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Research article

On the relationship between partial orders on multipartitions

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Abstract: In this paper, we studied the relationship between several partial orders on multipartitions, including dominance order, θ-dominance order, weak θ-dominance order, a-function order, Webster's ordering, and Lusztig's $\mathbf{a}_\sigma$-ordering.

Keywords: multipartitions; dominance order; θ-dominance order; a-function order; Webster's ordering; Lusztig's $\mathbf{a}_\sigma$-ordering

1. Introduction

A cellular algebra is an algebra with a cellular basis, which was first introduced by Graham and Lehrer [1]. The existence of a cellular basis implies rich information about representations of the algebra. For the cyclotomic Hecke algebras of type $G(\ell, 1, n)$, many authors have constructed different cellular bases [2–4]. The constructions of these cellular bases are closely related to the partial orders on multipartitions. Following Graham and Lehrer's cellular theory, different constructions of the cellular bases may give different parameterizations of simple modules of the cyclotomic Hecke algebras. To study the relationship between these parameterizations, it is necessary to understand the relationship between different partial orders on multipartitions.

Suppose $\geqslant_1$ and $\geqslant_2$ are partial orders on multipartitions. In this paper, we say "$\geqslant_2$ refines $\geqslant_1$" if "$\boldsymbol{\mu} \geqslant_1 \boldsymbol{\lambda}$ implies $\boldsymbol{\mu} \geqslant_2 \boldsymbol{\lambda}$". We say "$\geqslant_2$ refines the opposite $\geqslant_1$" if "$\boldsymbol{\mu} \geqslant_1 \boldsymbol{\lambda}$ implies $\boldsymbol{\lambda} \geqslant_2 \boldsymbol{\mu}$". We say "$\geqslant_2$ is equivalent to $\geqslant_1$" if "$\boldsymbol{\mu} \geqslant_1 \boldsymbol{\lambda}$ if, and only if, $\boldsymbol{\mu} \geqslant_2 \boldsymbol{\lambda}$".

Throughout this paper, we denote by $\mathbb{N}$ the set of natural numbers, $\mathbb{Z}$ the set of integers, and $\mathbb{Q}$ the set of rational numbers, respectively.

Suppose ℓ, n are positive integers and $\boldsymbol{\lambda}, \boldsymbol{\mu}$ are ℓ-partitions of n. Let $\mathbf{m} = (m_1, \ldots, m_\ell) \in \mathbb{Q}^\ell$, $\sigma = (e; \sigma_1, \ldots, \sigma_\ell) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$, and $\mathbf{u} = (u_1, \ldots, u_\ell) \in \mathbb{Q}^\ell$ with $0 < u_j - u_i < e$ for $i < j$. Let $\theta = (\theta_1, \ldots, \theta_\ell) \in \mathbb{Z}^\ell$ with $\theta_i - \theta_j \notin \ell\mathbb{Z}$ for $i < j$. To present the main contributions more clearly, we summarized the main results of this paper in Table 1, in which the third column gives the conclusions, the second column presents the corresponding conditions, and the first column indicates the numbering

of the corresponding conclusions in this paper.

Table 1. Main results.

Theorem 3.11	$m_i - m_j > n-1, \forall i < j$	$\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda} \Rightarrow \boldsymbol{\lambda} \geq_{\mathbf{m}} \boldsymbol{\mu}$
Corollary 3.17	$m_i = (-\theta_i + i)/\ell$ and $\theta_{i+1} - \theta_i > \ell n$	$\boldsymbol{\mu} \trianglerighteq_\theta \boldsymbol{\lambda} \Rightarrow \boldsymbol{\lambda} \geq_{\mathbf{m}} \boldsymbol{\mu}$
Theorem 4.8	$m_i = \sigma_i - u_i$	$\boldsymbol{\mu} \trianglerighteq_\sigma \boldsymbol{\lambda} \Rightarrow \boldsymbol{\lambda} \geq_{\mathbf{m}} \boldsymbol{\mu}$
Corollary 5.6	$m_i = \sigma_i - u_i$ and $\sigma_i - \sigma_{i+1} \geq n$	$\boldsymbol{\mu} \geq_\sigma \boldsymbol{\lambda} \Rightarrow \boldsymbol{\lambda} \geq_{\mathbf{m}} \boldsymbol{\mu}$.
Proposition 5.1	$\theta_i = \ell(-\sigma_i + \frac{i}{\ell})$	$\boldsymbol{\mu} \geq_\sigma \boldsymbol{\lambda} \Leftrightarrow \boldsymbol{\mu} \geq_\theta \boldsymbol{\lambda}$
Theorem 5.2	$\theta_i = \ell(-\sigma_i + \frac{i}{\ell})$	$\boldsymbol{\mu} \trianglerighteq_\sigma \boldsymbol{\lambda} \Leftrightarrow \boldsymbol{\mu} \trianglerighteq_\theta \boldsymbol{\lambda}$
Corollary 5.3	$\sigma_i - \sigma_{i+1} \geq n$	$\boldsymbol{\mu} \geq_\sigma \boldsymbol{\lambda} \Leftrightarrow \boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$
Corollary 5.4	$\sigma_i - \sigma_{i+1} \geq n$	$\boldsymbol{\mu} \trianglerighteq_\sigma \boldsymbol{\lambda} \Rightarrow \boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$

The content of this paper is organized as follows. In Section 2, we give notations and definitions. In Section 3, we first introduce the combinatorial construction of a-functions, then we discuss the relationship between dominance order and a-function order. As a corollary, we give the relationship between θ-dominance order and a-function order. In Section 4, we first give the relationship between Webster's ordering and a-function order, then we point out that [5, Proposition 5.6] is incorrect by giving a counterexample (see Example 4.11). In Section 5, we first give the relationship between Lusztig's $\mathbf{a}_\sigma$-ordering and weak θ-dominance order, then we give the relationship between Webster's ordering and θ-dominance order. As corollaries, we give the relationship between Lusztig's $\mathbf{a}_\sigma$-ordering and dominance order and the relationship between Webster's ordering and dominance order. We also give the relationship between Lusztig's $\mathbf{a}_\sigma$-ordering and a-function order.

2. Preliminaries

Let n be a natural number. Suppose $\lambda = (\lambda_1, \lambda_2, \dots)$ is a finite sequence of nonnegative integers with $\sum_i \lambda_i = n$. We call λ a composition of n and denote $|\lambda| = \sum_i \lambda_i$. For a composition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n, if the sequence is nonincreasing, then we call λ a partition of n and write $\lambda \vdash n$. Let $\mathcal{P}_n$ be the set of partitions of n. The Young diagram of $\boldsymbol{\lambda}$ is defined by $[\lambda] = \{(i, j) | 1 \leq j \leq \lambda_i, \forall i \geq 1\}$. Elements in $[\lambda]$ are called the nodes of λ. The Young diagram $[\lambda]$ corresponds to a tableau. For example, $\lambda = (4, 3, 2, 1)$ is a partition of 10, its Young diagram $[\lambda] = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2), (4,1)\}$. It corresponds to the tableau

$$[\lambda] = \begin{array}{|c|c|c|c|}\hline \multicolumn{1}{}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-4} \multicolumn{1}{}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array},$$

where (i, j) corresponds to the box in the i-th row and j-th column. For $\lambda \vdash n$, define its height $h(\lambda) = \max\{k \in \mathbb{N} | \lambda_k \neq 0\}$. In particular, $h(\emptyset) = 0$.

Suppose ℓ is a positive integer. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ be an ordered sequence of compositions with $|\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| = n$. Then we call $\boldsymbol{\lambda}$ a multicomposition of n with ℓ components. For simplicity, we also

call $\boldsymbol{\lambda}$ an ℓ-composition of n. For the ℓ-composition $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$, if each $\lambda^{(i)}$ is a partition, then we call $\boldsymbol{\lambda}$ a multipartition of n with ℓ components or an ℓ-partition of n and write $\boldsymbol{\lambda} \vdash_\ell n$. Let $\mathcal{P}_n^\ell$ be the set of multipartitions of n with ℓ components. In particular, if $\ell = 1$, then $\mathcal{P}_n^1 = \mathcal{P}_n$. The Young diagram $[\boldsymbol{\lambda}] = \{(i, j, s) | 1 \le s \le \ell, 1 \le j \le \lambda_i^{(s)}, \forall i \ge 1\}$. Elements in $[\boldsymbol{\lambda}]$ are called the nodes of $\boldsymbol{\lambda}$. The Young diagram $[\boldsymbol{\lambda}]$ corresponds to an ordered sequence of tableaux. For example, $\boldsymbol{\lambda} = ((3, 2), \emptyset, (2, 1))$ is a 3-partition of 8, the Young diagram $[\boldsymbol{\lambda}] = \{(1, 1, 1), (1, 2, 1), (1, 3, 1), (2, 1, 1), (2, 2, 1), (1, 1, 3), (1, 2, 3), (2, 1, 3)\}$, and it corresponds to the tableaux

$$[\boldsymbol{\lambda}] = \left(\begin{array}{|c|c|c|} \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} \\ \cline{1-2} \end{array}, \ \emptyset, \ \begin{array}{|c|c|} \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c|}{} \\ \cline{1-1} \end{array} \right),$$

where (i, j, s) corresponds to the box in the i-th row and j-th column of the s-th tableau.

Let $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. For $\alpha \in [\boldsymbol{\lambda}]$, if $[\boldsymbol{\lambda}] \setminus \{\alpha\}$ is still a Young diagram of some ℓ-partition of $n - 1$, then we call α a removable node of $\boldsymbol{\lambda}$ and denote by $\boldsymbol{\lambda} \setminus \alpha$ the ℓ-partition of $n - 1$. For $\beta \notin [\boldsymbol{\lambda}]$, if $[\boldsymbol{\lambda}] \cup \{\beta\}$ is still a Young diagram of some ℓ-partition of $n + 1$, then we call β an addable node of $\boldsymbol{\lambda}$ and denote by $\boldsymbol{\lambda} \cup \beta$ the ℓ-partition of $n + 1$.

For simplicity, we do not distinguish $\boldsymbol{\lambda}$ from its Young diagram $[\boldsymbol{\lambda}]$ when no confusion can arise.

Definition 2.1. *Suppose $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \mathcal{P}_n^\ell$ and $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(\ell)}) \in \mathcal{P}_n^\ell$. If*

$$\sum_{a=1}^{s-1} |\mu^{(a)}| + \sum_{i=1}^{t} \mu_i^{(s)} \ge \sum_{a=1}^{s-1} |\lambda^{(a)}| + \sum_{i=1}^{t} \lambda_i^{(s)} \quad \forall\, 1 \le s \le \ell \ \ \forall\, t \ge 1,$$

then we write $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$ and we say $\boldsymbol{\mu}$ dominates $\boldsymbol{\lambda}$. If $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$ and $\boldsymbol{\mu} \ne \boldsymbol{\lambda}$, we write $\boldsymbol{\mu} \triangleright \boldsymbol{\lambda}$. We call $\trianglerighteq$ the **dominance order**.

Let $\mathcal{N}_n^\ell = \{(r, c, l) | r, c, l \in \mathbb{N}_{\ge 1}, r + c \le 2(n + 1), 1 \le l \le \ell\}$. We also call the elements of $\mathcal{N}_n^\ell$ the nodes. So the Young diagram of an ℓ-composition of n is a subset of $\mathcal{N}_n^\ell$.

For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$, we call $\boldsymbol{\lambda}, \boldsymbol{\mu}$ neighbors with dominance order if $\boldsymbol{\mu} \triangleright \boldsymbol{\lambda}$ and there is no $\boldsymbol{\gamma} \in \mathcal{P}_n^\ell$ such that $\boldsymbol{\mu} \triangleright \boldsymbol{\gamma} \triangleright \boldsymbol{\lambda}$. The following lemma gives a characterization of neighbors with dominance order in $\mathcal{P}_n^\ell$.

Lemma 2.2 ([6, Lemma 6.3])**.** *Suppose $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(\ell)})$ are ℓ-partitions of n and $\boldsymbol{\mu} \triangleright \boldsymbol{\lambda}$. Then, $\boldsymbol{\lambda}, \boldsymbol{\mu}$ are neighbors with dominance order if, and only if, one of* (*a*)*,* (*b*)*, and* (*c*) *occurs, where*

(*a*) *there exists $s < \ell$ such that*

$$\begin{aligned} &\mu^{(r)} = \lambda^{(r)} \quad \forall r \ne s, s + 1 \\ &\mu^{(s)} = (\lambda_1^{(s)}, \ldots, \lambda_{k_s}^{(s)}, 1) \quad \textit{where} \quad k_s = h(\lambda^{(s)}) \\ &\mu_1^{(s+1)} = \lambda_1^{(s+1)} - 1 \quad \textit{and} \quad \mu_j^{(s+1)} = \lambda_j^{(s+1)} \quad \forall j > 1, \end{aligned}$$

(*b*) *there exist s and i such that*

$$\begin{aligned} &\mu^{(r)} = \lambda^{(r)} \quad \forall r \ne s \\ &\mu_j^{(s)} = \lambda_j^{(s)} \quad \forall j \ne i, i + 1 \\ &\mu_i^{(s)} = \lambda_i^{(s)} + 1 \quad \textit{and} \quad \mu_{i+1}^{(s)} = \lambda_{i+1}^{(s)} - 1, \end{aligned}$$

(*c*) *there exist* s *and* $i < i'$ *such that*

$$\begin{aligned}
&\mu^{(r)} = \lambda^{(r)} \quad \forall r \neq s\\
&\mu^{(s)}_j = \lambda^{(s)}_j \quad \forall j \neq i, i'\\
&\mu^{(s)}_i - 1 = \mu^{(s)}_{i'} + 1 = \lambda^{(s)}_i = \lambda^{(s)}_{i'}.
\end{aligned}$$

Example 2.3. $\boldsymbol{\lambda} = ((4,3,2),(4,2,1))$ *and* $\boldsymbol{\mu} = ((4,3,2,1),(3,2,1))$ *are neighbors. The Young diagram is*

$$[\boldsymbol{\lambda}] = \left(\begin{array}{|c|c|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} \\ \cline{1-3} \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \cline{1-2}\end{array}\right), \quad \left(\begin{array}{|c|c|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{*} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} \\ \cline{1-3} \multicolumn{1}{|c|}{} \\ \cline{1-1}\end{array}\right), \qquad [\boldsymbol{\mu}] = \left(\begin{array}{|c|c|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} \\ \cline{1-3} \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \cline{1-2} \multicolumn{1}{|c|}{*} \\ \cline{1-1}\end{array}\right), \quad \left(\begin{array}{|c|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \cline{1-2} \multicolumn{1}{|c|}{} \\ \cline{1-1}\end{array}\right).$$

We can obtain $[\boldsymbol{\mu}]$ *from* $[\boldsymbol{\lambda}]$ *by removing* $(1,4,2)$ *to* $(4,1,1)$.

Example 2.4. $\lambda = (4,3,2,1)$ *and* $\mu = (4,4,1,1)$ *are neighbors. The Young diagram is*

$$[\lambda] = \begin{array}{|c|c|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} \\ \cline{1-3} \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{*} \\ \cline{1-2} \multicolumn{1}{|c|}{} \\ \cline{1-1}\end{array}, \qquad [\mu] = \begin{array}{|c|c|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{*} \\ \cline{1-3} \multicolumn{1}{|c|}{} \\ \cline{1-1} \multicolumn{1}{|c|}{} \\ \cline{1-1}\end{array}.$$

We can obtain $[\mu]$ *from* $[\lambda]$ *by removing* $(3,2)$ *to* $(2,4)$.

Example 2.5. $\lambda = (4,3,3,3,1)$ *and* $\mu = (4,4,3,2,1)$ *are neighbors. The Young diagram is*

$$[\lambda] = \begin{array}{|c|c|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} \\ \cline{1-3} \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} \\ \cline{1-3} \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{*} \\ \cline{1-2} \multicolumn{1}{|c|}{} \\ \cline{1-1}\end{array}, \qquad [\mu] = \begin{array}{|c|c|c|c|}\hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} \\ \hline \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{*} \\ \cline{1-3} \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} \\ \cline{1-3} \multicolumn{1}{|c}{} & \multicolumn{1}{c|}{} \\ \cline{1-2} \multicolumn{1}{|c|}{} \\ \cline{1-1}\end{array}.$$

We can obtain $[\mu]$ *from* $[\lambda]$ *by removing* $(4,3)$ *to* $(2,4)$.

3. Relationship between a-function order and dominance order

In this section, we recall the combinatorial construction of a-functions. We refer the reader to [7, Section 5.5] for more details. Fix a sequence $\mathbf{m} = (m_1, m_2, \ldots, m_\ell) \in \mathbb{Q}^\ell$. For a rational number $t \geq 1$, let $[t]$ be the integral part of t. Let $q = (q_1, q_2, \ldots, q_k)$ be a sequence of rational numbers. Define

$$q(t) := (t - [t], t - [t] + 1, \ldots, t - 1, q_1 + t, \ldots, q_k + t).$$

So $q(t)$ is a sequence of rational numbers with length $k + [t]$.

Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ be an ℓ-partition of n and h_i be the height of $\lambda^{(i)}$. For each $i = 1, 2, \ldots, \ell$, consider the positive integer sequence

$$p_{\lambda^{(i)}} := (\lambda^{(i)}_{h_i} - h_i + h_i, \ldots, \lambda^{(i)}_j - j + h_i, \ldots, \lambda^{(i)}_1 - 1 + h_i).$$

It is clear $p_{\lambda^{(i)}}$ is a strictly increasing sequence since $\lambda^{(i)}$ is a partition.

Fix an integer s such that $s \geq n+1-\min\{m_i\}$. For each $i = 1, \ldots, \ell$, define the following column vector

$$\mathbf{B}(\boldsymbol{\lambda}) := \begin{pmatrix} B_1(\boldsymbol{\lambda}) \\ \vdots \\ B_\ell(\boldsymbol{\lambda}) \end{pmatrix}, \text{ where } B_i(\boldsymbol{\lambda}) = p_{\boldsymbol{\lambda}^{(i)}}(s - h_i + m_i).$$

By definition, $s - h_i + m_i \geq 1$ and $[s - h_i + m_i] = s - h_i + [m_i]$. Hence $B_i(\boldsymbol{\lambda})$ is a strictly increasing nonnegative rational sequence with length $s + [m_i]$.

Remark 3.1. *By convention, $\lambda_a^{(i)} = 0$ if $a > h_i$. Then,*

$$\begin{aligned} B_i(\boldsymbol{\lambda}) &= p_{\boldsymbol{\lambda}^{(i)}}(s - h_i + m_i) \\ &= (s - h_i + m_i - [s - h_i + m_i], \ldots, s - h_i + m_i - j, \ldots, s - h_i + m_i - 1, \\ &\quad \lambda_{h_i}^{(i)} - h_i + h_i + s - h_i + m_i, \ldots, \lambda_a^{(i)} - a + h_i + s - h_i + m_i, \ldots \lambda_1^{(i)} - 1 + h_i + s - h_i + m_i) \\ &= (\lambda_{s+[m_i]}^{(i)} - (s + [m_i]) + s + m_i, \ldots, \lambda_{h_i+j}^{(i)} - (h_i + j) + s + m_i, \ldots, \lambda_{h_i+1}^{(i)} - (h_i + 1) + s + m_i, \\ &\quad \lambda_{h_i}^{(i)} - h_i + s + m_i, \ldots, \lambda_a^{(i)} - a + s + m_i, \ldots \lambda_1^{(i)} - 1 + s + m_i). \end{aligned}$$

That is, elements in $B_i(\boldsymbol{\lambda})$ are $\{\lambda_a^{(i)} - a + s + m_i | a = 1, 2, \ldots, s + [m_i]\}$.

Example 3.2. *Let $n = 4$, $\ell = 2$, $(m_1, m_2) = (1, 2)$, and $s = 4$. Let $\boldsymbol{\lambda} = ((2, 1), (1))$, then $h_1 = 2, h_2 = 1$. By definition,*

$$p_{\boldsymbol{\lambda}^{(1)}} = (1, 3), \quad p_{\boldsymbol{\lambda}^{(2)}} = (1).$$

So we derive

$$\begin{aligned} B_1(\boldsymbol{\lambda}) &= p_{\boldsymbol{\lambda}^{(1)}}(s - h_1 + m_1) = p_{\boldsymbol{\lambda}^{(1)}}(3) = (0, 1, 2, 4, 6) \\ B_2(\boldsymbol{\lambda}) &= p_{\boldsymbol{\lambda}^{(2)}}(s - h_2 + m_2) = p_{\boldsymbol{\lambda}^{(2)}}(5) = (0, 1, 2, 3, 4, 6) \end{aligned}$$

and

$$\mathbf{B}(\boldsymbol{\lambda}) = \begin{pmatrix} B_1(\boldsymbol{\lambda}) \\ B_2(\boldsymbol{\lambda}) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 4 & 6 & \\ 0 & 1 & 2 & 3 & 4 & 6 \end{pmatrix}.$$

We write the elements in $\mathbf{B}(\boldsymbol{\lambda})$ with multiplicity in descending order

$$\kappa_1(\boldsymbol{\lambda}) \geq \kappa_2(\boldsymbol{\lambda}) \geq \cdots \geq \kappa_t(\boldsymbol{\lambda}), \tag{3.3}$$

where $t = \ell s + \sum_{1 \leq i \leq \ell} [m_i]$. Define

$$\kappa_{\mathbf{m}}(\boldsymbol{\lambda}) := (\kappa_1(\boldsymbol{\lambda}), \ldots, \kappa_t(\boldsymbol{\lambda})), \qquad n_{\mathbf{m}}(\boldsymbol{\lambda}) := \sum_{i=1}^{t} (i-1)\kappa_i(\boldsymbol{\lambda}).$$

Definition 3.4. *Define the a-function $a_{\mathbf{m}}(\boldsymbol{\lambda}) := n_{\mathbf{m}}(\boldsymbol{\lambda}) - n_{\mathbf{m}}(\underline{\emptyset})$.*

Definition 3.5. *For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$, write $\boldsymbol{\lambda} \geq_{\mathbf{m}} \boldsymbol{\mu}$ if $\boldsymbol{\lambda} = \boldsymbol{\mu}$ or $a_{\mathbf{m}}(\boldsymbol{\lambda}) > a_{\mathbf{m}}(\boldsymbol{\mu})$. We call $\geq_{\mathbf{m}}$ the a-***function order***. If $a_{\mathbf{m}}(\boldsymbol{\lambda}) > a_{\mathbf{m}}(\boldsymbol{\mu})$, we write $\boldsymbol{\lambda} >_{\mathbf{m}} \boldsymbol{\mu}$.*

Remark 3.6. *(i) For an ℓ-composition $\boldsymbol{\lambda}$, the sequence $\kappa_{\mathbf{m}}(\boldsymbol{\lambda})$ and the values $n_{\mathbf{m}}(\boldsymbol{\lambda})$, $a_{\mathbf{m}}(\boldsymbol{\lambda})$ are still well defined; see [7, Section 5.5].*

(ii) The value $a_{\mathbf{m}}(\boldsymbol{\lambda})$ is independent of the choice of the sufficiently large integer s; see [7, Section 5.5].

(iii) The a-function order is a partial order induced by the preorder "$a_{\mathbf{m}}(\boldsymbol{\lambda}) \geq a_{\mathbf{m}}(\boldsymbol{\mu})$", in fact, $\boldsymbol{\lambda} \not\geq_{\mathbf{m}} \boldsymbol{\mu}$ and $\boldsymbol{\mu} \not\geq_{\mathbf{m}} \boldsymbol{\lambda}$ happens exactly when $a_{\mathbf{m}}(\boldsymbol{\lambda}) = a_{\mathbf{m}}(\boldsymbol{\mu})$ with $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$.

For convenience of the discussion, we extend the definition of dominance order on partitions to the sequences of rational numbers, that is, for rational sequences $p = (p_1, \ldots, p_t)$ and $q = (q_1, \ldots, q_t)$, we write $p \trianglerighteq q$ if, and only if,

$$\sum_{i=1}^{r} p_i \geq \sum_{i=1}^{r} q_i \tag{3.7}$$

for each $r = 1, \ldots, t$. Moreover, we write $p \triangleright q$ whenever $p \trianglerighteq q$ and $p \neq q$.

Lemma 3.8. *Let s, t be positive integers. Suppose $p = (p_1, \ldots, p_t)$, $q = (q_1, \ldots, q_t)$ and $r = (r_1, \ldots, r_s)$ are rational sequences with descending order and $p \triangleright q$. Reorder the multisets $p \bigsqcup r$ and $q \bigsqcup r$ to obtain descending sequences $\widehat{p \bigsqcup r}$ and $\widehat{q \bigsqcup r}$, then $\widehat{p \bigsqcup r} \triangleright \widehat{q \bigsqcup r}$.*

Proof. We give the proof by induction on s.

If $s = 1$, that is, r is a rational number. Suppose

$$\widehat{p \bigsqcup r} = (p_1, \ldots, p_i, r, p_{i+1}, \ldots, p_t)$$

and

$$\widehat{q \bigsqcup r} = (q_1, \ldots, q_j, r, q_{j+1}, \ldots, q_t)$$

for some i and j. For covenience, we denote

$$p'_k = \begin{cases} p_k, & 1 \leq k \leq i \\ r, & k = i+1 \\ p_{k-1}, & i+2 \leq k \leq t+1 \end{cases} \quad \text{and} \quad q'_k = \begin{cases} q_k, & 1 \leq k \leq j \\ r, & k = j+1 \\ q_{k-1}, & j+2 \leq k \leq t+1 \end{cases}.$$

If $i = j$, it is clear $\widehat{p \bigsqcup r} \triangleright \widehat{q \bigsqcup r}$ by $p \triangleright q$ and (3.7).

If $i < j$, for $u \leq i$, by $p \triangleright q$ and (3.7), we have

$$\sum_{w=1}^{u} p'_w = \sum_{w=1}^{u} p_w \geq \sum_{w=1}^{u} q_w = \sum_{w=1}^{u} q'_w.$$

For $i < u \leq j$, because $\widehat{p \bigsqcup r}$ is a descending sequence, we have $r \geq p_u$, by $p \triangleright q$ and (3.7), then

$$\sum_{w=1}^{u} p'_w = r + \sum_{w=1}^{u-1} p_w \geq \sum_{w=1}^{u} p_w \geq \sum_{w=1}^{u} q_w = \sum_{w=1}^{u} q'_w.$$

For $u > j$, by $p \triangleright q$ and (3.7), we have

$$\sum_{w=1}^{u} p'_w = r + \sum_{w=1}^{u-1} p_w \geq r + \sum_{w=1}^{u-1} q_w = \sum_{w=1}^{u} q'_w.$$

Now, we have proved $\widehat{p \bigsqcup r} \rhd \widehat{q \bigsqcup r}$ whenever $i < j$.

For the case $i > j$, the discussion is similar. Hence, we completed the proof for $s = 1$.

Let $s > 1$, assume the conclusion holds for $s-1$. Denote $\hat{r} = (r_1, \ldots, r_{s-1})$. By assumption, we have $\widehat{p \bigsqcup \hat{r}} \rhd \widehat{q \bigsqcup \hat{r}}$. Apply the initial conclusion that we have proved to $\widehat{p \bigsqcup \hat{r}}$, $\widehat{q \bigsqcup \hat{r}}$ and r_s, and we obtain $\widehat{p \bigsqcup r} \rhd \widehat{q \bigsqcup r}$. Thus, by induction, the proof is completed. □

Proposition 3.9 ([7, Proposition 5.5.16]). *Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$. If $\kappa_{\mathbf{m}}(\boldsymbol{\mu}) \rhd \kappa_{\mathbf{m}}(\boldsymbol{\lambda})$, then $\boldsymbol{\lambda} >_{\mathbf{m}} \boldsymbol{\mu}$.*

Lemma 3.10. *Suppose $n \in \mathbb{N}_{\geq 1}$ and $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(\ell)})$ is an ℓ-partition of n. If $i \neq j$ and $\mu_a^{(i)} \neq 0$, $\mu_b^{(j)} \neq 0$ for some a, b, then $\mu_a^{(i)} - \mu_b^{(j)} + b - a \leq n - 2$.*

Proof. In fact,

$$\mu_a^{(i)} - \mu_b^{(j)} + b - a \leq \mu_a^{(i)} - 1 + b - 1 = \mu_a^{(i)} + b - 2 \leq \mu_a^{(i)} + \mu_1^{(j)} + \mu_2^{(j)} + \cdots + \mu_b^{(j)} - 2 \leq n - 2.$$

□

Suppose $n \in \mathbb{N}_{>1}$ and $m_i - m_j > n - 1$ for all $i < j$. Geck and Jacon stated $\boldsymbol{\lambda} >_{\mathbf{m}} \boldsymbol{\mu}$ whenever $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}$ in [7, Example 5.5.18], but no proof is given there. Now, we prove it for all $n \in \mathbb{N}$ explicitly.

Theorem 3.11. *Suppose $n \in \mathbb{N}$, $\mathbf{m} = (m_1, \ldots, m_\ell) \in \mathbb{Q}^\ell$ with $m_i - m_j > n - 1$ for all $i < j$. Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$. If $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}$, then $\boldsymbol{\lambda} >_{\mathbf{m}} \boldsymbol{\mu}$.*

Proof. For $n = 0$, the conclusion is trivial. Now we suppose $n > 0$, $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$, and $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}$. Fix an integer $s \geq n + 1 - \min\{m_i\}$. Let $1 \leq i < j \leq \ell$, by Remark 3.1, elements of $B_i(\boldsymbol{\mu})$ are $\{\mu_a^{(i)} - a + s + m_i | a = 1, 2, \ldots, s + [m_i]\}$, and elements of $B_j(\boldsymbol{\mu})$ are $\{\mu_b^{(j)} - b + s + m_j | b = 1, 2, \ldots, s + [m_j]\}$. Suppose $a \in \{1, 2, \ldots, s + [m_i]\}$, $b \in \{1, 2, \ldots, s + [m_j]\}$ with $\mu_a^{(i)} \neq 0, \mu_b^{(j)} \neq 0$. According to Lemma 3.10, $\mu_b^{(j)} - \mu_a^{(i)} + a - b \leq n - 2 < n - 1 < m_i - m_j$. Hence,

$$(\mu_a^{(i)} - a + s + m_i) - (\mu_b^{(j)} - b + s + m_j) > 1 \tag{3.12}$$

It is enough to prove the conclusion for the covering relations in the dominance order. Indeed, if

$$\boldsymbol{\mu} = \boldsymbol{\nu}_0 \rhd \boldsymbol{\nu}_1 \rhd \cdots \rhd \boldsymbol{\nu}_r = \boldsymbol{\lambda}$$

is a saturated chain and each step gives $\boldsymbol{\nu}_{i+1} >_{\mathbf{m}} \boldsymbol{\nu}_i$, then transitivity of $>_{\mathbf{m}}$ gives $\boldsymbol{\lambda} >_{\mathbf{m}} \boldsymbol{\mu}$. So we can suppose $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are neighbors with $\trianglerighteq$. By Proposition 3.9, to prove $\boldsymbol{\lambda} >_{\mathbf{m}} \boldsymbol{\mu}$, we should prove $\kappa_{\mathbf{m}}(\boldsymbol{\mu}) \rhd \kappa_{\mathbf{m}}(\boldsymbol{\lambda})$. Combining with Lemma 2.2, it is sufficient to prove $\kappa_{\mathbf{m}}(\boldsymbol{\mu}) \rhd \kappa_{\mathbf{m}}(\boldsymbol{\lambda})$ in the following three cases.

Case 1. Suppose there exists $t < \ell$ such that

$$\begin{aligned}
&\mu^{(r)} = \lambda^{(r)} \quad \forall r \neq t, t+1 \\
&\mu^{(t)} = (\lambda_1^{(t)}, \ldots, \lambda_{k_t}^{(t)}, 1), \quad \text{where } k_t = h(\lambda^{(t)}) \\
&\mu_1^{(t+1)} = \lambda_1^{(t+1)} - 1 \qquad \mu_j^{(t+1)} = \lambda_j^{(t+1)} \quad \forall j > 1.
\end{aligned}$$

Therefore, for all $r \neq t, t+1$, $B_r(\boldsymbol{\mu}) = B_r(\boldsymbol{\lambda})$. Let $k_{t+1} = h(\lambda^{(t+1)})$, and we have

$$\lambda^{(t)} = (\lambda_1^{(t)}, \ldots, \lambda_{k_t}^{(t)}), \quad \lambda^{(t+1)} = (\lambda_1^{(t+1)}, \lambda_2^{(t+1)}, \ldots, \lambda_{k_{t+1}}^{(t+1)}),$$

$$\mu^{(t)} = (\lambda_1^{(t)}, \ldots, \lambda_{k_t}^{(t)}, 1), \quad \mu^{(t+1)} = (\lambda_1^{(t+1)} - 1, \lambda_2^{(t+1)}, \ldots, \lambda_{k_{t+1}}^{(t+1)}).$$

According to Remark 3.1, we have

$$\begin{aligned}
B_t(\boldsymbol{\lambda}) = (&\underbrace{-(s + [m_t]) + s + m_t, \ldots, -(k_t + j) + s + m_t, \ldots, -(k_t + 2) + s + m_t}_{\text{(I)}},\\
&-(k_t + 1) + s + m_t, \underbrace{\lambda_{k_t}^{(t)} - k_t + s + m_t, \ldots, \lambda_a^{(t)} - a + s + m_t, \ldots, \lambda_1^{(t)} - 1 + s + m_t}_{\text{(II)}})\\
B_{t+1}(\boldsymbol{\lambda}) = (&\underbrace{-(s + [m_{t+1}]) + s + m_{t+1}, \ldots, -(k_{t+1} + j) + s + m_{t+1}, \ldots, -(k_{t+1} + 1) + s + m_{t+1}}_{\text{(III)}},\\
&\underbrace{\lambda_{k_{t+1}}^{(t+1)} - k_{t+1} + s + m^{t+1}, \ldots, \lambda_a^{(t+1)} - a + s + m_{t+1}, \ldots, \lambda_2^{(t+1)} - 2 + s + m_{t+1}}_{\text{(IV)}},\\
&\lambda_1^{(t+1)} - 1 + s + m_{t+1})\\
B_t(\boldsymbol{\mu}) = (&\underbrace{-(s + [m_t]) + s + m_t, \ldots, -(k_t + 1 + j) + s + m_t, \ldots, -(k_t + 1 + 1) + s + m_t}_{\text{(I}')},\\
&1 - (k_t + 1) + s + m_t, \underbrace{\lambda_{k_t}^{(t)} - k_t + s + m_t, \ldots, \lambda_a^{(t)} - a + s + m_t, \ldots, \lambda_1^{(t)} - 1 + s + m_t}_{\text{(II}')})\\
B_{t+1}(\boldsymbol{\mu}) = (&\underbrace{-(s + [m_{t+1}]) + s + m_{t+1}, \ldots, -(k_{t+1} + j) + s + m_{t+1}, \ldots, -(k_{t+1} + 1) + s + m_{t+1}}_{\text{(III}')},\\
&\underbrace{\lambda_{k_{t+1}}^{(t+1)} - k_{t+1} + s + m_{t+1}, \ldots, \lambda_a^{(t+1)} - a + s + m_{t+1}, \ldots, \lambda_2^{(t+1)} - 2 + s + m_{t+1}}_{\text{(IV}')},\\
&(\lambda_1^{(t+1)} - 1) - 1 + s + m_{t+1}).
\end{aligned}$$

Note that (I) = (I′), (II) = (II′), (III) = (III′), (IV) = (IV′). Now we consider the elements of $\mathbf{B}(\boldsymbol{\lambda})$

$$-(k_t + 1) + s + m_t, \quad \lambda_1^{(t+1)} - 1 + s + m_{t+1}$$

and elements of $\mathbf{B}(\boldsymbol{\mu})$

$$1 - (k_t + 1) + s + m_t, \quad (\lambda_1^{(t+1)} - 1) - 1 + s + m_{t+1}.$$

If $n > 1$ and $\mu_1^{(t+1)} = \lambda_1^{(t+1)} - 1 \neq 0$, apply (3.12) to $\boldsymbol{\mu}$, and we obtain

$$(\mu_{k_t+1}^{(t)} - (k_t + 1) + s + m_t) - (\mu_1^{(t+1)} - 1 + s + m_{t+1}) > 1.$$

That is,

$$\big((1 - (k_t + 1) + s + m_t) - \big((\lambda_1^{(t+1)} - 1) - 1 + s + m_{t+1}\big) > 1.$$

So we have

$$1 - (k_t + 1) + s + m_t > \lambda_1^{(t+1)} - 1 + s + m_{t+1}.$$

Hence,

$$(1-(k_t+1)+s+m_t,(\lambda_1^{(t+1)}-1)-1+s+m_{t+1})\rhd(-(k_t+1)+s+m_t,\lambda_1^{(t+1)}-1+s+m_{t+1}),$$
$$(1-(k_t+1)+s+m_t,(\lambda_1^{(t+1)}-1)-1+s+m_{t+1})\rhd(\lambda_1^{(t+1)}-1+s+m_{t+1},-(k_t+1)+s+m_t).$$

By Lemma 3.8, we obtain $\kappa_{\mathbf{m}}(\boldsymbol{\mu})\rhd\kappa_{\mathbf{m}}(\boldsymbol{\lambda})$.

If $n\geq 1$ and $\mu_1^{(t+1)}=\lambda_1^{(t+1)}-1=0$, then $k_{t+1}=1$, $\lambda_1^{(t+1)}=1$ and

$$\lambda^{(t)}=(\lambda_1^{(t)},\dots,\lambda_{k_t}^{(t)}),\quad \lambda^{(t+1)}=(1),\quad \mu^{(t)}=(\lambda_1^{(t)},\dots,\lambda_{k_t}^{(t)},1),\quad \mu^{(t+1)}=\underline{\emptyset}.$$

According to Remark 3.1, we have

$$B_t(\boldsymbol{\lambda})=(\underbrace{-(s+[m_t])+s+m_t,\dots,-(k_t+j)+s+m_t,\dots,-(k_t+2)+s+m_t}_{\text{(A)}},$$
$$-(k_t+1)+s+m_t,\underbrace{\lambda_{k_t}^{(t)}-k_t+s+m_t,\dots,\lambda_a^{(t)}-a+s+m_t,\dots,\lambda_1^{(t)}-1+s+m_t}_{\text{(B)}})$$
$$B_{t+1}(\boldsymbol{\lambda})=(\underbrace{-(s+[m_{t+1}])+s+m_{t+1},\dots,-(k_{t+1}+j)+s+m_{t+1},\dots,-(k_{t+1}+1)+s+m_{t+1}}_{\text{(C)}},$$
$$s+m_{t+1})$$
$$B_t(\boldsymbol{\mu})=(\underbrace{-(s+[m_t])+s+m_t,\dots,-(k_t+1+j)+s+m_t,\dots,-(k_t+1+1)+s+m_t}_{\text{(A$'$)}},$$
$$1-(k_t+1)+s+m_t,\underbrace{\lambda_{k_t}^{(t)}-k_t+s+m_t,\dots,\lambda_a^{(t)}-a+s+m_t,\dots,\lambda_1^{(t)}-1+s+m_t}_{\text{(B$'$)}})$$
$$B_{t+1}(\boldsymbol{\mu})=(\underbrace{-(s+[m_{t+1}])+s+m_{t+1},\dots,-(k_{t+1}+j)+s+m_{t+1},\dots,-(k_{t+1}+1)+s+m_{t+1}}_{\text{(C$'$)}},$$
$$-1+s+m_{t+1})$$

Note that (A) = (A′), (B) = (B′), and (C) = (C′). Now we consider the elements of $\mathbf{B}(\boldsymbol{\lambda})$

$$-(k_t+1)+s+m_t,\quad s+m_{t+1}$$

and elements of $\mathbf{B}(\boldsymbol{\mu})$

$$-k_t+s+m_t,\quad -1+s+m_{t+1}.$$

Since $m_t-m_{t+1}>n-1$,

$$-k_t+s+m_t-(-1+s+m_{t+1})=m_t-m_{t+1}+1-k_t>n-k_t\geq 1,$$

the last inequality follows from $\lambda_1^{(t+1)}=1$ and $k_t\leq n-1$. Hence, $-k_t+s+m_t>s+m_{t+1}$. Therefore,

$$(-k_t+s+m_t,-1+s+m_{t+1})\rhd(-(k_t+1)+s+m_t,s+m_{t+1}),$$
$$(-k_t+s+m_t,-1+s+m_{t+1})\rhd(s+m_{t+1},-(k_t+1)+s+m_t).$$

By Lemma 3.8, we obtain $\kappa_{\mathbf{m}}(\boldsymbol{\mu})\rhd\kappa_{\mathbf{m}}(\boldsymbol{\lambda})$.

Case 2. Suppose $n > 1$, and there exist t and i such that

$$\begin{aligned}&\mu^{(r)} = \lambda^{(r)}, \quad \forall r \neq t,\\ &\mu^{(t)}_j = \lambda^{(t)}_j, \quad \forall j \neq i, i+1,\\ &\mu^{(t)}_i = \lambda^{(t)}_i + 1 \text{ and } \mu^{(t)}_{i+1} = \lambda^{(t)}_{i+1} - 1.\end{aligned}$$

In this case, $B_r(\boldsymbol{\mu}) = B_r(\boldsymbol{\lambda})$ for all $r \neq t$. Suppose

$$\begin{aligned}&\lambda^{(t)} = (\lambda^{(t)}_1, \ldots, \lambda^{(t)}_{k_t}), \quad \text{where } k_t = h(\lambda^{(t)}), \text{ and}\\ &\mu^{(t)} = (\mu^{(t)}_1, \ldots, \mu^{(t)}_i, \mu^{(t)}_{i+1}, \ldots, \mu^{(t)}_{k_t}) = (\lambda^{(t)}_1, \ldots, \lambda^{(t)}_i + 1, \lambda^{(t)}_{i+1} - 1, \ldots, \lambda^{(t)}_{k_t}).\end{aligned}$$

By Remark 3.1, we have

$$\begin{aligned}B_t(\boldsymbol{\lambda}) &= (-(s + [m_t]) + s + m_t, \ldots, -(k_t + j) + s + m_t, \ldots, -(k_t + 1) + s + m_t,\\ &\lambda^{(t)}_{k_t} - k_t + s + m_t, \ldots, \underbrace{\lambda^{(t)}_{i+1} - (i+1) + s + m_t, \lambda^{(t)}_i - i + s + m_t}_{\text{(V)}}, \ldots, \lambda^{(t)}_1 - 1 + s + m_t),\\ B_t(\boldsymbol{\mu}) &= (-(s + [m_t]) + s + m_t, \ldots, -(k_t + j) + s + m_t, \ldots, -(k_t + 1) + s + m_t,\\ &\lambda^{(t)}_{k_t} - k_t + s + m_t, \ldots, \underbrace{\lambda^{(t)}_{i+1} - 1 - (i+1) + s + m_t, \lambda^{(t)}_i + 1 - i + s + m_t}_{\text{(V}')}, \ldots, \lambda^{(t)}_1 - 1 + s + m_t).\end{aligned}$$

On the one hand, $B_t(\boldsymbol{\lambda}) \setminus (\text{V}) = B_t(\boldsymbol{\mu}) \setminus (\text{V}')$. On the other hand, since $B_t(\boldsymbol{\lambda})$ and $B_t(\boldsymbol{\mu})$ are strictly increasing sequences, we have

$$\begin{aligned}\lambda^{(t)}_i - i + s + m_t &> \lambda^{(t)}_{i+1} - (i+1) + s + m_t,\\ \lambda^{(t)}_i + 1 - i + s + m_t &> \lambda^{(t)}_{i+1} - 1 - (i+1) + s + m_t,\end{aligned}$$

and hence

$$(\lambda^{(t)}_i + 1 - i + s + m_t, \lambda^{(t)}_{i+1} - 1 - (i+1) + s + m_t) \rhd (\lambda^{(t)}_i - i + s + m_t, \lambda^{(t)}_{i+1} - (i+1) + s + m_t).$$

By Lemma 3.8, we obtain $\kappa_{\mathbf{m}}(\boldsymbol{\mu}) \rhd \kappa_{\mathbf{m}}(\boldsymbol{\lambda})$ in this case.

Case 3. Suppose $n > 1$, and there exist t and $i < i'$ such that

$$\begin{aligned}&\mu^{(r)} = \lambda^{(r)}, \quad \forall r \neq t,\\ &\mu^{(t)}_j = \lambda^{(t)}_j, \quad \forall j \neq i, i',\\ &\mu^{(t)}_i - 1 = \mu^{(t)}_{i'} + 1 = \lambda^{(t)}_i = \lambda^{(t)}_{i'}.\end{aligned}$$

In this case, $B_r(\boldsymbol{\mu}) = B_r(\boldsymbol{\lambda})$ for all $r \neq t$. Suppose

$$\begin{aligned}&\lambda^{(t)} = (\lambda^{(t)}_1, \ldots, \lambda^{(t)}_{k_t}), \text{ where } k_t = h(\lambda^{(t)}), \text{ and}\\ &\mu^{(t)} = (\mu^{(t)}_1, \ldots, \mu^{(t)}_i, \ldots, \mu^{(t)}_{i'}, \ldots, \mu^{(t)}_{k_t}) = (\lambda^{(t)}_1, \ldots, \lambda^{(t)}_i + 1, \ldots, \lambda^{(t)}_{i'} - 1, \ldots, \lambda^{(t)}_{k_t}).\end{aligned}$$

By Remark 3.1,

$$B_t(\boldsymbol{\lambda}) = (-(s+[m_t])+s+m_t,\dots,-(k_t+j)+s+m_t,\dots,-(k_t+1)+s+m_t,$$
$$\lambda^{(t)}_{k_t}-k_t+s+m_t,\dots,\lambda^{(t)}_{i'}-i'+s+m_t,\dots,\lambda^{(t)}_{i}-i+s+m_t,\dots,\lambda^{(t)}_{1}-1+s+m_t),$$
$$B_t(\boldsymbol{\mu}) = (-(s+[m_t])+s+m_t,\dots,-(k_t+j)+s+m_t,\dots,-(k_t+1)+s+m_t,$$
$$\lambda^{(t)}_{k_t}-k_t+s+m_t,\dots,\lambda^{(t)}_{i'}-1-i'+s+m_t,\dots,\lambda^{(t)}_{i}+1-i+s+m_t,\dots,\lambda^{(t)}_{1}-1+s+m_t).$$

Note

$$\begin{aligned} &B_t(\boldsymbol{\lambda}) \setminus \{\lambda^{(t)}_i - i + s + m_t, \lambda^{(t)}_{i'} - i' + s + m_t\} \\ &= B_t(\boldsymbol{\mu}) \setminus \{\lambda^{(t)}_i + 1 - i + s + m_t, \lambda^{(t)}_{i'} - 1 - i' + s + m_t\}, \end{aligned}$$

and

$$(\lambda^{(t)}_i + 1 - i + s + m_t, \lambda^{(t)}_{i'} - 1 - i' + s + m_t) \rhd (\lambda^{(t)}_i - i + s + m_t, \lambda^{(t)}_{i'} - i' + s + m_t).$$

By Lemma 3.8, we obtain $\kappa_{\mathbf{m}}(\boldsymbol{\mu}) \rhd \kappa_{\mathbf{m}}(\boldsymbol{\lambda})$ in this case. □

Fix $\sigma = (e;\sigma_1,\dots,\sigma_\ell) \in \mathbb{N}_{>1}\times\mathbb{Z}^\ell$. For $\alpha = (r,c,l) \in \mathcal{N}^\ell_n$, we define its residue to be $\mathrm{res}(\alpha) = c - r + \sigma_l + e\mathbb{Z} \in \mathbb{Z}/e\mathbb{Z}$. A loading is a sequence of integers $\theta = (\theta_1,\dots,\theta_\ell)$ such that $\theta_i - \theta_j \notin \ell\mathbb{Z}$ for $i < j$. If $\theta_{i+1} - \theta_i > \ell n$ for each $i = 1,2,\dots,\ell-1$, we call θ a separated loading.

Definition 3.13. *[8, Definition 2.8] Let $\alpha = (r,c,l), \alpha' = (r',c',l') \in \mathcal{N}^\ell_n$. We write $\alpha >_\theta \alpha'$ if either*

(*i*) $\theta_l + \ell(r-c) < \theta_{l'} + \ell(r'-c')$ *or*
(*ii*) $\theta_l + \ell(r-c) = \theta_{l'} + \ell(r'-c')$ *and* $r + c < r' + c'$.

Moreover, if $\mathrm{res}(\alpha) = \mathrm{res}(\alpha')$*, then we write* $\alpha \rhd_\theta \alpha'$.

Definition 3.14. *[8, Definition 2.11] Let $\boldsymbol{\lambda},\boldsymbol{\mu} \in \mathcal{P}^\ell_n$. We write $\boldsymbol{\lambda} \trianglerighteq_\theta \boldsymbol{\mu}$ if*

$$|\{\beta \in \boldsymbol{\mu} | \beta \rhd_\theta \gamma\}| \le |\{\beta \in \boldsymbol{\lambda} | \beta \rhd_\theta \gamma\}| \quad \forall \gamma \in \mathcal{N}^\ell_n.$$

We call $\trianglerighteq_\theta$ the θ**-dominance order**.

Definition 3.15. *[9, Definition 2.3] Let $\boldsymbol{\lambda},\boldsymbol{\mu} \in \mathcal{P}^\ell_n$. We write $\boldsymbol{\lambda} \ge_\theta \boldsymbol{\mu}$ if*

$$|\{\beta \in \boldsymbol{\mu} | \beta >_\theta \gamma\}| \le |\{\beta \in \boldsymbol{\lambda} | \beta >_\theta \gamma\}| \quad \forall \gamma \in \mathcal{N}^\ell_n.$$

We call $\ge_\theta$ the **weak** θ**-dominance order**.

For $\boldsymbol{\lambda} \in \mathcal{P}^\ell_n$, we denote the residue multiset of $\boldsymbol{\lambda}$ by $\mathrm{res}(\boldsymbol{\lambda})$. Next, let's prove weak θ-dominance order refines θ-dominance order.

Proposition 3.16. *Let $\boldsymbol{\lambda},\boldsymbol{\mu} \in \mathcal{P}^\ell_n$. If $\boldsymbol{\lambda} \trianglerighteq_\theta \boldsymbol{\mu}$, then* $\mathrm{res}(\boldsymbol{\lambda}) = \mathrm{res}(\boldsymbol{\mu})$ *and* $\boldsymbol{\lambda} \ge_\theta \boldsymbol{\mu}$.

Proof. Suppose $\boldsymbol{\lambda} \trianglerighteq_\theta \boldsymbol{\mu}$. Let's prove $\mathrm{res}(\boldsymbol{\lambda}) = \mathrm{res}(\boldsymbol{\mu})$. First, we prove that the length of the diagonal of ℓ-partitions of n is less than that of $\mathcal{N}^\ell_n$. In other words, we just need to prove that in each row of

$\mathcal{N}_n^\ell$, the last two nodes cannot be the nodes of ℓ-partitions of n. For example, let $\ell = 1, n = 2$, then $\mathcal{N}_2^1 = \{(r, c)|r, c \in \mathbb{N}_{\geq 1}, r + c \leq 6\}$. The Young diagram of $\mathcal{N}_2^1$ is

$$[\mathcal{N}_2^1] = \begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & * & * \\ \hline \ & \ & * & * \\ \cline{1-4} \ & * & * \\ \cline{1-3} * & * \\ \cline{1-2} * \\ \cline{1-1} \end{array}$$

where the nodes marked with $*$ are the last node on their respective diagonals of $\mathcal{N}_2^1$. On the other hand, $\lambda = (2), \mu = (1, 1)$ are the only partitions of 2 and

$$[\lambda] = \begin{array}{|c|c|}\hline \ & \ \\ \hline \end{array}, \qquad [\mu] = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array}.$$

Obviously, all the nodes marked $*$ in $[\mathcal{N}_2^1]$ belong to neither $[\lambda]$ nor $[\mu]$. In general, let's prove that the last two nodes of each row of $\mathcal{N}_n^\ell$ cannot be the nodes of ℓ-partitions of n. Suppose k is a positive integer with $k \leq n$. Since $\mathcal{N}_n^\ell$ is defined with $r + c \leq 2(n + 1)$, and the last two nodes of the k-th row of each component of $\mathcal{N}_n^\ell$ are $(k, 2n + 2 - k)$ and $(k, 2n + 2 - k - 1)$. Since $(k, 2n + 2 - k)$ is after $(k, 2n + 2 - k - 1)$ in the same row, we just need to prove $(k, 2n + 2 - k - 1)$ cannot belong to any ℓ-partitions of n. Otherwise, assume $(k, 2n + 2 - k - 1)$ lies in an ℓ-partitions η of n, then we derive

$$|\eta| \geq k(2n + 2 - k - 1) = nk + (n - k + 1)k \geq nk + k > n.$$

This contradicts that η is an ℓ-partition of n.

For each residue $i \in \text{res}(\boldsymbol{\lambda}) \cup \text{res}(\boldsymbol{\mu})$, as we have proved, the length of the diagonal of $\boldsymbol{\lambda} \cup \boldsymbol{\mu}$ is strictly less than that of $\mathcal{N}_n^\ell$ and the nodes on the same diagonal have the same residue. Let γ be the smallest node with residue i under $\triangleright_\theta$ in $\mathcal{N}_n^\ell$. Then, γ must be the last node on a diagonal of $\mathcal{N}_n^\ell$. Hence, γ cannot belong to $\boldsymbol{\lambda} \cup \boldsymbol{\mu}$. Therefore, $\{\beta \in \boldsymbol{\mu} | \beta \triangleright_\theta \gamma\}$ is the set of i-nodes in $\boldsymbol{\mu}$ and $\{\alpha \in \boldsymbol{\lambda} | \alpha \triangleright_\theta \gamma\}$ is the set of i-nodes in $\boldsymbol{\lambda}$. By definition, $\boldsymbol{\lambda} \trianglerighteq_\theta \boldsymbol{\mu}$ implies

$$|\{\beta \in \boldsymbol{\lambda} | \beta \triangleright_\theta \gamma\}| \geq |\{\alpha \in \boldsymbol{\mu} | \alpha \triangleright_\theta \gamma\}|.$$

Moreover, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions of n, so we derive

$$|\{\beta \in \boldsymbol{\lambda} | \beta \triangleright_\theta \gamma\}| = |\{\alpha \in \boldsymbol{\mu} | \alpha \triangleright_\theta \gamma\}|,$$

hence, $\text{res}(\boldsymbol{\lambda}) = \text{res}(\boldsymbol{\mu})$.

Next, let's prove $\boldsymbol{\lambda} \geq_\theta \boldsymbol{\mu}$. Let $\gamma \in \mathcal{N}_n^\ell$, for each $i \in \text{res}(\boldsymbol{\lambda}) = \text{res}(\boldsymbol{\mu})$, and we define γ_i as follows:

- If $i = \text{res}(\gamma)$, then $\gamma_i := \gamma$.
- If $i \neq \text{res}(\gamma)$ and there exists no i-node α in $\boldsymbol{\lambda} \cup \boldsymbol{\mu}$ such that $\alpha >_\theta \gamma$, then $\gamma_i := \gamma$.
- If $i \neq \text{res}(\gamma)$ and there exists i-node α in $\boldsymbol{\lambda} \cup \boldsymbol{\mu}$ such that $\alpha >_\theta \gamma$, let α_i be the smallest node with residue i under $\triangleright_\theta$ in the set $\{\beta \in \boldsymbol{\lambda} \cup \boldsymbol{\mu} | \beta >_\theta \gamma\}$. We choose γ_i to be the last node on the same diagonal of $\mathcal{N}_n^\ell$ as α_i. Since the length of the diagonal of $\boldsymbol{\lambda} \cup \boldsymbol{\mu}$ is strictly less than that of $\mathcal{N}_n^\ell$, note that nodes on the same diagonal have the same residue, and we derive $\text{res}(\gamma_i) = i$ and $\alpha_i \triangleright_\theta \gamma_i >_\theta \gamma$.

By the choice of γ_i, we have

$$\{\beta \in \boldsymbol{\mu} | \beta >_\theta \gamma\} = \bigcup_{i \in \mathrm{res}(\boldsymbol{\lambda})} \{\beta \in \boldsymbol{\mu} | \beta \rhd_\theta \gamma_i\}$$

and

$$\{\beta \in \boldsymbol{\lambda} | \beta >_\theta \gamma\} = \bigcup_{i \in \mathrm{res}(\boldsymbol{\lambda})} \{\beta \in \boldsymbol{\lambda} | \beta \rhd_\theta \gamma_i\}.$$

Since $\boldsymbol{\lambda} \trianglerighteq_\theta \boldsymbol{\mu}$, we have

$$|\{\beta \in \boldsymbol{\lambda} | \beta \rhd_\theta \gamma_i\}| \geq |\{\beta \in \boldsymbol{\mu} | \beta \rhd_\theta \gamma_i\}| \quad \forall i \in \mathrm{res}(\boldsymbol{\lambda}).$$

Therefore, $|\{\beta \in \boldsymbol{\lambda} | \beta >_\theta \gamma\}| \geq |\{\beta \in \boldsymbol{\mu} | \beta >_\theta \gamma\}|$ and $\boldsymbol{\lambda} \geq_\theta \boldsymbol{\mu}$. □

Suppose $\theta = (\theta_1, \ldots, \theta_\ell)$ is separated. Let $\mathbf{m} = (m_1, \ldots, m_\ell)$, where $m_i = (-\theta_i + i)/\ell$, then

$$m_i - m_{i+1} = \tfrac{\theta_{i+1} - \theta_i}{\ell} - \tfrac{1}{\ell} > n - \tfrac{1}{\ell} > n - 1.$$

According to Proposition 3.16, Theorem 3.11, and [9, Theorem 4.1], we obtain the relationship between θ-dominance order and a-function order.

Corollary 3.17. *Suppose $\theta = (\theta_1, \ldots, \theta_\ell)$ is separated. If $\boldsymbol{\mu} \trianglerighteq_\theta \boldsymbol{\lambda}$, then $\boldsymbol{\lambda} \geq_{\mathbf{m}} \boldsymbol{\mu}$.*

4. Relationship between Webster's ordering and a-function order

Fix $\sigma = (e; \sigma_1, \ldots, \sigma_\ell) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$. For $\alpha = (r, c, l) \in \mathcal{N}_n^\ell$, we define its content to be $\mathbf{ct}(\alpha) = c - r + \sigma_l$. By definition, $\mathrm{res}(\alpha) = \mathbf{ct}(\alpha) + e\mathbb{Z} \in \mathbb{Z}/e\mathbb{Z}$.

Definition 4.1 ([4, Definition 1.1]). *Given $\sigma = (e; \sigma_1, \ldots, \sigma_\ell) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$, $\alpha = (r, c, l) \in \mathcal{N}_n^\ell$, and $\beta = (r', c', l') \in \mathcal{N}_n^\ell$, we write $\beta >_\sigma \alpha$ if either*

(*i*) $\mathbf{ct}(\alpha) < \mathbf{ct}(\beta)$, *or*
(*ii*) $\mathbf{ct}(\alpha) = \mathbf{ct}(\beta)$ *and* $l > l'$.

Moreover, if $\mathrm{res}(\alpha) = \mathrm{res}(\beta)$, *then we write $\beta \rhd_\sigma \alpha$.*

Definition 4.2 ([4, Definition 1.2]). *For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$, we write $\boldsymbol{\lambda} \geq_\sigma \boldsymbol{\mu}$ if there is a bijection $f : [\boldsymbol{\lambda}] \longrightarrow [\boldsymbol{\mu}]$ such that either $\alpha >_\sigma f(\alpha)$ or $f(\alpha) = \alpha$ for all $\alpha \in [\boldsymbol{\lambda}]$. We call $\geq_\sigma$* **Lusztig's $\mathbf{a}_\sigma$-ordering**.

Definition 4.3 ([4, Definition 1.3]). *For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$, we write $\boldsymbol{\lambda} \trianglerighteq_\sigma \boldsymbol{\mu}$ if there is a residue preserving bijection $f : [\boldsymbol{\lambda}] \longrightarrow [\boldsymbol{\mu}]$ such that either $\alpha \rhd_\sigma f(\alpha)$ or $f(\alpha) = \alpha$ for all $\alpha \in [\boldsymbol{\lambda}]$. We call $\trianglerighteq_\sigma$* **Webster's ordering**.

The Webster's ordering was originally defined by Webster via diagrammatic form [10, Definition 2.8]. Bowman rephrased it in the form of Definition 4.3. It is clear that Lusztig's $\mathbf{a}_\sigma$-ordering refines Webster's ordering.

Fix a sequence $\mathbf{u} = (u_1, \ldots, u_\ell) \in \mathbb{Q}^\ell$ with $0 < u_j - u_i < e,\ \forall i < j$. Let

$$\mathbf{m} = (m_1, \ldots, m_\ell) \in \mathbb{Q}^\ell, \text{ where } m_i = \sigma_i - u_i,\ \forall 1 \leq i \leq \ell. \tag{4.4}$$

Remark 4.5. *For any rational sequence* $\mathbf{m} = (m_1, \ldots, m_\ell) \in \mathbb{Q}^\ell$*, we can always choose a suitable sequence* $(e; \sigma_1, \sigma_2, \ldots, \sigma_\ell) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ *and a strictly increasing rational sequence* $\mathbf{u} = (u_1, \ldots, u_\ell) \in \mathbb{Q}^\ell$ *with* $0 < u_j - u_i < e$ *for each* $i < j$ *such that* $m_i = \sigma_i - u_i$*,* $\forall 1 \le i \le \ell$*. For example,* $(m_1, m_2, m_3) = (2.4, 1.5, 3.8)$*. To construct the suitable* e*,* $(\sigma_1, \sigma_2, \sigma_3)$ *and* (u_1, u_2, u_3)*, we first choose sequences*

$$([2.4] + 1, [1.5] + 1, [3.8] + 1) = (3, 2, 4)$$

and

$$([2.4] + 1 - 2.4, [1.5] + 1 - 1.5, [3.8] + 1 - 3.8) = (0.6, 0.5, 0.2),$$

where $[\cdot]$ *is the rounding function. To make the second sequence strictly increasing, we add 1 to the second number and 2 to the third number in both sequences, then we get sequences*

$$(\sigma_1, \sigma_2, \sigma_3) = (3, 3, 6)$$

and

$$(u_1, u_2, u_3) = (0.6, 1.5, 2.2).$$

Since $u_3 - u_1 = 1.6 < 2$*, we choose* $e = 2$*.*

For $\alpha = (r, c, l) \in \mathcal{N}_n^\ell$, define $\eta(\alpha) = \mathbf{ct}(\alpha) - u_l = c - r + \sigma_l - u_l$.

The next lemma is proved in [5, Lemma 5.5], and we provide the details of the proof here for the reader's convenience.

Lemma 4.6 ([5, Lemma 5.5])**.** *Suppose* $\boldsymbol{\mu}, \boldsymbol{\mu}'$ *are* ℓ*-compositions of* n *with* $\kappa_{\mathbf{m}}(\boldsymbol{\mu}') \trianglerighteq \kappa_{\mathbf{m}}(\boldsymbol{\mu})$*. Let* $\boldsymbol{\lambda}$ *(respectively,* $\boldsymbol{\lambda}'$*) be an* ℓ*-composition obtained from* $\boldsymbol{\mu}$ *(respectively,* $\boldsymbol{\mu}'$*) by adding an extra node* β *(respectively,* β'*). If* $\eta(\beta') > \eta(\beta)$*, then* $\kappa_{\mathbf{m}}(\boldsymbol{\lambda}') \rhd \kappa_{\mathbf{m}}(\boldsymbol{\lambda})$*.*

Proof. Suppose $\kappa_{\mathbf{m}}(\boldsymbol{\mu}) = (\kappa_1(\boldsymbol{\mu}), \kappa_2(\boldsymbol{\mu}), \ldots, \kappa_t(\boldsymbol{\mu}))$ and $\kappa_{\mathbf{m}}(\boldsymbol{\mu}') = (\kappa_1(\boldsymbol{\mu}'), \kappa_2(\boldsymbol{\mu}'), \ldots, \kappa_t(\boldsymbol{\mu}'))$, where $\kappa_{\mathbf{m}}(\boldsymbol{\mu}), \kappa_{\mathbf{m}}(\boldsymbol{\mu}')$ are descending sequences by (3.3). Since $\kappa_{\mathbf{m}}(\boldsymbol{\mu}') \trianglerighteq \kappa_{\mathbf{m}}(\boldsymbol{\mu})$, we have

$$\sum_{i=1}^{r} \kappa_i(\boldsymbol{\mu}') \ge \sum_{i=1}^{r} \kappa_i(\boldsymbol{\mu}), \qquad \forall 1 \le r \le t.$$

The nodes β and β' are added to the end of a row (which may be empty) of the compositions $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ respectively. According to Remark 3.1, there exist $1 \le j \le t$ and $1 \le j' \le t$ such that, as multi-sets,

$$\kappa_{\mathbf{m}}(\boldsymbol{\lambda}) = (\kappa_{\mathbf{m}}(\boldsymbol{\mu}) \setminus \{\kappa_j(\boldsymbol{\mu})\}) \cup \{\kappa_j(\boldsymbol{\mu}) + 1\}, \qquad \kappa_{\mathbf{m}}(\boldsymbol{\lambda}') = (\kappa_{\mathbf{m}}(\boldsymbol{\mu}') \setminus \{\kappa_{j'}(\boldsymbol{\mu}')\}) \cup \{\kappa_{j'}(\boldsymbol{\mu}') + 1\}.$$

Suppose $\beta = (r, c, l), \beta' = (r', c', l')$, according to Remark 3.1 and note $\lambda_r^{(l)} = c, \lambda'^{(l')}_{r'} = c'$, then we have

$$\kappa_j(\boldsymbol{\mu}) + 1 = \lambda_r^{(l)} - r + s + m_l = c - r + \sigma_l - u_l = \eta(\beta) + s$$

and

$$\kappa_{j'}(\boldsymbol{\mu}') + 1 = \lambda'^{(l')}_{r'} - r' + s + m_{l'} = c' - r' + \sigma_{l'} - u_{l'} = \eta(\beta') + s.$$

Since $\eta(\beta) < \eta(\beta')$, we have $\kappa_j(\boldsymbol{\mu}) < \kappa_{j'}(\boldsymbol{\mu}')$. Let $\kappa_0(\boldsymbol{\mu}) = \kappa_0(\boldsymbol{\mu}') = \infty$, and there exist $1 \le l \le j$ and $1 \le l' \le j'$ such that

$$\kappa_{l-1}(\boldsymbol{\mu}) > \kappa_j(\boldsymbol{\mu}) + 1 \ge \kappa_l(\boldsymbol{\mu}) \ge \kappa_j(\boldsymbol{\mu}), \qquad \kappa_{l'-1}(\boldsymbol{\mu}') > \kappa_{j'}(\boldsymbol{\mu}') + 1 \ge \kappa_{l'}(\boldsymbol{\mu}') \ge \kappa_{j'}(\boldsymbol{\mu}').$$

Then we have descending sequences

$$\kappa_{\mathbf{m}}(\boldsymbol{\lambda}) = (\kappa_1(\boldsymbol{\lambda}), \kappa_2(\boldsymbol{\lambda}), \ldots, \kappa_t(\boldsymbol{\lambda})) \quad \text{and} \quad \kappa_{\mathbf{m}}(\boldsymbol{\lambda}') = (\kappa_1(\boldsymbol{\lambda}'), \kappa_2(\boldsymbol{\lambda}'), \ldots, \kappa_t(\boldsymbol{\lambda}')),$$

where

$$\kappa_i(\boldsymbol{\lambda}) = \begin{cases} \kappa_i(\boldsymbol{\mu}), \text{ for } i < l \text{ or } i \geq j+1; \\ \kappa_j(\boldsymbol{\mu}) + 1, \text{ for } i = l; \\ \kappa_{i-1}(\boldsymbol{\mu}), \text{ for } l < i < j+1, \end{cases} \quad \text{and} \quad \kappa_i(\boldsymbol{\lambda}') = \begin{cases} \kappa_i(\boldsymbol{\mu}'), \text{ for } i < l' \text{ or } i \geq j'+1; \\ \kappa_{j'}(\boldsymbol{\mu}') + 1, \text{ for } i = l'; \\ \kappa_{i-1}(\boldsymbol{\mu}'), \text{ for } l < i < j'+1. \end{cases}$$

Next, to prove $\kappa_{\mathbf{m}}(\boldsymbol{\lambda}') \rhd \kappa_{\mathbf{m}}(\boldsymbol{\lambda})$, we just need to prove that

$$\sum_{i=1}^{r} \kappa_i(\boldsymbol{\lambda}') \geq \sum_{i=1}^{r} \kappa_i(\boldsymbol{\lambda}), \qquad \forall 1 \leq r \leq t,$$

and there exist some $1 \leq q \leq t$ such that

$$\sum_{i=1}^{q} \kappa_i(\boldsymbol{\lambda}') > \sum_{i=1}^{q} \kappa_i(\boldsymbol{\lambda}).$$

We can prove this by distinguishing the follow six cases:

$$l' \leq l \leq j \leq j'; \quad l \leq j \leq l' \leq j'; \quad l' \leq j' \leq l \leq j; \quad l' \leq l \leq j' \leq j; \quad l \leq l' \leq j \leq j'; \quad l \leq l' \leq j' \leq j.$$

To make the proof more readable, we only give the proof for the case $l' \leq l \leq j \leq j'$. The proof for the remaining five cases is similar. If $l' \leq l \leq j \leq j'$, then the descending sequences $\kappa_{\mathbf{m}}(\boldsymbol{\lambda})$ and $\kappa_{\mathbf{m}}(\boldsymbol{\lambda}')$ can be rewritten as:

$$\begin{aligned} &\kappa_1(\boldsymbol{\mu}) \geq \cdots \geq \kappa_{l'-1}(\boldsymbol{\mu}) \geq \kappa_{l'}(\boldsymbol{\mu}) \geq \kappa_{l'+1}(\boldsymbol{\mu}) \geq \cdots \geq \kappa_{l-1}(\boldsymbol{\mu}) > \kappa_j(\boldsymbol{\mu}) + 1 \geq \kappa_l(\boldsymbol{\mu}) \geq \ldots \\ &\cdots \geq \kappa_{j-1}(\boldsymbol{\mu}) \leq \kappa_{j+1}(\boldsymbol{\mu}) \geq \kappa_{j+2}(\boldsymbol{\mu}) \geq \cdots \geq \kappa_{j'-1}(\boldsymbol{\mu}) \geq \kappa_{j'}(\boldsymbol{\mu}) \geq \kappa_{j'+1}(\boldsymbol{\mu}) \geq \ldots, \end{aligned}$$

and

$$\begin{aligned} &\kappa_1(\boldsymbol{\mu}') \geq \cdots \geq \kappa_{l'-1}(\boldsymbol{\mu}') > \kappa_{j'}(\boldsymbol{\mu}') + 1 \geq \kappa_{l'}(\boldsymbol{\mu}') \geq \cdots \geq \kappa_{l-2}(\boldsymbol{\mu}') \geq \kappa_{l-1}(\boldsymbol{\mu}') \geq \kappa_l(\boldsymbol{\mu}') \geq \ldots \\ &\cdots \geq \kappa_{j-1}(\boldsymbol{\mu}') \leq \kappa_j(\boldsymbol{\mu}') \geq \kappa_{j+1}(\boldsymbol{\mu}') \geq \cdots \geq \kappa_{j'-1}(\boldsymbol{\mu}') \geq \kappa_{j'+1}(\boldsymbol{\mu}') \geq \kappa_{j'+2}(\boldsymbol{\mu}') \geq \ldots. \end{aligned}$$

For $1 \leq p < l'$, we have

$$\sum_{i=1}^{p} \kappa_i(\boldsymbol{\lambda}') = \sum_{i=1}^{p} \kappa_i(\boldsymbol{\mu}') \geq \sum_{i=1}^{p} \kappa_i(\boldsymbol{\mu}) = \sum_{i=1}^{p} \kappa_i(\boldsymbol{\lambda}).$$

For $l' \leq p < l$, since $\sum_{i=1}^{p-1} \kappa_i(\boldsymbol{\mu}') \geq \sum_{i=1}^{p-1} \kappa_i(\boldsymbol{\mu})$ and $\kappa_{j'}(\boldsymbol{\mu}') + 1 \geq \kappa_{l'}(\boldsymbol{\mu}') \geq \kappa_p(\boldsymbol{\mu}')$, we have

$$\sum_{i=1}^{p} \kappa_i(\boldsymbol{\lambda}') = \sum_{i=1}^{p-1} \kappa_i(\boldsymbol{\mu}') + (\kappa_{j'}(\boldsymbol{\mu}') + 1) \geq \sum_{i=1}^{p} \kappa_i(\boldsymbol{\mu}') \geq \sum_{i=1}^{p} \kappa_i(\boldsymbol{\mu}) = \sum_{i=1}^{p} \kappa_i(\boldsymbol{\lambda}).$$

For $l \le p \le j$, since $\sum_{i=1}^{p-1} \kappa_i(\boldsymbol{\mu}') \ge \sum_{i=1}^{p-1} \kappa_i(\boldsymbol{\mu})$ and $\kappa_{j'}(\boldsymbol{\mu}') > \kappa_j(\boldsymbol{\mu})$, we have the strict inequality

$$\sum_{i=1}^{p} \kappa_i(\boldsymbol{\lambda}') = \sum_{i=1}^{p-1} \kappa_i(\boldsymbol{\mu}') + (\kappa_{j'}(\boldsymbol{\mu}') + 1) > \sum_{i=1}^{p-1} \kappa_i(\boldsymbol{\mu}) + (\kappa_j(\boldsymbol{\mu}) + 1) = \sum_{i=1}^{p} \kappa_i(\boldsymbol{\lambda}). \tag{4.7}$$

For $j < p \le j'$, since $\sum_{i=1}^{p-1} \kappa_i(\boldsymbol{\mu}') \ge \sum_{i=1}^{p-1} \kappa_i(\boldsymbol{\mu})$ and $\kappa_{j'}(\boldsymbol{\mu}') > \kappa_j(\boldsymbol{\mu}) \ge \kappa_p(\boldsymbol{\mu})$, we have the strict inequality

$$\sum_{i=1}^{p} \kappa_i(\boldsymbol{\lambda}') = \sum_{i=1}^{p-1} \kappa_i(\boldsymbol{\mu}') + (\kappa_{j'}(\boldsymbol{\mu}') + 1) > \sum_{i=1}^{p} \kappa_i(\boldsymbol{\mu}) + 1 = \sum_{i=1}^{p} \kappa_i(\boldsymbol{\lambda}).$$

For $p > j'$, since $\sum_{i=1}^{p} \kappa_i(\boldsymbol{\mu}') \ge \sum_{i=1}^{p} \kappa_i(\boldsymbol{\mu})$, we have

$$\sum_{i=1}^{p} \kappa_i(\boldsymbol{\lambda}') = \sum_{i=1}^{p} \kappa_i(\boldsymbol{\mu}') + 1 \ge \sum_{i=1}^{p} \kappa_i(\boldsymbol{\mu}) + 1 = \sum_{i=1}^{p} \kappa_i(\boldsymbol{\lambda}).$$

Note there must exist some $l \le p \le j$ such that the strict inequality (4.7) holds. Hence, we completed the proof.

□

The following conclusion gives the relationship between the Webster's ordering and a-function order.

Theorem 4.8. *Let $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{P}_n^\ell$. If $\boldsymbol{\lambda}' \trianglerighteq_\sigma \boldsymbol{\lambda}$, then $\boldsymbol{\lambda} \ge_{\mathbf{m}} \boldsymbol{\lambda}'$.*

Proof. If $\boldsymbol{\lambda}' \trianglerighteq_\sigma \boldsymbol{\lambda}$, then we have residue preserving bijection $f : [\boldsymbol{\lambda}'] \longrightarrow [\boldsymbol{\lambda}]$ such that $\alpha' \triangleright_\sigma f(\alpha')$ or $f(\alpha') = \alpha'$ for all $\alpha' \in [\boldsymbol{\lambda}']$. In other words, there exist an ordering of $[\boldsymbol{\lambda}]$

$$\alpha_1, \alpha_2, \ldots, \alpha_n,$$

and an ordering of $[\boldsymbol{\lambda}']$

$$\alpha_1', \alpha_2', \ldots, \alpha_n',$$

such that

$$\alpha_i' \triangleright_\sigma \alpha_i \quad \text{or} \quad \alpha_i = \alpha_i' \quad i = 1, 2, \ldots, n.$$

Suppose $\alpha_i = (r, c, l), \alpha_i' = (r', c', l')$, and $\alpha_i' \triangleright_\sigma \alpha_i$, then $\mathrm{res}(\alpha_i) = \mathrm{res}(\alpha_i')$ and $\mathbf{ct}(\alpha_i) \le \mathbf{ct}(\alpha_i')$. Therefore, there exists nonnegative integer m such that

$$\mathbf{ct}(\alpha_i') - \mathbf{ct}(\alpha_i) = me.$$

If $\mathbf{ct}(\alpha_i) < \mathbf{ct}(\alpha_i')$, then $m \ge 1$ and

$$\eta(\alpha_i') - \eta(\alpha_i) = (\mathbf{ct}(\alpha_i') - u_{l'}) - (\mathbf{ct}(\alpha_i) - u_l) = me - (u_{l'} - u_l) > 0.$$

If $\mathbf{ct}(\alpha_i) = \mathbf{ct}(\alpha_i')$ and $l > l'$, then $u_l > u_{l'}$ and

$$\eta(\alpha_i') - \eta(\alpha_i) = (\mathbf{ct}(\alpha_i') - u_{l'}) - (\mathbf{ct}(\alpha_i) - u_l) = u_l - u_{l'} > 0.$$

Therefore, $\eta(\alpha_i) < \eta(\alpha'_i)$ whenever $\alpha_i \neq \alpha'_i$.

By the above arguments, if there exist $i \neq j$ satisfying

$$\alpha_i \neq \alpha'_i, \quad \alpha_j \neq \alpha'_j, \quad \alpha'_i = \alpha_j,$$

then we have $\eta(\alpha_i) < \eta(\alpha'_i) = \eta(\alpha_j) < \eta(\alpha'_j)$. Then we can obtain a new ordering of $[\boldsymbol{\lambda}']$

$$\gamma'_1, \gamma'_2 \ldots, \gamma'_n$$

by exchanging α'_i and α'_j. We have $\eta(\alpha_i) < \eta(\alpha'_j) = \eta(\gamma'_i)$ and $\alpha_j = \alpha'_i = \gamma'_j$. Therefore, by reordering $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\alpha'_1, \alpha'_2, \ldots, \alpha'_n$, we can obtain an ordering of $[\boldsymbol{\lambda}]$

$$\beta_1, \ldots, \beta_n,$$

and an ordering of $[\boldsymbol{\lambda}']$

$$\beta'_1, \ldots, \beta'_n,$$

such that there exists $r \in \{0, 1, \ldots, n\}$ satisfying:

- $\beta_i = \beta'_i$, $i = 1, \ldots, r$,
- $\eta(\beta_i) < \eta(\beta'_i)$, $i = r+1, \ldots, n$,
- $\{\beta_{r+1}, \ldots, \beta_n\} \cap \{\beta'_{r+1}, \ldots, \beta'_n\} = \emptyset$.

In fact, $\{\beta_1, \ldots, \beta_r\} = [\boldsymbol{\lambda}] \cap [\boldsymbol{\lambda}']$. Since $\boldsymbol{\lambda}, \boldsymbol{\lambda}'$ are ℓ-partitions of n, hence $[\boldsymbol{\lambda}] \cap [\boldsymbol{\lambda}']$ is a Young diagram of some ℓ-partition $\boldsymbol{\mu}$ of r. If $r = n$, then $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$. Now we assume $r < n$, then

$$[\boldsymbol{\lambda}] = [\boldsymbol{\mu}] \cup \{\beta_{r+1}, \ldots, \beta_n\}, \quad [\boldsymbol{\lambda}'] = [\boldsymbol{\mu}] \cup \{\beta'_{r+1}, \ldots, \beta'_n\},$$

and

$$\eta(\beta_i) < \eta(\beta'_i), \quad \forall r+1 \leq i \leq n. \tag{4.9}$$

By reordering $\beta_{r+1}, \ldots, \beta_n$, we obtain a sequence $b_1, b_2, \ldots, b_{n-r}$ satisfying

$$\eta(b_1) \leq \eta(b_2) \leq \cdots \leq \eta(b_{n-r}).$$

Similarly, by reordering $\beta'_{r+1}, \ldots, \beta'_n$, we obtain a sequence $b'_1, b'_2, \ldots, b'_{n-r}$ satisfying

$$\eta(b'_1) \leq \eta(b'_2) \leq \cdots \leq \eta(b'_{n-r}).$$

Note that if $\alpha = (r, c, m)$ and $\alpha' = (r, c', m)$ lie in the same row of $[\boldsymbol{\lambda}]$, then

$$\eta(\alpha) < \eta(\alpha') \text{ if and only if } c < c'.$$

We can add the nodes $b_1, b_2, \ldots, b_{n-r}$ to $[\boldsymbol{\mu}]$ in order and get a sequence of Young diagrams of ℓ-compositions

$$[\boldsymbol{\mu}] \cup \{b_1\}, \ [\boldsymbol{\mu}] \cup \{b_1, b_2\}, \ \ldots, \ [\boldsymbol{\mu}] \cup \{b_1, b_2, \ldots, b_{n-r}\}.$$

Similarly, we can add the nodes $b'_1, b'_2, \ldots, b'_{n-r}$ to $[\boldsymbol{\mu}]$ in order and get a sequence of Young diagrams of ℓ-compositions

$$[\boldsymbol{\mu}] \cup \{b'_1\}, \ [\boldsymbol{\mu}] \cup \{b'_1, b'_2\}, \ \ldots, \ [\boldsymbol{\mu}] \cup \{b'_1, b'_2, \ldots, b'_{n-r}\}.$$

By (4.9), we derive $\eta(b_i) < \eta(b'_i),\ i = 1, 2, \ldots, n-r$. In fact, for any $1 \le t \le n-r$, the original partners of the t primed nodes having the smallest η-values all have η-value smaller than $\eta(b'_t)$ by (4.9). Hence, at least t unprimed values are smaller than $\eta(b'_t)$, which gives $\eta(b_t) < \eta(b'_t)$. For each $1 \le i \le n-r$, let $\boldsymbol{\mu} \cup \{b_1, \ldots, b_i\}$ be the ℓ-composition corresponding to the Young diagram $[\boldsymbol{\mu}] \cup \{b_1, \ldots, b_i\}$, and let $\boldsymbol{\mu} \cup \{b'_1, \ldots, b'_i\}$ be the ℓ-composition corresponding to the Young diagram $[\boldsymbol{\mu}] \cup \{b'_1, \ldots, b'_i\}$. Apply Lemma 4.6 to the ℓ-composition $\boldsymbol{\mu}$ and nodes b_1, b'_1, and we derive $\kappa_{\mathbf{m}}(\boldsymbol{\mu} \cup \{b'_1\}) \rhd \kappa_{\mathbf{m}}(\boldsymbol{\mu} \cup \{b_1\})$. Suppose $\kappa_{\mathbf{m}}(\boldsymbol{\mu} \cup \{b'_1, \ldots, b'_i\}) \rhd \kappa_{\mathbf{m}}(\boldsymbol{\mu} \cup \{b_1, \ldots, b_i\})$. According to Lemma 4.6, we get $\kappa_{\mathbf{m}}(\boldsymbol{\mu} \cup \{b'_1, \ldots, b'_i, b'_{i+1}\}) \rhd \kappa_{\mathbf{m}}(\boldsymbol{\mu} \cup \{b_1, \ldots, b_i, b_{i+1}\})$. Finally, by induction, we derive $\kappa_{\mathbf{m}}(\boldsymbol{\mu} \cup \{b'_1, \ldots, b'_{n-r}\}) \rhd \kappa_{\mathbf{m}}(\boldsymbol{\mu} \cup \{b_1, \ldots, b_{n-r}\})$, that is, $\kappa_{\mathbf{m}}(\boldsymbol{\lambda}') \rhd \kappa_{\mathbf{m}}(\boldsymbol{\lambda})$. Therefore, by Proposition 3.9, we have $\boldsymbol{\lambda} >_{\mathbf{m}} \boldsymbol{\lambda}'$. □

Remark 4.10. *We remind the reader that the a-function $a_{\mathbf{m}}$ in this paper is the same as $a_{\mathbf{t}}$ in [5, 5.4]. The function η in this paper is the same as that in [5, 5.4]. The vector-valued function $\kappa_{\mathbf{m}}$ is the same as $\kappa_{\mathbf{t}}$ in [5, 5.4]. The function* **ct** *in this paper is the same as ϑ in [5, 5.4], hence the order $>_\sigma$ on nodes given in Definition 4.1 in this paper is the same as $\succ$ in [5, 5.6]. Therefore, the relation given in the hypothesis of [5, Proposition 5.6] is exactly the definition of Lusztig's $\boldsymbol{a}_\sigma$-ordering in this paper. In [5, Proposition 5.6], M. Chlouveraki, I. Gordon, and S. Griffeth claim the following conclusion: Let $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \mathcal{P}^\ell_n$, if there exist ordering of $[\boldsymbol{\lambda}]$*

$$\gamma_1, \gamma_2, \ldots, \gamma_n,$$

and ordering of $[\boldsymbol{\lambda}']$

$$\gamma'_1, \gamma'_2, \ldots, \gamma'_n,$$

such that

$$\gamma'_i >_\sigma \gamma_i \text{ or } \gamma_i = \gamma'_i \quad \forall i = 1, \ldots, n.$$

Then, $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$ or $a_{\mathbf{m}}(\boldsymbol{\lambda}) > a_{\mathbf{m}}(\boldsymbol{\lambda}')$, that is, if $\boldsymbol{\lambda}' \ge_\sigma \boldsymbol{\lambda}$, then $\boldsymbol{\lambda} \ge_{\mathbf{m}} \boldsymbol{\lambda}'$. We point out that [5, Proposition 5.6] is incorrect. In fact, the first sentence in the proof of [5, Proposition 5.6] is, "For all $1 \le i \le n$, we have $\eta(\gamma_i) < \eta(\gamma'_i)$, unless $\gamma_i = \gamma'_i$,..." is false. The following example gives a counterexample. Moreover, besides Theorem 4.8 which can be regarded as a revised version of [5, Proposition 5.6], we can also add the condition that σ is asymptotic to make [5, Proposition 5.6] hold; see Corollary 5.6.

Example 4.11. *Suppose $e = 4$, $\ell = 2$, $(\sigma_1, \sigma_2) = (4, 5)$, $\mathbf{u} = (u_1, u_2) = (1, 4)$, $0 < u_2 - u_1 < 4$. Let $m_i = \sigma_i - u_i$, then $\mathbf{m} = (m_1, m_2) = (3, 1)$. For $\boldsymbol{\lambda} = ((2), (2))$ and $\boldsymbol{\mu} = ((1), (3))$, the Young diagram is*

$$[\boldsymbol{\lambda}] = (\ydiagram{2}\,\alpha,\ \ydiagram{2}) \qquad [\boldsymbol{\mu}] = (\ydiagram{1},\ \ydiagram{3}\,\beta),$$

where $\alpha = (1, 2, 1) \in [\boldsymbol{\lambda}]$ and $\beta = (1, 3, 2) \in [\boldsymbol{\mu}]$. We have

$$\mathbf{ct}(\alpha) = 2 - 1 + 4 = 5 < \mathbf{ct}(\beta) = 3 - 1 + 5 = 7.$$

Hence, $\beta >_\sigma \alpha$, but

$$\eta(\alpha) = \mathbf{ct}(\alpha) - 1 = 4 > \eta(\beta) = \mathbf{ct}(\beta) - 4 = 3.$$

This gives a counterexample of the sentence, "For all $1 \le i \le n$, we have $\eta(\gamma_i) < \eta(\gamma'_i)$, unless $\gamma_i = \gamma'_i$,..." in the proof of [5, Proposition 5.6]. Moreover, since $\boldsymbol{\lambda} \setminus \alpha = \boldsymbol{\mu} \setminus \beta$, we have $\boldsymbol{\mu} \succ_\sigma \boldsymbol{\lambda}$, that is, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ satisfy all the hypotheses of [5, Proposition 5.6]. Next let's compute $a_{\mathbf{m}}(\boldsymbol{\lambda})$ and $a_{\mathbf{m}}(\boldsymbol{\mu})$. Let $s = 4$. For $\boldsymbol{\lambda} = ((2), (2))$, we have

$$p_{\lambda^{(1)}} = (2), \quad p_{\lambda^{(2)}} = (2).$$

Then

$$B_1(\boldsymbol{\lambda}) = p_{\lambda^{(1)}}(s - h_1 + m_1) = (0, 1, 2, 3, 4, 5, 8),$$
$$B_2(\boldsymbol{\lambda}) = p_{\lambda^{(2)}}(s - h_2 + m_2) = (0, 1, 2, 3, 6).$$

So we have $\kappa_{\mathbf{m}}(\boldsymbol{\lambda}) = (8, 6, 5, 4, 3, 3, 2, 2, 1, 1, 0, 0)$*, and hence* $n_{\mathbf{m}}(\boldsymbol{\lambda}) = 98$*. For* $\boldsymbol{\mu} = ((1), (3))$*, we have*

$$p_{\mu^{(1)}} = (1), \quad p_{\mu^{(2)}} = (3).$$

Then

$$B_1(\boldsymbol{\mu}) = p_{\mu^{(1)}}(s - h_1 + m_1) = (0, 1, 2, 3, 4, 5, 7),$$
$$B_2(\boldsymbol{\mu}) = p_{\mu^{(2)}}(s - h_2 + m_2) = (0, 1, 2, 3, 7).$$

So, we have $\kappa_{\mathbf{m}}(\boldsymbol{\mu}) = (7, 7, 5, 4, 3, 3, 2, 2, 1, 1, 0, 0)$ *and, hence,* $n_{\mathbf{m}}(\boldsymbol{\mu}) = 99$*. For the empty multipartition* $\underline{\emptyset} = (\emptyset, \emptyset)$*, we have*

$$p_{\underline{\emptyset}^{(1)}} = \emptyset, \quad p_{\underline{\emptyset}^{(2)}} = \emptyset.$$

Then

$$B_1(\underline{\emptyset}) = p_{\underline{\emptyset}^{(1)}}(s - h_1 + m_1) = (0, 1, 2, 3, 4, 5, 6),$$
$$B_2(\underline{\emptyset}) = p_{\underline{\emptyset}^{(2)}}(s - h_2 + m_2) = (0, 1, 2, 3, 4).$$

So, we have $\kappa_{\mathbf{m}}(\underline{\emptyset}) = (6, 5, 4, 4, 3, 3, 2, 2, 1, 1, 0, 0)$ *and, hence,* $n_{\mathbf{m}}(\underline{\emptyset}) = 95$*. Finally,*

$$a_{\mathbf{m}}(\boldsymbol{\lambda}) = n_{\mathbf{m}}(\boldsymbol{\lambda}) - n_{\mathbf{m}}(\underline{\emptyset}) = 98 - 95 = 3,$$
$$a_{\mathbf{m}}(\boldsymbol{\mu}) = n_{\mathbf{m}}(\boldsymbol{\mu}) - n_{\mathbf{m}}(\underline{\emptyset}) = 99 - 95 = 4.$$

Therefore, $a_{\mathbf{m}}(\boldsymbol{\mu}) > a_{\mathbf{m}}(\boldsymbol{\lambda})$*, that is,* $\boldsymbol{\mu} >_{\mathbf{m}} \boldsymbol{\lambda}$*. This gives a counterexample of [5, Proposition 5.6].*

Remark 4.12. *The authors of [5] proved [5, Theorem 5.7] using [5, Proposition 5.6], since we have shown [5, Proposition 5.6] is incorrect and we presented a corrected version of it, which is Proposition 4.8 in this paper. It remains to check whether the use of [5, Proposition 5.6] in the proof of [5, Theorem 5.7] can be replaced by Proposition 4.8 of our paper. For this purpose, for the sequences* $\gamma_1, \gamma_2, \ldots, \gamma_n$ *and* $\gamma'_1, \gamma'_2, \ldots, \gamma'_n$ *appearing in the last paragraph of the proof of [5, Theorem 5.7], one must verify whether* $\mathrm{res}(\gamma_i) = \mathrm{res}(\gamma'_i)$ *holds for each i. This is exactly what I cannot determine at the moment.*

5. Relationship between Lusztig's $\mathbf{a}_\sigma$-ordering, Webster's ordering, and weak θ-dominance order, θ-dominance order, dominance order

In this section, we fix $\sigma = (e; \sigma_1, \sigma_2, \ldots, \sigma_\ell) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ and $\theta = (\theta_1, \ldots, \theta_\ell) \in \mathbb{Z}^\ell$, where $\theta_l = \ell(-\sigma_l + \frac{l}{\ell})$ for each $1 \leq l \leq \ell$. It is clear $\ell \nmid (\theta_i - \theta_j)$ for all $1 \leq i \neq j \leq \ell$.

Proposition 5.1. *Let* $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}^\ell_n$*, then* $\boldsymbol{\mu} \geq_\sigma \boldsymbol{\lambda}$ *if, and only if,* $\boldsymbol{\mu} \geq_\theta \boldsymbol{\lambda}$*.*

Proof. Let's prove the necessity. Suppose $\boldsymbol{\mu} \geq_\sigma \boldsymbol{\lambda}$. By the definition of σ-dominance order, there exists bijection $A : \boldsymbol{\mu} \longrightarrow \boldsymbol{\lambda}$ such that

$$\alpha >_\sigma A(\alpha) \text{ or } A(\alpha) = \alpha \quad \forall \alpha \in \boldsymbol{\mu}.$$

For $(r, c, l) \in \boldsymbol{\mu}$ with $(r, c, l) >_\sigma A(r, c, l)$, we write $A(r, c, l) = (r', c', l')$.

If $\mathbf{ct}(r, c, l) > \mathbf{ct}(r', c', l')$, then

$$\begin{aligned}\theta_{l'} + \ell(r' - c') - (\theta_l + \ell(r - c)) &= \ell(-\sigma_{l'} + \sigma_l + \tfrac{l'-l}{\ell}) + \ell(r' - r + c - c') \\ &= \ell(\sigma_l + c - r) - \ell(\sigma_{l'} + c' - r') + (l' - l) \\ &= \ell(\mathbf{ct}(r, c, l) - \mathbf{ct}(r', c', l')) + (l' - l) > 0,\end{aligned}$$

and the last inequality follows from $|l' - l| < \ell$. Hence, $(r, c, l) >_\theta A(r, c, l)$.

If $\mathbf{ct}(r, c, l) = \mathbf{ct}(r', c', l')$ and $l' > l$, then

$$\begin{aligned}\theta_{l'} + \ell(r' - c') - (\theta_l + \ell(r - c)) &= \ell(\mathbf{ct}(r, c, l) - \mathbf{ct}(r', c', l')) + (l' - l) \\ &= l' - l > 0.\end{aligned}$$

Hence, $(r, c, l) >_\theta A(r, c, l)$.

Therefore, we have $A^{-1}(\beta) \geq_\theta \beta$ for each $\beta \in \boldsymbol{\lambda}$. So we derive

$$|\{\beta \in \boldsymbol{\lambda} | \beta >_\theta \gamma\}| \leq |\{\alpha \in \boldsymbol{\mu} | \alpha >_\theta \gamma\}| \quad \forall \gamma \in \mathcal{N}_n^\ell.$$

Hence, $\boldsymbol{\mu} \geq_\theta \boldsymbol{\lambda}$.

Next let's prove the sufficiency. Suppose $\boldsymbol{\mu} \geq_\theta \boldsymbol{\lambda}$. According to the counting definition of weak θ-dominance order, we have $\boldsymbol{\mu} \setminus \boldsymbol{\lambda} \geq_\theta \boldsymbol{\lambda} \setminus \boldsymbol{\mu}$. Let $k := |\boldsymbol{\lambda} \setminus \boldsymbol{\mu}| = |\boldsymbol{\mu} \setminus \boldsymbol{\lambda}|$. Since $(\mathcal{N}_n^\ell, \geq_\theta)$ is a totally ordered set, we can give an ordering of the nodes in $\boldsymbol{\mu} \setminus \boldsymbol{\lambda}$,

$$\gamma_1 >_\theta \gamma_2 >_\theta \cdots >_\theta \gamma_k.$$

Similarly, we can give an ordering of the nodes in $\boldsymbol{\lambda} \setminus \boldsymbol{\mu}$,

$$\gamma'_1 >_\theta \gamma'_2 >_\theta \cdots >_\theta \gamma'_k.$$

We claim $\gamma_i >_\theta \gamma'_i$ for each $1 \leq i \leq k$. Otherwise, if there exists i such that $\gamma'_i >_\theta \gamma_i$, then

$$|\{\beta \in \boldsymbol{\lambda} \setminus \boldsymbol{\mu} | \beta >_\theta \gamma_i\}| \geq i > i - 1 = |\{\alpha \in \boldsymbol{\mu} \setminus \boldsymbol{\lambda} | \alpha >_\theta \gamma_i\}|.$$

This contradicts $\boldsymbol{\mu} \setminus \boldsymbol{\lambda} \geq_\theta \boldsymbol{\lambda} \setminus \boldsymbol{\mu}$.

Now let's prove $\gamma_i >_\sigma \gamma'_i$. Suppose $\gamma_i = (r_i, c_i, l_i)$, $\gamma'_i = (r'_i, c'_i, l'_i)$. Since $\gamma_i \in \boldsymbol{\mu} \setminus \boldsymbol{\lambda}$ and $\gamma'_i \in \boldsymbol{\lambda} \setminus \boldsymbol{\mu}$, γ_i and γ'_i lie in different diagonals. In fact, on each fixed diagonal, the nodes of $\boldsymbol{\mu}$ and of $\boldsymbol{\lambda}$ each form an initial segment; these two initial segments are nested. Consequently, $\boldsymbol{\mu} \setminus \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \setminus \boldsymbol{\mu}$ cannot both contain a node on the same diagonal. Since $\gamma_i >_\theta \gamma'_i$, so

$$\theta_{l'_i} + \ell(r'_i - c'_i) - (\theta_{l_i} + \ell(r_i - c_i)) = \ell(\mathbf{ct}(\gamma_i) - \mathbf{ct}(\gamma'_i)) + (l'_i - l_i) > 0.$$

Moreover, $|l'_i - l_i| < \ell$, so $\mathbf{ct}(\gamma_i) > \mathbf{ct}(\gamma'_i)$ or $\mathbf{ct}(\gamma_i) = \mathbf{ct}(\gamma'_i)$ and $l'_i > l_i$, therefore $\gamma_i >_\sigma \gamma'_i$. Finally, define the bijection $A : \boldsymbol{\mu} \longrightarrow \boldsymbol{\lambda}$ such that

$$\begin{aligned}A(\alpha) &= \alpha, \quad \forall \alpha \in \boldsymbol{\lambda} \cap \boldsymbol{\mu}, \\ A(\gamma_i) &= \gamma'_i, \quad i = 1, \ldots, k.\end{aligned}$$

Because $\gamma_i >_\sigma \gamma'_i = A(\gamma_i)$, so $\boldsymbol{\mu} \geq_\sigma \boldsymbol{\lambda}$. □

Theorem 5.2. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$. Then $\boldsymbol{\mu} \trianglerighteq_\sigma \boldsymbol{\lambda}$ if, and only if, $\boldsymbol{\mu} \trianglerighteq_\theta \boldsymbol{\lambda}$.*

Proof. The proof of necessity is analogous to that of Proposition 5.1. Suppose $\boldsymbol{\mu} \trianglerighteq_\sigma \boldsymbol{\lambda}$. By the definition of Webster order, there exists bijection $A : \boldsymbol{\mu} \longrightarrow \boldsymbol{\lambda}$ such that

$$\alpha \rhd_\sigma A(\alpha) \text{ or } A(\alpha) = \alpha \quad \forall \alpha \in \boldsymbol{\mu}.$$

For $(r, c, l) \in \boldsymbol{\mu}$ with $(r, c, l) \rhd_\sigma A(r, c, l)$, we write $A(r, c, l) = (r', c', l')$.

If $\mathbf{ct}(r, c, l) > \mathbf{ct}(r', c', l')$, by the same calculations as in Proposition 5.1, we derive $(r, c, l) \rhd_\theta A(r, c, l)$.

If $\mathbf{ct}(r, c, l) = \mathbf{ct}(r', c', l')$ and $l' > l$, by the same calculations as in Proposition 5.1, we derive $(r, c, l) \rhd_\theta A(r, c, l)$.

Therefore, we have $A^{-1}(\beta) \trianglerighteq_\theta \beta$ for each $\beta \in \boldsymbol{\lambda}$, so we derive

$$|\{\beta \in \boldsymbol{\lambda} | \beta \rhd_\theta \gamma\}| \leq |\{\alpha \in \boldsymbol{\mu} | \alpha \rhd_\theta \gamma\}| \quad \forall \gamma \in \mathcal{N}_n^\ell.$$

Hence, $\boldsymbol{\mu} \trianglerighteq_\theta \boldsymbol{\lambda}$.

Next let's prove the sufficiency. According to Proposition 3.16, we have $\mathrm{res}(\boldsymbol{\mu}) = \mathrm{res}(\boldsymbol{\lambda})$. Now we fix a residue $i \in \mathrm{res}(\boldsymbol{\mu}) = \mathrm{res}(\boldsymbol{\lambda})$, and we denote the set of i-nodes of $\boldsymbol{\mu}$ by $\boldsymbol{\mu}[i]$ and that of $\boldsymbol{\lambda}$ by $\boldsymbol{\lambda}[i]$. According to the counting definition of θ-dominance order, we have $\boldsymbol{\mu} \setminus \boldsymbol{\lambda} \trianglerighteq_\theta \boldsymbol{\lambda} \setminus \boldsymbol{\mu}$. It is clear the set of i-nodes of $\boldsymbol{\mu} \setminus \boldsymbol{\lambda}$ is $\boldsymbol{\mu}[i] \setminus \boldsymbol{\lambda}[i]$ and that of $\boldsymbol{\lambda} \setminus \boldsymbol{\mu}$ is $\boldsymbol{\lambda}[i] \setminus \boldsymbol{\mu}[i]$. Let $k := |\boldsymbol{\lambda}[i] \setminus \boldsymbol{\mu}[i]| = |\boldsymbol{\mu}[i] \setminus \boldsymbol{\lambda}[i]|$. Since $(\mathcal{N}_n^\ell, \geq_\theta)$ is a totally ordered set, its restriction to a fixed residue subset of $\mathcal{N}_n^\ell$ is also a totally ordered set, denoted by $\trianglerighteq_\theta$. So, we can give an ordering of the nodes in $\boldsymbol{\mu}[i] \setminus \boldsymbol{\lambda}[i]$,

$$\gamma_1 \trianglerighteq_\theta \gamma_2 \trianglerighteq_\theta \cdots \trianglerighteq_\theta \gamma_k.$$

Similarly, we can give an ordering of the nodes in $\boldsymbol{\lambda}[i] \setminus \boldsymbol{\mu}[i]$,

$$\gamma_1' \trianglerighteq_\theta \gamma_2' \trianglerighteq_\theta \cdots \trianglerighteq_\theta \gamma_k'.$$

By a discussion similar to that of Proposition 5.1, we can prove $\gamma_j \trianglerighteq_\sigma \gamma_j'$ for each $1 \leq j \leq k$. Finally, define the bijection $A : \boldsymbol{\mu} \longrightarrow \boldsymbol{\lambda}$ such that

$$\begin{aligned} A(\alpha) &= \alpha, \quad \forall \alpha \in \boldsymbol{\lambda} \cap \boldsymbol{\mu}, \\ A(\gamma_j) &= \gamma_j', \quad j = 1, \ldots, k. \end{aligned}$$

where i runs over all possible residues of the nodes of $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$. Because $\gamma_j \trianglerighteq_\sigma \gamma_j' = A(\gamma_j)$, $\boldsymbol{\mu} \trianglerighteq_\sigma \boldsymbol{\lambda}$. □

If $\sigma_i - \sigma_{i+1} \geq n,\ i = 1, \ldots, \ell - 1$, we call σ asymptotic.

Corollary 5.3. *Suppose $\sigma = (e; \sigma_1, \ldots, \sigma_\ell)$ is asymptotic, $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$. Then $\boldsymbol{\mu} \geq_\sigma \boldsymbol{\lambda}$ if, and only if, $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$.*

Proof. Since σ is asymptotic, $\theta_{i+1} - \theta_i = \ell(\sigma_i - \sigma_{i+1}) + 1 > \ell n$, that is, θ is separated. Therefore, the conclusion follows from [9, Theorem 4.1] and Proposition 5.1. □

By Theorem 5.2, Proposition 3.16, and [9, Theorem 4.1], we obtain the following corollary.

Corollary 5.4. *Suppose $\sigma = (e; \sigma_1, \ldots, \sigma_\ell)$ is asymptotic, $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$. If $\boldsymbol{\mu} \trianglerighteq_\sigma \boldsymbol{\lambda}$, then $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$ and* $\mathrm{res}(\boldsymbol{\lambda}) = \mathrm{res}(\boldsymbol{\mu})$.

The inverse of Corollary 5.4 is not true. We give a counterexample as follows.

Example 5.5. *Let $\ell = 2$, $n = 6$, $\sigma = (e; \sigma_1, \sigma_2) = (2; 14, 1)$. Let $\boldsymbol{\lambda} = ((2,1),(2,1))$, $\boldsymbol{\mu} = ((3),(3))$, and the Young diagrams with residue are*

$$\boldsymbol{\mu} = \Big(\boxed{0}\boxed{1}\boxed{0},\ \boxed{1}\boxed{0}\boxed{1} \Big), \qquad \boldsymbol{\lambda} = \Big(\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 \\ \cline{1-1} \end{array},\ \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 \\ \cline{1-1} \end{array} \Big).$$

It is clear that $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}$ and $\mathrm{res}(\boldsymbol{\lambda}) = \mathrm{res}(\boldsymbol{\mu})$. The 1-nodes in $\boldsymbol{\mu}$ are

$$\{\alpha_1 = (1,2,1), \alpha_2 = (1,3,2), \alpha_3 = (1,1,2)\},$$

and

$$\mathbf{ct}(\alpha_1) = 15 \quad \mathbf{ct}(\alpha_2) = 3 \quad \mathbf{ct}(\alpha_3) = 1.$$

The 1-nodes in $\boldsymbol{\lambda}$ are

$$\{\alpha_1' = (1,2,1), \alpha_2' = (2,1,1), \alpha_3' = (1,1,2)\},$$

and

$$\mathbf{ct}(\alpha_1') = 15 \quad , \mathbf{ct}(\alpha_2') = 13 \quad \mathbf{ct}(\alpha_3') = 1.$$

So,

$$\alpha_1 = \alpha_1' \rhd_\sigma \alpha_2' \rhd_\sigma \alpha_2 \rhd_\sigma \alpha_3 = \alpha_3'.$$

Suppose $A : \boldsymbol{\mu} \longrightarrow \boldsymbol{\lambda}$ is a residue-preserving bijection such that $\alpha \rhd_\sigma A(\alpha)$ or $A(\alpha) = \alpha$, $\forall \alpha \in \boldsymbol{\mu}$. Then, $A(\alpha_3) = \alpha_3'$, hence $A(\alpha_2) = \alpha_1'$ or $A(\alpha_2) = \alpha_2'$, which contradicts to the assumption of A. Hence, $\boldsymbol{\mu} \not\rhd_\sigma \boldsymbol{\lambda}$.

Recall the choice of $\mathbf{m}$ in (4.4), and fix $\mathbf{u} = (u_1, \ldots, u_\ell) \in \mathbb{Q}^\ell$ with $0 < u_j - u_i < e$, $\forall i < j$. Let $\mathbf{m} = (m_1, \ldots, m_\ell) \in \mathbb{Q}^\ell$, where $m_i = \sigma_i - u_i$ for all $1 \le i \le \ell$. Suppose that σ is asymptotic, then

$$m_i - m_{i+1} = (\sigma_i - \sigma_{i+1}) + (u_{i+1} - u_i) > n > n - 1 \quad \forall 1 \le i < \ell.$$

By Theorem 3.11 and Corollary 5.3, we give the relationship between Lusztig's $\mathbf{a}_\sigma$-ordering and a-function order.

Corollary 5.6. *Suppose $\sigma = (e; \sigma_1, \ldots, \sigma_\ell)$ is asymptotic, $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$. If $\boldsymbol{\mu} \succeq_\sigma \boldsymbol{\lambda}$, then $\boldsymbol{\lambda} \ge_{\mathbf{m}} \boldsymbol{\mu}$.*

6. Main results

Now we summarize the main results of this paper in the following theorem. More explicitly, in the following theorem, we summarize Theorem 3.11 as (a), Corollary 3.17 as (b), Theorem 4.8 and Corollary 5.6 as (c), Proposition 5.1 and Theorem 5.2 as (d), and Corollary 5.3 and Corollary 5.4 as (e). Let $\mathbf{m} = (m_1, \ldots, m_\ell) \in \mathbb{Q}^\ell$, $\sigma = (e; \sigma_1, \ldots, \sigma_\ell) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ and $\mathbf{u} = (u_1, \ldots, u_\ell) \in \mathbb{Q}^\ell$ with $0 < u_j - u_i < e$ for $i < j$. Let $\theta = (\theta_1, \ldots, \theta_\ell) \in \mathbb{Z}^\ell$ with $\theta_i - \theta_j \notin \ell\mathbb{Z}$ for $i < j$.

Theorem 6.1. *Suppose $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}_n^\ell$,*

- (a) *Suppose $\mathbf{m} \in \mathbb{Q}^\ell$ with $m_i - m_j > n - 1, \forall i < j$. If $\boldsymbol{\mu} \rhd \boldsymbol{\lambda}$, then $\boldsymbol{\lambda} >_{\mathbf{m}} \boldsymbol{\mu}$.*
- (b) *Suppose θ is separated and $\mathbf{m} \in \mathbb{Q}^\ell$ with $m_i = (-\theta_i + i)/\ell$. If $\boldsymbol{\mu} \trianglerighteq_\theta \boldsymbol{\lambda}$, then $\boldsymbol{\lambda} \ge_{\mathbf{m}} \boldsymbol{\mu}$.*
- (c) *Suppose $\mathbf{m} \in \mathbb{Q}^\ell$ with $m_i = \sigma_i - u_i$. If $\boldsymbol{\mu} \trianglerighteq_\sigma \boldsymbol{\lambda}$, then $\boldsymbol{\lambda} \ge_{\mathbf{m}} \boldsymbol{\mu}$. Moreover, suppose σ is asymptotic, if $\boldsymbol{\mu} \succeq_\sigma \boldsymbol{\lambda}$, then $\boldsymbol{\lambda} \ge_{\mathbf{m}} \boldsymbol{\mu}$.*
- (d) *Suppose $\theta \in \mathbb{Z}^\ell$ with $\theta_i = \ell(-\sigma_i + \frac{i}{\ell})$. Then $\boldsymbol{\mu} \succeq_\sigma \boldsymbol{\lambda}$ if, and only if, $\boldsymbol{\mu} \succeq_\theta \boldsymbol{\lambda}$. Moreover, $\boldsymbol{\mu} \trianglerighteq_\sigma \boldsymbol{\lambda}$ if, and only if, $\boldsymbol{\mu} \trianglerighteq_\theta \boldsymbol{\lambda}$.*
- (e) *Suppose σ is asymptotic. Then $\boldsymbol{\mu} \succeq_\sigma \boldsymbol{\lambda}$ if, and only if, $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$. Moreover, if $\boldsymbol{\mu} \trianglerighteq_\sigma \boldsymbol{\lambda}$, then $\boldsymbol{\mu} \trianglerighteq \boldsymbol{\lambda}$ and $\mathrm{res}(\boldsymbol{\lambda}) = \mathrm{res}(\boldsymbol{\mu})$.*

7. Conclusions

In this paper, if $\mathbf{m} = (m_1, \ldots, m_\ell) \in \mathbb{Q}^\ell$ with $m_i - m_j > n - 1, \forall i < j$, we proved that a-function order refines the opposite dominance order. If $\theta = (\theta_1, \ldots, \theta_\ell)$ is separated and $\mathbf{m} = (m_1, \ldots, m_\ell)$ with $m_i = (-\theta_i + i)/\ell$, we proved that a-function order refines the opposite θ-dominance order. If $\sigma = (e; \sigma_1, \ldots, \sigma_\ell) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$, $\mathbf{u} = (u_1, \ldots, u_\ell) \in \mathbb{Q}^\ell$ with $0 < u_j - u_i < e, \forall i < j$, and $\mathbf{m} = (m_1, \ldots, m_\ell)$ with $m_i = \sigma_i - u_i$, we proved that a-function order refines the opposite Webster's ordering. Moreover, if $\sigma = (e; \sigma_1, \sigma_2, \ldots, \sigma_\ell)$ is asymptotic, we proved that a-function order also refines the opposite Lusztig's $\mathbf{a}_\sigma$-ordering. If $\sigma = (e; \sigma_1, \sigma_2, \ldots, \sigma_\ell) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ and $\theta = (\theta_1, \ldots, \theta_\ell)$ with $\theta_i = \ell(-\sigma_i + \frac{i}{\ell})$, we proved that Lusztig's $\mathbf{a}_\sigma$-ordering is equivalent to weak θ-dominance order and Webster's ordering is equivalent to θ-dominance order. If $\sigma = (e; \sigma_1, \sigma_2, \ldots, \sigma_\ell) \in \mathbb{N}_{>1} \times \mathbb{Z}^\ell$ is asymptotic, we proved that Lusztig's $\mathbf{a}_\sigma$-ordering is equivalent to dominance order and dominance order refines Webster's ordering.

We hope that the main results of this paper may contribute to the study of the relationship among different parameterizations of simple modules of the cyclotomic Hecke algebras, which also constitutes one of our goals for future work.

Use of AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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