac{\sqrt{2}}{(2\pi)^{d/2}}\, C \sin(\ell \bullet x + \phi) \quad \text{for } x \in \mathbf{T}^d, \tag{6.20}$$

$$(V, C \in \mathbf{R}^d,\ k, \ell \in \mathbf{Z}^d \setminus \{0\},\ \phi \in \mathbf{R})\,.$$

We claim the following:

‡‡Just for completeness, let us mention that reference [49] considers Eqs (1.2) with $\nu = 0$, $\eta = 0$ on a toroidal domain $\Omega \subset \mathbf{R}^3$ (not to be confused with the torus $\mathbf{T}^3$), and investigates the stationary solutions where the pressure p is piecewise constant, $v(x, t) := 0$ and $b(x, t) := b_0(x)$, where b_0 is a piecewise Beltrami flow.

(i) Both v_0, c_0 are C^∞ with vanishing mean, i.e., $v_0, c_0 \in \mathbb{H}^\infty_0$ (recall Eq (2.8)).

(ii) We have

$$\Delta v_0 = -|k|^2 v_0\,, \qquad \Delta c_0 = -|\ell|^2 c_0\,. \tag{6.21}$$

and $\|v_0\|_{L^2} = |V|$, $\|c_0\|_{L^2} = |C|$, whence

$$\|v_0\|_p = |V|\,|k|^p\,, \qquad \|c_0\|_p = |C|\,|\ell|^p \quad \text{for all } p \in \mathbf{R}\,. \tag{6.22}$$

(iii) The pair $\mathbf{v}_0 = (v_0, c_0)$ is a generalized Beltrami pair (in the sense of Definition 6.10) if and only if

$$V\bullet k = 0\,, \qquad C\bullet \ell = 0\,, \qquad (V\bullet\ell)C = 0\,, \qquad (C\bullet k)V = 0 \tag{6.23}$$

(of course, in this case, the relations involving Δ in (6.12) hold with $\kappa = |k|$ and $\lambda = |\ell|$ due to (6.21)).

(iv) If $V, C \neq 0$, conditions (6.23) to get a generalized Beltrami pair are equivalent to $V\bullet k = V\bullet\ell = 0$ and $C\bullet k = C\bullet\ell = 0$, i.e.,

$$V, C \in \{k, \ell\}^\perp\,, \tag{6.24}$$

with $^\perp$ indicating the orthogonal complement in $\mathbf{R}^d$.

We proceed to the verification of the above claims. Statements (i) and (ii) are elementary, and in fact correspond to already mentioned facts: see Example 6.3, noting that both v_0 and c_0 have the form assumed therein for the vector field w_0. Statement (iv) is obvious, and we are left with the verification of (iii).

In order to check (iii), we first note that Example 6.3 with $w_0 = v_0$ or $w_0 = c_0$ gives

$$\begin{aligned} v_0, c_0 \in \mathbb{H}^\infty_{\Sigma 0} \quad &\Longleftrightarrow \quad V\bullet k = 0,\ C\bullet\ell = 0 \\ \Longleftrightarrow \quad &v_0, c_0 \text{ are generalized Beltrami flows, and the conditions on } \Delta v_0, \Delta c_0 \text{ in (6.12)} \\ &\text{hold with } \kappa = |k|\,,\ \lambda = |\ell|\,. \end{aligned} \tag{6.25}$$

We now discuss the validity conditions for Eq (6.13) $(v_0\bullet\nabla)c_0 = (c_0\bullet\nabla)v_0$. To this purpose we note that writing $((v_0\bullet\nabla)c_0)_r = \sum_{s=1}^d v_{0s}\partial_s c_{0r}$ we get $((v_0\bullet\nabla)c_0)(x) = 2(2\pi)^{-d}(V\bullet\ell)C\sin(k\bullet x)\cos(\ell\bullet x + \phi)$; we now express sin, cos in terms of exponentials and recall that $e^{ih\bullet x} = (2\pi)^{d/2}e_h(x)$ for all $h \in \mathbf{Z}^d$, thus concluding that

$$\begin{aligned} &(v_0\bullet\nabla)c_0 = Pe_{k+\ell} + Qe_{k-\ell} + \overline{P}e_{-(k+\ell)} + \overline{Q}e_{-(k-\ell)}\,, \\ &P := \frac{e^{i\phi}}{2(2\pi)^{d/2}i}(V\bullet\ell)C\,, \quad Q := \frac{e^{-i\phi}}{2(2\pi)^{d/2}i}(V\bullet\ell)C\,. \end{aligned} \tag{6.26}$$

In a similar way we obtain $((c_0\bullet\nabla)v_0)(x) = 2(2\pi)^{-d}(C\bullet k)V\sin(\ell\bullet x + \phi)\cos(k\bullet x)$, i.e.,

$$\begin{aligned} &(c_0\bullet\nabla)v_0 = Re_{k+\ell} + S\,e_{k-\ell} + \overline{R}e_{-(k+\ell)} + \overline{S}\,e_{-(k-\ell)}\,, \\ &R := \frac{e^{i\phi}}{2(2\pi)^{d/2}i}(C\bullet k)V\,, \quad S := -\frac{e^{-i\phi}}{2(2\pi)^{d/2}i}(C\bullet k)V\,. \end{aligned} \tag{6.27}$$

Thus $(v_0\bullet\nabla)c_0 = (c_0\bullet\nabla)v_0 \Longleftrightarrow P = R$ and $Q = S \Longleftrightarrow (V\bullet\ell)C = (C\bullet k)V$ and $(V\bullet\ell)C = -(C\bullet k)V$; so,

$$(v_0\bullet\nabla)c_0 = (c_0\bullet\nabla)v_0 \qquad \Longleftrightarrow \qquad (V\bullet\ell)C = 0,\ (C\bullet k)V = 0\,. \tag{6.28}$$

Comparing Definition 6.10 with Eqs (6.25) and (6.28), we conclude that (iii) is true.

6.15. Remark. If $d = 3$, $V \bullet k = 0, C \bullet \ell = 0$ and $V, C \neq 0$, both vector fields v_0, c_0 in the previous Example 6.14 are generalized Beltrami flows, but *not* Beltrami flows: see Remark 6.8, noting again that v_0 and c_0 have the form of the vector field w_0 considered therein.

Of course, for $d = 3$ one would like to build a generalized Beltrami pair $\mathbf{v}_0 = (v_0, c_0)$ in which both components are Beltrami flows. According to Example 6.13, a more or less trivial pair of this kind can be obtained putting $\mathbf{v}_0 = (w_0, \alpha w_0)$ or $\mathbf{v}_0 = (\alpha w_0, w_0)$, where w_0 is a Beltrami flow and $\alpha \in \mathbf{R}$. Hereafter we present another generalized Beltrami pair, made of two Beltrami flows.

6.16. Example. Let $v_0, c_0 : \mathbf{T}^3 \to \mathbf{R}^3$ be the C^∞ vector fields defined as follows:

$$\begin{aligned} v_0(x) &= \frac{1}{(2\pi)^{3/2}}\Big[\epsilon\,(\alpha,\beta,0)\sin(\kappa x_3) + (-\beta,\alpha,0)\cos(\kappa x_3)\Big], \\ c_0(x) &= \frac{1}{(2\pi)^{3/2}}\Big[\sigma(\gamma,\delta,0)\sin(\lambda x_3) + (-\delta,\gamma,0)\cos(\lambda x_3)\Big] \\ \text{for } x = (x_1,&x_2,x_3) \in \mathbf{T}^3 \quad (\alpha,\beta,\gamma,\delta \in \mathbf{R}, \epsilon,\sigma \in \{\pm 1\}, \kappa,\lambda \in \{1,2,3,...\})\,. \end{aligned} \tag{6.29}$$

Both v_0 and c_0 have the form of the vector field w_0 in Example 6.7. Thus, both v_0 and c_0 are Beltrami flows; more precisely,

$$\operatorname{rot} v_0 = \epsilon\,\kappa\, v_0\,, \qquad \operatorname{rot} c_0 = \sigma\lambda\, c_0\,. \tag{6.30}$$

According to Proposition 6.5, v_0 and c_0 fulfill Eq (6.12) in Definition 6.10 about generalized Beltrami pairs; it is readily checked that Eq (6.13) therein holds as well, since $(v_0 \bullet \nabla)c_0 = (c_0 \bullet \nabla)v_0 = 0$. Summing up, $\mathbf{v}_0 := (v_0, c_0)$ *is a generalized Beltrami pair.*

Let us also mention that, according to Example 6.7, we have $\|v_0\|_{L^2} = \sqrt{\alpha^2+\beta^2}$, $\|c_0\|_{L^2} = \sqrt{\gamma^2+\delta^2}$ and

$$\|v_0\|_p = \kappa^p \sqrt{\alpha^2+\beta^2}\,, \qquad \|c_0\|_p = \lambda^p \sqrt{\gamma^2+\delta^2} \qquad \text{for all } p \in \mathbf{R}\,. \tag{6.31}$$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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