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Research article

A mixed Darcy–power-law mobility model for pressure diffusion in complex fluids: Constitutive structure, global well-posedness, regularization, and asymptotics

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Abstract: A scalar parabolic model was proposed for pressure equalization in a fluid-saturated complex medium in which two dissipative mobility channels act in parallel. The pressure flux contains a persistent Darcy contribution and a gradient-activated power-law contribution. In dimensional variables, the closure is

$$Q = -K_1 \nabla P - K_p \, |\nabla P|^{p-2} \, \nabla P, \qquad K_1, K_p > 0, \quad p > 2,$$

and the corresponding nondimensional equation is

$$\partial_t u - \operatorname{div}\Big(\nu \nabla u + \kappa \, |\nabla u|^{p-2} \, \nabla u\Big) = f.$$

This is a standard monotone operator with $(2, p)$-growth. No novelty was claimed for the algebraic sum of the Laplacian and the p-Laplacian. The model-level contribution was the constitutive interpretation and its measurable crossover observables. On bounded connected Lipschitz domains, including homogeneous mixed boundaries, an implicit-Euler/Browder–Minty construction proved global well-posedness, energy balance, comparison, positivity, continuous dependence, and pure-Neumann mass conservation. The convex gradient-flow structure gave positive-time energy regularization and sign-preserving L^2-to-L^∞ smoothing. A separate whole-space theorem proved conditional Gaussian asymptotics with a first-moment correction. For $p \geq 3$ and $L^1 \cap L^2$ initial data, a new lemma verified the basic flux hypotheses for $1 \leq r \leq p' = p/(p-1)$. The high-gradient similarity and crossover-time predictions were explicitly heuristic, not transition-time bounds.

Keywords: nonlinear parabolic equation; Darcy law; power-law fluid; monotone operator; gradient flow; heat kernel; asymptotic expansion

1. Introduction

Throughout the article, the space dimension is an arbitrary integer $d \geq 1$; no restriction such as $d > 2$ is imposed. The macroscopic description of slow flow through a rigid porous matrix begins with Darcy's empirical pressure–flux relation [1]. Its continuum interpretation, including storage and permeability effects, is standard in porous-media mechanics [2]. When the saturating material is non-Newtonian, however, a constant linear mobility may be insufficient. Generalized Newtonian rheology, power-law response, yield effects, and pore-scale constrictions can produce nonlinear relations between the driving pressure gradient and the averaged flux; useful accounts are given in [3–5]. Modern upscaling and thermodynamic studies likewise show that effective porous transport laws can depend on the operating regime and local rheological state [6–8].

Motivated by this observation, we study the constitutive law

$$Q = -K_1 \nabla P - K_p \left|\nabla P\right|^{p-2} \nabla P, \qquad K_1 > 0, \quad K_p > 0, \quad p > 2. \tag{1.1}$$

The two terms are interpreted as parallel mobility and dissipation channels. The first remains active at arbitrarily small gradients and produces classical pressure diffusion. The second dominates at high gradients. Writing $G := |\nabla P|$ for the magnitude of the pressure gradient, the apparent mobility $|Q|/G$ increases with G when $p > 2$; in conventional power-law terminology, this is closer to a shear-thinning than to a shear-thickening response. The interpretation is deliberately phenomenological: the scalar closure does not replace a pore-scale momentum system, and its parameters must be calibrated for a specified material and geometry.

After nondimensionalization, the equation reads

$$\partial_t u - \operatorname{div} \mathcal{A}(\nabla u) = f, \qquad \mathcal{A}(\xi) = \nu \xi + \kappa \left|\xi\right|^{p-2} \xi, \qquad \nu, \kappa > 0, \quad p > 2. \tag{1.2}$$

The associated convex dissipation-potential density (not the fluid mass density)

$$\Phi(\xi) = \frac{\nu}{2} |\xi|^2 + \frac{\kappa}{p} |\xi|^p$$

has $(2, p)$-growth. Mixed-growth and nonstandard-growth energies are established mathematical objects; foundational elliptic results include [9], while double-phase regularity was developed in [10]. Accordingly, no priority claim is made for the bare differential operator $-\nu\Delta u - \kappa\Delta_p u$.

The mathematical operator and the monotone-operator tools are therefore standard. The model-level contribution is narrower: a phenomenological pressure-flux closure is paired with observable crossover quantities. The analysis then has two logically separate components: a complete bounded-domain well-posedness theory and a conditional whole-space Gaussian asymptotic theorem. The former is not used to infer the latter.

The analysis uses the monotonicity principles of Minty and Browder [11, 12], the Leray–Lions framework [13], and nonlinear evolution theory in Gelfand triples [14–17]. Sobolev truncations, weak chain rules, and compactness were used in the form developed in [18, 19]. The pure evolutionary p-Laplacian is degenerate at zero gradient; its qualitative and regularity theory is described in [20, 21]. Its finite-mass self-similar solutions and asymptotic scaling differ from those of the heat equation [22–24]. In the present mixed equation, the positive coefficient ν removes zero-gradient degeneracy, while the p-term retains stronger dissipation for steep profiles.

The long-time argument uses Gaussian kernel estimates and moment expansions. Classical fundamental-solution bounds originate in [25]. Mass-driven expansions and the role of spatial moments are closely related to [26, 27]. A technical point is essential: the spacetime condition $F \in L_t^1 L_x^1$ for $F = |\nabla u|^{p-2}\nabla u$ controls the early Duhamel contribution but, by itself, does not control the singular heat-gradient kernel as $s \uparrow t$ in every L^r space. The asymptotic theorem therefore states an additional, explicit L^r decay hypothesis. This prevents a hidden terminal-layer assumption from entering the proof.

1.1. Positioning, contributions, and organization

The heat equation has infinite propagation speed and Gaussian asymptotics, the porous-medium equation is degenerate in the state variable and has compactly supported Barenblatt profiles, and the evolutionary p-Laplacian is degenerate in the gradient and possesses a different mass-preserving similarity exponent. Here $\nu\nabla u$ preserves uniform small-gradient diffusion, while the power-law term supplies stronger high-gradient dissipation. Thus a Barenblatt-type scaling can at most describe an intermediate core, whereas the conditional whole-space endpoint is Gaussian.

The bounded-domain theorem gives existence, uniqueness, exact energy balance, continuous dependence, comparison, positivity, pure-Neumann mass conservation, and positive-time regularization. Its proof uses implicit time discretization, Browder–Minty solvability, and a Minty limit identification. The separate whole-space theorem gives Gaussian asymptotics and a first moment correction under explicit flux assumptions; a data-based lemma verifies the basic assumptions for $p \geq 3$ and $1 \leq r \leq p'$. Sections 2 and 3 derive the model and record the operator estimates, Section 4 proves the bounded-domain results, and Sections 5 and 6 treat the whole-space problem and the formal crossover scaling. The appendices are cited at the points where their chain rule, dimensional table, and regime map are used.

2. Constitutive derivation and nondimensional form

Let $P = P(X, T)$ denote excess pressure in a rigid or weakly compliant fluid-saturated region. Assume a constant storage coefficient $S > 0$, so local conservation is

$$S\,\partial_T P + \nabla_X \cdot Q = 0. \tag{2.1}$$

We postulate that two mobility and dissipation channels act in parallel. The first is a Darcy channel, $Q_D = -K_1 \nabla_X P$, with $K_1 > 0$. The second is a cooperative power-law channel,

$$Q_N = -K_p\,|\nabla_X P|^{p-2}\,\nabla_X P, \qquad K_p > 0,$$

which may represent the effective response of a pore network whose local conductance or viscous resistance changes with the imposed gradient. Adding both fluxes gives

$$S\,\partial_T P = \nabla_X \cdot \left(K_1 \nabla_X P + K_p\,|\nabla_X P|^{p-2}\,\nabla_X P\right). \tag{2.2}$$

Choose a length L, pressure P_*, and time $T_* = S L^2/K_1$. Set $x = X/L$, $t = T/T_*$, and $u = P/P_*$. Then

$$\partial_t u = \Delta u + \Lambda\,\mathrm{div}\left(|\nabla u|^{p-2}\,\nabla u\right), \qquad \Lambda = \frac{K_p P_*^{p-2}}{K_1 L^{p-2}}. \tag{2.3}$$

Retaining general coefficients gives (1.2). The ratio

$$\Re(x,t) = \frac{\kappa\,|\nabla u|^{p-1}}{\nu\,|\nabla u|} = \frac{\kappa}{\nu}\,|\nabla u|^{p-2} \tag{2.4}$$

measures the nonlinear-to-linear flux ratio. At zero gradient, $\Re$ is defined by its continuous extension, with a value of zero. Darcy-dominated mobility corresponds to $\Re \ll 1$, while the nonlinear mobility contribution dominates when $\Re \gg 1$. The dimensions of all constitutive quantities are collected in Appendix B.

2.1. Thermodynamic consistency

Define the convex dissipation-potential density

$$\Phi(\xi) = \frac{\nu}{2}\,|\xi|^2 + \frac{\kappa}{p}\,|\xi|^p\,. \tag{2.5}$$

Then $\mathcal{A} = \nabla_\xi \Phi$. Consequently

$$q \cdot \nabla u = -\mathcal{A}(\nabla u) \cdot \nabla u = -\nu\,|\nabla u|^2 - \kappa\,|\nabla u|^p \le 0.$$

The constitutive law therefore dissipates the quadratic pressure energy. Strict convexity of Φ is also the source of uniqueness.

2.2. Heuristic crossover scaling (non-rigorous)

The calculations in this subsection are formal similarity predictions. They are not used in the well-posedness or asymptotic proofs, and they do not define a sharp transition time for the full equation. Equating the magnitudes of the two fluxes gives the crossover gradient

$$G_c = \left(\frac{\nu}{\kappa}\right)^{1/(p-2)}.$$

For a localized disturbance with characteristic amplitude $U(t)$ and width $L(t)$, the gradient scale is $G(t) \sim U(t)/L(t)$. In a regime whose core is approximated by the pure p-Laplacian, mass conservation gives $UL^d \sim M$, and the formal balance $U/t \sim \kappa U^{p-1}/L^p$ yields, after normalizing the mass and the p-mobility prefactors,

$$L(t) \sim t^{1/[d(p-2)+p]}, \qquad U(t) \sim t^{-d/[d(p-2)+p]}.$$

Hence $G(t) \sim t^{-(d+1)/[d(p-2)+p]}$. This suggests the crossover time

$$t_c \asymp G_c^{-\frac{d(p-2)+p}{d+1}}, \tag{2.6}$$

up to mass- and coefficient-dependent prefactors, which are restored in (6.24). Equation (2.6) is a dimensional scaling diagnostic, not a bound on a transition time. Since the ratio $\Re(x,t)$ varies in space, different parts of a solution cross the level $\Re = 1$ at different times. Appendix C provides a schematic representation.

3. Functional framework and operator inequalities

Let $\Omega \subset \mathbb{R}^d$ be a bounded connected Lipschitz domain, $T > 0$, and $p > 2$. For $1 \le q < \infty$, $L^q(\Omega)$ is the space of measurable functions with finite norm $\|v\|_q = (\int_\Omega |v|^q \, dx)^{1/q}$, and $W^{1,q}(\Omega)$ consists of those L^q functions whose first weak derivatives also belong to $L^q(\Omega)$. The space $W_0^{1,q}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{1,q}(\Omega)$; in particular, its members have zero Sobolev trace on the whole boundary. Moreover, $W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))^*$, where $p' = p/(p-1)$.

To include mixed boundary conditions, let Γ_D and Γ_N be disjoint relatively open parts of $\partial\Omega$ whose closures cover $\partial\Omega$ up to an $\mathcal{H}^{d-1}$-null interface, and set

$$W^{1,p}_{\Gamma_D}(\Omega) := \{v \in W^{1,p}(\Omega) : \operatorname{Tr} v = 0 \ \mathcal{H}^{d-1} \text{ a.e. on } \Gamma_D\}.$$

We take

$$V = W^{1,p}_{\Gamma_D}(\Omega), \qquad H = L^2(\Omega), \qquad V^* = V'.$$

The choices $\Gamma_D = \partial\Omega$ and $\Gamma_D = \varnothing$ give the homogeneous Dirichlet and pure no-flux cases, respectively; for a genuine mixed problem, $\mathcal{H}^{d-1}(\Gamma_D) > 0$ and the natural condition on Γ_N is $\mathcal{A}(\nabla u) \cdot n = 0$. Since $p > 2$ and Ω is bounded, $V \hookrightarrow H \hookrightarrow V^*$ continuously and densely. If $\mathcal{H}^{d-1}(\Gamma_D) > 0$, the Poincaré inequality holds on V. In the pure no-flux case, coercivity of the gradient part is understood on the mean-zero subspace, and constants are retained in V.

Define $A : V \to V^*$ by

$$\langle A(v), \varphi\rangle_{V^*,V} = \int_\Omega \left(\nu \nabla v + \kappa\, |\nabla v|^{p-2}\, \nabla v\right) \cdot \nabla\varphi \, dx. \tag{3.1}$$

Lemma 3.1 (Pointwise monotonicity inequalities)**.** *For $p \ge 2$ and all $\xi, \eta \in \mathbb{R}^d$, set*

$$c_p = \frac{1}{p 2^{p-2}}.$$

Then

$$\left(|\xi|^{p-2}\,\xi - |\eta|^{p-2}\,\eta\right) \cdot (\xi - \eta) \ge c_p\, |\xi - \eta|^p\,, \tag{3.2}$$

$$\left||\xi|^{p-2}\,\xi - |\eta|^{p-2}\,\eta\right| \le (p-1)(|\xi| + |\eta|)^{p-2}\, |\xi - \eta|\,. \tag{3.3}$$

Proof. Set $\psi(\zeta) = |\zeta|^p / p$, reserving Φ for the full dissipation-potential density in (2.5). Then $\nabla\psi(\zeta) = |\zeta|^{p-2}\,\zeta$. For $\zeta \ne 0$,

$$\mathrm{D}^2\psi(\zeta) = |\zeta|^{p-2}\, I + (p-2)\, |\zeta|^{p-4}\, \zeta \otimes \zeta,$$

where $\mathrm{D}^2\psi$ is the Hessian matrix and $(a \otimes b)_{ij} = a_i b_j$. The formula extends continuously to $\zeta = 0$ for $p > 2$ (with value zero there), while for $p = 2$, the Hessian is I. Its operator norm satisfies

$$\left\|\mathrm{D}^2\psi(\zeta)\right\|_{\mathrm{op}} \le (p-1)\, |\zeta|^{p-2}\,.$$

For $s \in [0,1]$, apply the fundamental theorem of calculus to the segment $s \mapsto \eta + s(\xi - \eta)$:

$$\nabla\psi(\xi) - \nabla\psi(\eta) = \int_0^1 \mathrm{D}^2\psi(\eta + s(\xi - \eta))(\xi - \eta)\, ds.$$

Since $|\eta + s(\xi - \eta)| \le |\xi| + |\eta|$, taking Euclidean norms proves (3.3).

The standard uniform convexity of ψ already implies a p-coercive monotonicity estimate. We retain the following short proof in order to exhibit a definite constant. Define

$$g(s) = \psi(\eta + s(\xi - \eta)), \qquad s \in [0, 1].$$

Then

$$g'(1) - g'(0) = \left(|\xi|^{p-2}\,\xi - |\eta|^{p-2}\,\eta\right) \cdot (\xi - \eta).$$

Because g' is nondecreasing,

$$g(1) - g(1/2) = \int_{1/2}^{1} g'(s)\,ds \le \frac{1}{2} g'(1),$$
$$g(1/2) - g(0) = \int_{0}^{1/2} g'(s)\,ds \ge \frac{1}{2} g'(0).$$

Subtracting the second inequality from the first gives

$$g'(1) - g'(0) \ge 2[g(1) + g(0) - 2g(1/2)]. \tag{3.4}$$

Clarkson's inequality for the Euclidean norm and $p \ge 2$ states

$$\left|\frac{\xi + \eta}{2}\right|^p + \left|\frac{\xi - \eta}{2}\right|^p \le \frac{|\xi|^p + |\eta|^p}{2}.$$

After division by p and rearrangement, this becomes

$$g(1) + g(0) - 2g(1/2) \ge \frac{|\xi - \eta|^p}{p\,2^{p-1}}. \tag{3.5}$$

Combining (3.4) and (3.5) yields

$$g'(1) - g'(0) \ge \frac{|\xi - \eta|^p}{p\,2^{p-2}} = c_p\,|\xi - \eta|^p\,,$$

which proves (3.2). The constant is not sharp, but it is explicit and sufficient for the estimates below.

□

Lemma 3.2 (Properties of the evolution operator)**.** *The operator A defined by* (3.1) *is bounded on bounded subsets of V, hemicontinuous, and monotone. More precisely,*

$$\|A(v)\|_{V^*} \le C\left(\nu\,\|v\|_V + \kappa\,\|v\|_V^{p-1}\right), \tag{3.6}$$
$$\langle A(v), v\rangle = \nu\,\|\nabla v\|_2^2 + \kappa\,\|\nabla v\|_p^p\,, \tag{3.7}$$
$$\langle A(v) - A(w), v - w\rangle \ge \nu\,\|\nabla(v - w)\|_2^2 + \kappa c_p\,\|\nabla(v - w)\|_p^p\,. \tag{3.8}$$

If $\mathcal{H}^{d-1}(\Gamma_D) > 0$, then A is coercive and strictly monotone on V. In the pure no-flux case, it is coercive and strictly monotone on the mean-zero subspace or, equivalently, modulo constants. The resolvent $\tau^{-1}I + A$ is coercive and strictly monotone on all of V.

Proof. For any $\varphi \in V$, Hölder's inequality gives

$$|\langle A(v), \varphi\rangle| \le \nu \|\nabla v\|_2 \|\nabla\varphi\|_2 + \kappa \|\nabla v\|_p^{p-1} \|\nabla\varphi\|_p .$$

Because $p > 2$ and Ω is bounded, $L^p(\Omega) \hookrightarrow L^2(\Omega)$. When $\mathcal{H}^{d-1}(\Gamma_D) > 0$, Poincaré's inequality identifies the gradient norm with an equivalent norm on V. Taking the supremum over $\|\varphi\|_V \le 1$ proves (3.6).

For fixed $v, w, \varphi \in V$, the map $s \mapsto \langle A(v + sw), \varphi\rangle$ is continuous by pointwise continuity of $\mathcal{A}$ and dominated convergence, using (3.3) and the integrability of $(|\nabla v| + |\nabla w|)^{p-1} |\nabla\varphi|$. Hence A is hemicontinuous.

Identity (3.7) follows by choosing $\varphi = v$. If $\mathcal{H}^{d-1}(\Gamma_D) > 0$, Poincaré's inequality then implies

$$\frac{\langle A(v), v\rangle}{\|v\|_V} \to \infty \quad \text{as } \|v\|_V \to \infty,$$

so A is coercive. Finally, subtract the two weak forms, test by $v - w$, and apply Lemma 3.1 pointwise to obtain (3.8).

If the right-hand side vanishes, then $v - w$ is constant. The trace condition forces that constant to be zero when $\mathcal{H}^{d-1}(\Gamma_D) > 0$. In the pure no-flux case, the inequality is strict whenever $v - w$ is nonconstant. Coercivity on the mean-zero subspace follows from Poincaré–Wirtinger. The final resolvent assertion follows because $\tau^{-1} \|v\|_2^2$ controls the constant component.

□

Definition 3.3 (Weak solution)**.** Given $u_0 \in H$ and $f \in L^{p'}(0, T; V^*)$, a weak solution of

$$u_t + A(u) = f \tag{3.9}$$

is a function

$$u \in L^p(0, T; V) \cap C([0, T]; H), \qquad u_t \in L^{p'}(0, T; V^*),$$

satisfying

$$\int_0^T \langle u_t, \varphi\rangle \, dt + \int_0^T \int_\Omega \mathcal{A}(\nabla u) \cdot \nabla\varphi \, dxdt = \int_0^T \langle f, \varphi\rangle \, dt$$

for every $\varphi \in L^p(0, T; V)$ and $u(0) = u_0$ in H.

4. Global well-posedness and positive-time regularization

Theorem 4.1 (Global weak solution, structural estimates, and regularization)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded connected Lipschitz domain, let $p > 2$, and let $\nu, \kappa > 0$. Let Γ_D, Γ_N, and $V = W^{1,p}_{\Gamma_D}(\Omega)$ be as in Section 3; thus the theorem includes pure Dirichlet, pure no-flux, and homogeneous mixed Dirichlet–no-flux conditions. Let $H = L^2(\Omega)$, identify H with its dual, and use $V \hookrightarrow H \hookrightarrow V^*$. Assume*

$$u_0 \in H, \qquad f \in L^{p'}(0, T; V^*), \qquad p' = \frac{p}{p-1}.$$

Then there exists a unique

$$u \in L^p(0, T; V) \cap C([0, T]; H), \qquad u_t \in L^{p'}(0, T; V^*),$$

that satisfies $u_t + A(u) = f$ in $L^{p'}(0,T;V^*)$ and $u(0) = u_0$ in H. For every $0 \le s \le t \le T$, the solution satisfies the exact balance

$$\frac{1}{2}\|u(t)\|_2^2 + \nu\int_s^t \|\nabla u(\tau)\|_2^2\,d\tau + \kappa\int_s^t \|\nabla u(\tau)\|_p^p\,d\tau = \frac{1}{2}\|u(s)\|_2^2 + \int_s^t \langle f(\tau), u(\tau)\rangle\,d\tau. \tag{4.1}$$

If u and v correspond to data (u_0, f) and (v_0, g), respectively, then

$$\frac{1}{2}\|u(t)-v(t)\|_2^2 + \nu\int_0^t \|\nabla(u-v)\|_2^2\,ds + \kappa c_p\int_0^t \|\nabla(u-v)\|_p^p\,ds \le \frac{1}{2}\|u_0-v_0\|_2^2 + \int_0^t \langle f-g, u-v\rangle\,ds. \tag{4.2}$$

Consequently, the solution is unique and the unforced solution operator is contractive in L^2. If $u_0 \le v_0$ a.e. and $f \le g$ in the dual order, then $u \le v$ a.e.; in particular, nonnegative initial data and forcing generate a nonnegative solution. Under pure no-flux conditions,

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx + \int_0^t \langle f(s), 1\rangle\,ds, \tag{4.3}$$

and the mass is conserved when $\langle f, 1\rangle = 0$. Identity (4.3) *is asserted only when $\Gamma_D = \varnothing$. If $\Gamma_D \ne \varnothing$, then $1 \notin V$; the weak formulation neither defines $\langle f, 1\rangle$ for a general $f \in V^*$ nor excludes mass exchange through the Dirichlet part of the boundary.*

For the unforced problem $f = 0$, define the convex functional on H by

$$\mathcal{E}(v) = \begin{cases} \displaystyle\int_\Omega \left(\frac{\nu}{2}|\nabla v|^2 + \frac{\kappa}{p}|\nabla v|^p\right)dx, & v \in V, \\ +\infty, & v \notin V. \end{cases} \tag{4.4}$$

Then u is the H-gradient flow of $\mathcal{E}$. For every $t > 0$, $u(t) \in V$ and

$$\mathcal{E}(u(t)) \le \frac{\|u_0\|_2^2}{2t}. \tag{4.5}$$

Moreover, for every $0 < \delta < T$,

$$u \in H^1(\delta, T; H), \qquad A(u) = -u_t \in L^2(\delta, T; H), \tag{4.6}$$

and for almost every $\delta \le s \le t \le T$,

$$\mathcal{E}(u(t)) + \int_s^t \|u_t(\tau)\|_2^2\,d\tau = \mathcal{E}(u(s)). \tag{4.7}$$

Thus arbitrary L^2 initial pressure is instantaneously regularized into the energy domain with a quantitative t^{-1} bound.

Proof. Fix an integer $N \geq 1$, set $\tau = T/N$, $t_k = k\tau$, and let

$$f^k = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} f(s)\,ds \in V^*.$$

Starting from $u^0 = u_0 \in H \hookrightarrow V^*$, define $u^k \in V$ by

$$\left(\frac{u^k - u^{k-1}}{\tau}, \varphi\right)_H + \left\langle A(u^k), \varphi\right\rangle = \left\langle f^k, \varphi\right\rangle \qquad \text{for every } \varphi \in V. \tag{4.8}$$

For fixed k, introduce $B_\tau : V \to V^*$ by

$$\langle B_\tau(v), \varphi\rangle = \frac{1}{\tau}(v, \varphi)_H + \langle A(v), \varphi\rangle\,.$$

The embedding $V \hookrightarrow H$ is continuous, so the first term is bounded and hemicontinuous. Lemma 3.2 gives the same properties for A. For $v, w \in V$,

$$\langle B_\tau(v) - B_\tau(w), v - w\rangle \geq \frac{1}{\tau}\|v - w\|_2^2 + \nu\|\nabla(v - w)\|_2^2 + \kappa c_p\|\nabla(v - w)\|_p^p\,,$$

so B_τ is strictly monotone. Furthermore,

$$\frac{\langle B_\tau(v), v\rangle}{\|v\|_V} = \frac{\|v\|_2^2}{\tau\|v\|_V} + \frac{\nu\|\nabla v\|_2^2 + \kappa\|\nabla v\|_p^p}{\|v\|_V} \longrightarrow \infty$$

as $\|v\|_V \to \infty$; in the pure no-flux case, the H term controls constants, while the gradient term controls the mean-zero component; if $\mathcal{H}^{d-1}(\Gamma_D) > 0$, Poincaré's inequality controls the full norm. Hence B_τ is coercive. The Browder–Minty theorem therefore gives a unique u^k solving (4.8) because the right-hand side $f^k + u^{k-1}/\tau$ belongs to V^*.

Taking $\varphi = u^k$ in (4.8) and using

$$(u^k - u^{k-1}, u^k)_H = \frac{1}{2}\left(\left\|u^k\right\|_2^2 - \left\|u^{k-1}\right\|_2^2 + \left\|u^k - u^{k-1}\right\|_2^2\right)$$

yields

$$\frac{1}{2\tau}\left(\left\|u^k\right\|_2^2 - \left\|u^{k-1}\right\|_2^2 + \left\|u^k - u^{k-1}\right\|_2^2\right) + \nu\left\|\nabla u^k\right\|_2^2 + \kappa\left\|\nabla u^k\right\|_p^p = \left\langle f^k, u^k\right\rangle. \tag{4.9}$$

If $\mathcal{H}^{d-1}(\Gamma_D) > 0$, Poincaré's inequality and Young's inequality give, for every $\varepsilon > 0$,

$$\left\langle f^k, u^k\right\rangle \leq \varepsilon\left\|\nabla u^k\right\|_p^p + C_\varepsilon\left\|f^k\right\|_{V^*}^{p'}.$$

In the pure no-flux case, first choose $\varphi = 1$ in (4.8). Writing $m^k = |\Omega|^{-1}\int_\Omega u^k dx$, one obtains

$$m^k = m^{k-1} + \frac{\tau}{|\Omega|}\left\langle f^k, 1\right\rangle.$$

Hölder's inequality in time and Jensen's inequality for the averages f^k imply $\max_k \left|m^k\right| \le C(\left|m^0\right| + \|f\|_{L^{p'}(V^*)})$. The Poincaré–Wirtinger inequality then gives

$$\left\|u^k\right\|_V \le C(\left\|\nabla u^k\right\|_p + \left|m^k\right|),$$

and consequently,

$$\left\langle f^k, u^k \right\rangle \le \varepsilon \left\|\nabla u^k\right\|_p^p + C_\varepsilon \left\|f^k\right\|_{V^*}^{p'} + C\left|m^k\right| \left\|f^k\right\|_{V^*}.$$

The final term is summable because the means are uniformly bounded and $L^{p'}(0,T) \hookrightarrow L^1(0,T)$ on a finite interval. Choosing $\varepsilon < \kappa/2$, summing (4.9), and using Jensen's inequality for f^k yields

$$\begin{aligned} &\max_{0\le k\le N} \left\|u^k\right\|_2^2 + \sum_{k=1}^N \left\|u^k - u^{k-1}\right\|_2^2 + \tau \sum_{k=1}^N \left(\left\|\nabla u^k\right\|_2^2 + \left\|\nabla u^k\right\|_p^p\right) \\ &\qquad \le C\left(1 + \|u_0\|_2^2 + \|f\|_{L^{p'}(V^*)}^{p'} + \|f\|_{L^{p'}(V^*)}^2\right). \end{aligned} \tag{4.10}$$

Jensen's inequality was used to estimate $\tau \sum_k \left\|f^k\right\|_{V^*}^{p'}$ by $\|f\|_{L^{p'}(V^*)}^{p'}$. The quadratic forcing term accounts for the possible growth of the Neumann mean. The constant C depends on the fixed domain, time interval, and coefficients, but not on N or the data.

Let $\bar{u}_\tau$ and $\bar{f}_\tau$ be the right-continuous piecewise constant interpolants with values u^k and f^k on $(t_{k-1}, t_k]$, and let $\widehat{u}_\tau$ be the affine interpolant joining u^{k-1} to u^k. The discrete equation becomes

$$\partial_t \widehat{u}_\tau + A(\bar{u}_\tau) = \bar{f}_\tau \quad \text{in } L^{p'}(0,T;V^*). \tag{4.11}$$

The growth estimate (3.6) and (4.10) show that $A(\bar{u}_\tau)$ is bounded in $L^{p'}(0,T;V^*)$. Since $\bar{f}_\tau \to f$ in $L^{p'}(0,T;V^*)$, Eq (4.11) bounds $\partial_t \widehat{u}_\tau$ in that space. Hence, along a sequence $\tau \downarrow 0$,

$$\begin{aligned} \bar{u}_\tau &\rightharpoonup u && \text{in } L^p(0,T;V), \\ \bar{u}_\tau &\overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,T;H), \\ \partial_t \widehat{u}_\tau &\rightharpoonup u_t && \text{in } L^{p'}(0,T;V^*), \\ A(\bar{u}_\tau) &\rightharpoonup \chi && \text{in } L^{p'}(0,T;V^*). \end{aligned}$$

One small initial-time precaution is necessary: u_0 belongs to H, not necessarily to V, so the affine interpolant need not lie in $L^p(0,T;V)$ on its first time cell. Define $\widetilde{u}_\tau = u^1$ on $[0,\tau]$ and $\widetilde{u}_\tau = \widehat{u}_\tau$ on $[\tau,T]$. Then $\widetilde{u}_\tau$ is bounded in $L^p(0,T;V)$, its time derivative is bounded in $L^{p'}(0,T;V^*)$, and

$$\left\|\widetilde{u}_\tau - \widehat{u}_\tau\right\|_{L^2(0,T;H)}^2 \le \frac{\tau}{3}\left\|u^1 - u^0\right\|_2^2 \longrightarrow 0.$$

The compact embedding $V \Subset H$ and Aubin–Lions applied to $\widetilde{u}_\tau$ therefore give $\widehat{u}_\tau \to u$ strongly in $L^2(0,T;H)$. Moreover,

$$\int_0^T \left\|\bar{u}_\tau - \widehat{u}_\tau\right\|_2^2 dt \le \frac{\tau}{3}\sum_{k=1}^N \left\|u^k - u^{k-1}\right\|_2^2 \longrightarrow 0,$$

so $\bar{u}_\tau \to u$ strongly in the same space. Passing to the limit in the linear terms of (4.11) gives

$$u_t + \chi = f \quad \text{in } L^{p'}(0,T;V^*). \tag{4.12}$$

For each fixed $\varphi \in V$, the scalar functions $(\widehat{u}_\tau(t),\varphi)_H$ are uniformly bounded and equicontinuous: their increments are bounded by $C\,\|\varphi\|_V\,|t-s|^{1/p}$. A density argument, the uniform H bound, and the preceding strong convergence identify the weak H limit at every time. In particular, $\widehat{u}_\tau(0) \rightharpoonup u(0) = u_0$ and $\widehat{u}_\tau(T) \rightharpoonup u(T)$ in H. Since $u \in L^p(V)$ and $u_t \in L^{p'}(V^*)$, the Lions–Magenes chain rule gives $u \in C([0,T];H)$. This also justifies the terminal lower semicontinuity used next.

It remains to prove $\chi = A(u)$. Monotonicity gives, for arbitrary $v \in L^p(0,T;V)$,

$$\int_0^T \langle A(\bar{u}_\tau) - A(v), \bar{u}_\tau - v\rangle \, dt \geq 0. \tag{4.13}$$

From (4.11),

$$\int_0^T \langle A(\bar{u}_\tau), \bar{u}_\tau\rangle \, dt = \int_0^T \left\langle \bar{f}_\tau, \bar{u}_\tau \right\rangle dt - \int_0^T (\partial_t \widehat{u}_\tau, \bar{u}_\tau)_H dt.$$

On each interval, direct integration gives

$$\int_{t_{k-1}}^{t_k} (\partial_t \widehat{u}_\tau, \bar{u}_\tau)_H dt = \frac{1}{2}\left(\left\|u^k\right\|_2^2 - \left\|u^{k-1}\right\|_2^2 + \left\|u^k - u^{k-1}\right\|_2^2\right).$$

Therefore, weak lower semicontinuity of the terminal norm and strong convergence of $\bar{f}_\tau$ imply

$$\begin{aligned}\limsup_{\tau\downarrow 0} \int_0^T \langle A(\bar{u}_\tau), \bar{u}_\tau\rangle \, dt &\leq \int_0^T \langle f, u\rangle \, dt + \frac{1}{2}\,\|u_0\|_2^2 - \frac{1}{2}\,\|u(T)\|_2^2 \\ &= \int_0^T \langle \chi, u\rangle \, dt,\end{aligned}$$

where the last equality follows by pairing (4.12) with u and applying the evolution-triple chain rule. Passing to the limit in (4.13) yields

$$\int_0^T \langle \chi - A(v), u - v\rangle \, dt \geq 0 \qquad \text{for every } v \in L^p(0,T;V).$$

Choose $v = u - \lambda w$ with arbitrary $w \in L^p(0,T;V)$ and $\lambda > 0$. Division by λ, hemicontinuity, and the growth bound permit $\lambda \downarrow 0$, giving $\int_0^T \langle \chi - A(u), w\rangle \, dt \geq 0$. Replacing w by $-w$ proves $\chi = A(u)$ almost everywhere. Thus $u_t + A(u) = f$.

Pairing the equation with u is legitimate because $u \in L^p(V)$ and $u_t \in L^{p'}(V^*)$. The chain rule gives

$$\frac{d}{dt}\frac{1}{2}\,\|u(t)\|_2^2 + \nu\,\|\nabla u(t)\|_2^2 + \kappa\,\|\nabla u(t)\|_p^p = \langle f(t), u(t)\rangle$$

in $L^1(0,T)$, and integration proves (4.1). The evolution-triple chain rule used here is recalled in Appendix A. For two solutions, subtract the equations, pair with $u - v$, and use (3.8); this proves (4.2) and uniqueness.

For comparison, set $z = u - v$ and use $T_\varepsilon(z^+)$ as a test function, where $T_\varepsilon(r) = \min\{r, \varepsilon\}$. The Sobolev trace commutes with Lipschitz composition, so $T_\varepsilon(z^+) \in V$: it has zero trace on Γ_D, while the no-flux condition on Γ_N is already encoded in the weak formulation. Consequently, neither a boundary term nor a boundary-layer correction occurs, including at a mixed-boundary interface. The chain rule for Sobolev truncations, the dual-order assumption, and monotonicity give

$$\frac{d}{dt}\int_\Omega \int_0^{z(x,t)} T_\varepsilon(s^+)\,ds\,dx + \int_{\{0<z<\varepsilon\}} (\mathcal{A}(\nabla u) - \mathcal{A}(\nabla v)) \cdot \nabla z\,dx \le \langle f - g, T_\varepsilon(z^+)\rangle \le 0.$$

The diffusion term is nonnegative. Since $z^+(0) = 0$, integration in time forces the nonnegative primitive to vanish; letting $\varepsilon \downarrow 0$ gives $z^+ = 0$. Positivity follows by comparing with the zero solution. In the pure no-flux formulation, and only there, $1 \in V$; testing by 1 gives (4.3).

Assume now $f = 0$. The functional (4.4) is proper, convex, and lower semicontinuous on H, and its Hilbert-space subdifferential is the realization of A in H. The discrete equation is equivalently the Euler condition for the unique minimizer

$$u^k = \operatorname*{argmin}_{v\in H} \left\{ \frac{1}{2\tau} \left\| v - u^{k-1} \right\|_2^2 + \mathcal{E}(v) \right\}.$$

The subgradient inequality, with arbitrary $v \in V$, reads

$$\frac{1}{\tau}(u^k - u^{k-1}, v - u^k)_H + \mathcal{E}(v) - \mathcal{E}(u^k) \ge 0.$$

Using the polarization identity and summing from 1 to m gives

$$\frac{1}{2}\|u^m - v\|_2^2 + \tau \sum_{k=1}^{m} (\mathcal{E}(u^k) - \mathcal{E}(v)) \le \frac{1}{2}\|u_0 - v\|_2^2.$$

Because $\mathcal{E}(u^k)$ is nonincreasing for $k \ge 1$, one obtains

$$\mathcal{E}(u^m) \le \mathcal{E}(v) + \frac{\|u_0 - v\|_2^2}{2m\tau}.$$

Choosing $v = 0$ and passing to the limit proves (4.5). Comparing the minimizer with u^{k-1} also gives

$$\frac{1}{2\tau}\left\|u^k - u^{k-1}\right\|_2^2 + \mathcal{E}(u^k) \le \mathcal{E}(u^{k-1}).$$

Starting the sum at a time level approximating $\delta/2$ and using the preceding t^{-1} energy estimate yields a bound for the affine derivatives in $L^2(\delta, T; H)$, uniform in τ. Passage to the limit proves (4.6). Since $u_t + A(u) = 0$, the operator value belongs to $L^2(\delta, T; H)$. Finally, the Hilbert-space chain rule for convex subdifferentials gives

$$\frac{d}{dt}\mathcal{E}(u(t)) = (A(u(t)), u_t(t))_H = -\|u_t(t)\|_2^2$$

for almost every positive time. Integration proves (4.7) and completes the proof. □

Remark 4.2 (Why the regularization statement is unforced)**.** The existence, comparison, and balance assertions allow $f \in L^{p'}(0,T;V^*)$. The stronger conclusions (4.5)–(4.7) use $f = 0$ precisely when the Euler step is identified with the autonomous minimizing movement for $\mathcal{E}$ and when $A(u) = -u_t$ yields $\frac{d}{dt}\mathcal{E}(u) = -\|u_t\|_2^2$. For a time-independent $f_0 \in H$, the same argument applies to the tilted energy $\mathcal{E}_{f_0}(v) = \mathcal{E}(v) - (f_0, v)_H$, with a lower-bound condition if one wants the corresponding global dissipation estimate. The variational estimate then reads

$$\mathcal{E}_{f_0}(u(t)) \le \mathcal{E}_{f_0}(v) + \frac{\|u_0 - v\|_2^2}{2t}, \qquad v \in V,$$

not the unforced bound (4.5). The tilted functional is bounded below if $\mathcal{H}^{d-1}(\Gamma_D) > 0$; in the pure Neumann case, this requires $\int_\Omega f_0\,dx = 0$ because constants have zero energy. For a general time-dependent $f \in L^{p'}(V^*)$, there is no fixed Lyapunov functional and the pairing $(f, u_t)_H$ need not even be defined.

A variational alternative is the minimizing-movement approach and its evolution variational inequality. If the energy itself is approximated, Mosco convergence of convex energies provides an appropriate route to resolvent and gradient-flow convergence. Here the energy is fixed and only the time step tends to zero: static Γ-convergence alone is not a substitute for identifying the evolution. The direct Minty argument above supplies that identification under the stated hypotheses.

Corollary 4.3 (Exponential stability for identical forcing)**.** *Let u and v be two weak solutions driven by the same forcing $f \in L^{p'}(0,T;V^*)$, with initial data $u_0, v_0 \in H$. If $\mathcal{H}^{d-1}(\Gamma_D) > 0$, then, for every $t \in [0,T]$,*

$$\|u(t) - v(t)\|_2 \le e^{-\nu\lambda_1 t}\,\|u_0 - v_0\|_2\,,$$

where

$$\lambda_1 := \inf_{\substack{\varphi \in W^{1,2}_{\Gamma_D}(\Omega)\\ \varphi \neq 0}} \frac{\|\nabla\varphi\|_2^2}{\|\varphi\|_2^2} > 0.$$

In the pure no-flux case, set

$$\overline{z} := \frac{1}{|\Omega|}\int_\Omega (u_0 - v_0)\,dx$$

and let $\lambda_1^N > 0$ be the first nonzero Neumann eigenvalue. Then

$$\|u(t) - v(t) - \overline{z}\|_2 \le e^{-\nu\lambda_1^N t}\,\|u_0 - v_0 - \overline{z}\|_2\,.$$

In particular, pure no-flux solutions with equal initial mass contract exponentially toward one another.

Proof. Set

$$z := u - v.$$

When $\mathcal{H}^{d-1}(\Gamma_D) > 0$, the difference has zero trace on Γ_D. Thus, for almost every $t > 0$,

$$z(t) \in W^{1,2}_{\Gamma_D}(\Omega).$$

Moreover, the regularity of the weak solutions implies that the map

$$t \longmapsto \frac{1}{2}\,\|z(t)\|_2^2$$

is absolutely continuous on every bounded time interval. Consequently, the difference equation may be tested with $z(t)$. Subtracting the weak formulation satisfied by v from that satisfied by u, and choosing $z = u - v$ as a test function, gives, for almost every $t > 0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_2^2 + \nu\|\nabla z(t)\|_2^2 + \kappa\int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot(\nabla u - \nabla v)\,\mathrm{d}x = 0.$$

No forcing term appears because the two identical forcing terms cancel on subtraction; neither solution needs to be unforced. The Dirichlet trace on Γ_D and the natural zero flux on Γ_N eliminate all boundary contributions. The nonlinear mapping

$$A_p : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \qquad A_p(\xi) := |\xi|^{p-2}\xi,$$

is monotone for every $p \geq 2$. In particular,

$$(A_p(\xi) - A_p(\eta))\cdot(\xi - \eta) \geq 0 \qquad \text{for all } \xi, \eta \in \mathbb{R}^d.$$

Applying this pointwise with

$$\xi = \nabla u(x,t), \qquad \eta = \nabla v(x,t),$$

and then integrating over Ω, we obtain

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot(\nabla u - \nabla v)\,\mathrm{d}x \geq 0.$$

Therefore, the nonlinear diffusion term can be omitted from the left-hand side while preserving the inequality. It follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_2^2 + \nu\|\nabla z(t)\|_2^2 \leq 0$$

for almost every $t > 0$. Because $z(t) \in W^{1,2}_{\Gamma_D}(\Omega)$, the variational characterization of λ_1 yields the Poincaré inequality

$$\|\nabla z(t)\|_2^2 \geq \lambda_1\|z(t)\|_2^2.$$

Substitution into the preceding differential inequality gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_2^2 + \nu\lambda_1\|z(t)\|_2^2 \leq 0.$$

Equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_2^2 + 2\nu\lambda_1\|z(t)\|_2^2 \leq 0.$$

Define

$$Y(t) := \|z(t)\|_2^2.$$

Then Y is nonnegative and absolutely continuous, and it satisfies

$$Y'(t) + 2\nu\lambda_1 Y(t) \leq 0$$

for almost every $t > 0$. Multiplication by the integrating factor $e^{2\nu\lambda_1 t}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{2\nu\lambda_1 t}Y(t)\right) = e^{2\nu\lambda_1 t}\left(Y'(t) + 2\nu\lambda_1 Y(t)\right) \leq 0.$$

Hence the function

$$t \longmapsto e^{2\nu\lambda_1 t} Y(t)$$

is nonincreasing. Integrating between 0 and t, and using the continuity of the solutions in $L^2(\Omega)$ at the initial time, gives

$$e^{2\nu\lambda_1 t} Y(t) \le Y(0).$$

Since

$$Y(0) = \|u_0 - v_0\|_2^2 ,$$

we conclude that

$$\|u(t) - v(t)\|_2^2 \le e^{-2\nu\lambda_1 t} \|u_0 - v_0\|_2^2 .$$

Taking the nonnegative square root on both sides yields

$$\|u(t) - v(t)\|_2 \le e^{-\nu\lambda_1 t} \|u_0 - v_0\|_2 ,$$

which proves the first estimate.

For pure no-flux data, (4.3) shows that the spatial mean of z is the constant $\bar{z}$. Testing the difference equation with $z - \bar{z}$, whose gradient equals ∇z, and applying the Poincaré–Wirtinger inequality gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|z - \bar{z}\|_2^2 + \nu\lambda_1^N \|z - \bar{z}\|_2^2 \le 0.$$

The integrating-factor argument proves the no-flux estimate. If the forcings are different, the difference identity instead contains $\langle f - g, z\rangle$ on its right-hand side; it yields the usual forced stability estimate after Hölder–Young and Gronwall, but not the homogeneous contraction displayed above.

□

Proposition 4.4 (Sign-preserving L^2–L^∞ smoothing)**.** *Assume $f = 0$ and $u_0 \in L^2(\Omega)$ with $u_0 \ge 0$ a.e. For any of the pure or mixed homogeneous boundary conditions described above, the solution of Theorem 4.1 satisfies $u(t) \ge 0$ and*

$$\|u(t)\|_\infty \le C_{\Omega,d} \left[1 + (\nu t)^{-d/4}\right] \|u_0\|_2 , \qquad t > 0. \tag{4.14}$$

Thus the positive-time regularization preserves the sign and includes boundedness, not only membership in the energy domain. No higher-order boundary regularity at Dirichlet–Neumann junctions is claimed.

Proof. Positivity is the comparison assertion of Theorem 4.1. We initially take bounded initial data. Testing with $(u - \|u_0\|_\infty)^+$ gives the maximum bound, so powers of u can be used through bounded Lipschitz extensions. For $q \ge 2$, we test with u^{q-1}. The trace condition is stable under these compositions, and the natural no-flux part contributes no boundary term. The weak chain rule gives, for almost every positive time,

$$\begin{aligned} \frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_q^q + \nu(q-1) \int_\Omega u^{q-2} |\nabla u|^2 \, dx \\ + \kappa(q-1) \int_\Omega u^{q-2} |\nabla u|^p \, dx = 0. \end{aligned} \tag{4.15}$$

Since $u^{q-2}\,|\nabla u|^2 = (4/q^2)\left|\nabla(u^{q/2})\right|^2$, dropping the nonnegative p-term gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u\|_q^q + \frac{4\nu(q-1)}{q}\left\|\nabla(u^{q/2})\right\|_2^2 \le 0.$$

On a bounded Lipschitz domain, the Nash inequality with the lower-order term,

$$\|w\|_2^{2+4/d} \le C_{\Omega,d}(\|\nabla w\|_2^2 + \|w\|_2^2)\,\|w\|_1^{4/d},$$

holds for $w \in H^1(\Omega)$. To make the iteration explicit, let $q \ge 4$, $s < t$, $0 < \nu(t-s) \le 1$, and $M = \|u(s)\|_{q/2}$. The $L^{q/2}$ norm is nonincreasing by (4.15). With $Y = \|u\|_q^q$, the Nash inequality and $2 \le 4(q-1)/q \le 4$ give

$$Y' \le -c\nu M^{-2q/d} Y^{1+2/d} + 4\nu Y,$$

where $c > 0$ depends only on Ω, d; if $M = 0$, the solution is zero and there is nothing to prove. Applying this inequality to $e^{-4\nu(\cdot - s)}Y$ and integrating the resulting scalar differential inequality yields

$$\|u(t)\|_q \le C^{1/q}[\nu(t-s)]^{-d/(2q)}\,\|u(s)\|_{q/2},$$

with C independent of q. Fix $h = \min\{t, \nu^{-1}\}$ and iterate with $q_j = 2^{j+1}$ and $s_j = t - h2^{-j}$, $j \ge 0$. Here $s_j - s_{j-1} = h2^{-j}$ and

$$\sum_{j=1}^{\infty} \frac{1}{q_j} = \frac{1}{2}, \qquad \sum_{j=1}^{\infty} \frac{j}{q_j} = 1.$$

Using the L^{q_j} contraction between s_j and t and then letting $j \to \infty$, we obtain

$$\|u(t)\|_\infty \le C_{\Omega,d}(\nu h)^{-d/4}\,\|u(t-h)\|_2 \le C_{\Omega,d}\max\{1, (\nu t)^{-d/4}\}\,\|u_0\|_2.$$

This proves (4.14) for bounded data. Finally, we use $u_{0,n} = \min\{u_0, n\}$, the L^2 contraction, and the uniform bound to pass to arbitrary nonnegative $u_0 \in L^2(\Omega)$. □

5. Whole-space formulation and heat-kernel preliminaries

On $\mathbb{R}^d$, with arbitrary $d \ge 1$, consider

$$u_t - \nu\Delta u = \kappa\,\mathrm{div}\,F, \qquad F = |\nabla u|^{p-2}\,\nabla u, \qquad u(0) = u_0. \tag{5.1}$$

Let

$$G_\nu(x,t) = (4\pi\nu t)^{-d/2}\exp\!\left(-\frac{|x|^2}{4\nu t}\right).$$

Definition 5.1 (Global mild solution)**.** In the finite-mass class considered here, given $u_0 \in L^1(\mathbb{R}^d)$, a global mild solution of (5.1) is a function $u \in C([0,\infty); L^1(\mathbb{R}^d))$, with $u(0) = u_0$, having a weak spatial gradient such that $F = |\nabla u|^{p-2}\,\nabla u$ is measurable and, for every $t > 0$,

$$s \longmapsto \sum_{i=1}^{d} \partial_i G_\nu(t-s) * F_i(s) \quad \text{belongs to } L^1((0,t); L^1(\mathbb{R}^d)).$$

The identity

$$u(t) = G_\nu(t) * u_0 + \kappa \int_0^t \nabla G_\nu(t-s) * F(s)\, ds \tag{5.2}$$

is required in $L^1(\mathbb{R}^d)$; the vector convolution denotes the sum displayed above. When estimating this formula in L^r, convergence of the same integral in L^r is justified separately by the assumptions of Theorem 6.1 for $t \geq 2$.

The sign is positive because $G_\nu * \operatorname{div} F = \operatorname{div}(G_\nu * F) = \nabla G_\nu * F$. For $1 \leq r \leq \infty$, set

$$\alpha_r = \frac{d}{2}\left(1 - \frac{1}{r}\right).$$

This exponent is defined in every dimension $d \geq 1$. Below $C_{d,r,\nu} > 0$ denotes a generic constant depending only on the displayed parameters and may change from line to line. Scaling gives

$$\|G_\nu(t)\|_r = C_{d,r,\nu} t^{-\alpha_r}, \tag{5.3}$$

$$\|\nabla G_\nu(t)\|_r = C_{d,r,\nu} t^{-\alpha_r - 1/2}, \tag{5.4}$$

$$\left\|D^2 G_\nu(t)\right\|_r = C_{d,r,\nu} t^{-\alpha_r - 1}, \tag{5.5}$$

$$\|\partial_t \nabla G_\nu(t)\|_r = C_{d,r,\nu} t^{-\alpha_r - 3/2}. \tag{5.6}$$

Lemma 5.2 (A data-based criterion for nonlinear-flux integrability)**.** *Let $d \geq 1$, $p \geq 3$, and $u_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Let u be the whole-space L^2-gradient-flow solution associated with*

$$\mathcal{E}_{\mathbb{R}^d}(v) := \int_{\mathbb{R}^d} \left(\frac{\nu}{2}|\nabla v|^2 + \frac{\kappa}{p}|\nabla v|^p\right) dx,$$

whose effective domain is

$$\mathcal{D}(\mathcal{E}_{\mathbb{R}^d}) = \{v \in L^2(\mathbb{R}^d) : \nabla v \in L^2(\mathbb{R}^d;\mathbb{R}^d) \cap L^p(\mathbb{R}^d;\mathbb{R}^d)\}.$$

Then the solution is L^1-contractive and, with $F = |\nabla u|^{p-2}\nabla u$,

$$F \in L^1((0,\infty); L^1(\mathbb{R}^d;\mathbb{R}^d)). \tag{5.7}$$

Moreover, for $1 \leq r \leq p' = p/(p-1)$ and almost every $t \geq 1$,

$$\|F(t)\|_r \leq C_{d,p,r,\nu,\kappa} \|u_0\|_1^{2/r} t^{-(1+d/2)/r}. \tag{5.8}$$

Thus the local decay assumption (6.2) *below follows for $1 \leq r \leq p'$ with $\gamma_r = (1+d/2)/r$, since*

$$\gamma_r - \alpha_r - \frac{1}{2} = (d+1)\left(\frac{1}{r} - \frac{1}{2}\right) > 0.$$

If $u_0 \geq 0$, the solution remains nonnegative and its conserved mass is $\int_{\mathbb{R}^d} u_0\, dx$.

Proof. The functional $\mathcal{E}_{\mathbb{R}^d}$ is proper, convex, lower semicontinuous on L^2, and has a dense effective domain. Its subdifferential is maximal monotone and generates the global gradient-flow solution [15]. For the L^1 estimate, let $v + \lambda A(v) = h$ and $w + \lambda A(w) = k$ be two resolvent equations with $h, k \in L^1 \cap L^2$. Test their difference with $T_\varepsilon(v - w) = \max\{-1, \min\{1, (v - w)/\varepsilon\}\}$. This is an admissible energy-space test: it belongs to L^2 and its gradient is a bounded multiple of $\nabla(v - w)$. Monotonicity gives

$$\int_{\mathbb{R}^d} (v - w) T_\varepsilon(v - w)\, dx \le \int_{\mathbb{R}^d} (h - k) T_\varepsilon(v - w)\, dx \le \|h - k\|_1 .$$

Letting $\varepsilon \downarrow 0$ shows that the resolvent is contractive in L^1. The contraction-semigroup construction and density of C_c^∞ in L^1 give the consistent L^1-continuous extension and hence

$$\|u(t)\|_1 \le \|u_0\|_1 , \qquad t \ge 0.$$

The same truncation argument proves positivity.

Testing the evolution by u and using the gradient-flow energy identity give

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 + D(t) = 0, \tag{5.9}$$
$$D(t) := \nu \|\nabla u(t)\|_2^2 + \kappa \|\nabla u(t)\|_p^p , \qquad D(t) \ge 2\mathcal{E}_{\mathbb{R}^d}(u(t)).$$

The Nash inequality [28]

$$\|v\|_2^{2+4/d} \le C_d \|v\|_1^{4/d} \|\nabla v\|_2^2$$

and L^1 contraction imply, by solving the resulting differential inequality for $\|u(t)\|_2^2$,

$$\|u(t)\|_2^2 \le C_d (\nu t)^{-d/2} \|u_0\|_1^2 , \qquad t > 0. \tag{5.10}$$

The map $t \mapsto \mathcal{E}_{\mathbb{R}^d}(u(t))$ is nonincreasing for $t > 0$. Integrating (5.9) from $t/2$ to t therefore gives

$$t\, \mathcal{E}_{\mathbb{R}^d}(u(t)) \le \int_{t/2}^{t} D(s)\, ds \le \frac{1}{2}\|u(t/2)\|_2^2 ,$$

and hence, by (5.10),

$$\mathcal{E}_{\mathbb{R}^d}(u(t)) \le C_{d,\nu} \|u_0\|_1^2\, t^{-1-d/2}. \tag{5.11}$$

For $p \ge 3$ and $1 \le r \le p'$, one has $2 \le r(p-1) \le p$, so $a^{r(p-1)} \le a^2 + a^p$ for every $a \ge 0$. Consequently,

$$\|F(t)\|_r^r = \int_{\mathbb{R}^d} |\nabla u(t)|^{r(p-1)}\, dx \le \|\nabla u(t)\|_2^2 + \|\nabla u(t)\|_p^p \le \left(\frac{2}{\nu} + \frac{p}{\kappa}\right)\mathcal{E}_{\mathbb{R}^d}(u(t)),$$

which proves (5.8). Taking $r = 1$, integrating the same pointwise inequality in time, and using (5.9) yields

$$\int_0^\infty \|F(t)\|_1\, dt \le \left(\frac{1}{\nu} + \frac{1}{\kappa}\right)\int_0^\infty D(t)\, dt \le \frac{1}{2}\left(\frac{1}{\nu} + \frac{1}{\kappa}\right)\|u_0\|_2^2 ,$$

which proves (5.7). For each $t > 0$, the variation-of-constants integral converges in L^1: on $0 < s < t/2$, its norm is bounded by $Ct^{-1/2}\int_0^{t/2} \|F(s)\|_1\, ds$, and on $t/2 < s < t$, (5.11) bounds $\|F(s)\|_1$ uniformly

while $\|\nabla G_\nu(t-s)\|_1 = C(t-s)^{-1/2}$ is integrable. Testing the weak equation against backward heat evolutions of smooth compactly supported functions gives (5.2), first in distributions and then in L^1. Thus this gradient flow is a mild solution in the sense of Definition 5.1. Fubini's theorem applies to its Bochner integral, whose spatial integral vanishes because $\int_{\mathbb{R}^d} \nabla G_\nu(x, t-s)\,dx = 0$. This proves mass conservation, also for signed data. Notice that for $2 < p < 3$, the energy controls the L^2 and L^p norms of ∇u, but on $\mathbb{R}^d$, it does not by itself control the lower power $|\nabla u|^{p-1}$ in L^1. Additional spatial information is therefore genuinely needed in that range. □

Lemma 5.3 (Mass and moment expansion for the linear heat flow)**.** *Let $u_0 \in L^1(\mathbb{R}^d)$ and $M = \int_{\mathbb{R}^d} u_0\,dx$. Then*

$$t^{\alpha_r}\,\|G_\nu(t) * u_0 - MG_\nu(t)\|_r \longrightarrow 0 \qquad (t \to \infty)$$

for every $1 \le r \le \infty$. If $\int_{\mathbb{R}^d} |x|\,|u_0(x)|\,dx < \infty$ and $m_0 = \int_{\mathbb{R}^d} xu_0(x)\,dx$, then

$$\|G_\nu(t) * u_0 - MG_\nu(t)\|_r \le Ct^{-\alpha_r - 1/2} \int_{\mathbb{R}^d} |x|\,|u_0(x)|\,dx, \tag{5.12}$$

$$t^{\alpha_r+1/2}\,\|G_\nu(t) * u_0 - MG_\nu(t) + m_0 \cdot \nabla G_\nu(t)\|_r \longrightarrow 0. \tag{5.13}$$

If, in addition, $\int_{\mathbb{R}^d} |x|^2\,|u_0(x)|\,dx < \infty$, then

$$\|G_\nu(t) * u_0 - MG_\nu(t) + m_0 \cdot \nabla G_\nu(t)\|_r \le Ct^{-\alpha_r - 1} \int_{\mathbb{R}^d} |x|^2\,|u_0(x)|\,dx. \tag{5.14}$$

Proof. By convolution and the definition of M,

$$G_\nu(t) * u_0 - MG_\nu(t) = \int_{\mathbb{R}^d} [G_\nu(\cdot - y, t) - G_\nu(\cdot, t)]u_0(y)dy.$$

For fixed y, scaling reduces the expression multiplied by t^{α_r} to a translation difference of the time-one Gaussian by $y/\sqrt{t}$. Translation continuity in L^r for finite r, and uniform continuity for $r = \infty$, give pointwise convergence in y. The scaled norm is bounded by $2C_{d,r,\nu}$, so dominated convergence proves the leading mass statement.

If the first absolute moment is finite, the fundamental theorem of calculus along $\theta \mapsto x - \theta y$ gives

$$G_\nu(x-y,t) - G_\nu(x,t) = -\int_0^1 y \cdot \nabla G_\nu(x - \theta y, t)d\theta.$$

Minkowski's inequality, translation invariance, and (5.4) prove (5.12). Adding $y \cdot \nabla G_\nu(x,t)$ inside the integral produces the first-order translation remainder. After multiplication by $t^{\alpha_r+1/2}$ and scaling to time one, differentiability of translations in L^r (and uniform differentiability of the Gaussian for $r = \infty$) gives convergence for every fixed y; the first-moment bound supplies domination. This proves (5.13). A second application of the fundamental theorem of calculus gives

$$|G_\nu(x-y,t) - G_\nu(x,t) + y \cdot \nabla G_\nu(x,t)| \le |y|^2 \int_0^1 (1-\theta)\left|D^2 G_\nu(x - \theta y, t)\right| d\theta,$$

and (5.5) yields (5.14). □

6. Gaussian asymptotics and the first nonlinear correction

6.1. Conditional Gaussian asymptotics

Theorem 6.1 (Gaussian endpoint and dipole correction)**.** *Fix $1 \le r \le \infty$ and let u be a global mild solution of* (5.1) *satisfying* (5.2)*. Assume*

$$u_0 \in L^1(\mathbb{R}^d), \qquad F \in L^1((0,\infty); L^1(\mathbb{R}^d;\mathbb{R}^d)), \tag{6.1}$$

and suppose that for some $\gamma_r > \alpha_r + \frac{1}{2}$,

$$\|F(t)\|_r \le C_F(1+t)^{-\gamma_r} \qquad \text{for almost every } t \ge 1. \tag{6.2}$$

Then the mass $M = \int_{\mathbb{R}^d} u_0(x)\,dx$ is conserved and

$$t^{\alpha_r}\,\|u(t) - MG_\nu(t)\|_r \longrightarrow 0 \qquad (t \to \infty). \tag{6.3}$$

The assumptions have distinct roles: $L^1_t L^1_x$ controls the portion of the Duhamel integral separated from $s = t$, while (6.2) *controls the singular terminal layer through the L^1 norm of ∇G_ν.*

Assume further that

$$\int_{\mathbb{R}^d} |x|\,|u_0(x)|\,dx < \infty, \qquad \int_0^\infty \int_{\mathbb{R}^d} (1+s+|x|)\,|F(x,s)|\;dxds < \infty, \tag{6.4}$$

and that (6.2) *holds with the stronger exponent $\gamma_r > \alpha_r + 1$. Define*

$$m_0 = \int_{\mathbb{R}^d} x u_0(x) dx, \qquad B = \kappa \int_0^\infty \int_{\mathbb{R}^d} F(x,s)\,dxds. \tag{6.5}$$

Then

$$t^{\alpha_r+1/2}\,\|u(t) - MG_\nu(t) - (B-m_0)\cdot\nabla G_\nu(t)\|_r \longrightarrow 0. \tag{6.6}$$

Thus the leading profile is determined only by mass, whereas the first dipole correction records the initial first moment and the accumulated nonlinear flux with opposite signs.

Proof. Integrating (5.2) over $\mathbb{R}^d$ gives $\int_{\mathbb{R}^d} u(x,t)\,dx = M$. Indeed, the heat term preserves mass and, for almost every $s < t$, Fubini's theorem is justified by $F(s) \in L^1(\mathbb{R}^d;\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} (\nabla G_\nu(t-s) * F(s))(x) dx = \left(\int_{\mathbb{R}^d} \nabla G_\nu(z, t-s) dz\right) \cdot \left(\int_{\mathbb{R}^d} F(y,s) dy\right) = 0.$$

Write

$$u(t) - MG_\nu(t) = L(t) + N(t),$$

where $L(t) = G_\nu(t) * u_0 - MG_\nu(t)$ and

$$N(t) = \kappa \int_0^t \nabla G_\nu(t-s) * F(s) ds.$$

Lemma 5.3 gives $t^{\alpha_r}\|L(t)\|_r \to 0$. Split the nonlinear term at $t/2$. Young's inequality and (5.4) give

$$t^{\alpha_r}\left\|\kappa\int_0^{t/2}\nabla G_\nu(t-s)*F(s)ds\right\|_r \le Ct^{\alpha_r}(t/2)^{-\alpha_r-1/2}\int_0^{t/2}\|F(s)\|_1\,ds$$
$$\le Ct^{-1/2}\|F\|_{L_t^1L_x^1}\longrightarrow 0.$$

For the terminal half (and $t \ge 2$), use $\|\nabla G_\nu(\tau)\|_1 = C\tau^{-1/2}$ and the L^r decay of F:

$$t^{\alpha_r}\left\|\kappa\int_{t/2}^{t}\nabla G_\nu(t-s)*F(s)ds\right\|_r \le Ct^{\alpha_r}\int_{t/2}^{t}(t-s)^{-1/2}(1+s)^{-\gamma_r}ds$$
$$\le Ct^{\alpha_r-\gamma_r}\int_0^{t/2}\tau^{-1/2}d\tau \le Ct^{\alpha_r+1/2-\gamma_r}\longrightarrow 0.$$

This proves (6.3) without invoking an unquantified regularity assumption near $s = t$.

For the first correction, Lemma 5.3 gives

$$t^{\alpha_r+1/2}\|L(t)+m_0\cdot\nabla G_\nu(t)\|_r \to 0.$$

It remains to prove

$$t^{\alpha_r+1/2}\|N(t)-B\cdot\nabla G_\nu(t)\|_r \to 0.$$

Set $b(s) = \int_{\mathbb{R}^d}F(y,s)dy$. Decompose exactly as

$$N(t)-B\cdot\nabla G_\nu(t) = \kappa\int_0^{t/2}[\nabla G_\nu(t-s)*F(s)-b(s)\cdot\nabla G_\nu(t)]\,ds$$
$$+\kappa\int_{t/2}^{t}\nabla G_\nu(t-s)*F(s)ds-\kappa\nabla G_\nu(t)\cdot\int_{t/2}^{\infty}b(s)ds. \tag{6.7}$$

For $s \le t/2$, insert and subtract $b(s)\cdot\nabla G_\nu(t-s)$. The spatial translation part satisfies

$$\|\nabla G_\nu(t-s)*F(s)-b(s)\cdot\nabla G_\nu(t-s)\|_r$$
$$\le\int_{\mathbb{R}^d}\|\nabla G_\nu(\cdot-y,t-s)-\nabla G_\nu(\cdot,t-s)\|_r\,|F(y,s)|\,dy$$
$$\le C(t-s)^{-\alpha_r-1}\int_{\mathbb{R}^d}|y|\,|F(y,s)|\,dy,$$

where the final estimate follows from the fundamental theorem of calculus and (5.5). The time-shift part satisfies, by (5.6),

$$|b(s)|\,\|\nabla G_\nu(t-s)-\nabla G_\nu(t)\|_r \le |b(s)|\int_{t-s}^{t}\|\partial_\tau\nabla G_\nu(\tau)\|_r\,d\tau$$
$$\le Cst^{-\alpha_r-3/2}\|F(s)\|_1.$$

After integration over $0 < s < t/2$ and multiplication by $t^{\alpha_r+1/2}$, the two contributions are bounded by

$$Ct^{-1/2}\int_0^\infty\int_{\mathbb{R}^d}|y|\,|F|\;dyds + Ct^{-1}\int_0^\infty s\,\|F(s)\|_1\,ds,$$

which tends to zero by (6.4). The last term in (6.7) satisfies

$$t^{\alpha_r+1/2}\,\|\nabla G_\nu(t)\|_r \int_{t/2}^{\infty} \|F(s)\|_1\,ds \le C \int_{t/2}^{\infty} \|F(s)\|_1\,ds \longrightarrow 0.$$

Finally, the middle term in (6.7) is estimated exactly as the previous terminal layer, now at the dipole scale:

$$t^{\alpha_r+1/2}\left\|\int_{t/2}^{t} \nabla G_\nu(t-s) * F(s)ds\right\|_r \le Ct^{\alpha_r+1-\gamma_r} \longrightarrow 0$$

because $\gamma_r > \alpha_r + 1$. Combining these estimates with the linear expansion proves (6.6). □

Corollary 6.2 (Gaussian endpoint from $L^1 \cap L^2$ data for $p \ge 3$)**.** *Under the hypotheses of Lemma 5.2, let* $M = \int_{\mathbb{R}^d} u_0(x)\,dx$. *For every* $1 \le r \le p'$, *one has*

$$t^{\alpha_r}\,\|u(t) - MG_\nu(t)\|_r \longrightarrow 0 \qquad (t \to \infty).$$

Proof. Lemma 5.2 verifies (6.1) and (6.2) with $\gamma_r = (1 + d/2)/r > \alpha_r + 1/2$ for $1 \le r \le p'$. The conclusion follows from Theorem 6.1. □

Remark 6.3 (Dipole terminology)**.** In a heat-kernel moment expansion, the mass multiple MG_ν is the zeroth-order, or monopole, term. A vector coefficient multiplying ∇G_ν is the first spatial-moment, or dipole, term; this is the standard multipole terminology for heat asymptotics [27]. Thus *first dipole correction* in (6.6) refers to its derivative-of-Gaussian form, not to an additional physical dipole postulate.

Remark 6.4 (Interpretation and scope)**.** The decay hypothesis (6.2) is sufficient, explicit, and experimentally or numerically checkable. Lemma 5.2 derives it from $L^1 \cap L^2$ data for $p \ge 3$ and $1 \le r \le p'$; no claim is made that minimal L^1 data imply it for every r, dimension, and $2 < p < 3$. The theorem isolates the general asymptotic mechanism without concealing the terminal-layer regularity required by the heat-gradient convolution.

6.2. Formal intermediate power-law similarity regime

The similarity formulas below are exact for the reduced pure p-Laplacian source solution. Their use to describe an intermediate core of the full mixed equation, and every crossover-time prediction, is heuristic: no error-controlled approximation or transition-time bound for the full equation is asserted.

The coexistence of the linear and nonlinear fluxes introduces a natural gradient-dependent separation of regimes. Recalling that

$$\Re(x,t) = \frac{\kappa\,|\nabla u(x,t)|^{p-1}}{\nu\,|\nabla u(x,t)|} = \frac{\kappa}{\nu}\,|\nabla u(x,t)|^{p-2},$$

the crossover gradient is obtained by imposing equality between the two flux contributions:

$$\nu G_c = \kappa G_c^{p-1}.$$

Since $p > 2$, this relation has the unique positive solution

$$G_c = \left(\frac{\nu}{\kappa}\right)^{1/(p-2)}. \tag{6.8}$$

Consequently,

$$\Re(x,t) = \left(\frac{|\nabla u(x,t)|}{G_c} \right)^{p-2}. \tag{6.9}$$

Thus, in regions where $|\nabla u| \gg G_c$, one has $\Re \gg 1$, and the leading-order equation is formally

$$\partial_t u = \kappa \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right). \tag{6.10}$$

This reduction is local in gradient space rather than necessarily global in physical space: even when the nonlinear flux dominates in the central high-gradient region of a disturbance, the linear contribution may remain relevant in the low-gradient outer region. The source-type and asymptotic solutions of the evolutionary p-Laplacian are well-understood [20–23]. To recover their scaling in the present notation, consider the mass-preserving ansatz

$$u(x,t) = t^{-\alpha} U(y), \qquad y = x t^{-\beta}. \tag{6.11}$$

Assume that $U \in L^1(\mathbb{R}^d)$ and that

$$M := \int_{\mathbb{R}^d} u(x,t)\, dx$$

is finite and independent of time. Changing variables according to $x = t^\beta y$ gives

$$\int_{\mathbb{R}^d} u(x,t)\, dx = t^{-\alpha+d\beta} \int_{\mathbb{R}^d} U(y)\, dy.$$

Hence conservation of a nonzero mass requires

$$\alpha = d\beta. \tag{6.12}$$

The time derivative of the ansatz is

$$\partial_t u(x,t) = t^{-\alpha-1} \left(-\alpha U(y) - \beta y \cdot \nabla U(y) \right),$$

whereas

$$\nabla_x u(x,t) = t^{-\alpha-\beta} \nabla U(y).$$

It follows that

$$|\nabla_x u|^{p-2} \nabla_x u = t^{-(p-1)(\alpha+\beta)} |\nabla U|^{p-2} \nabla U.$$

Since one further spatial derivative contributes the factor $t^{-\beta}$,

$$\operatorname{div}_x \left(|\nabla_x u|^{p-2} \nabla_x u \right) = t^{-(p-1)(\alpha+\beta)-\beta} \operatorname{div}_y \left(|\nabla U|^{p-2} \nabla U \right).$$

Balancing the temporal powers in (6.10) therefore gives

$$\alpha + 1 = (p-1)(\alpha+\beta) + \beta.$$

Combining this identity with $\alpha = d\beta$, one obtains

$$1 = [d(p-2) + p]\beta.$$

It is convenient to introduce

$$\Lambda_p := d(p-2) + p.$$

The similarity exponents are then

$$\boxed{\beta_p = \frac{1}{\Lambda_p} = \frac{1}{d(p-2)+p}, \qquad \alpha_p = \frac{d}{\Lambda_p} = \frac{d}{d(p-2)+p}.} \tag{6.13}$$

With these exponents, substitution of (6.11) into (6.10) gives the stationary similarity equation

$$-\alpha_p U - \beta_p y \cdot \nabla U = \kappa \operatorname{div}_y \left(|\nabla U|^{p-2} \nabla U \right). \tag{6.14}$$

Because $\alpha_p = d\beta_p$,

$$\alpha_p U + \beta_p y \cdot \nabla U = \beta_p \operatorname{div}_y (yU),$$

and hence (6.14) can be rewritten in the conservative form

$$\operatorname{div}_y \left[\kappa |\nabla U|^{p-2} \nabla U + \beta_p y U \right] = 0. \tag{6.15}$$

For a nonnegative radial profile $U(y) = U(r)$, $r = |y|$, Eq (6.15) becomes

$$\frac{1}{r^{d-1}} \frac{d}{dr} \left\{ r^{d-1} \left[\kappa |U'(r)|^{p-2} U'(r) + \beta_p r U(r) \right] \right\} = 0.$$

Regularity at $r = 0$ excludes a singular radial flux, and therefore the quantity inside the square brackets must vanish:

$$\kappa |U'(r)|^{p-2} U'(r) + \beta_p r U(r) = 0. \tag{6.16}$$

For a decreasing profile, $U'(r) \le 0$, so

$$-U'(r) = \left(\frac{\beta_p}{\kappa} \right)^{1/(p-1)} r^{1/(p-1)} U(r)^{1/(p-1)}.$$

Separating variables gives

$$U(r)^{-1/(p-1)} \, dU = -\left(\frac{\beta_p}{\kappa} \right)^{1/(p-1)} r^{1/(p-1)} \, dr.$$

Integration yields

$$\frac{p-1}{p-2} U(r)^{(p-2)/(p-1)} = C_0 - \frac{p-1}{p} \left(\frac{\beta_p}{\kappa} \right)^{1/(p-1)} r^{p/(p-1)}.$$

After absorbing the harmless multiplicative constant into C_0, the nonnegative radial profile can be written as

$$U_M(y) = \left[C_M - a_{p,\kappa} |y|^{p/(p-1)} \right]_+^{(p-1)/(p-2)}, \qquad a_{p,\kappa} := \frac{p-2}{p} \left(\frac{\beta_p}{\kappa} \right)^{1/(p-1)}, \tag{6.17}$$

where $[s]_+ := \max\{s, 0\}$. The constant $C_M > 0$ is uniquely fixed by the prescribed mass

$$\int_{\mathbb{R}^d} U_M(y) \, dy = M.$$

Indeed, setting

$$q_p := \frac{p}{p-1}, \qquad m_p := \frac{p-1}{p-2},$$

one obtains

$$M = \frac{\left|\mathbb{S}^{d-1}\right|}{q_p} a_{p,\kappa}^{-d/q_p} C_M^{m_p+d/q_p} \mathrm{B}\left(\frac{d}{q_p}, m_p+1\right). \tag{6.18}$$

Here $\left|\mathbb{S}^{d-1}\right| = \mathcal{H}^{d-1}(\mathbb{S}^{d-1})$ denotes the surface measure of the unit $(d-1)$-sphere, and B denotes the Euler beta function. Since the right-hand side is strictly increasing in C_M, Eq (6.18) determines C_M uniquely. The corresponding source-type solution of the reduced equation is

$$\mathcal{B}_M(x,t) = t^{-\alpha_p}\left[C_M - a_{p,\kappa}\,|x|^{p/(p-1)}\, t^{-\beta_p p/(p-1)}\right]_+^{(p-1)/(p-2)}. \tag{6.19}$$

Its support radius is

$$R_M(t) = \left(\frac{C_M}{a_{p,\kappa}}\right)^{(p-1)/p} t^{\beta_p}. \tag{6.20}$$

After eliminating C_M through the mass constraint, the characteristic scales satisfy

$$R_M(t) \asymp \left(\kappa M^{p-2} t\right)^{1/\Lambda_p}, \tag{6.21}$$

$$\|\mathcal{B}_M(t)\|_\infty \asymp M^{p/\Lambda_p}\,(\kappa t)^{-d/\Lambda_p}, \tag{6.22}$$

$$\|\nabla \mathcal{B}_M(t)\|_\infty \asymp M^{2/\Lambda_p}\,(\kappa t)^{-(d+1)/\Lambda_p}. \tag{6.23}$$

Here and below, the implicit constants depend only on d and p. These relations show explicitly that the mass affects not only the amplitude but also the propagation scale of the nonlinear regime. The compact support of $\mathcal{B}_M$ reflects the finite-propagation property of the degenerate equation (6.10) for $p > 2$ [20, 22]. The full mixed equation, however, contains the uniformly parabolic term $\nu\Delta u$. Consequently, the Barenblatt profile should not be interpreted as a global exact approximation to the full solution. The linear term is never negligible in portions of the solution where the gradient is small, particularly in the far field. The resulting heuristic picture is a nonlinear Barenblatt-like core surrounded by a low-amplitude linear halo. The core can exhibit the power-law scaling (6.13), while the outer tail already displays Gaussian features. In this picture, the linear region gradually invades the core and the Barenblatt-like approximation ceases to be dominant. A refined heuristic crossover scale follows from (6.23). Defining the transition by

$$\|\nabla \mathcal{B}_M(t_c)\|_\infty \asymp G_c,$$

one obtains

$$t_c \asymp \frac{1}{\kappa} M^{2/(d+1)} G_c^{-\Lambda_p/(d+1)}. \tag{6.24}$$

Using (6.8), this may equivalently be written as

$$t_c \asymp \frac{1}{\kappa} M^{2/(d+1)} \left(\frac{\nu}{\kappa}\right)^{-\Lambda_p/[(d+1)(p-2)]}. \tag{6.25}$$

Equation (6.24) refines (2.6) by displaying the dependence on the mass and on the nonlinear mobility coefficient. It remains a heuristic similarity prediction. A bound for a transition time defined by the

partial differential equation (PDE) would require a separate analysis because the crossover is spatially distributed and occurs continuously. The flux ratio evaluated at the nonlinear similarity scale satisfies

$$\mathfrak{R}_{\mathrm{core}}(t) \asymp \left[\frac{M^{2/\Lambda_p}(\kappa t)^{-(d+1)/\Lambda_p}}{G_c}\right]^{p-2}. \tag{6.26}$$

In particular,

$$\mathfrak{R}_{\mathrm{core}}(t) \propto t^{-\theta_p}, \qquad \theta_p := \frac{(d+1)(p-2)}{d(p-2)+p} = \frac{(d+1)(p-2)}{\Lambda_p}.$$

Since

$$\Lambda_p = (d+1)(p-2)+2,$$

one has $0 < \theta_p < 1$. Thus nonlinear dominance is eroded algebraically rather than abruptly. A clearly observable intermediate regime requires a nonempty temporal window

$$t_{\mathrm{relax}} \ll t \ll t_c,$$

where t_{relax} denotes the time required for the solution to lose the detailed shape of its initial data and approach a source-type nonlinear core. If t_c is comparable to or smaller than this relaxation time, a clean p-Laplacian similarity plateau should not be expected. Several experimentally and numerically testable consequences follow. Let $L(t)$ be a characteristic width that is homogeneous under spatial dilations, such as a fixed-mass quantile radius or, when finite, the square root of a second moment. Define the instantaneous logarithmic spreading exponent by

$$\beta_{\mathrm{eff}}(t) := \frac{d\log L(t)}{d\log t}. \tag{6.27}$$

During a well-developed nonlinear intermediate regime,

$$\beta_{\mathrm{eff}}(t) \approx \beta_p = \frac{1}{d(p-2)+p},$$

whereas Gaussian similarity suggests

$$\beta_{\mathrm{eff}}(t) \longrightarrow \frac{1}{2}$$

when the Gaussian regime becomes dominant. This last derivative limit is a heuristic prediction, not a conclusion of Theorem 6.1. For nonnegative mass $M > 0$, its L^1 conclusion does imply that a fixed-mass quantile radius, divided by $\sqrt{t}$, tends to the corresponding Gaussian quantile radius: the rescaled distribution functions converge uniformly, and the Gaussian radial distribution is continuous and strictly increasing. Convergence of $L(t)/\sqrt{t}$ does not imply convergence of $tL'(t)/L(t)$. Likewise, a second-moment width requires weighted uniform integrability, which is not assumed in that theorem. Since

$$\frac{1}{2} - \beta_p = \frac{(d+1)(p-2)}{2[d(p-2)+p]} > 0,$$

the two predicted endpoint exponents are ordered. A monotone increase between them is a testable scenario, not a consequence of the present analysis. The corresponding heuristic prediction for the peak amplitude is a change from

$$\|u(t)\|_\infty \sim t^{-\alpha_p}, \qquad \alpha_p = d\beta_p,$$

to Gaussian decay

$$\|u(t)\|_\infty \sim t^{-d/2}.$$

The late peak law follows rigorously only when the hypotheses of Theorem 6.1 hold at $r = \infty$ and $M > 0$. Under differentiable similarity behavior, simultaneous measurements of width and peak amplitude should satisfy the relation

$$\alpha_{\text{eff}}(t) \approx d\,\beta_{\text{eff}}(t)$$

within either mass-preserving similarity regime. The observed intermediate spreading exponent also provides a direct estimate of the constitutive exponent. Solving

$$\beta_p^{-1} = (d+1)p - 2d$$

for p gives

$$p = \frac{\beta_p^{-1} + 2d}{d+1}. \tag{6.28}$$

Thus p may be inferred independently from transient spreading data and compared with the large-gradient slope $p - 1$ obtained from the steady pressure–flux relation (7.1). Agreement between these two estimates would provide a nontrivial validation of the proposed closure. Once p is known, the early-time spreading prefactor in (6.21) provides information about κ, while the late Gaussian width identifies ν. The observed transition scale then supplies an independent estimate of

$$G_c = \left(\frac{\nu}{\kappa}\right)^{1/(p-2)}.$$

As a consistency check, when $p \downarrow 2$,

$$\beta_p \longrightarrow \frac{1}{2}, \qquad \alpha_p \longrightarrow \frac{d}{2}.$$

At $p = 2$, the two constitutive channels are both linear and the equation reduces to

$$\partial_t u = (\nu + \kappa)\Delta u.$$

There is then no distinct nonlinear similarity regime: the equation is the heat equation with effective diffusivity $\nu + \kappa$. Hence the appearance of two distinct spreading exponents is a specifically nonlinear prediction of the model for $p > 2$, and constitutes one of its clearest falsifiable transient signatures.

7. Scope of the contribution and its relation to applications

The operator $-\nu\Delta - \kappa\Delta_p$ is the sum of two standard monotone operators; neither that sum nor the proof technology is presented as new. The contribution lies in the following model-level synthesis:

(1) a phenomenological two-term pressure-flux closure for a complex fluid;
(2) a dimensionless crossover gradient G_c that can be measured from steady pressure–flux data;
(3) explicitly heuristic transient scaling laws, separated from the rigorous conditional Gaussian endpoint; and
(4) a self-contained analytical treatment of bounded-domain well-posedness, mixed boundary conditions, and whole-space asymptotics.

This distinction separates the physical interpretation proposed here from the established mathematical structure used to analyze it.

7.1. Parameter identification

Consider a source-free, one-dimensional steady problem on a sample of length ℓ. In nondimensional variables, the flux is constant and obeys

$$q = -\nu u_x - \kappa |u_x|^{p-2} u_x.$$

The flux map is strictly monotone, so the gradient is constant. With $G = |u(\ell) - u(0)| / \ell$, its magnitude satisfies

$$|q| = \nu G + \kappa G^{p-1}. \tag{7.1}$$

Thus ν is the small-gradient slope, $p - 1$ is the large-gradient log-log slope, and κ is the corresponding amplitude. Dimensional calibration uses $|Q| = K_1 G_{\mathrm{dim}} + K_p G_{\mathrm{dim}}^{p-1}$ instead. The crossover G_c is the intersection of the two normalized contributions. Equation (7.1) provides a direct falsification test: if measured pressure-flux curves cannot be represented by the sum of these powers over the intended range, the closure should be rejected or generalized.

7.2. Relation to Darcy–Forchheimer pressure reductions

There is a structural connection with generalized Darcy–Forchheimer models for Newtonian flow at velocities for which inertial resistance is non-negligible. A generalized momentum relation is commonly written as

$$g(|Q|)Q = -\nabla P, \qquad g(s) > 0,$$

and its inversion gives

$$Q = -K(|\nabla P|)\nabla P.$$

Combining this relation with storage conservation produces a scalar pressure equation of the same divergence-form architecture as (2.2); see [29, 30]. The constitutive directions must nevertheless be distinguished. For the usual inertial Forchheimer law, g increases with $|Q|$, so the inverted mobility K decreases with $|\nabla P|$. In (1.1), by contrast, $K_{\mathrm{mix}}(G) = K_1 + K_p G^{p-2}$ increases for $p > 2$. Thus elimination of velocity leads to related pressure PDEs, but the standard inertial Forchheimer correction is not identical to the present high-gradient mobility law. Which closure is appropriate is an empirical question. A derivation from a specified microgeometry would require an explicit homogenization limit of the pore-scale equations; standard frameworks and non-Newtonian examples are described in [31].

7.3. Limitations

The scalar equation neglects inertia, vorticity, tensorial stresses, memory, temperature coupling, and deformation of the matrix. It is appropriate only after velocity has been eliminated and pressure is the dominant state variable. For $p > 2$, the flux grows superlinearly with the pressure gradient, so the effective mobility increases at high gradients; the terminology must therefore not be confused with a shear-thickening viscosity law. A subquadratic term with $1 < p < 2$ instead has a decreasing mobility contribution. Its singular behavior at zero gradient requires a different analysis and is not covered by the present results.

Spatially varying scalar coefficients are natural extensions, but anisotropic permeability calls for tensor coefficients. One convex generalization is

$$\Phi(x, \xi) = \frac{1}{2}\xi \cdot K(x)\xi + \frac{\kappa(x)}{p}(\xi \cdot B(x)\xi)^{p/2},$$

where $K(x)$ and $B(x)$ are bounded measurable, symmetric, uniformly positive definite matrices, and $0 < \kappa_* \le \kappa(x) \le \kappa^* < \infty$. Its flux is

$$\nabla_\xi \Phi(x, \xi) = K(x)\xi + \kappa(x)(\xi \cdot B(x)\xi)^{(p-2)/2} B(x)\xi,$$

which retains monotonicity while encoding directional permeability.

8. Conclusions

The mixed law illustrates a general selection principle: the persistent quadratic channel controls the conditional large-time Gaussian universality class, while the higher-growth channel can affect transients and the first moment correction. The bounded-domain well-posedness, comparison, mixed boundary, and smoothing results are unconditional under the stated data. On $\mathbb{R}^d$, Lemma 5.2 removes the basic flux assumptions for $p \ge 3$ and $1 \le r \le p'$; larger r and the range $2 < p < 3$ remain conditional. The crossover time is deliberately left as a heuristic, spatially distributed similarity scale rather than a theorem. The next analytical step is therefore a sharp whole-space gradient-smoothing theory, while the next modeling step is to distinguish the proposed increasing-mobility law experimentally from an inverted Darcy–Forchheimer law and, where appropriate, derive its coefficients by a specified homogenization procedure [31].

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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Appendix

A. Auxiliary chain rule in an evolution triple

For completeness, let $V \hookrightarrow H \hookrightarrow V^*$ be an evolution triple, $u \in L^p(0,T;V)$, and $u_t \in L^{p'}(0,T;V^*)$. Then u has an H-continuous representative and

$$\frac{1}{2}\|u(t)\|_H^2 - \frac{1}{2}\|u(s)\|_H^2 = \int_s^t \langle u_\tau(\tau), u(\tau)\rangle \, d\tau. \tag{A.1}$$

One proves this by temporal convolution. Extend u slightly outside $(0,T)$, set $u_\varepsilon = \rho_\varepsilon * u$, and note that u_ε is smooth in time with values in V. The classical identity holds for u_ε. Since $u_\varepsilon \to u$ in $L^p(V)$ and $(u_\varepsilon)_t \to u_t$ in $L^{p'}(V^*)$, the duality products converge in L^1. The scalar functions $\|u_\varepsilon(t)\|_H^2$ therefore converge to an absolutely continuous representative, yielding (A.1) and the H continuity.

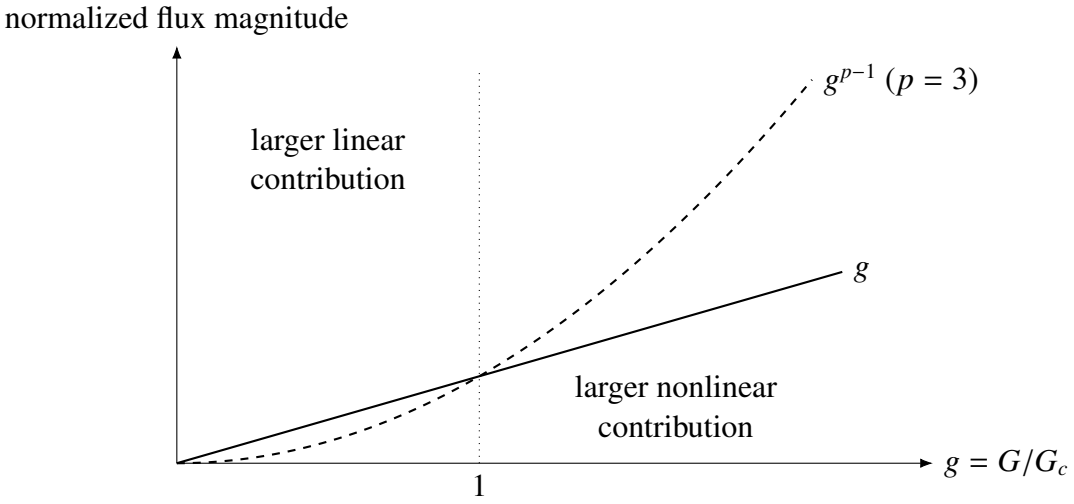
B. Dimensional consistency table

In the compact dimension notation below, L, T, and P denote length, time, and pressure, respectively.

Quantity	Dimension	Interpretation
P	pressure	excess pressure or hydraulic potential
S	inverse pressure	storage/compressibility coefficient in normalized volume units
Q	length/time	volumetric flux per unit area
K_1	$L^2T^{-1}P^{-1}$	linear mobility coefficient
K_p	$L^pT^{-1}P^{1-p}$	nonlinear gradient mobility; equivalently, $K_p\,\lvert\nabla P\rvert^{p-2}\,\nabla P$ has the dimension of Q
$G_{c,\mathrm{dim}}$	pressure/length	$(K_1/K_p)^{1/(p-2)}$; dimensional counterpart of G_c
Λ	dimensionless	nonlinear-to-linear mobility ratio at the reference gradient

C. Schematic regime map

Set $g = G/G_c$ and normalize fluxes by νG_c. The two magnitudes are then g and g^{p-1}, and they coincide for positive g exactly at $g = 1$. The schematic below uses $p = 3$; it represents a constitutive comparison, not a time-dependent transition theorem.



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