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Research article

Hausdorff measure of the free boundary for the p-obstacle problem with subcritical exponents

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Abstract: This paper investigates a class of p-obstacle problems with subcritical exponents having the form

$$\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) = m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}} \text{ in } \Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 2)$, m_1 and m_2 are positive constants, the coefficient function $a \in C^2(\Omega)$ has a positive lower bound, and $2 \leq p < \lambda < p^* := \frac{Np}{N-p}$ when $p < N$ and $N \geq 3$ or $2 \leq p < \lambda < +\infty$ when $N = 2$. By using the mountain-pass lemma, combined with the penalty method, we first establish the existence of non-negative weak solutions. Then, using the De Giorgi-Nash iteration, we prove the L^∞ bound and local $C^{1,\alpha}$ continuity for the solutions. In addition, we prove local porosity of the free boundary based on the optimal growth and non-degeneracy of solutions near the free boundary. Furthermore, by means of Lebesgue measure estimates for gradient level sets, we show that at least one solution corresponds to the free boundary having a locally finite $(N-1)$-dimensional Hausdorff measure.

Keywords: obstacle problem; free boundary; Hausdorff measure; porosity; subcritical exponents

1. Introduction

Free boundary problems described by partial differential equations (PDEs) of an elliptic type naturally arise in a wide range of physical models and variational settings, and constitute a central subject in the theory of PDEs. Typical examples include the obstacle problem, Bernoulli-type free boundary problems, chemical reaction equations, among others. A crucial theme in the study of the free boundary problems is proving the existence and regularity of solutions and analyzing the geometric properties of the free boundary such as its porosity and Hausdorff measure.

Recently, the authors of [1] studied a superlinear obstacle-type free boundary problem of the

following form:

$$\begin{cases} \Delta u = \left(1 - u^{\lambda-1}\right)\chi_{\{u>0\}} \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 2)$, $\chi_{\{u>0\}}(\cdot)$ is the standard characteristic function, and the exponent λ satisfies

$$2 < \lambda < \begin{cases} 2^* := \frac{2N}{N-2}, & N \geq 3, \\ +\infty, & N = 2. \end{cases}$$

The main challenge in studying the regularity theory for such a problem comes from the coupling of the superlinear term $u^{\lambda-1}$ and the characteristic function $\chi_{\{u>0\}}$. This coupling disrupts the coercivity of the associated energy functional (thus ruling out standard minimization arguments for existence). As a consequence, classical variational methods cannot be applied straightforwardly. To overcome this obstacle, the authors of [1] employed a penalty method, combined with the mountain pass lemma, to establish the existence of nontrivial solutions, and further demonstrated that these solutions possess locally optimal regularity by means of a suitable monotonicity formula combined with a bootstrap iteration argument. Moreover, non-degeneracy of solutions near the free boundary and local porosity of the free boundary were established. Furthermore, it was shown that the free boundary possesses a locally finite $(N-1)$-dimensional Hausdorff measure and a decomposition property of the free boundary was provided; see [1].

Motivated by the superlinear obstacle-type free boundary problem studied in [1], the aim of the present paper is to investigate a class of obstacle problems with more involved operators and nonlinear structures. In particular, in this paper, quasilinear operators of the p-Laplace type with variable coefficients are considered and the nonlinear term on the right-hand side of the equation allows for subcritical exponents, both of which bring more complexities in the establishment of solutions and analysis of free boundaries. More precisely, we study the following problem:

$$\begin{cases} \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) = m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}} \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 2)$, the coefficient function $a \in C^2(\Omega)$ satisfies

$$0 < a_0 \leq a \text{ in } \Omega \text{ and } \|a\|_{C^2(\Omega)} \leq a_1$$

with positive constants a_0 and a_1, m_1, m_2 are positive constants, and the exponent λ satisfies

$$2 \leq p < \lambda < \begin{cases} p^* := \frac{Np}{N-p}, & N \geq 3, p < N, \\ +\infty, & N = 2. \end{cases} \tag{1.2}$$

Formally, Eq (1.1) corresponds to the following energy functional:

$$\mathcal{J}(u) = \int_\Omega \frac{a(x)}{p}|\nabla u|^p + m_1 u^+ - \frac{m_2}{\lambda}(u^+)^\lambda \mathrm{d}x,$$

where $u^+ = \max\{u, 0\}$.

It is worth mentioning that the authors of [2–5] and [6, 7] investigated the porosity and Hausdorff measure of the free boundary for the p-Laplace type obstacle problem with $m_2 = 0$ in Sobolev spaces, and for the A-Laplace type obstacle problem with $m_2 = 0$ in Orlicz spaces, respectively. The porosity and Hausdorff measure of the free boundary for a p-Laplace obstacle problem associated with the Ginzburg-Landau equation, for which the coefficient of the superlinear term is non-negative, were studied in [8]. Additionally, it is worth mentioning that the porosity of the free boundary for the p-Laplace non-zero obstacle problems was established in [9]. Nevertheless, the regularity, in particular, the Hausdorff measure of the free boundary, for the p-Laplace obstacle problem with variable coefficients and general subcritical terms remains unexplored.

In this paper, we first prove the existence and regularity of weak solutions of the p-Laplace type obstacle problem (1.1), and then study the local porosity and finite $(N-1)$-dimensional Hausdorff measure of the free boundary $\Gamma^+ := \partial\{u > 0\} \cap \Omega$. Compared with [1], the problem considered in this paper is more challenging in the following three aspects.

- Lack of the Alt-Caffarelli-Friedman (ACF) monotonicity formula. For the case where $p = 2$, the ACF monotonicity formula serves as a powerful tool for proving the optimal growth of solutions near the free boundary, which is a key step in establishing the regularity of the free boundary. However, for $p \neq 2$, this formula is no longer applicable. To overcome this difficulty, we will employ the method of [2, 3, 5], which combines a blow-up argument with the homogeneity of the p-Laplacian, to prove the optimal growth of solutions.
- Non-uniqueness of solutions. Unlike the setting in [1], solutions of Problem (1.1) may not be unique. This imposes difficulties in establishing the regularity of the free boundary for every solution under consideration.
- The presence of the variable coefficient $a(x)$. The appearance of $a(x)$ complicates the regularity analysis of both the solution and the free boundary.

Overall, the main contribution of this paper is twofold.

- By using a regularization and penalty method, we prove the existence and regularity of weak solutions for a wider class of p-Laplace type obstacle problems, which extends relevant results obtained in [1] for the obstacle problem governed by the Laplacian.
- We overcome the technical difficulties arising from the general p, the variable coefficients, and the nonlinearity of subcritical terms, and show that for each solution the free boundary is locally porous and that at least one solution corresponds to the free boundary having a locally finite $(N-1)$-dimensional Hausdorff measure, extending the work in the literature [1–5, 8, 9].

The paper is organized as follows: in Section 2, we present some notations and technical lemmas used for the proofs of main results; in Section 3, we prove the existence and uniform L^∞ boundedness of solutions of Eq (1.1); in Section 4, we establish the local $C^{0,\alpha}$ and $C^{1,\alpha}$ regularities of the solutions; in Section 5, we prove the local porosity of the free boundary based on the optimal growth and non-degeneracy of solutions near the free boundary; in Section 6, we provide some auxiliary results for a corresponding penalized problem; and we show that for at least one solution, the free boundary has a locally finite $(N-1)$-dimensional Hausdorff measure in Section 7.

2. Notations and technical Lemmas

$\mathbb{R}$ denotes the set of real numbers, $\mathbb{N} := \{0, 1, 2, \dots\}$, and $\mathbb{N}^+ := \{1, 2, 3, \dots\}$. $B_r(x)$ denotes the ball in $\mathbb{R}^N$ with a center of x and a radius $r > 0$. When the center is not emphasized, we simply write B_r for a ball with radius r. For a function u defined on Ω, let $\{u > 0\} := \{x \in \Omega : u(x) > 0\}$ and $\Gamma^+ := (\partial\{x \in \Omega : u(x) > 0\}) \cap \Omega$. For $m \in \mathbb{N}^+$ and measurable set $E \subset \mathbb{R}^m$, $|E| := \mathcal{L}^m(E)$ denotes its m-dimensional Lebesgue measure, and $\mathcal{H}^m(E)$ denotes its m-dimensional Hausdorff measure.

The following inequality ensures the monotonicity of the p-Laplacian.

Lemma 1 ([10, p.13]). *For $p \geq 2$, it holds that*

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta) \geq \frac{1}{2^p}|\xi - \eta|^p, \forall \xi, \eta \in \mathbb{R}^N.$$

The following result, which is called the Du Bois-Reymond lemma, is a fundamental lemma in the calculus of variations.

Lemma 2 ([11, Corollary 4.24]). *Let $f \in L^1_{\text{loc}}(\Omega)$ satisfy $\int_\Omega f g \mathrm{d}x = 0$ for all $g \in C_0^\infty(\Omega)$. Then, $f = 0$ a.e. in Ω.*

The following lemma will be used in the proof of the L^∞ estimate of weak solutions of the obstacle problem (1.1).

Lemma 3 ([12]). *Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative numbers satisfying*

$$g_{n+1} \leq CD^n g_n^{1+\zeta}, \quad n \in \mathbb{N},$$

where $C, \zeta > 0$ and $D > 1$ are constants not depending on n. If

$$g_0 \leq C^{-\frac{1}{\zeta}} D^{-\frac{1}{\zeta^2}},$$

then $g_n \to 0$ as $n \to +\infty$.

The following three results will be used in delicate iterations and comparison arguments with appropriate competitors. Moreover, they will be employed to obtain interior regularity of the gradients of weak solutions of the obstacle problem (1.1).

Lemma 4 ([13, Lemma A.3]). *Let τ be a non-negative function on an interval $(0, R_*]$ with $R_* \leq 1$. Suppose that for all r and R satisfying $0 < r \leq R \leq R_*$, there holds*

$$\tau(r) \leq A\left(\frac{r}{R}\right)^\alpha \tau(R) + BR^\beta,$$

where $A > 1, B, \alpha$, and β are positive constants with $\alpha > \beta$. Fix $\delta \in (\beta, \alpha)$ and consider $\theta \in (0, 1)$ with $A\theta^\alpha = \theta^\delta$. Suppose that there exists a constant $d > 0$ such that $\tau(r) \leq d\tau(\theta^k R)$ for all non-negative integers k and $r \in [\theta^{k+1}R, \theta^k R]$. Then, there is a positive constant C which only depends on A, α, β, δ, and d such that

$$\tau(r) \leq C\left(\frac{r}{R}\right)^\sigma (\tau(R) + BR^\sigma)$$

holds true for any r, R, and σ satisfying $0 < r \le R \le R_*$ and $0 < \sigma \le \beta$. Furthermore, there is a positive constant D which only depends on B, C, β, and $\tau(R_*)$ such that

$$\tau(r) \le Dr^{\sigma}$$

holds true for any $r \in (0, R_*]$ and $\sigma \in (0, \beta]$.

Lemma 5 ([14, Lemma 3.1]). *Let $u \in W^{1,p}(\Omega)$, $B_R \subset \Omega$. If v is a bounded weak solution of the p-harmonic equation*

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0 \text{ in } B_R, \quad v - u \in W_0^{1,p}(B_R),$$

then for any $\mu \in (0, N)$, there exists a positive constant C which only depends on N, p, μ, and $\|v\|_{L^\infty(B_R)}$ such that

$$\int_{B_R} |\nabla u - \nabla v|^p \mathrm{d}x \le C \int_{B_R} (|\nabla u|^p - |\nabla v|^p)\mathrm{d}x + CR^{\frac{\mu}{2}} \left(\int_{B_R} (|\nabla u|^p - |\nabla v|^p)\mathrm{d}x \right)^{\frac{1}{2}}.$$

Lemma 6 ([15, Lemma 4.1]). *Let $u \in W^{1,p}(B_R)$, $B_R \subset \Omega$. If $v \in W^{1,p}(B_R)$ is a weak solution of the p-harmonic equation*

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0 \text{ in } B_R,$$

then there exist a constant $\delta \in (0, 1)$ and a positive constant C which only depends on N and p such that

$$\int_{B_r} |\nabla u - (\nabla u)_r|^p \mathrm{d}x \le C \left(\frac{r}{R}\right)^{N+\delta} \int_{B_R} |\nabla u - (\nabla u)_R|^p \mathrm{d}x + C \int_{B_R} |\nabla u - \nabla v|^p \mathrm{d}x, \forall r \in (0, R].$$

3. Existence and uniform boundedness of solutions

In this section, we establish the existence and uniform L^∞ boundedness of solutions to the obstacle problem (1.1).

The existence result is stated as below.

Proposition 1. *The Eq* (1.1) *admits at least one weak solution.*

Proof. We employ the penalty method and the mountain pass lemma (see [16]) to prove this result. More specifically, we consider the associated penalized problem, namely, for $0 < \varepsilon < 1$, we consider the equation

$$\begin{cases} \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) = m_1 \chi_\varepsilon(u) - m_2 (u^+)^{\lambda-1} \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\chi_\varepsilon(s)$ is a smooth monotone approximation of the Heaviside function $\chi_{\{s>0\}}$ satisfying

$$\chi_\varepsilon'(s) \ge 0, \quad \chi_\varepsilon(s) = 0 \text{ for } s \le 0, \quad \chi_\varepsilon(s) = 1 \text{ for } s \ge \varepsilon.$$

We claim that a critical point, denoted by u_ε, of the functional

$$\mathcal{J}_\varepsilon(u) = \int_\Omega \frac{a(x)}{p}|\nabla u|^p + m_1\Phi_\varepsilon(u) - \frac{m_2}{\lambda}(u^+)^\lambda \mathrm{d}x \tag{3.2}$$

is a solution of Eq (3.1), where $\Phi_\varepsilon(s) = \int_{-\infty}^{s}\chi_\varepsilon(t)\mathrm{d}t \geq 0$.

Indeed, noting that $\mathcal{J}_\varepsilon \in C^1\left(W_0^{1,p}(\Omega), \mathbb{R}\right)$, it suffices to show that for a sufficiently small $\varepsilon > 0$, the functional $\mathcal{J}_\varepsilon(u)$ admits a mountain pass geometry.

First, it is clear that $\mathcal{J}_\varepsilon(0) = 0$. Let S be the optimal embedding constant of $W_0^{1,p}(\Omega) \hookrightarrow L^\lambda(\Omega)$. For any $u \in W_0^{1,p}(\Omega)$ satisfying $\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)^{\frac{1}{p}} = \left(\frac{a_0}{m_2}\right)^{\frac{1}{\lambda-p}} S^{\frac{\lambda}{\lambda-p}}$, by the Sobolev embedding theorem, we deduce that

$$\begin{aligned}
\mathcal{J}_\varepsilon(u) \geq& \int_\Omega \frac{a_0}{p}|\nabla u|^p - \frac{m_2}{\lambda}(u^+)^\lambda \mathrm{d}x \\
\geq& \frac{a_0}{p}\int_\Omega |\nabla u|^p \mathrm{d}x - \frac{m_2}{\lambda S^\lambda}\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)^{\frac{\lambda}{p}} \\
=& \frac{1}{p}a_0^{\frac{\lambda}{\lambda-p}} m_2^{\frac{p}{p-\lambda}} S^{\frac{\lambda p}{\lambda-p}} - \frac{1}{\lambda}a_0^{\frac{\lambda}{\lambda-p}} m_2^{\frac{p}{p-\lambda}} S^{\frac{\lambda p}{\lambda-p}} \\
=& \frac{\lambda-p}{\lambda p} a_0^{\frac{\lambda}{\lambda-p}} m_2^{\frac{p}{p-\lambda}} S^{\frac{\lambda p}{\lambda-p}}.
\end{aligned}$$

Next, using the properties of $\chi_\varepsilon(s)$, we obtain

$$\begin{aligned}
\mathcal{J}_\varepsilon(tu) =& \int_\Omega \frac{a(x)t^p}{p}|\nabla u|^p + m_1\Phi_\varepsilon(tu) - \frac{m_2 t^\lambda}{\lambda}(u^+)^\lambda \mathrm{d}x \\
\leq& \frac{a_1 t^p}{p}\int_\Omega |\nabla u|^p \mathrm{d}x + m_1\int_\Omega (|tu| + \varepsilon)\mathrm{d}x - \frac{m_2 t^\lambda}{\lambda}\int_\Omega (u^+)^\lambda \mathrm{d}x, \forall t > 0,
\end{aligned} \tag{3.3}$$

which, along with $\lambda > p$, implies that $\mathcal{J}_\varepsilon(tu) \to -\infty$ as $t \to +\infty$.

Therefore, the mountain pass lemma ensures that there exists a Palais-Smale sequence for $\mathcal{J}_\varepsilon(u)$ at the level $c \geq \frac{\lambda-p}{\lambda p} a_0^{\frac{\lambda}{\lambda-p}} m_2^{\frac{p}{p-\lambda}} S^{\frac{\lambda p}{\lambda-p}}$.

In addition, we need to verify the Palais-Smale compactness condition. Suppose that there exists a Palais-Smale sequence $\{u_n\}$ such that $\mathcal{J}_\varepsilon(u_n)$ is bounded (i.e., $|\mathcal{J}_\varepsilon(u_n)| \leq M$ with some constant $M > 0$) and $\mathcal{J}'_\varepsilon(u_n) \to 0$ in $\left(W_0^{1,p}(\Omega)\right)^*$. First, we prove that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, and then show the existence of a strongly convergent subsequence. Indeed, from (3.2), we obtain the following:

$$\mathcal{J}_\varepsilon(u_n) = \int_\Omega \frac{a(x)}{p}|\nabla u_n|^p + m_1\Phi_\varepsilon(u_n) - \frac{m_2}{\lambda}(u_n^+)^\lambda \mathrm{d}x. \tag{3.4}$$

From (3.1), we obtain

$$\frac{1}{\lambda}\left\langle \mathcal{J}'_\varepsilon(u_n), u_n\right\rangle = \int_\Omega \frac{a(x)}{\lambda}|\nabla u_n|^p + \frac{m_1}{\lambda}\chi_\varepsilon(u_n)u_n - \frac{m_2}{\lambda}(u_n^+)^\lambda \mathrm{d}x. \tag{3.5}$$

Combining (3.4) and (3.5) yields

$$\mathcal{J}_\varepsilon(u_n) - \frac{1}{\lambda}\langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle \geq \int_\Omega \left(\frac{a(x)}{p} - \frac{a(x)}{\lambda}\right)|\nabla u_n|^p \mathrm{d}x - \int_\Omega \frac{m_1}{\lambda}\chi_\varepsilon(u_n)u_n \mathrm{d}x + \int_\Omega m_1\Phi_\varepsilon(u_n)\mathrm{d}x$$

$$\geq \int_\Omega \left(\frac{a_0}{p} - \frac{a_0}{\lambda}\right)|\nabla u_n|^p \mathrm{d}x - \int_\Omega \frac{m_1}{\lambda}\chi_\varepsilon(u_n)u_n \mathrm{d}x$$
$$\geq \left(\frac{a_0}{p} - \frac{a_0}{\lambda}\right)\int_\Omega |\nabla u_n|^p \mathrm{d}x - C\int_\Omega |u_n|\mathrm{d}x, \tag{3.6}$$

where C is a positive constant which only depends on N, λ, m_1, and Ω. Since $\mathcal{J}'_\varepsilon(u_n) \to 0$, for any $\varphi \in W_0^{1,p}(\Omega)$, it holds that

$$|\langle \mathcal{J}'_\varepsilon(u_n), \varphi\rangle| \leq \|\mathcal{J}'_\varepsilon(u_n)\|_{\left(W_0^{1,p}(\Omega)\right)^*}\|\varphi\|_{W_0^{1,p}(\Omega)} = o(1)\|\varphi\|_{W_0^{1,p}(\Omega)}. \tag{3.7}$$

Substituting $\varphi = u_n$ into (3.7) leads to the following:

$$\frac{1}{\lambda}|\langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle| \leq o(1)\|u_n\|_{W_0^{1,p}(\Omega)}. \tag{3.8}$$

Using (3.8) and the condition $|\mathcal{J}_\varepsilon(u_n)| \leq M$, we infer that

$$\mathcal{J}_\varepsilon(u_n) - \frac{1}{\lambda}\langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle \leq |\mathcal{J}_\varepsilon(u_n)| + \frac{1}{\lambda}|\langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle|$$
$$\leq o(1)\|u_n\|_{W_0^{1,p}(\Omega)} + M. \tag{3.9}$$

Combining (3.6), (3.9), and Young's inequality with $\varepsilon_1, \varepsilon_2 > 0$, we obtain the following:

$$\left(\frac{a_0}{p} - \frac{a_0}{\lambda}\right)\int_\Omega |\nabla u_n|^p \mathrm{d}x \leq C\int_\Omega |u_n|\mathrm{d}x + o(1)\|u_n\|_{W_0^{1,p}(\Omega)} + C$$
$$\leq C\varepsilon_1 \int_\Omega |u_n|^p \mathrm{d}x + o(1)\varepsilon_2\left(\int_\Omega |u_n|^p \mathrm{d}x + \int_\Omega |\nabla u_n|^p \mathrm{d}x\right) + C_{\varepsilon_1,\varepsilon_2}$$
$$\leq C(\varepsilon_1 + o(1)\varepsilon_2)\int_\Omega |\nabla u_n|^p \mathrm{d}x + C_{\varepsilon_1,\varepsilon_2}, \tag{3.10}$$

where C is a positive constant which only depends on N, p, λ, m_1, M, and Ω, and $C_{\varepsilon_1,\varepsilon_2}$ is a positive constant depending additionally on ε_1 and ε_2. By choosing sufficiently small ε_1 and ε_2 and using (3.10), we obtain the following:

$$\int_\Omega |\nabla u_n|^p \mathrm{d}x \leq C,$$

where C is a positive constant which only depends on N, p, λ, m_1, a_0, M, and Ω. Therefore, the sequence $\{u_n\}$ is uniformly bounded in $W_0^{1,p}(\Omega)$. By the embedding theorem, there exist a subsequence, still denoted by $\{u_n\}$, and a function $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$u_n \rightharpoonup u_\varepsilon \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u_\varepsilon \text{ in } L^{p^*}(\Omega). \tag{3.11}$$

Since $\mathcal{J}'_\varepsilon(u_n) \to 0$, it holds that

$$\left|\langle \mathcal{J}'_\varepsilon(u_n), u_n - u_\varepsilon\rangle\right| \to 0. \tag{3.12}$$

By (3.12) and Lemma 1, we deduce that

$$\begin{aligned}
0 \leftarrow & \left|\langle \mathcal{J}_\varepsilon'(u_n), u_n - u_\varepsilon\rangle\right| \\
&= \int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n(\nabla u_n - \nabla u_\varepsilon) + m_1\chi_\varepsilon(u_n)(u_n - u_\varepsilon) - m_2(u_n^+)^{\lambda-2}u_n^+(u_n - u_\varepsilon)\mathrm{d}x \\
&= \int_\Omega a(x)\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right)(\nabla u_n - \nabla u_\varepsilon)\mathrm{d}x \\
&\quad + \int_\Omega a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon(\nabla u_n - \nabla u_\varepsilon)\mathrm{d}x \\
&\quad + \int_\Omega m_1\chi_\varepsilon(u_n)(u_n - u_\varepsilon) - m_2(u_n^+)^{\lambda-2}u_n^+(u_n - u_\varepsilon)\mathrm{d}x \\
&\geq \int_\Omega \left(\frac{1}{2}\right)^p a_0|\nabla u_n - \nabla u_\varepsilon|^p\mathrm{d}x + \int_\Omega a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon(\nabla u_n - \nabla u_\varepsilon)\mathrm{d}x \\
&\quad + \int_\Omega m_1\chi_\varepsilon(u_n)(u_n - u_\varepsilon)\mathrm{d}x - \int_\Omega m_2(u_n^+)^{\lambda-2}u_n^+(u_n - u_\varepsilon)\mathrm{d}x. \qquad (3.13)
\end{aligned}$$

In view of (3.11) and the boundedness of $a(x)$, we obtain the following:

$$\lim_{n\to\infty}\int_\Omega a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon(\nabla u_n - \nabla u_\varepsilon)\mathrm{d}x = 0, \qquad (3.14a)$$

$$\lim_{n\to\infty}\int_\Omega m_1\chi_\varepsilon(u_n)(u_n - u_\varepsilon)\mathrm{d}x = 0, \qquad (3.14b)$$

$$\lim_{n\to\infty}\int_\Omega m_2(u_n^+)^{\lambda-2}u_n^+(u_n - u_\varepsilon)\mathrm{d}x = 0. \qquad (3.14c)$$

From (3.13) and (3.14), we infer that

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n - \nabla u_\varepsilon|^p\mathrm{d}x = 0, \qquad (3.15)$$

which, along with (3.11), implies that $u_n \to u_\varepsilon$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$; hence, the Palais-Smale condition is satisfied.

We verified that all conditions of the mountain pass lemma are satisfied. Therefore, there exists a nonzero critical point u_ε such that $\mathcal{J}_\varepsilon'(u_\varepsilon) = 0$, and the corresponding critical value can be expressed as

$$\mathcal{J}_\varepsilon(u_\varepsilon) = c_\varepsilon := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{J}_\varepsilon(\gamma(t)) \qquad (3.16)$$

with

$$\Gamma := \left\{\gamma \in C\left([0,1], W_0^{1,p}(\Omega)\right) : \gamma(0) = 0,\ \mathcal{J}_\varepsilon(\gamma(1)) < 0\right\}.$$

Therefore, u_ε is a mountain pass solution of Eq (3.1) at the level c_ε.

Now, we prove the existence of a weak solution of Eq (1.1). Indeed, in view of (3.3), for a fixed $u_0 \in W_0^{1,p}(\Omega) \setminus \{0\}$, there exists $t_0 > 0$ independent of ε such that $\mathcal{J}_\varepsilon(t_0u_0) < 0$. Setting $w_0 = t_0u_0$, we have $tw_0 \in \Gamma$ for $t \in [0,1]$, and there exists a constant $B(p, \lambda, m_1, m_2, a_1, w_0, \Omega) > 0$ independent of ε such that

$$c_\varepsilon \leq \max_{t\in[0,1]} J_\varepsilon(tw_0)$$

$$
\begin{aligned}
&= \max_{t\in[0,1]}\left[\int_\Omega \frac{a(x)t^p}{p}|\nabla w_0|^p \mathrm{d}x + m_1\int_\Omega \Phi_\varepsilon(tw_0)\mathrm{d}x - \frac{m_2 t^\lambda}{\lambda}\int_\Omega |w_0^+|^\lambda \mathrm{d}x\right]\\
&\le \max_{t\in[0,1]}\left[\frac{a_1 t^p}{p}\int_\Omega |\nabla w_0|^p \mathrm{d}x + m_1\int_\Omega (|tw_0|+1)\mathrm{d}x + \frac{m_2 t^\lambda}{\lambda}\int_\Omega |w_0^+|^\lambda \mathrm{d}x\right]\\
&=: B(p,\lambda,m_1,m_2,a_1,w_0,\Omega). \qquad (3.17)
\end{aligned}
$$

From (3.6), (3.16), and (3.17), we infer that

$$
\begin{aligned}
B(p,\lambda,m_1,m_2,a_1,w_0,\Omega) \ge& c_\varepsilon\\
=&\mathcal{J}_\varepsilon(u_\varepsilon) - \frac{1}{\lambda}\langle \mathcal{J}'_\varepsilon(u_\varepsilon), u_\varepsilon\rangle\\
\ge&\left(\frac{a_0}{p} - \frac{a_0}{\lambda}\right)\int_\Omega |\nabla u_\varepsilon|^p \mathrm{d}x - C\int_\Omega |u_\varepsilon|\mathrm{d}x,
\end{aligned}
$$

which, along with Young's inequality with $\varepsilon_1 > 0$, implies that

$$
\left(\frac{a_0}{p} - \frac{a_0}{\lambda}\right)\int_\Omega |\nabla u_\varepsilon|^p \mathrm{d}x \le C\int_\Omega |u_\varepsilon|\mathrm{d}x + C \le C\varepsilon_1\int_\Omega |u_\varepsilon|^p \mathrm{d}x + C_{\varepsilon_1}. \qquad (3.18)
$$

Choosing a sufficiently small ε_1, we deduce from (3.18) that

$$
\int_\Omega |\nabla u_\varepsilon|^p \mathrm{d}x \le C,
$$

where C is a positive constant which only depends on p, λ, m_1, m_2, a_0, a_1, w_0, and Ω. Since the family $\{u_\varepsilon\}$ is uniformly bounded in $W_0^{1,p}(\Omega)$ with respect to ε, there exist a subsequence, still denoted by $\{u_\varepsilon\}$, and a function $u \in W_0^{1,p}(\Omega)$ such that

$$
u_\varepsilon \rightharpoonup u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_\varepsilon \to u \text{ in } L^{p^*}(\Omega).
$$

Then, proceeding in the same way as for (3.15), we obtain the following:

$$
\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon - \nabla u|^p \mathrm{d}x = 0.
$$

Thus, $u_\varepsilon \to u$ in $W_0^{1,p}(\Omega)$. Now, multiplying Eq (3.1) by any $\varphi \in C_0^\infty(\Omega)$ and integrating over Ω, we obtain the following:

$$
\int_\Omega a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla\varphi + m_1\chi_\varepsilon(u_\varepsilon)\varphi - m_2(u_\varepsilon^+)^{\lambda-1}\varphi \mathrm{d}x = 0. \qquad (3.19)
$$

By virtue of $u_\varepsilon \to u$ in $W_0^{1,p}(\Omega)$ and the subcritical growth condition (1.2), by taking the limit $\varepsilon \to 0$ in (3.19), we obtain the following:

$$
\int_\Omega a(x)|\nabla u|^{p-2}\nabla u\nabla\varphi + m_1\chi_{\{u>0\}}\varphi - m_2(u^+)^{\lambda-1}\varphi \mathrm{d}x = 0,
$$

which, along with density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$, implies that u satisfies the first equation of (1.1) in the weak sense. Furthermore, one can show that such u is non-negative, and hence u is a weak solution of Eq (1.1). Indeed, using $u - u^+ \in W_0^{1,p}(\Omega)$ as a test function for Eq (1.1), we obtain the following:

$$\int_\Omega a|\nabla u|^{p-2}\nabla u(\nabla u - \nabla u^+)\mathrm{d}x + \int_\Omega (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u - u^+)\mathrm{d}x = 0.$$

It follows that

$$\begin{aligned}\int_\Omega a|\nabla u|^p\mathrm{d}x &= \int_\Omega a|\nabla u|^{p-2}\nabla u\nabla u^+\mathrm{d}x - \int_\Omega (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u - u^+)\mathrm{d}x\\ &\le \frac{p-1}{p}\int_\Omega a|\nabla u|^p\mathrm{d}x + \frac{1}{p}\int_\Omega a|\nabla u^+|^p\mathrm{d}x - \int_\Omega (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u - u^+)\mathrm{d}x,\end{aligned}$$

where we used Young's inequality in the last inequality. Therefore, we have the following:

$$\begin{aligned}\int_\Omega a|\nabla u|^p\mathrm{d}x &= \int_\Omega a|\nabla u|^{p-2}\nabla u\nabla u^+\mathrm{d}x - \int_\Omega (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u - u^+)\mathrm{d}x\\ &\le \int_\Omega a|\nabla u^+|^p\mathrm{d}x - p\int_\Omega (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u - u^+)\mathrm{d}x\\ &= \int_{\Omega\cap\{u>0\}} a|\nabla u^+|^p\mathrm{d}x - p\int_{\Omega\cap\{u>0\}} (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u - u^+)\mathrm{d}x\\ &\quad + \int_{\Omega\cap\{u<0\}} a|\nabla u^+|^p\mathrm{d}x - p\int_{\Omega\cap\{u<0\}} (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u - u^+)\mathrm{d}x\\ &= \int_{\Omega\cap\{u>0\}} a|\nabla u|^p\mathrm{d}x,\end{aligned}$$

which implies that

$$\int_{\Omega\cap\{u<0\}} a|\nabla u|^p\mathrm{d}x = 0.$$

Thus, either $u \equiv C$ or $u \ge 0$ a.e. in Ω. Since $u \in W_0^{1,p}(\Omega)$, it follows that $u \ge 0$ a.e. in Ω. □

The following result gives a uniform L^∞ bound for the solutions of Eq (1.1).

Proposition 2. *All weak solutions, still denoted by u, of Eq* (1.1) *are uniformly bounded, having the estimate*

$$\|u\|_{L^\infty(\Omega)} \le C,$$

where C is a positive constant which only depends on N, p, λ, m_1, m_2, a_0, and Ω.

Proof. Fix $k_0 > 0$. For every $k \ge k_0$, define the function $u_k : \Omega \to \mathbb{R}$ by the following:

$$u_k(x) = \begin{cases} k, & \text{if } u(x) > k,\\ u(x), & \text{if } u(x) \le k.\end{cases}$$

Let $S_k = \{x \in \Omega : u(x) > k\}$. Then, for all $k \ge k_0$, we have the following:

$$u_k = u \text{ in } S_k^c \text{ and } u_k = k \text{ in } S_k.$$

Using $u - u_k \in W_0^{1,p}(\Omega)$ as a test function for Eq (1.1), we obtain the following:

$$-\int_\Omega a|\nabla u|^{p-2}\nabla u(\nabla u - \nabla u_k)\mathrm{d}x = \int_\Omega (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u-u_k)\mathrm{d}x.$$

It follows that

$$\begin{aligned}\int_\Omega a|\nabla u|^p\mathrm{d}x =& \int_\Omega a|\nabla u|^{p-2}\nabla u\nabla u_k\mathrm{d}x - \int_\Omega (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u-u_k)\mathrm{d}x\\ \le& \frac{p-1}{p}\int_\Omega a|\nabla u|^p\mathrm{d}x + \frac{1}{p}\int_\Omega a|\nabla u_k|^p\mathrm{d}x - \int_\Omega (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u-u_k)\mathrm{d}x,\end{aligned}$$

where we used Young's inequality in the last inequality. Therefore, we have

$$\begin{aligned}\int_\Omega a|\nabla u|^p\mathrm{d}x =& \int_\Omega a|\nabla u|^{p-2}\nabla u\nabla u_k\mathrm{d}x - \int_\Omega (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u-u_k)\mathrm{d}x\\ \le& \int_\Omega a|\nabla u_k|^p\mathrm{d}x - p\int_\Omega (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u-u_k)\mathrm{d}x,\end{aligned}$$

which, along with the non-negativity of solutions, implies that

$$\begin{aligned}\int_{S_k} a_0|\nabla u|^p\mathrm{d}x \le& \int_{S_k} a|\nabla u|^p\mathrm{d}x\\ =& \int_\Omega a|\nabla u|^p\mathrm{d}x - \int_\Omega a|\nabla u_k|^p\mathrm{d}x\\ \le& -p\int_\Omega (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u-u_k)\mathrm{d}x\\ =& -p\int_{S_k} (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u-u_k)\mathrm{d}x\\ =& -p\int_{S_k} (m_1 - m_2 u^{\lambda-1})(u-k)\mathrm{d}x\\ \le& p\int_{S_k} m_2 u^{\lambda-1}(u-k)\mathrm{d}x\\ \le& p\int_{S_k} m_2 u^{\lambda}\mathrm{d}x\\ \le& pm_2 2^{\lambda-1}\int_{S_k} (u-k)^\lambda + k^\lambda\mathrm{d}x\\ \le& pm_2 2^{\lambda-1}\int_{S_k} |u-k|^\lambda\,\mathrm{d}x + pm_2 2^{\lambda-1}k^\lambda|S_k|.\end{aligned}$$

Thus, it holds that

$$\int_{S_k} |\nabla u|^p\mathrm{d}x \le C\int_{S_k} (u-k)^\lambda\,\mathrm{d}x + Ck^\lambda|S_k|, \tag{3.20}$$

where C is a positive constant which only depends on p, λ, m_2, and a_0.

Define $K_n := \frac{K}{2}\left(1 - \frac{1}{2^{n+1}}\right)$ for all $n \in \mathbb{N}$, where $K_n, K \geq k_0$. Set $g_n := \int_{S_{K_n}} ((u - K_n)^+)^\lambda \mathrm{d}x$ for all $n \in \mathbb{N}$. We claim that

$$g_{n+1} \leq CD^n g_n^{1+\zeta}, \forall n \in \mathbb{N}, \tag{3.21}$$

where $C > 0$, $\zeta > 0$, and $D > 1$ are constants independent of n.

Let

$$q = \begin{cases} p^*, & N \geq 3, \\ p + \lambda, & N = 2. \end{cases}$$

By Hölder's inequality and the embedding inequality, it follows that

$$\begin{aligned} g_{n+1} =& \int_{S_{K_{n+1}}} ((u - K_{n+1})^+)^\lambda \, \mathrm{d}x \\ \leq& \left(\int_{S_{K_{n+1}}} ((u - K_{n+1})^+)^q \, \mathrm{d}x \right)^{\frac{\lambda}{q}} |S_{K_{n+1}}|^{1-\frac{\lambda}{q}} \\ \leq& \left(\int_{\Omega} ((u - K_{n+1})^+)^q \, \mathrm{d}x \right)^{\frac{\lambda}{q}} |S_{K_{n+1}}|^{1-\frac{\lambda}{q}} \\ \leq& C \left(\int_{\Omega} \left| \nabla (u - K_{n+1})^+ \right|^p \, \mathrm{d}x \right)^{\frac{\lambda}{p}} |S_{K_{n+1}}|^{1-\frac{\lambda}{q}} \\ \leq& C \left(\int_{S_{K_{n+1}}} |\nabla u|^p \mathrm{d}x \right)^{\frac{\lambda}{p}} |S_{K_{n+1}}|^{1-\frac{\lambda}{q}}, \end{aligned} \tag{3.22}$$

where C is a positive constant which only depends on N, p, λ, and Ω.

Noting that $K_{n+1} - K_n = \frac{K}{2^{n+3}}$, we obtain the following:

$$\begin{aligned} \left(\frac{K}{2^{n+3}} \right)^\lambda |S_{K_{n+1}}| =& (K_{n+1} - K_n)^\lambda |S_{K_{n+1}}| \\ =& \int_{S_{K_{n+1}}} |K_{n+1} - K_n|^\lambda \mathrm{d}x \\ \leq& \int_{S_{K_{n+1}}} ((u - K_n)^+)^\lambda \, \mathrm{d}x \\ \leq& \int_{S_{K_n}} ((u - K_n)^+)^\lambda \, \mathrm{d}x \\ =& g_n, \end{aligned}$$

which implies that

$$|S_{K_{n+1}}| \leq \left(\frac{2^{n+3}}{K} \right)^\lambda g_n. \tag{3.23}$$

Using (3.23), we deduce that

$$\begin{aligned}\int_{S_{K_{n+1}}} |u-K_{n+1}|^\lambda \,\mathrm{d}x \leq & 2^\lambda \int_{S_{K_{n+1}}} |u-K_n|^\lambda \,\mathrm{d}x + 2^\lambda \int_{S_{K_{n+1}}} |K_n - K_{n+1}|^\lambda \mathrm{d}x \\ \leq & 2^\lambda \int_{S_{K_n}} |u-K_n|^\lambda \,\mathrm{d}x + 2^\lambda |K_n - K_{n+1}|^\lambda |S_{K_{n+1}}| \\ \leq & 2^\lambda g_n + 2^\lambda \left(\frac{2^{n+3}}{K}\right)^\lambda \left(\frac{K}{2^{n+3}}\right)^\lambda g_n \\ = & 2^{\lambda+1} g_n. \end{aligned} \tag{3.24}$$

Then, we have the following:

$$K_{n+1}^\lambda |S_{K_{n+1}}| \leq \left(\frac{K}{2}\left(1-\frac{1}{2^{n+2}}\right)\right)^\lambda \left(\frac{2^{n+3}}{K}\right)^\lambda g_n = \left(\frac{2^{n+3}}{2}\left(1-\frac{1}{2^{n+2}}\right)\right)^\lambda g_n \leq 2^{2n+4} g_n. \tag{3.25}$$

Combining (3.20) and (3.22)–(3.25), we obtain the following:

$$\begin{aligned} g_{n+1} \leq & C\left(\int_{S_{K_{n+1}}} |u-K_{n+1}|^\lambda \,dx + K_{n+1}^\lambda |S_{K_{n+1}}|\right)^{\frac{\lambda}{p}} |S_{K_{n+1}}|^{1-\frac{\lambda}{q}} \\ \leq & C\left(2^{\lambda+1} g_n + 2^{\lambda(n+2)} g_n\right)^{\frac{\lambda}{p}} \left(\left(\frac{2^{n+3}}{K}\right)^\lambda g_n\right)^{1-\frac{\lambda}{q}} \\ \leq & C\left(2^{\lambda+1} + 2^{\lambda(n+2)}\right)^{\frac{\lambda}{p}} g_n^{\frac{\lambda}{p}} \left(\frac{2^{n+3}}{K}\right)^{\lambda-\frac{\lambda^2}{q}} g_n^{1-\frac{\lambda}{q}} \\ \leq & C\left(2^{\lambda(n+2)}\right)^{\frac{\lambda}{p}} \left(2^{n+3}\right)^{\lambda-\frac{\lambda^2}{q}} g_n^{1+\frac{\lambda}{p}-\frac{\lambda}{q}} \\ \leq & C 2^{\frac{n\lambda^2}{p}+\frac{2\lambda^2}{p}} 2^{n\lambda-\frac{n\lambda^2}{q}} g_n^{1+\frac{\lambda}{p}-\frac{\lambda}{q}} \\ \leq & C 2^{n\lambda\left(1+\frac{\lambda}{p}-\frac{\lambda}{q}\right)} g_n^{1+\frac{\lambda}{p}-\frac{\lambda}{q}} \\ \leq & C\left(2^{\lambda+\lambda^2}\right)^n g_n^{1+\zeta}, \end{aligned}$$

where $\zeta := \frac{\lambda}{p} - \frac{\lambda}{q} > 0$. Therefore, inequality (3.21) holds with $D := 2^{\lambda+\lambda^2}$ and some constant $C := C(N, p, \lambda, m_2, a_0, K, \Omega)$. In particular, we have the following:

$$\left(\frac{K}{4}\right)^q |S_{K_0}| = (K_0)^q |S_{K_0}| \leq \int_{S_{K_0}} |u|^q \mathrm{d}x \leq \int_\Omega |u|^q \mathrm{d}x = \|u\|_{L^q(\Omega)}^q. \tag{3.26}$$

From (3.26) and the embedding theorem, we infer that

$$g_0 = \int_{S_{K_0}} ((u-K_0)^+)^\lambda \,\mathrm{d}x \leq \int_{S_{K_0}} |u|^\lambda \mathrm{d}x \leq \|u\|_{L^q(\Omega)}^\lambda |S_{K_0}|^{1-\frac{\lambda}{q}} \leq \left(\frac{4}{K}\right)^{q-\lambda} \|u\|_{L^q(\Omega)}^q.$$

Therefore, by choosing a sufficiently large K which depends on $\|u\|_{L^q(\Omega)}$, we obtain the following:

$$g_0 \leq \left(\frac{4}{K}\right)^{q-\lambda} \|u\|_{L^q(\Omega)}^q \leq C^{-\frac{1}{\zeta}} D^{-\frac{1}{\zeta^2}}. \tag{3.27}$$

In view of (3.21) and (3.27), by Lemma 3, we deduce that $g_n \to 0$ as $n \to \infty$; therefore,

$$\int_{S_{K/2}} \left(\left(u - \frac{K}{2} \right)^+ \right)^p \mathrm{d}x = \lim_{n \to +\infty} \int_{S_{K_n}} ((u - K_n)^+)^p \, \mathrm{d}x = 0.$$

Thus,

$$0 \le u \le \frac{K}{2} \text{ a.e. in } \Omega, \tag{3.28}$$

where K is a positive constant which depends on $\|u\|_{L^q(\Omega)}$.

Now, we estimate $\|u\|_{L^q(\Omega)}$. Indeed, by the embedding theorem, we obtain the following:

$$\|u\|_{L^q(\Omega)} \le C \|\nabla u\|_{L^p(\Omega)}, \tag{3.29}$$

where C is a positive constant which only depends on N, p, q, and Ω, and hence, only on N, p, λ, and Ω. Furthermore, taking u as test function for Eq (1.1), then applying Young's inequality with $\varepsilon_1, \varepsilon_2 > 0$, the embedding inequality, and the Poincaré's inequality, we obtain the following:

$$\begin{aligned}
\int_\Omega a_0 |\nabla u|^p \mathrm{d}x \le & \int_\Omega a(x) |\nabla u|^p \mathrm{d}x \\
= & - \int_\Omega m_1 \chi_{\{u>0\}} \, u \mathrm{d}x + \int_\Omega m_2 u^{\lambda - 1} \chi_{\{u>0\}} u \mathrm{d}x \\
\le & \int_\Omega m_1 |u| \mathrm{d}x + \int_\Omega m_2 |u|^\lambda \mathrm{d}x \\
\le & \varepsilon_1 \int_\Omega |u|^p \mathrm{d}x + \varepsilon_2 \int_\Omega |u|^q \mathrm{d}x + C_{\varepsilon_1, \varepsilon_2} \\
\le & C \varepsilon_1 \int_\Omega |\nabla u|^p \mathrm{d}x + C \varepsilon_2 \int_\Omega |\nabla u|^p \mathrm{d}x + C_{\varepsilon_1, \varepsilon_2},
\end{aligned} \tag{3.30}$$

where C is a positive constant which only depends on p, N, and Ω, while $C_{\varepsilon_1, \varepsilon_2}$ is a positive constant which only depends on ε_1, ε_2, N, p, λ, m_1, m_2, and Ω. Let ε_1 and ε_2 be sufficiently small. It follows from (3.30) that

$$\int_\Omega |\nabla u|^p \mathrm{d}x \le C, \tag{3.31}$$

where C is a positive constant which only depends on N, p, λ, m_1, m_2, a_0, and Ω.

From (3.28), (3.29), and (3.31), we conclude that all solutions have the same bound, which only depends on N, p, λ, m_1, m_2, a_0, and Ω. □

4. Uniformly local Hölder continuity of solutions and their gradients

In this section, we establish uniformly local Hölder continuity of solutions and their gradients for the obstacle problem (1.1).

First, we have the following result regarding the uniformly local Hölder continuity of solutions.

Proposition 3. *All solutions, denoted by u, of Eq* (1.1) *belong to* $C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ *with the same constant* $\alpha \in (0,1)$. *Moreover, for any* $\Omega' \subset\subset \Omega$, *there exists a positive constant C depending on N, p,* λ, m_1, m_2, a_0, a_1, μ, Ω', *and* Ω *such that*

$$\|u\|_{C^{0,\alpha}(\Omega')} \leq C.$$

Proof. For any $x_0 \in \Omega$, let $B_R = B_R(x_0) \subset\subset \Omega$ with some $R > 0$. Let v be a p-harmonic function which satisfies the following:

$$\begin{cases} \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) = 0 \text{ in } B_R, \\ v = u \text{ on } \partial B_R. \end{cases} \tag{4.1}$$

By the maximum principle, we obtain the following:

$$\|v\|_{L^\infty(B_R)} \leq \|v\|_{L^\infty(\partial B_R)} = \|u\|_{L^\infty(\partial B_R)} \leq \|u\|_{L^\infty(\Omega)}. \tag{4.2}$$

Using $u - v \in W^{1,p}_0(B_R)$ as a test function for Equation (1.1), we obtain the following:

$$\int_{B_R} a|\nabla u|^{p-2}\nabla u(\nabla u - \nabla v)\mathrm{d}x + \int_{B_R} (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u-v)\mathrm{d}x = 0.$$

It follows that

$$\begin{aligned} \int_{B_R} a|\nabla u|^p \mathrm{d}x =& \int_{B_R} a|\nabla u|^{p-2}\nabla u\nabla v\mathrm{d}x - \int_{B_R} (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u-v)\mathrm{d}x \\ \leq& \frac{p-1}{p}\int_{B_R} a|\nabla u|^p\mathrm{d}x + \frac{1}{p}\int_{B_R} a|\nabla v|^p\mathrm{d}x - \int_{B_R} (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u-v)\mathrm{d}x, \end{aligned}$$

where we used Young's inequality in the last inequality. Therefore, we have the following:

$$\begin{aligned} \int_{B_R} a|\nabla u|^p\mathrm{d}x - \int_{B_R} a|\nabla v|^p\mathrm{d}x \leq& -p\int_{B_R} (m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}})(u-v)\mathrm{d}x \\ \leq& p\left(\int_{B_R} m_1|v-u|\mathrm{d}x + \int_{B_R} m_2\|u\|^{\lambda-1}_{L^\infty(\Omega)}|v-u|\mathrm{d}x\right) \\ \leq& C\int_{B_R}|v-u|\mathrm{d}x \qquad (4.3) \\ \leq& CR^N, \qquad (4.4) \end{aligned}$$

where, by virtue of (4.2) and Proposition 2, C is a positive constant which only depends on N, p, λ, m_1, m_2, and Ω.

By (4.4) and [13, Lemma A.14], for any $\kappa \in (0, N)$, there holds

$$\begin{aligned} \int_{B_R}|\nabla u|^p\mathrm{d}x \leq& \frac{a}{a_0}\int_{B_R}|\nabla u|^p\mathrm{d}x \\ \leq& \frac{a}{a_0}\int_{B_R}|\nabla v|^p\mathrm{d}x + CR^N \end{aligned}$$

$$\begin{aligned}&\leq\frac{a_1}{a_0}\int_{B_R}|\nabla v|^p\mathrm{d}x+CR^N\\&\leq CR^{\kappa},\end{aligned}\tag{4.5}$$

where we let $\kappa\in(N-1,N)$. Applying [13, Lemma A.13], we conclude that $u\in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha\in(0,1)$. □

The local Hölder continuity for the solutions' gradients is given below.

Theorem 1. *All solutions, denoted by u, of Eq* (1.1) *belong to* $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ *with the same constant* $\alpha\in(0,1)$. *Moreover, for any* $\Omega'\subset\subset\Omega$, *there exists a positive constant C depending on N, p,* λ, m_1, m_2, a_0, a_1, μ, Ω', *and* Ω *such that*

$$\|u\|_{C^{1,\alpha}(\Omega')}\leq C.\tag{4.6}$$

Proof. For any $x_0\in\Omega$, let $B_R=B_R(x_0)\subset\subset\Omega$ with some $R\leq R_*\leq 1$. Let v be a p-harmonic function satisfying (4.1). For any $B_r\subset\subset B_R$, by Lemmas 5 and 6, there exists a constant $\delta\in(0,1)$ such that

$$\begin{aligned}\int_{B_r}|\nabla u-(\nabla u)_r|^p\mathrm{d}x\leq&C\left(\frac{r}{R}\right)^{N+\delta}\int_{B_R}|\nabla u-(\nabla u)_R|^p\mathrm{d}x+C\int_{B_R}|\nabla u-\nabla v|^p\mathrm{d}x\\\leq&C\left(\frac{r}{R}\right)^{N+\delta}\int_{B_R}|\nabla u-(\nabla u)_R|^p\mathrm{d}x+C\int_{B_R}\left(|\nabla u|^p-|\nabla v|^p\right)\mathrm{d}x\\&+CR^{\frac{\mu}{2}}\left(\int_{B_R}\left(|\nabla u|^p-|\nabla v|^p\right)\mathrm{d}x\right)^{\frac{1}{2}},\end{aligned}\tag{4.7}$$

where $\mu\in(0,N)$ is an arbitrary constant, and C is a positive constant which only depends on N, p, μ, and $\|v\|_{L^\infty(B_R)}$.

By (4.3), (4.5), and the C^1-continuity of a, we obtain the following:

$$\begin{aligned}a(x_0)\int_{B_R}\left(|\nabla u|^p-|\nabla v|^p\right)\mathrm{d}x=&\int_{B_R}\left(a(x_0)-a(x)\right)|\nabla u|^p\mathrm{d}x+\int_{B_R}a(x)\left(|\nabla u|^p-|\nabla v|^p\right)\mathrm{d}x\\&+\int_{B_R}\left(a(x)-a(x_0)\right)|\nabla v|^p\mathrm{d}x\\\leq&C\int_{B_R}|x_0-x||\nabla u|^p\mathrm{d}x+\int_{B_R}a(x)\left(|\nabla u|^p-|\nabla v|^p\right)\mathrm{d}x+C\int_{B_R}|x_0-x||\nabla v|^p\mathrm{d}x\\\leq&CR\int_{B_R}\left(|\nabla u|^p+|\nabla v|^p\right)\mathrm{d}x+\int_{B_R}a(x)\left(|\nabla u|^p-|\nabla v|^p\right)\mathrm{d}x\\\leq&CR^{1+\kappa}+C\int_{B_R}|v-u|\mathrm{d}x,\end{aligned}\tag{4.8}$$

where C is a positive constant which only depends on N, p, λ, m_1, m_2, a, and Ω.

By Young's inequality with $\varepsilon_1>0$, we obtain the following:

$$\int_{B_R}|v-u|\mathrm{d}x\leq|B_R|^{\frac{1}{N}}\|v-u\|_{L^{1^*}(B_R)}$$

$$\leq CR\int_{B_R}|\nabla(v-u)|\mathrm{d}x$$
$$\leq C\varepsilon_1 R\int_{B_R}|\nabla v-\nabla u|^p\mathrm{d}x+C_{\varepsilon_1}R^{N+1}, \tag{4.9}$$

where C is a positive constant which only depends on N, p, and Ω, and where $\varepsilon_1>0$ will be chosen later.

We infer from Lemma 5, (4.8), and (4.9) that

$$\begin{aligned}\int_{B_R}|\nabla u-\nabla v|^p\mathrm{d}x \leq& C\int_{B_R}\left(|\nabla u|^p-|\nabla v|^p\right)\mathrm{d}x+CR^{\frac{\mu}{2}}\left(\int_{B_R}\left(|\nabla u|^p-|\nabla v|^p\right)\mathrm{d}x\right)^{\frac{1}{2}}\\ \leq& CR^{1+\kappa}+C\varepsilon_1 R\int_{B_R}|\nabla u-\nabla v|^p\mathrm{d}x+C_{\varepsilon_1}R^{N+1}+C\varepsilon_1^{\frac{1}{2}}R^{\frac{\mu+1}{2}}\left(\int_{B_R}|\nabla u-\nabla v|^p\mathrm{d}x\right)^{\frac{1}{2}}\\ &+C_{\varepsilon_1}R^{\frac{\mu+1+N}{2}}\\ \leq& CR^{1+\kappa}+C\varepsilon_1 R\int_{B_R}|\nabla u-\nabla v|^p\mathrm{d}x+C_{\varepsilon_1}R^{N+1}+\varepsilon_1\int_{B_R}|\nabla u-\nabla v|^p\mathrm{d}x+C_{\varepsilon_1}R^{\frac{\mu+1+N}{2}}\\ &+CR^{\mu+1}.\end{aligned}$$

Considering a sufficiently small ε_1, we obtain the following:

$$\int_{B_R}|\nabla u-\nabla v|^p\mathrm{d}x\leq CR^m, \tag{4.10}$$

where $m:=\min\left\{1+\kappa,N+1,\mu+1,\frac{\mu+1+N}{2}\right\}>N$, and C is a positive constant which only depends on N, p, λ, m_1, m_2, a, μ, and Ω.

From (4.8)–(4.10), and the choice of ε_1, we obtain the following:

$$\begin{aligned}\int_{B_R}\left(|\nabla u|^p-|\nabla v|^p\right)\mathrm{d}x\leq& C\int_{B_R}|\nabla u-\nabla v|^p\mathrm{d}x+CR^{N+1}\\ \leq& CR^{m+1},\end{aligned} \tag{4.11}$$

where C is a positive constant which only depends on N, p, λ, m_1, m_2, a, μ, and Ω.

Substituting (4.11) into (4.7), we obtain that

$$\int_{B_r}|\nabla u-(\nabla u)_r|^p\mathrm{d}x\leq C\left(\frac{r}{R}\right)^{N+\delta}\int_{B_R}|\nabla u-(\nabla u)_R|^p\mathrm{d}x+CR^{m+1}+CR^{\frac{\mu+N+1}{2}}+CR^{\frac{\mu+m+1}{2}},\forall r\in(0,R].$$

Since $\mu\in(0,N)$ is arbitrary, we may choose $\mu\in(N-1,N)$, which then implies that $\min\left\{m+1,\frac{\mu+N+1}{2},\frac{\mu+m+1}{2}\right\}>N$. Consequently, there exists a constant $\alpha_1>0$ such that

$$\int_{B_r}|\nabla u-(\nabla u)_r|^p\mathrm{d}x\leq C\left(\frac{r}{R}\right)^{N+\delta}\int_{B_R}|\nabla u-(\nabla u)_R|^p\mathrm{d}x+CR^{N+\alpha_1},$$

where C is a positive constant which only depends on N, p, λ, m_1, m_2, a, μ, and Ω.

By inequality (5.1) in [17] it holds that

$$\int_{B_r}|\nabla u-(\nabla u)_r|^p\mathrm{d}x\leq C\int_{B_r}|\nabla u-(\nabla u)_R|^p\mathrm{d}x$$

$$\leq C \int_{B_R} |\nabla u - (\nabla u)_R|^p \mathrm{d}x,$$

where C is a positive constant which only depends on N and p.

Applying Lemma 4 with $\tau(r) := \int_{B_r} |\nabla u - (\nabla u)_r|^p \mathrm{d}x$, we find that there exists a constant $\alpha_2 \in (0, 1)$ such that

$$\int_{B_r} |\nabla u - (\nabla u)_r|^p \mathrm{d}x \leq C r^{N+\alpha_2},$$

where C is a positive constant which only depends on N, p, λ, m_1, m_2, a_0, μ, and Ω. It follows that

$$\begin{aligned}\int_{B_r} |\nabla u - (\nabla u)_r| \mathrm{d}x \leq & \left(\int_{B_r} |\nabla u - (\nabla u)_r|^p \mathrm{d}x \right)^{\frac{1}{p}} |B_r|^{\frac{p-1}{p}} \\ \leq & C r^{N+\alpha}\end{aligned}$$

with $\alpha := \frac{\alpha_2}{p} \in (0, 1)$, where C is a positive constant which only depends on N, p, λ, m_1, m_2, a, μ, and Ω. Thus, ∇u belongs to the Campanato space. According to the Campanato's embedding theorem, we conclude that u is locally $C^{1,\alpha}$ continuous and (4.6) holds true. □

5. Porosity of the free boundary

This section first presents some results on the optimal growth and non-degeneracy of solutions of Eq (1.1), and then states the local porosity of the free boundary. Without special statements, u always denotes the weak solution of Eq (1.1).

The first result is concerned with the optimal growth of solutions near the free boundary.

Proposition 4. *Let $y \in \Gamma^+$ and $B_{r_1}(y) \subset\subset \Omega$ for some $r_1 > 0$. Then, there exists a positive constant C_1 which only depends on N, p, λ, m_1, m_2, and Ω such that*

$$|u(x)| \leq C_1 |x - y|^{\frac{p}{p-1}}, \forall x \in B_r(y) \tag{5.1}$$

and

$$|\nabla u(x)| \leq C_1 |x - y|^{\frac{1}{p-1}}, \forall x \in B_r(y), \tag{5.2}$$

hold for every $r \in (0, r_1)$.

Proof. Once the regularity of solutions has been established, the proof can proceed in a standard manner as in [2, 3, 5]. For the readers' convenience, we only present the main steps.

Step 1: Normalize the problem. Without a loss of generality, we assume that $B_1(y) \subset\subset \Omega$ with $y := 0 \in \Gamma^+$. For simplicity, let $B_r(y) := B_r$. Since u is bounded (Proposition 2), it follows that

$$\left\| \mathrm{div}\left(a(x) |\nabla u|^{p-2} \nabla u \right) \right\|_{L^\infty(B_1)} = \left\| \left(m_1 \chi_{\{u>0\}} - m_2 u^{\lambda-1} \chi_{\{u>0\}} \right) \right\|_{L^\infty(B_1)} \leq M := \max\left\{ 1, m_1 + m_2 \|u\|^{\lambda}_{L^\infty(\Omega)} \right\} < +\infty.$$

Furthermore, we can assume that $M = 1$ and $\|u\|_{L^\infty(B_1)} \leq 1$. Therefore,

$$u \in \mathbb{U} := \left\{ v \in W^{1,p}(B_1) : \left\| \mathrm{div}\left(a(x) |\nabla v|^{p-2} \nabla v \right) \right\|_{L^\infty(B_1)} \leq 1, \ \ 0 \leq v \leq 1 \text{ in } B_1, \ \ v(0) = 0 \right\}. \tag{5.3}$$

Define $S(r, v) := \sup_{x\in B_r} v(x)$, and for $v \in \mathbb{U}$, define the set

$$\mathbb{M} := \left\{ j \in \mathbb{N}^+ : 2^{\frac{p-1}{p}} S\left(2^{-j-1}, v\right) \geq S\left(2^{-j}, v\right) \right\}.$$

Step 2: We show that for any $v \in \mathbb{U}$,

$$S\left(2^{-j-1}, v\right) \leq K\left(2^{-j}\right)^{\frac{p}{p-1}}, \forall j \in \mathbb{N}^+. \tag{5.4}$$

Indeed, if (5.4) is not true, then we have the following:

$$\forall k \in \mathbb{N}^+,\ \exists v_k \in \mathbb{U},\ \exists j_k \in \mathbb{M} \text{ such that } S(2^{-j_k-1}, v_k) \geq k(2^{-j_k})^{\frac{p}{p-1}}. \tag{5.5}$$

Note that (5.5) and the boundedness of v_k ensure that $j_k \to \infty$ as $k \to \infty$. Let $\tilde{v}_k(x) := \frac{v_k(2^{-j_k}x)}{S(2^{-j_k-1}, v_k)}$ and $a_k(x) := a(2^{-j_k}x)$ be defined in B_1. As in the proof of [5, Lemma 2.3], we deduce that

$$\mathrm{div}_x\left(a_k(x)\left|\nabla\tilde{v}_k(x)\right|^{p-2}\nabla\tilde{v}_k(x)\right) = \left(\frac{2^{-j_k}}{S(2^{-j_k-1}, v_k)}\right)^{p-1} 2^{-j_k}\mathrm{div}_z\left(a(z)\left|(\nabla v_k)(z)\right|^{p-2}(\nabla v_k)(z)\right).$$

Using (5.3) and (5.5), we obtain

$$\left\|\mathrm{div}_x\left(a_k(x)\left|\nabla\tilde{v}_k(x)\right|^{p-2}\nabla\tilde{v}_k(x)\right)\right\|_{L^\infty(B_1)} \leq \frac{1}{k^{p-1}},$$

which implies that

$$1 = \sup_{B_{\frac{1}{2}}} \tilde{v}_k(x) \leq \frac{C}{k} \to 0 \ \text{ as } \ k \to \infty,$$

where C is a positive constant which only depends on N, p, λ, m_1, and m_2. We get a contradiction. Thus, (5.4) holds true.

Step 3: We show that (5.1) holds true. Indeed, as in the proof of [5, Theorem 2.2], whether $j \in \mathbb{M}$ or $j \notin \mathbb{M}$, we always have the following for $v \in \mathbb{U}$:

$$S\left(2^{-j-1}, v\right) \leq C_1\left(2^{-j-2}\right)^{\frac{p}{p-1}},$$

where $C_1 := \max\left\{2^{\frac{p}{p-1}}, K2^{\frac{2p}{p-1}}\right\}$. Furthermore, noting that $u \in \mathbb{U}$, a standard argument leads to (5.1).

Step 4: We show that (5.2) holds true. Indeed, by virtue of Steps 1–3, it suffices to prove that there exists a constant $C^* = C^*(N, p, m_1, m_2, \lambda)$ such that for all $v \in \mathbb{U}$,

$$S\left(2^{-j-1}, |\nabla v|\right) \leq \max\left\{C^*\left(2^{-j}\right)^{\frac{1}{p-1}}, 2^{-\frac{1}{p-1}}S\left(2^{-j}, |\nabla v|\right)\right\}, \forall j \in \mathbb{N}^+. \tag{5.6}$$

Indeed, if (5.6) is not true, then,

$$\forall k \in \mathbb{N}^+,\ v_k \in \mathbb{F} \ \text{ such that } S\left(2^{-j_k-1}, |\nabla v_k|\right) \geq \max\left\{k\left(2^{-j_k}\right)^{\frac{1}{p-1}}, 2^{-\frac{1}{p-1}}S\left(2^{-j}, |\nabla v_k|\right)\right\}.$$

Let $\tilde{v}_k(x) = \frac{v_k(2^{-j_k}x)}{2^{-j_k}S\left(2^{-j_k-1},|\nabla v_k|\right)}$ and $a_k(x) = a(2^{-j_k}x)$ be defined in B_1. As in the proof of [3, Lemma 2.3], we have

$$|\tilde{v}_k| \le \frac{C_1}{k} \text{ and } \left\|\mathrm{div}_x\left(a_k(x)\,|\nabla\tilde{v}_k(x)|^{p-2}\,\nabla\tilde{v}_k(x)\right)\right\|_{L^\infty(B_1)} \le \frac{1}{k^{p-1}},$$

which implies that

$$1 = \sup_{B_{\frac{1}{2}}} \tilde{v}_k(x) \le C\left(\frac{1}{k} + \frac{1}{k^{p-1}}\right) \to 0 \text{ as } k \to \infty,$$

where C is a positive constant which only depends on N, p, λ, m_1, and m_2. We get a contradiction. Therefore, (5.6) holds true. Furthermore, noting that $u \in \mathbb{U}$, a standard argument leads to (5.2). □

The following result gives the non-degeneracy of weak solutions of Eq (1.1) near the free boundary Γ^+.

Proposition 5. *Let $y \in \Gamma^+$, and let $B_{r_2} \subset\subset \Omega$ with $r_2 \le \min\left\{r_1, \frac{1}{a_1}, \left(\frac{m_1}{2m_2C_1^{\lambda-1}}\right)^{\frac{p-1}{p(\lambda-1)}}\right\}$. Then, there exists a positive constant C_2 which only depends on N, p, m_1, a_1, and r_2 such that*

$$\sup_{\partial B_r(y)\cap\{u>0\}} u \ge C_2 r^{\frac{p}{p-1}}, \forall r \in \left(0, \frac{r_2}{2}\right),$$

where r_1 and C_1 are positive constants the same as in Proposition 4.

Proof. Let $z \in \{u > 0\}$. For any $x_0 \in \Gamma^+$ satisfying $|z - x_0| \le \frac{r_2}{2}$, we have $B_r(z) \subset B_{r_2}(x_0)$ with $r < \frac{r_2}{2}$. Define $C_2 := \frac{p-1}{p}\left(\frac{m_1}{2N(a_1+1)}\right)^{\frac{1}{p-1}}$ and $v(x) := C_2|x - z|^{\frac{p}{p-1}}$ in $B_r(z)$. By direct computations, we have the following:

$$\begin{aligned}
\nabla v &= C_2\frac{p}{p-1}|x-z|^{\frac{2-p}{p-1}}(x-z),\\
|\nabla v| &= C_2\frac{p}{p-1}|x-z|^{\frac{1}{p-1}},\\
|\nabla v|^{p-2} &= C_2^{p-2}\left(\frac{p}{p-1}\right)^{p-2}|x-z|^{\frac{p-2}{p-1}}.
\end{aligned}$$

Noting that $r_2 < \frac{1}{a_1}$ and using the definition of C_2, we obtain the following:

$$\begin{aligned}
\mathrm{div}\left(a(x)|\nabla v|^{p-2}\nabla v\right) =& \mathrm{div}\left(C_2^{p-1}\left(\frac{p}{p-1}\right)^{p-1}a(x)(x-z)\right)\\
=& C_2^{p-1}\left(\frac{p}{p-1}\right)^{p-1}\mathrm{div}\,(a(x)(x-z))\\
=& NC_2^{p-1}\left(\frac{p}{p-1}\right)^{p-1}a(x) + C_2^{p-1}\left(\frac{p}{p-1}\right)^{p-1}(x-z)\nabla a(x)\\
\le& Na_1C_2^{p-1}\left(\frac{p}{p-1}\right)^{p-1} + Na_1r_2C_2^{p-1}\left(\frac{p}{p-1}\right)^{p-1}
\end{aligned}$$

$$\begin{aligned}
&\leq Na_1C_2^{p-1}\left(\frac{p}{p-1}\right)^{p-1} + NC_2^{p-1}\left(\frac{p}{p-1}\right)^{p-1}\\
&= N(a_1+1)C_2^{p-1}\left(\frac{p}{p-1}\right)^{p-1}\\
&= \frac{m_1}{2} \text{ in } B_r(z). \qquad (5.7)
\end{aligned}$$

Using the fact that $B_r(z) \subset B_{r_2}(x_0)$ and Proposition 4, we deduce that

$$\begin{aligned}
\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) &= m_1 - m_2 u^{\lambda-1}\\
&\geq m_1 - m_2 C_1^{\lambda-1} r_2^{\frac{p(\lambda-1)}{p-1}}\\
&\geq \frac{m_1}{2} \text{ in } B_r(z)\cap\{u>0\},
\end{aligned}$$

which, along with (5.7), implies that

$$-\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) \leq -\operatorname{div}\left(a(x)|\nabla v|^{p-2}\nabla v\right) \text{ in } B_r(z)\cap\{u>0\}.$$

Note that the definitions of u and v ensure that

$$u = 0 \leq v \text{ on } B_r(z)\cap\Gamma^+.$$

If $u \leq v$ on $\partial B_r(z)\cap\{u>0\}$, then by the comparison principle, we obtain

$$u \leq v \text{ in } B_r(z)\cap\{u>0\}.$$

However, $u(z) > 0 = v(z)$, which yields a contradiction. Therefore, there exists $y_0 \in \partial B_r(z)\cap\{u>0\}$ such that

$$u(y_0) \geq v(y_0) = C_2 r^{\frac{p}{p-1}},$$

so we have

$$\sup_{\partial B_r(z)\cap\{u>0\}} u \geq C_2 r^{\frac{p}{p-1}}. \qquad (5.8)$$

Now, for $y \in \partial\{u>0\}$, we take a sequence $\{z_i\} \subset \{u>0\}$ such that $z_i \to y$ as $i \to \infty$. By (5.8), we have

$$\sup_{\partial B_r(z_i)\cap\{u>0\}} u \geq C_2 r^{\frac{p}{p-1}},$$

which, along with the continuity of u, implies the desired result. □

Recall that a set $P \subset \mathbb{R}^N$ is called porous with the porosity constant δ if there exist constants $r_0 > 0$ and $\delta > 0$ such that

$$\forall x \in P, \forall r \in (0, r_0) \quad \Rightarrow \quad \exists y \in \mathbb{R}^N \text{ such that } B_{\delta r}(y) \subset B_r(x)\setminus P.$$

It is known that the Hausdorff dimension of a porous set is no more than $N - c\delta^N$ with c only depending on N; for example, see [18, 19].

For the obstacle problem (1.1), the optimal growth and non-degeneracy of solutions imply the local porosity of the free boundary, which is stated as the following theorem.

Theorem 2. *Let u be a solution of Eq* (1.1)*. Then, for every compact set $K \subset \Omega$, the intersection $K \cap \partial\{u>0\}$ is porous with the porosity constant δ which only depends on N, p, λ, m_1, m_2, a_0, a_1, and Ω.*

Proof. The proof is standard; for example, see [2, 6, 20]. □

6. The corresponding penalized problem

In this section, we study properties of solutions u_ε to the penalty problem (3.1), for which the limit of u_ε in $W_0^{1,p}(\Omega)$ is a solution, denoted by u, to the obstacle problem (1.1) (see the proof of Proposition 1). Note that $u \geq 0$ a.e. in Ω; thus, $u_\varepsilon \geq 0$ a.e. in Ω. Moreover, proceeding in the same way as in the proof of Proposition 2, it can be shown that u_ε is uniformly bounded with $\|u_\varepsilon\|_{L^\infty(\Omega)}$ only depending on N, p, λ, m_1, m_2, a_0, and Ω. Furthermore, a similar proof of Theorem 1 ensures that $u_\varepsilon \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ with some $\alpha \in (0,1)$ and that for any $\Omega' \subset\subset \Omega$, $\|u_\varepsilon\|_{C^{1,\alpha}(\Omega')}$ only depends on N, p, λ, m_1, m_2, a, Ω', and Ω. In particular, there exists a positive constant c_1 independent of ε such that

$$\|u_\varepsilon\|_{L^\infty(\Omega')} + \|\nabla u_\varepsilon\|_{L^\infty(\Omega')} \leq c_1. \tag{6.1}$$

In addition, for any $\beta \in (0,\alpha)$ and any relatively compact set $\Omega'' \subset\subset \Omega'$, $u_\varepsilon \to u$ in $C^{1,\beta}(\Omega'')$. In the sequel, we always denote a non-negative weak solution of Eq (3.1) which satisfies (6.1) by u_ε.

The following result indicates that every weak solution of the penalty problem (3.1) is actually a strong solution.

Proposition 6. *The weak solution u_ε satisfies* (3.1) *almost everywhere in* Ω.

Proof. By Lemma 2, it suffices to show that

$$\left(a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right)_{x_i} \in \left(L^1_{\mathrm{loc}}(\Omega)\right)^N.$$

Indeed, a direct computation yields the following:

$$\begin{aligned}\left|\left(a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right)_{x_i}\right| &= \left|a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon + (p-1)a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_{\varepsilon x_i}\right| \\ &\leq |a_{x_i}(x)||\nabla u_\varepsilon|^{p-1} + (p-1)a(x)|\nabla u_\varepsilon|^{p-2}|D^2 u_\varepsilon|. \end{aligned}\tag{6.2}$$

Define $\Psi := u_{\varepsilon x_i}\phi^2$, with $\phi \in C_0^\infty(B_{3r/5})$ satisfying

$$\begin{cases} 0 \leq \phi \leq 1 \text{ in } B_{3r/5}, \\ \phi = 1 \text{ in } B_{r/2}, \\ |\nabla\phi| \leq \frac{4}{r} \text{ in } B_{3r/5}, \end{cases}$$

where $B_r \subset\subset \Omega$ is a ball with some fixed radius $r > 0$.

Differentiating (3.1) with respect to x_i, then multiplying the result by Ψ and integrating over $B_{3r/5}$, we obtain the following:

$$\begin{aligned} I_1 &:= \int_{B_{3r/5}} \left(a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right)_{x_i} \nabla\Psi \mathrm{d}x \\ &= \int_{B_{3r/5}} \left(-m_1\chi_\varepsilon(u_\varepsilon) + m_2(u_\varepsilon^+)^{\lambda-1}\right)_{x_i} \Psi \mathrm{d}x \\ &= \int_{B_{3r/5}} m_2\left((u_\varepsilon^+)^{\lambda-1}\right)_{x_i}\Psi \mathrm{d}x - \int_{B_{3r/5}} m_1\left(\chi_\varepsilon(u_\varepsilon)\right)_{x_i}\Psi \mathrm{d}x \\ &=: I_2 + I_3. \end{aligned}\tag{6.3}$$

Using the definition of Ψ, the term I_1 expands as follows:

$$\begin{aligned}
I_1 &= \int_{B_{3r/5}} (a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)_{x_i}\nabla\Psi \mathrm{d}x \\
&= \int_{B_{3r/5}} \Big((p-1)a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_{\varepsilon x_i} + a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\Big)\Big(\nabla u_{\varepsilon x_i}\phi^2 + 2u_{\varepsilon x_i}\phi\nabla\phi\Big)\mathrm{d}x \\
&= \int_{B_{3r/5}} (p-1)a(x)|\nabla u_\varepsilon|^{p-2}|\nabla u_{\varepsilon x_i}|^2\phi^2\mathrm{d}x + \int_{B_{3r/5}} 2(p-1)a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_{\varepsilon x_i}u_{\varepsilon x_i}\phi\nabla\phi\mathrm{d}x \\
&\quad + \int_{B_{3r/5}} a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla u_{\varepsilon x_i}\phi^2\mathrm{d}x + \int_{B_{3r/5}} 2a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon u_{\varepsilon x_i}\phi\nabla\phi\mathrm{d}x \\
&=: I_{11} + I_{12} + I_{13} + I_{14}.
\end{aligned} \tag{6.4}$$

Since $p \geq 2$, it follows that

$$\begin{aligned}
I_{11} &= \int_{B_{3r/5}} (p-1)a(x)|\nabla u_\varepsilon|^{p-2}|\nabla u_{\varepsilon x_i}|^2\phi^2\mathrm{d}x \\
&\geq \int_{B_{3r/5}} a_0|\nabla u_\varepsilon|^{p-2}|\nabla u_{\varepsilon x_i}|^2\phi^2\mathrm{d}x.
\end{aligned} \tag{6.5}$$

Combining the definition of Ψ, Young's inequality, and (6.1), we obtain the following:

$$\begin{aligned}
|I_{12}| \leq& \int_{B_{3r/5}} 2(p-1)a_1|\nabla u_\varepsilon|^{p-2}|\nabla u_{\varepsilon x_i}||u_{\varepsilon x_i}||\phi||\nabla\phi|\mathrm{d}x \\
\leq& \frac{8}{r}\int_{B_{3r/5}} (p-1)a_1|\nabla u_\varepsilon|^{p-2}|\nabla u_{\varepsilon x_i}||u_{\varepsilon x_i}||\phi|\mathrm{d}x \\
\leq& \frac{a_0}{2}\int_{B_{3r/5}} |\nabla u_\varepsilon|^{p-2}|\nabla u_{\varepsilon x_i}|^2|\phi|^2\mathrm{d}x + \frac{32(p-1)^2a_1^2}{a_0r^2}\int_{B_{3r/5}} |\nabla u_\varepsilon|^{p-2}|u_{\varepsilon x_i}|^2\mathrm{d}x \\
\leq& \frac{a_0}{2}\int_{B_{3r/5}} |\nabla u_\varepsilon|^{p-2}|\nabla u_{\varepsilon x_i}|^2|\phi|^2\mathrm{d}x + \frac{32(p-1)^2a_1^2}{a_0r^2}\int_{B_{3r/5}} |\nabla u_\varepsilon|^{p}\mathrm{d}x \\
\leq& \frac{a_0}{2}\int_{B_{3r/5}} |\nabla u_\varepsilon|^{p-2}|\nabla u_{\varepsilon x_i}|^2|\phi|^2\mathrm{d}x + Cr^{N-2},
\end{aligned} \tag{6.6}$$

where C is a positive constant which only depends on p, a_0, a_1, and c_1.

By Young's inequality with $\varepsilon_1 > 0$, and (6.1), we have the following:

$$\begin{aligned}
|I_{13}| =& \int_{B_{3r/5}} |a_{x_i}(x)||\nabla u_\varepsilon|^{p-1}|\nabla u_{\varepsilon x_i}||\phi|^2\mathrm{d}x \\
\leq& \int_{B_{3r/5}} \frac{\varepsilon_1}{2}|\nabla u_\varepsilon|^{p-2}|\nabla u_{\varepsilon x_i}|^2|\phi|^2\mathrm{d}x + \int_{B_{3r/5}} \frac{1}{2\varepsilon_1}|a_{x_i}(x)|^2|\nabla u_\varepsilon|^p||\phi|^2\mathrm{d}x \\
\leq& \frac{\varepsilon_1}{2}\int_{B_{3r/5}} |\nabla u_\varepsilon|^{p-2}|\nabla u_{\varepsilon x_i}|^2|\phi|^2\mathrm{d}x + \frac{a_1^2}{2\varepsilon_1}\int_{B_{3r/5}} |\nabla u_\varepsilon|^p|\phi|^2\mathrm{d}x.
\end{aligned} \tag{6.7}$$

Choosing a suitable ε_1, Eq (6.7) becomes the following:

$$|I_{13}| \leq \frac{a_0}{4}\int_{B_{3r/5}} |\nabla u_\varepsilon|^{p-2}|\nabla u_{\varepsilon x_i}|^2|\phi|^2\mathrm{d}x + Cr^N, \tag{6.8}$$

where C is a positive constant which only depends on a_0, a_1, and c_1.

Using the definition of ϕ and (6.1), we obtain the following:

$$\begin{aligned} |I_{14}| =& \int_{B_{3r/5}} 2a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon u_{\varepsilon x_i}\phi\nabla\phi \mathrm{d}x \\ \leq& \int_{B_{3r/5}} \frac{8a_1}{r}|\nabla u_\varepsilon|^p \mathrm{d}x \\ \leq& Cr^{N-1}, \end{aligned} \tag{6.9}$$

where C is a positive constant which only depends on a_1 and c_1.

From the definition of ϕ, (6.1), and the Poincaré's inequality, we deduce that

$$\begin{aligned} I_2 =& \int_{B_{3r/5}} m_2\left((u_\varepsilon^+)^{\lambda-1}\right)_{x_i}\Psi \mathrm{d}x \\ \leq& \int_{B_{3r/5}} m_2(\lambda-1)(u_\varepsilon)^{\lambda-2}(u_{\varepsilon x_i})^2\phi^2 \mathrm{d}x \\ \leq& \int_{B_{3r/5}} m_2(\lambda-1)c_1^\lambda \mathrm{d}x \\ \leq& Cr^N, \end{aligned} \tag{6.10}$$

where C is a positive constant which only depends on N, λ, m_2, c_1, and r.

Note that $\chi'_\varepsilon(s) \geq 0$. It follows that

$$\begin{aligned} I_3 =& -\int_{B_{3r/5}} m_1\left(\chi_\varepsilon(u_\varepsilon)\right)_{x_i}\Psi \mathrm{d}x \\ =& -\int_{B_{3r/5}} m_1\chi'_\varepsilon(u_\varepsilon)(u_{\varepsilon x_i})^2\phi^2 \mathrm{d}x \\ \leq& 0. \end{aligned} \tag{6.11}$$

Using (6.3), (6.10), and (6.11), we have the following:

$$I_1 = I_2 + I_3 \leq Cr^N, \tag{6.12}$$

where C is a positive constant which only depends on N, λ, m_2, c_1, and r.

From (6.4)–(6.6), (6.8), and (6.9), we infer that

$$\begin{aligned} I_1 \geq& I_{11} - |I_{12}| - |I_{13}| - |I_{14}| \\ \geq& \int_{B_{3r/5}} \frac{a_0}{4}|\nabla u_\varepsilon|^{p-2}|\nabla u_{\varepsilon x_i}|^2\phi^2 \mathrm{d}x - Cr^{N-2} \\ \geq& \int_{B_{r/2}} \frac{a_0}{4}|\nabla u_\varepsilon|^{p-2}|\nabla u_{\varepsilon x_i}|^2\phi^2 \mathrm{d}x - Cr^{N-2} \\ =& \int_{B_{r/2}} \frac{a_0}{4}|\nabla u_\varepsilon|^{p-2}|\nabla u_{\varepsilon x_i}|^2 \mathrm{d}x - Cr^{N-2}, \end{aligned} \tag{6.13}$$

where C is a positive constant which only depends on p, a_0, a_1, and c_1.

By (6.12) and (6.13), we obtain the following:

$$\int_{B_{r/2}} |\nabla u_\varepsilon|^{p-2} |\nabla u_{\varepsilon x_i}|^2 \mathrm{d}x \le C r^{N-2}, \tag{6.14}$$

where C is a positive constant which only depends on N, p, λ, m_2, c_1, a_0, a_1, and r.

Summing (6.14) over $i = 1$ to N leads to the following:

$$\int_{B_{r/2}} |\nabla u_\varepsilon|^{p-2} |D^2 u_\varepsilon| \mathrm{d}x \le \int_{B_{r/2}} \frac{1}{2} |\nabla u_\varepsilon|^{p-2} |D^2 u_\varepsilon|^2 \mathrm{d}x + \int_{B_{r/2}} \frac{1}{2} |\nabla u_\varepsilon|^{p-2} \mathrm{d}x \le C r^{N-2}. \tag{6.15}$$

Note that (6.1) ensures that

$$\int_{B_{r/2}} |a_{x_i}(x)| |\nabla u_\varepsilon|^{p-1} \le a_1 \int_{B_{r/2}} |\nabla u_\varepsilon|^{p-1} \le C r^N, \tag{6.16}$$

where C is a positive constant which only depends on p, a_1, and c_1.

From (6.15), (6.16), and the boundedness of a, we deduce that

$$|a_{x_i}(x)| |\nabla u_\varepsilon|^{p-1} + (p-1) a(x) |\nabla u_\varepsilon|^{p-2} |D^2 u_\varepsilon| \in L^1_{\mathrm{loc}}(\Omega),$$

which, along with (6.2), implies that

$$\left(a(x) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \right)_{x_i} \in \left(L^1_{\mathrm{loc}}(\Omega) \right)^N.$$

□

The following proposition presents a quantitative lower bound that characterizes the non-degeneracy of both the gradient and the Hessian of the penalized solution.

Proposition 7. *It holds that*

$$\frac{1}{C_3} \left| m_1 \chi_\varepsilon (u_\varepsilon) - m_2 (u_\varepsilon^+)^{\lambda-1} \right|^2 \le |\nabla u_\varepsilon|^{2(p-1)} + \left(|\nabla u_\varepsilon|^{p-2} |D^2 u_\varepsilon| \right)^2 \ \text{a.e. in } \Omega,$$

where C_3 is a positive constant which only depends on N, p, and a_1.

Proof. By direct calculation, we obtain the following:

$$\begin{aligned}
&\left| m_1 \chi_\varepsilon (u_\varepsilon) - m_2 (u_\varepsilon^+)^{\lambda-1} \right|^2 \\
&= \left(\operatorname{div} \left(a(x) |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \right) \right)^2 \\
&= \left(\sum_{i=1}^N \left(\frac{\partial \left(a(x) |\nabla u_\varepsilon|^{p-2} \right)}{\partial x_i} u_{\varepsilon x_i} + a(x) |\nabla u_\varepsilon|^{p-2} u_{\varepsilon x_i x_i} \right) \right)^2 \\
&= \left(\sum_{i=1}^N \left(a_{x_i}(x) |\nabla u_\varepsilon|^{p-2} u_{\varepsilon x_i} + (p-2) a(x) |\nabla u_\varepsilon|^{p-4} \left(\sum_{j=1}^N u_{\varepsilon x_j} u_{\varepsilon x_j x_i} \right) u_{\varepsilon x_i} + a(x) |\nabla u_\varepsilon|^{p-2} u_{\varepsilon x_i x_i} \right) \right)^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\sum_{i=1}^{N} \left(a_1 |\nabla u_\varepsilon|^{p-1} + (p-2) a_1 |\nabla u_\varepsilon|^{p-2} |\nabla u_{\varepsilon x_i}| + a_1 |\nabla u_\varepsilon|^{p-2} |u_{\varepsilon x_i x_i}| \right) \right)^2 \\
&\leq \left(N a_1 |\nabla u_\varepsilon|^{p-1} + \sum_{i=1}^{N} \left((p-2) a_1 |\nabla u_\varepsilon|^{p-2} |\nabla u_{\varepsilon x_i}| + a_1 |\nabla u_\varepsilon|^{p-2} |u_{\varepsilon x_i x_i}| \right) \right)^2 \\
&\leq \left(N a_1 |\nabla u_\varepsilon|^{p-1} + N a_1 (p-1) |\nabla u_\varepsilon|^{p-2} |D^2 u_\varepsilon| \right)^2 \\
&\leq C_3 \left(|\nabla u_\varepsilon|^{2(p-1)} + \left(|\nabla u_\varepsilon|^{p-2} |D^2 u_\varepsilon| \right)^2 \right).
\end{aligned} \tag{6.17}
$$

This completes the proof. □

7. Hausdorff measure of the free boundary

In this section, we prove that at least one solution u to the obstacle problem (1.1) corresponds to the free boundary having a locally finite $(N-1)$-dimensional Hausdorff measure. More specifically, we consider $u_\varepsilon \to u$ in $C^{1,\alpha}_{\text{loc}}(\Omega)$ with u_ε and u being the solutions of the penalty problem (3.1) and the obstacle problem (1.1), respectively. First, by virtue of Proposition 4, for $\sigma \in (0,1)$, we define the following:

$$
O_\sigma := \left\{ |\nabla u| \leq \sigma^{\frac{1}{p-1}} \right\}, \quad O_{\sigma i} := \left\{ |u_{x_i}| \leq \sigma^{\frac{1}{p-1}} \right\}.
$$

The following proposition provides a crucial measure-theoretic estimate, which serves as a foundation to investigate the Hausdorff measure of the free boundary.

Proposition 8. *Let C_1 and r_2 be the positive constants determined by Propositions 4 and 5, respectively. Let $B_{r_4} \subset\subset \Omega$ be a ball whose radius r_4 is sufficiently small and satisfies $r_4 \leq r_2$. Then, for every $x_0 \in \Gamma^+ \cap B_{r_4}$, any $\sigma \in (0,1)$, and any $r \in (0, r_4)$ with $B_r(x_0) \subset B_{r_4}$, it holds that*

$$
\int_0^1 \mathcal{L}^N \left(O_\sigma \cap B_{rs}(x_0) \cap \{u > 0\} \right) \mathrm{d}s \leq C \sigma r^{N-1},
$$

where C is a positive constant which only depends on N, p, λ, m_1, m_2, a_0, a_1, c_1, and Ω.

Proof. Define the following:

$$
O_\varepsilon := \left\{ |\nabla u_\varepsilon| \leq 2\sigma^{\frac{1}{p-1}} \right\}, \quad O_{\varepsilon i} := \left\{ |u_{\varepsilon x_i}| \leq 2\sigma^{\frac{1}{p-1}} \right\}.
$$

We claim that $(O_\sigma \cap B_{r_4}) \subset (O_\varepsilon \cap B_{r_4})$. Indeed, there exists $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1)$,

$$
\left\| |\nabla u_\varepsilon| - |\nabla u| \right\|_{L^\infty(\overline{B}_{r_4})} \leq \sigma^{\frac{1}{p-1}}.
$$

Then, for any $x \in O_\sigma \cap B_{r_4}$, we have

$$
\|\nabla u_\varepsilon\|_{L^\infty(\overline{B}_{r_4})} \leq \left\| |\nabla u_\varepsilon| - |\nabla u| \right\|_{L^\infty(\overline{B}_{r_4})} + \|\nabla u\|_{L^\infty(\overline{B}_{r_4})} \leq 2\sigma^{\frac{1}{p-1}},
$$

which implies that $x \in O_\varepsilon \cap B_{r_4}$, or equivalently, $(O_\sigma \cap B_{r_4}) \subset (O_\varepsilon \cap B_{r_4})$.

Let H be defined by the following:

$$H(\eta) := \begin{cases} 2^{p-1}\sigma, & \eta > 2\sigma^{\frac{1}{p-1}}, \\ 2^{p-2}\sigma^{\frac{p-2}{p-1}}\eta, & |\eta| \le 2\sigma^{\frac{1}{p-1}}, \\ -2^{p-1}\sigma, & \eta < -2\sigma^{\frac{1}{p-1}}. \end{cases}$$

It is clear that

$$|H(\eta)| \le 2^{p-1}\sigma. \tag{7.1}$$

Differentiating (3.1) with respect to x_i, we obtain the following:

$$-\mathrm{div}\left(a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon + (p-1)a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_{\varepsilon x_i}\right) + m_1\chi'_\varepsilon(u_\varepsilon)\,u_{\varepsilon x_i} - m_2(\lambda-1)\left(u_\varepsilon^+\right)^{\lambda-2}u_{\varepsilon x_i} = 0. \tag{7.2}$$

Multiplying (7.2) by $H(u_{\varepsilon x_i})$ and integrating over $B_{rs}(x)$, we obtain the following:

$$\begin{aligned} &\int_{B_{rs}(x_0)}\left(a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon + (p-1)a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_{\varepsilon x_i}\right)\nabla H(u_{\varepsilon x_i})\mathrm{d}x \\ &+ \int_{B_{rs}(x_0)} m_1\chi'_\varepsilon(u_\varepsilon)\,u_{\varepsilon x_i}H(u_{\varepsilon x_i})\mathrm{d}x - \int_{B_{rs}(x_0)} m_2(\lambda-1)\left(u_\varepsilon^+\right)^{\lambda-2}u_{\varepsilon x_i}H(u_{\varepsilon x_i})\mathrm{d}x \\ =&\int_{\partial B_{rs}(x_0)}\left(a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon + (p-1)a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_{\varepsilon x_i}\right)H(u_{\varepsilon x_i})\nu\mathrm{d}S, \end{aligned}$$

where ν is the unit outward normal vector.

Let

$$\begin{aligned} I_1 &:= \int_{B_{rs}(x_0)}\left(a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon + (p-1)a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_{\varepsilon x_i}\right)\nabla H(u_{\varepsilon x_i})\mathrm{d}x, \\ I_2 &:= \int_{B_{rs}(x_0)} m_1\chi'_\varepsilon(u_\varepsilon)\,u_{\varepsilon x_i}H(u_{\varepsilon x_i})\mathrm{d}x, \\ I_3 &:= \int_{B_{rs}(x_0)} m_2(\lambda-1)\left(u_\varepsilon^+\right)^{\lambda-2}u_{\varepsilon x_i}H(u_{\varepsilon x_i})\mathrm{d}x, \\ I_4 &:= \int_{\partial B_{rs}(x_0)}\left(a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon + (p-1)a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_{\varepsilon x_i}\right)H(u_{\varepsilon x_i})\nu\mathrm{d}S. \end{aligned}$$

It follows that

$$I_1 + I_2 - I_3 = I_4. \tag{7.3}$$

Now, we estimate I_1, I_2, I_3, and I_4. From the definition of $H(\eta)$, it is clear that $H(\eta)\eta \ge 0$. Consequently, for I_2, we obtain the following:

$$I_2 = \int_{B_{rs}(x_0)} m_1\chi'_\varepsilon(u_\varepsilon)\,u_{\varepsilon x_i}H(u_{\varepsilon x_i})\mathrm{d}x \ge 0. \tag{7.4}$$

Combining (6.1) and (7.1), we find that I_3 satisfies the following:

$$\begin{aligned} I_3 =& \int_{B_{rs}(x_0)} m_2(\lambda-1)\left(u_\varepsilon^+\right)^{\lambda-2} u_{\varepsilon x_i} H(u_{\varepsilon x_i})\mathrm{d}x \\ \leq& \int_{B_{rs}(x_0)} m_2(\lambda-1)\left|u_\varepsilon^+\right|^{\lambda-2} |u_{\varepsilon x_i}||H(u_{\varepsilon x_i})|\mathrm{d}x \\ \leq& m_2(\lambda-1)c_1^{\lambda-1}\int_{B_{rs}(x_0)} 2^{p-1}\sigma \mathrm{d}x \\ \leq& C\sigma r^N s^N, \end{aligned} \tag{7.5}$$

where C is a positive constant which only depends on p, λ, m_2, and c_1.

By Hölder's inequality and (6.15), we find that I_4 satisfies the following:

$$\begin{aligned} \int_0^1 I_4 \mathrm{d}s =& \int_0^1 \int_{\partial B_{rs}(x_0)} \left(a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon + (p-1)a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_{\varepsilon x_i}\right) H(u_{\varepsilon x_i})\nu \mathrm{d}S\,\mathrm{d}s \\ \leq& \int_{B_r(x_0)} a_1|\nabla u_\varepsilon|^{p-1}|H(u_{\varepsilon x_i})|\mathrm{d}x + \int_{B_r(x_0)} (p-1)a_1|\nabla u_\varepsilon|^{p-2}|D^2 u||H(u_{\varepsilon x_i})|\mathrm{d}x \\ \leq& \int_{B_r(x_0)} a_1|\nabla u_\varepsilon|^{p-1}2^{p-1}\sigma \mathrm{d}x + \int_{B_r(x_0)} (p-1)a_1 2^{p-1}\sigma|\nabla u_\varepsilon|^{p-2}|D^2 u|\mathrm{d}x \\ \leq& C\sigma r^N + C\sigma\left(\int_{B_r(x_0)} |\nabla u_\varepsilon|^{p-2}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{B_r(x_0)} |\nabla u_\varepsilon|^{p-2}|D^2 u|^2\mathrm{d}x\right)^{\frac{1}{2}} \\ \leq& C\sigma r^N + C\sigma r^{N-1} \\ \leq& C\sigma r^{N-1}, \end{aligned} \tag{7.6}$$

where C is a positive constant which only depends on p, a_1, c_1, and Ω.

For the term I_1, we have the following:

$$\begin{aligned} I_1 =& \int_{B_{rs}(x_0)} \left(a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon + (p-1)a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_{\varepsilon x_i}\right)\nabla H(u_{\varepsilon x_i})\mathrm{d}x \\ =& \int_{B_{rs}(x_0)} a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla H(u_{\varepsilon x_i})\mathrm{d}x + \int_{B_{rs}(x_0)} (p-1)a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_{\varepsilon x_i}\nabla H(u_{\varepsilon x_i})\mathrm{d}x. \end{aligned}$$

Let

$$\begin{aligned} I_{11} :=& \int_{B_{rs}(x_0)} a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla H(u_{\varepsilon x_i})\mathrm{d}x, \\ I_{12} :=& \int_{B_{rs}(x_0)} (p-1)a(x)|\nabla u_\varepsilon|^{p-2}\nabla u_{\varepsilon x_i}\nabla H(u_{\varepsilon x_i})\mathrm{d}x. \end{aligned}$$

Thus,

$$I_1 = I_{11} + I_{12}. \tag{7.7}$$

By [21, P.711] and (6.17), we find that I_{11} satisfies the following:

$$\sum_{i=1}^N \int_0^1 |I_{11}|\,\mathrm{d}s = \sum_{i=1}^N \int_0^1 \left|\int_{B_{rs}(x_0)} a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla H(u_{\varepsilon x_i})\mathrm{d}x\right| \mathrm{d}s$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \int_0^1 \Big| - \int_{B_{rs}(x_0)} \operatorname{div}\left(a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right) H(u_{\varepsilon x_i})\mathrm{d}x \\
&\qquad + \int_{\partial B_{rs}(x_0)} a_{x_i}(x) H(u_{\varepsilon x_i})|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nu \mathrm{d}S \Big| \mathrm{d}s \\
&\le \sum_{i=1}^{N} \int_0^1 \left| \int_{B_{rs}(x_0)} \operatorname{div}\left(a_{x_i}(x)|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right) H(u_{\varepsilon x_i})\mathrm{d}x \right| \mathrm{d}s \\
&\quad + \sum_{i=1}^{N} \int_0^1 \left| \int_{\partial B_{rs}(x_0)} a_{x_i}(x) H(u_{\varepsilon x_i})|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nu \mathrm{d}S \right| \mathrm{d}s \\
&\le \sum_{i=1}^{N} \int_0^1 \int_{B_{rs}(x_0)} \left(N a_1 |\nabla u_\varepsilon|^{p-1} + N a_1 (p-1)|\nabla u_\varepsilon|^{p-2}|D^2 u_\varepsilon| \right) |H(u_{\varepsilon x_i})| \mathrm{d}x \mathrm{d}s \\
&\quad + \sum_{i=1}^{N} \int_{B_r(x_0)} |a_{x_i}(x)||H(u_{\varepsilon x_i})||\nabla u_\varepsilon|^{p-1} \mathrm{d}x \\
&\le \sum_{i=1}^{N} \int_{B_r(x_0)} \left(N a_1 |\nabla u_\varepsilon|^{p-1} + N a_1 (p-1)|\nabla u_\varepsilon|^{p-2}|D^2 u_\varepsilon| \right) |H(u_{\varepsilon x_i})| \mathrm{d}x \mathrm{d}s \\
&\quad + \sum_{i=1}^{N} \int_{B_r(x_0)} a_1 |\nabla u_\varepsilon|^{p-1} 2^{p-1} \sigma \mathrm{d}x \\
&\le C\sigma r^{N-1}, \qquad (7.8)
\end{aligned}
$$

where we used (7.6) in the last inequality, and C is a positive constant which only depends on N, p, a_1, c_1, and Ω.

Since $\nabla H(u_{\varepsilon x_i}) = 0$ in $O_{\varepsilon i}^c$ and $O_\sigma \subset O_\varepsilon \subset O_{\varepsilon i}$, the term I_{12} satisfies the following:

$$
\begin{aligned}
\sum_{i=1}^{N} I_{12} &= \sum_{i=1}^{N} \int_{B_{rs}(x_0)} (p-1) a(x) |\nabla u_\varepsilon|^{p-2} \nabla u_{\varepsilon x_i} \nabla H(u_{\varepsilon x_i}) \mathrm{d}x \\
&= \sum_{i=1}^{N} \int_{B_{rs}(x_0)\cap O_{\varepsilon i}} (p-1) a(x) |\nabla u_\varepsilon|^{p-2} \nabla u_{\varepsilon x_i} \nabla H(u_{\varepsilon x_i}) \mathrm{d}x \\
&\quad + \sum_{i=1}^{N} \int_{B_{rs}(x_0)\cap O_{\varepsilon i}^c} (p-1) a(x) |\nabla u_\varepsilon|^{p-2} \nabla u_{\varepsilon x_i} \nabla H(u_{\varepsilon x_i}) \mathrm{d}x \\
&= \sum_{i=1}^{N} \int_{B_{rs}(x_0)\cap O_{\varepsilon i}} (p-1) a(x) |\nabla u_\varepsilon|^{p-2} \nabla u_{\varepsilon x_i} \nabla H(u_{\varepsilon x_i}) \mathrm{d}x \\
&= \sum_{i=1}^{N} \int_{B_{rs}(x_0)\cap O_{\varepsilon i}} (p-1) a(x) |\nabla u_\varepsilon|^{p-2} |\nabla u_{\varepsilon x_i}|^2 2^{p-2} \sigma^{\frac{p-2}{p-1}} \mathrm{d}x \\
&\ge \sum_{i=1}^{N} \int_{B_{rs}(x_0)\cap O_{\varepsilon}} (p-1) a(x) |\nabla u_\varepsilon|^{p-2} |\nabla u_{\varepsilon x_i}|^2 2^{p-2} \sigma^{\frac{p-2}{p-1}} \mathrm{d}x \\
&= \int_{B_{rs}(x_0)\cap O_{\varepsilon}} (p-1) a(x) |\nabla u_\varepsilon|^{p-2} |D^2 u_\varepsilon|^2 2^{p-2} \sigma^{\frac{p-2}{p-1}} \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\geq \int_{B_{rs}(x_0)\cap O_\varepsilon} (p-1)a(x)|\nabla u_\varepsilon|^{2(p-2)}|D^2u_\varepsilon|^2\mathrm{d}x\\
&\geq C\int_{B_{rs}(x_0)\cap O_\varepsilon} \left(|\nabla u_\varepsilon|^{(p-2)}|D^2u_\varepsilon|\right)^2\mathrm{d}x\\
&\geq C\int_{B_{rs}(x_0)\cap O_\sigma} \left(|\nabla u_\varepsilon|^{(p-2)}|D^2u_\varepsilon|\right)^2\mathrm{d}x,
\end{aligned}
\tag{7.9}
$$

where C is a positive constant which only depends on p and a_0.

By (7.3), (7.4), and (7.7), we obtain

$$
I_{12} = I_4 + I_3 - I_2 - I_{11} \leq I_4 + I_3 + |I_{11}|,
$$

which, along with (7.5), (7.6), and (7.8), gives

$$
\begin{aligned}
\sum_{i=1}^{N}\int_0^1 I_{12}\mathrm{d}s &\leq \sum_{i=1}^{N}\int_0^1 I_4\mathrm{d}s + \sum_{i=1}^{N}\int_0^1 I_3\mathrm{d}s + \sum_{i=1}^{N}\int_0^1 |I_{11}|\mathrm{d}s\\
&\leq C\sigma r^{N-1} + \sum_{i=1}^{N}\int_0^1 C\sigma r^N s^N\mathrm{d}s\\
&\leq C\sigma r^{N-1},
\end{aligned}
\tag{7.10}
$$

where C is a positive constant which only depends on N, p, λ, m_2, a_1, c_1, and Ω.

Combining (7.9) and (7.10), we obtain the following:

$$
\int_0^1\int_{B_{rs}(x_0)\cap O_\sigma} \left(|\nabla u_\varepsilon|^{(p-2)}|D^2u_\varepsilon|\right)^2\mathrm{d}x\mathrm{d}s \leq C\sigma r^{N-1},
\tag{7.11}
$$

where C is a positive constant which only depends on N, p, λ, m_2, a_0, a_1, c_1, and Ω.

Using Proposition 7, (7.11), the definition of O_ε, and the inclusion $O_\sigma \subset O_\varepsilon$, we deduce that

$$
\begin{aligned}
&\frac{1}{C_3}\int_0^1\int_{B_{rs}(x_0)\cap O_\sigma} \left|m_1\chi_\varepsilon(u_\varepsilon) - m_2\left(u_\varepsilon^+\right)^{\lambda-1}\right|^2\mathrm{d}x\mathrm{d}s\\
\leq &\int_0^1\int_{B_{rs}(x_0)\cap O_\sigma} |\nabla u_\varepsilon|^{2(p-1)}\mathrm{d}x\mathrm{d}s + \int_0^1\int_{B_{rs}(x_0)\cap O_\sigma} \left(|\nabla u_\varepsilon|^{p-2}|D^2u_\varepsilon|\right)^2\mathrm{d}x\mathrm{d}s\\
\leq &C\sigma r^{N-1} + \int_0^1\int_{B_{rs}(x_0)\cap O_\varepsilon} |\nabla u_\varepsilon|^{2(p-1)}\mathrm{d}x\mathrm{d}s\\
\leq &C\sigma r^{N-1} + \int_0^1\int_{B_{rs}(x_0)\cap O_\varepsilon} 2^{p-1}\sigma|\nabla u_\varepsilon|^{(p-1)}\mathrm{d}x\mathrm{d}s\\
\leq &C\sigma r^{N-1},
\end{aligned}
$$

where C is a positive constant which only depends on N, p, λ, m_2, a_0, a_1, c_1, and Ω.

Letting $\varepsilon \to 0$, we obtain the following:

$$
\frac{1}{C_3}\int_0^1\int_{B_{rs}(x_0)\cap O_\sigma} \left|m_1\chi_{\{u>0\}} - m_2u^{\lambda-1}\chi_{\{u>0\}}\right|^2\mathrm{d}x\mathrm{d}s \leq C\sigma r^{N-1},
\tag{7.12}
$$

where C is a positive constant which only depends on N, p, λ, m_2, a_0, a_1, c_1, and Ω.

Since $r < r_2$, by Proposition 4, we have

$$\begin{aligned} m_1\chi_{\{u>0\}} - m_2 u^{\lambda-1}\chi_{\{u>0\}} &= m_1 - m_2 u^{\lambda-1} \\ &\geq m_1 - m_2 C_1^{\lambda-1} r_2^{\frac{p(\lambda-1)}{p-1}} \\ &\geq \frac{m_1}{2} \text{ in } B_r(x_0)\cap\{u>0\}, \end{aligned}$$

which, along with (7.14), implies that

$$\int_0^1 \int_{B_{rs}(x_0)\cap O_\sigma\cap\{u>0\}} \frac{m_1^2}{4C_3} \mathrm{d}x\mathrm{d}s \leq C\sigma r^{N-1},$$

or equivalently,

$$\int_0^1 \mathcal{L}^N\left(O_\sigma \cap B_{rs}(x_0)\cap\{u>0\}\right)\mathrm{d}s \leq C\sigma r^{N-1},$$

where C is a positive constant which only depends on N, p, λ, m_1, m_2, a_0, a_1, c_1, and Ω. □

Now, we recall the definition of the Hausdorff measure of a set in $\mathbb{R}^N$.

Definition 1 ([22, p.60]). *(i) Let $A \subset \mathbb{R}^N$, $0 \leq t < \infty$, $0 < \delta \leq \infty$. Define*

$$\mathcal{H}_\delta^t(A) \equiv \inf\left\{\sum_{j=1}^\infty \alpha(t)\left(\frac{\operatorname{diam} B_j}{2}\right)^t : A \subset \bigcup_{j=1}^\infty B_j,\ \operatorname{diam} B_j \leq \delta\right\},$$

where $\alpha(t) \equiv \frac{\pi^{\frac{t}{2}}}{\Gamma\left(\frac{t}{2}+1\right)}$; here, $\Gamma(t) \equiv \int_0^\infty e^{-x}x^{t-1}\mathrm{d}x$, $(0 < t < \infty)$, is the usual gamma function.

(i) For A and t as above, define

$$\mathcal{H}^t(A) \equiv \lim_{\delta\to 0}\mathcal{H}_\delta^t(A) = \sup_{\delta>0}\mathcal{H}_\delta^t(A).$$

We call that $\mathcal{H}^t(A)$ is the t-dimensional Hausdorff measure of A in $\mathbb{R}^N$.

Based on Proposition 8, we show that the free boundary Γ^+ possesses a locally finite $(N-1)$-dimensional Hausdorff measure.

Theorem 3. *Let r_4 be the same as in Proposition 8. Then, for every $x_0 \in \Gamma^+ \cap B_{r_4}$ and $r \in \left(0, \frac{r_4}{2}\right)$, there holds*

$$\mathcal{H}^{N-1}\left(B_r(x_0)\cap\Gamma^+\right) \leq Cr^{N-1},$$

where C is a positive constant which only depends on N, p, λ, m_1, m_2, a_0, a_1, c_1, and Ω.

Proof. First, we claim the following:

$$\mathcal{L}^N\left(O_\sigma \cap B_{rs}(x_0)\cap\{u>0\}\right) \leq C\sigma r^{N-1}, \forall r < \frac{r_4}{2}, \tag{7.13}$$

where C is a positive constant which only depends on N, p, λ, m_1, m_2, a_0, a_1, c_1, and Ω.

By Proposition 8, we obtain the following:

$$\begin{aligned} C\sigma r^{N-1} &\geq \int_0^1 \mathcal{L}^N\left(O_\sigma \cap B_{2rs}(x_0) \cap \{u>0\}\right) \mathrm{d}s \\ &= \int_0^{\frac{1}{2}} \mathcal{L}^N\left(O_\sigma \cap B_{2rs}(x_0) \cap \{u>0\}\right) \mathrm{d}s + \int_{\frac{1}{2}}^1 \mathcal{L}^N\left(O_\sigma \cap B_{2rs}(x_0) \cap \{u>0\}\right) \mathrm{d}s \\ &\geq \frac{1}{2} \mathcal{L}^N\left(O_\sigma \cap B_r(x_0) \cap \{u>0\}\right); \end{aligned}$$

therefore, (7.13) holds.

Now, according to the optimal growth and non-degeneracy properties of solutions (see Propositions 4 and 5, respectively), there exists a constant $c>0$ such that for any $x_0 \in B_{1-\sigma} \cap \Gamma^+$, there exist $y_0 \in \{u>0\}$ such that

$$B_{c\sigma}(y_0) \subset B_\sigma(x_0) \cap O_\sigma \cap \{u>0\}. \tag{7.14}$$

The proof of (7.14) is standard (for example, see [5]) and hence is omitted.

By virtue of the Besicovitch covering theorem, let $\{B_\sigma(x^i)\}_{i\in I}$ be finite coverings of $B_r(x_0) \cap \Gamma^+$ with $x^i \in \Gamma^+$ and at most $n(N)$ overlapping at each point. Then, we obtain the following:

$$\sum_{i\in I} C(N)(c\sigma)^N \leq \sum_{i\in I} \mathcal{L}^N\left(O_\sigma \cap B_\sigma(x^i) \cap \{u>0\}\right) \leq n\, \mathcal{L}^N\left(O_\sigma \cap B_r(x_0) \cap \{u>0\}\right) \leq Cn\sigma r^{N-1},$$

where C is a positive constant which only depends on N, p, λ, m_1, m_2, a_0, a_1, c_1, and Ω. Therefore, it follows that

$$\mathcal{H}^{N-1}\left(B_r(x_0) \cap \Gamma^+\right) \leq \liminf_{\sigma\to 0} C(N)\sigma^{N-1} \leq Cr^{N-1}.$$

□

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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