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*Theory article*

## **Embeddings of $\mathbf{B}$ -valued Lorentz-Karamata spaces associated with martingale transforms**

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**Abstract:** In this article, the embeddings of Banach space valued (briefly by  $\mathbf{B}$ -valued) Lorentz-Karamata martingale spaces are established. Different from the previous approach, we employ the technique of martingale transforms defined by some operator sequences, which can often supply a clear and succinct method in the research of martingale theory and Harmonic analysis. It's worth mentioning that the results obtained here are connected closely with the geometrical properties of the Banach spaces in which the martingales take values, such as  $p$ -uniform smoothness and  $q$ -uniform convexity.

**Keywords:** martingale transform; Lorentz-Karamata space; embedding;  $\mathbf{B}$ -valued martingale; uniform convexity; uniform smoothness

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### **1. Introduction**

In 1975, Pisier's notable work [1] opened the prelude to the study of  $\mathbf{B}$ -valued martingales, in which he gave a new proof for Enflo's theorem that every super-reflexive space admits an equivalent uniformly convex norm using martingale inequality. It revealed that the conclusions for classical martingale setting are not naturally valid in the  $\mathbf{B}$ -valued martingale situation. Whether these conclusions hold rely on some geometrical properties of the Banach spaces in which martingales take values, such as uniform smoothness, uniform convexity, and the Radon-Nikodym property. From then on, the research of  $\mathbf{B}$ -valued martingales flushed rapidly in the last several decades. It can be seen in [2–5] and the references therein.

Recently, Jiao et al. [6] characterized the predual of classical Lorentz martingale space by ingeniously introducing a new generalized martingale bounded mean oscillation ( $BMO$ ) space via the stopping time sequences. Then the results above were extended to the Lorentz-Karamata martingale spaces by them [7], which further improved the conclusions in Ho [8], where the duality for these martingale spaces was completely instructed, including the small index situation. Hao et al. [9, 10]

considered the Orlicz-Lorentz-Karamata spaces, martingale inequalities, and real interpolations, and their applications in Fourier analysis were studied.

For the  $\mathbf{B}$ -valued martingale setting, Yu [11–13] studied the duality of  $\mathbf{B}$ -valued martingale Hardy space and the (weak) Orlicz-Hardy martingale space. Inspired by [6], Liu et al. [14] extended [7] to the  $\mathbf{B}$ -valued martingale setting, in which atomic decompositions, duality, and martingale inequalities were obtained for the  $\mathbf{B}$ -valued martingale Hardy-Lorentz-Karamata spaces. Very recently, Hao et al. [15] discussed the dual space and inequalities to  $\mathbf{B}$ -valued martingale Hardy-Lorentz-Karamata spaces for the two-parameter setting.

As early as 1966, Burkholder [16] first introduced martingale transforms as a useful method, which have been extensively researched; see [17–19] and so on. In 2000, Martinez and Torrea [20] initiated the study of the operator-valued martingale transform. Then applying martingale transforms, Jiao et al. [21] obtained the embedding theorems for the  $\mathbf{B}$ -valued martingale Lorentz spaces. This study aims to establish the embeddings of  $\mathbf{B}$ -valued martingale Hardy-Lorentz-Karamata spaces also by martingale transforms, which can be regarded as a new generalization of classical martingale inequalities. Without exception, the main results obtained here are closely connected with the geometrical properties of the Banach spaces in which martingales take values.

In the whole paper, the set of integers and non-negative integers is denoted respectively by  $\mathbb{Z}$ ,  $\mathbb{N}$ . The symbol  $\hookrightarrow$  means the continuous embedding, and  $C$  stands for a positive constant that may change from line to line.

## 2. Preliminaries

We first give the definitions of the Lorentz-Karamata space associated with the slowly varying functions, and some properties for these spaces are presented. The reader is referred to [22, 23] for more details on these function spaces.

**Definition 2.1.** ([7]) A Lebesgue measurable function  $b : [1, \infty) \rightarrow (0, \infty)$  is called a slowly varying function if, for any given  $\epsilon > 0$ , the function  $t^\epsilon b(t)$  is equivalent to a non-decreasing function, and the function  $t^{-\epsilon} b(t)$  is equivalent to a non-increasing function on  $[1, \infty)$ .

Let  $b$  be a slowly varying function on  $[1, \infty)$ , and define

$$\gamma_b(t) = b(\max(t, t^{-1})), \quad t > 0.$$

**Remark 2.2.** (i) Obviously, if  $b$  is a non-decreasing function, then  $\gamma_b$  is non-increasing on  $(0, 1]$ .

(ii) For any given  $\epsilon > 0$ , the function  $t^\epsilon \gamma_b(t)$  is equivalent to a non-decreasing function, and the function  $t^{-\epsilon} \gamma_b(t)$  is equivalent to a non-increasing function on  $[1, \infty)$ .

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space and  $\mathbf{B}$  denote a Banach space with norm  $\|\cdot\|$ . For any measurable function  $f$  defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  with values in  $\mathbf{B}$ , denote its distribution function

$$\lambda_f(s) = \mathbb{P}\{x \in \Omega : \|f(x)\| > s\}, \quad s \geq 0$$

and the decreasing rearrangement function

$$f^*(t) = \inf\{s \geq 0 : \lambda_f(s) \leq t\}, \quad t \geq 0, \quad \inf \emptyset = \infty.$$

**Definition 2.3.** ([7]) Let  $0 < p \leq \infty$ ,  $0 < q \leq \infty$ , and  $b$  be a slowly varying function. The Lorentz-Karamata space  $L_{p,q,b}$  consists of all measurable functions  $f$  satisfying  $\|f\|_{p,q,b} < \infty$ , where

$$\|f\|_{p,q,b} = \left( \int_0^1 (t^{1/p} \gamma_b(t) f^*(t))^q \frac{dt}{t} \right)^{1/q}, \quad 0 < q < \infty,$$

$$\|f\|_{p,q,b} = \sup_{0 \leq t \leq 1} t^{1/p} \gamma_b(t) f^*(t), \quad q = \infty.$$

**Remark 2.4.** (i) The Lorentz-Karamata spaces  $L_{p,q,b}$  are rearrangement-invariant (r.-i.) quasi-Banach function spaces. If  $1 \leq p, q < \infty$ ,  $L_{p,q,b}$  is a Banach space. If we set  $b \equiv 1$ , the Lorentz-Karamata spaces  $L_{p,q,b}$  become the usual Lorentz spaces  $L_{p,q}$ .

(ii) The quasi-norm  $\|\cdot\|_{p,q,b}$  also has another equivalent characterization as follows:

$$\|f\|_{p,q,b} \approx \left( \int_0^\infty (\lambda_f(s)^{1/p} \gamma_b(\lambda_f(s)) s)^q \frac{ds}{s} \right)^{1/q}, \quad 0 < q < \infty,$$

$$\|f\|_{p,q,b} \approx \sup_{s>0} \lambda_f(s)^{1/p} \gamma_b(\lambda_f(s)) s, \quad q = \infty.$$

Let  $\{\mathcal{F}_n\}_{n \geq 0}$  be a nondecreasing sub- $\sigma$ -algebra sequence of  $\mathcal{F}$  with  $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$ . The expectation operator and the conditional expectation operator relative to  $\mathcal{F}_n$  are denoted by  $\mathbb{E}$  and  $\mathbb{E}_n$ , respectively.  $\mathbf{B}$ -valued random variables sequence  $f = (f_n)_{n \geq 0}$  is called a  $\mathbf{B}$ -valued martingale if  $\mathbb{E}_n(f_{n+1}) = f_n$  for arbitrary  $n \geq 0$  (with convention  $f_0 = 0$ ). For a  $\mathbf{B}$ -valued martingale  $f = (f_n)_{n \geq 0}$  and a stopping time  $\tau \in \mathcal{T}$ , we denote the stopped martingale by  $f^\tau = (f_{n \wedge \tau})_{n \geq 0}$ . Denote  $df_n := f_n - f_{n-1}$  ( $n \geq 1$ ) as the martingale difference of  $f$ . The maximal function and  $q$ -variation of  $f$  are defined as follows:

$$M_n(f) = \sup_{1 \leq k \leq n} \|f_k\|, \quad M(f) = \sup_{n \geq 1} \|f_n\|,$$

$$S_n^{(q)}(f) = \left( \sum_{k=1}^n \|df_k\|^q \right)^{1/q}, \quad S^{(q)}(f) = \left( \sum_{k=1}^\infty \|df_k\|^q \right)^{1/q}, \quad 1 \leq q < \infty.$$

For convenience, we use the symbol  $M_n(df)$  to denote  $\sup_{1 \leq k \leq n} \|df_k\|$  in the sequel (see [24] Remark 2.4).

**Definition 2.5.** Let  $0 < r < \infty$ ,  $0 < s \leq \infty$ ,  $1 \leq q < \infty$ ,  $b$  be a slowly varying function and,  $\mathbf{B}$  be a Banach space, and define the  $\mathbf{B}$ -valued martingale Hardy-Lorentz-Karamata spaces as follows:

$$H_{r,s,b}^*(\mathbf{B}) = \left\{ f = (f_n)_{n \geq 0} : \|f\|_{H_{r,s,b}^*(\mathbf{B})} := \|M(f)\|_{r,s,b} < \infty \right\},$$

$${}_q H_{r,s,b}^S(\mathbf{B}) = \left\{ f = (f_n)_{n \geq 0} : \|f\|_{{}_q H_{r,s,b}^S(\mathbf{B})} := \|S^{(q)}(f)\|_{r,s,b} < \infty \right\}.$$

**Definition 2.6.** ([24]) Let  $\mathbf{B}_1, \mathbf{B}_2$  be Banach spaces and the space of bounded linear operators from  $\mathbf{B}_1$  to  $\mathbf{B}_2$  denoted by  $\mathcal{L}(\mathbf{B}_1, \mathbf{B}_2)$ . Let  $v = \{v_n\}_{n \geq 1}$  be a sequence of  $\mathcal{L}(\mathbf{B}_1, \mathbf{B}_2)$ -valued mappings defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  such that  $v_n$  is  $\mathcal{F}_{n-1}$ -measurable, satisfying  $v_n \in L_\infty(\mathcal{L}(\mathbf{B}_1, \mathbf{B}_2))$  and  $\sup_{n \geq 1} \|v_n\|_{L_\infty(\mathcal{L}(\mathbf{B}_1, \mathbf{B}_2))} \leq 1$ . Given a  $\mathbf{B}_1$ -valued martingale  $f = (f_n)_{n \geq 0}$ , define a martingale operator  $T$  associated with the sequence  $v$  below.

$$(Tf)_n = \sum_{k=1}^n v_k df_k,$$

then  $Tf$  is called the martingale transform of  $f$  defined by the sequence  $v$ .

We now introduce the following lemma proved by Pisier [1, 4], which also can be seen in [3]. The lemma unveils the fact that there exists essential difference between the scalar-valued martingale and  $\mathbf{B}$ -valued martingale setting; in other words, the results for the  $\mathbf{B}$ -valued setting are closely relying on the geometrical properties of the underlying spaces, such as the uniform smoothness, uniform convexity, and so on.

**Lemma 2.7.** ([4]) (i) Let  $1 < p \leq 2$  and  $\mathbf{B}$  be a Banach space.  $\mathbf{B}$  is  $p$ -uniformly smooth if and only if there exists a constant  $C > 0$  such that for any  $\mathbf{B}$ -valued martingale  $f = (f_n)_{n \geq 0}$ ,

$$\sup_{n \geq 0} \mathbb{E} \|f_n\|^p \leq C \sum_{n=1}^{\infty} \mathbb{E} \|df_n\|^p.$$

(ii) Let  $2 \leq q < \infty$  and  $\mathbf{B}$  be a Banach space.  $\mathbf{B}$  is  $q$ -uniformly convex if and only if there exists a constant  $C > 0$  such that for any  $\mathbf{B}$ -valued martingale  $f = (f_n)_{n \geq 0}$ ,

$$C \sup_{n \geq 0} \mathbb{E} \|f_n\|^q \geq \sum_{n=1}^{\infty} \mathbb{E} \|df_n\|^q.$$

Applying Davis decomposition on a martingale, we give the following lemma, which can be regarded as a generalization of Lemma 2 in [21]. The reader can be referred to [3, 18, 25] for more information on the Davis decomposition.

**Lemma 2.8.** Let  $1 < p < \infty$ ,  $1 \leq q \leq \infty$ ,  $\mathbf{B}$  be a Banach space, and  $b$  be a slowly varying function. Then for each  $\mathbf{B}$ -valued martingale,  $f = (f_n)_{n \geq 0} \in H_{p,q,b}^*(\mathbf{B})$  can be decomposed as  $f = g + h$ , where  $g, h$  are martingales such that

- (i)  $\|dg_n\| \leq 4M_{n-1}(df)$ ,
- (ii)  $\left\| \sum_{n=1}^{\infty} \|dh_n\| \right\|_{p,q,b} \leq C \|M(f)\|_{p,q,b}$ .

*Proof.* Set a martingale  $f = (f_n)_{n \geq 1} \in H_{p,q,b}^*(\mathbf{B})$  and  $f_n = \sum_{k=1}^n df_k$ . Set  $\Delta'_n = df_n \chi_{\{\|df_n\| \leq 2M_{n-1}(df)\}}$ ,  $\Delta''_n = df_n \chi_{\{\|df_n\| > 2M_{n-1}(df)\}}$ , then

$$dg_0 = 0, \quad dg_n = \Delta'_n - \mathbb{E}_{n-1}(\Delta'_n), \quad g_n = \sum_{k=1}^n dg_k,$$

$$dh_0 = 0, \quad dh_n = \Delta''_n - \mathbb{E}_{n-1}(\Delta''_n), \quad h_n = \sum_{k=1}^n dh_k.$$

For each  $n \geq 1$ , we have  $\mathbb{E}_{n-1}(dg_n) = \mathbb{E}_{n-1}(dh_n) = 0$ , so  $g = (g_n)_{n \geq 0}$  and  $h = (h_n)_{n \geq 0}$  are martingales.

Since  $dg_n + dh_n = df_n - \mathbb{E}_{n-1}(df_n) = df_n$ , then  $f_n = g_n + h_n$ .

Furthermore,

$$\|dg_n\| \leq \|\Delta'_n\| + \mathbb{E}_{n-1}(\|\Delta'_n\|) \leq 4M_{n-1}(df).$$

Since on the set  $\{\|df_n\| > 2M_{n-1}(df)\}$ ,  $\|df_n\| - 2M_{n-1}(df) \geq 0$ , therefore

$$\|\Delta''_n\| \leq 2(\|df_n\| - M_{n-1}(df)) \chi_{\{\|df_n\| > 2M_{n-1}(df)\}} \leq 2(M_n(df) - M_{n-1}(df)),$$

$$\sum_{n=1}^{\infty} \|dh_n\| \leq 2M(df) + 2 \sum_{n=1}^{\infty} \mathbb{E}_{n-1}(M_n(df) - M_{n-1}(df)).$$

Using the norm definition of  $\|\cdot\|_{p,q,b}$ , we get

$$\begin{aligned} \left\| \sum_{n=1}^{\infty} \|dh_n\| \right\|_{p,q,b} &\leq 2\|M(df)\|_{p,q,b} + 2 \left\| \sum_{n=1}^{\infty} \mathbb{E}_{n-1}(M_n(df) - M_{n-1}(df)) \right\|_{p,q,b} \\ &\leq C\|M(df)\|_{p,q,b} \leq C\|M(f)\|_{p,q,b}. \end{aligned}$$

□

**Lemma 2.9.** Let  $\mathbf{B}_1, \mathbf{B}_2$  be Banach spaces,  $f = (f_n)_{n \geq 0}$  be a  $\mathbf{B}$ -valued martingale,  $b$  be a slowly varying function, and  $T$  be the martingale transform operator defined as above, then the following conditions are equivalent:

(i) For any  $\lambda > 0$ , exists a positive constant  $C$  satisfying  $\lambda \mathbb{P}\{M(Tf) > \lambda\} \leq C\|M(f)\|_1$ .

(ii) For arbitrary  $1 < p < \infty$ ,  $1 \leq q \leq \infty$ , exists a positive constant  $C$  depending on  $p, q, b$  such that  $\|M(Tf)\|_{p,q,b} \leq C\|M(f)\|_{p,q,b}$ .

*Proof.* (i)  $\Rightarrow$  (ii) Set a  $\mathbf{B}$ -valued martingale  $f = (f_n)_{n \geq 0} \in H_{p,q,b}^*(\mathbf{B})$ . According to Lemma 2.8,  $f$  can be written by  $f = g + h$ . Owing to the sublinearity of the operator  $M$ , we obtain  $M(Tf) \leq M(Tg) + M(Th)$ , then

$$\|M(Tf)\|_{p,q,b} \leq C(\|M(Tg)\|_{p,q,b} + \|M(Th)\|_{p,q,b}).$$

Because of the boundedness of the sequence  $\{v_k\}$ , we get

$$\|M(Th)\|_{p,q,b} \leq \left\| \sup_n \sum_{k=1}^n \|dh_k\|_{\mathbf{B}_1} \right\|_{p,q,b} = \left\| \sum_{k=1}^{\infty} \|dh_k\|_{\mathbf{B}_1} \right\|_{p,q,b} \leq C\|M(f)\|_{p,q,b}.$$

Let  $W = (W_n)$ , where  $W_n = 4M_{n-1}(df)$ , then the sequence  $(W_n)$  is non-decreasing, and for each  $n \geq 0$ ,  $W_n$  is  $\mathcal{F}_{n-1}$  measurable. For any positive number  $\lambda, \beta, \delta$ , define the following stopping times:

$$\mu = \inf \{n : \|(Tg)_n\|_{\mathbf{B}_2} > \lambda\}, \quad \nu = \inf \{n : \|(Tg)_n\|_{\mathbf{B}_2} > \beta\lambda\},$$

$$\sigma = \inf \{n : \max \{\|g_n\|_{\mathbf{B}_1}, W_{n+1}\} > \delta\lambda\} \quad (\inf \emptyset = \infty).$$

Set  $u_k = \chi_{A_k}$ , where  $A_k = \{\omega : \mu < k \leq \min(\nu, \sigma)\}$ , then  $A_k$  is  $\mathcal{F}_{k-1}$  measurable. Consider a martingale  $a = (a_n)_{n \geq 0}$ , where  $a_n = \sum_{k=1}^n u_k dg_k$ , its martingale transform being  $(Ta)_n = \sum_{k=1}^n v_k u_k dg_k$ . Observe that  $\|dg_n\| \leq W_n$ .

On the set  $\{\mu < \infty\}$ , set

$$u_n = \chi_{\{\mu < n \leq \min(\nu, \sigma)\}} = \begin{cases} 1, & \mu < n \leq \min(\nu, \sigma), \\ 0, & \text{others.} \end{cases}$$

Then there exists a  $k_0$  with  $1 \leq k_0 < n$  such that  $\omega \notin A_{k_0}$  and  $\omega \in A_{k_0+1}$ , thus

$$a_n = \sum_{k=k_0+1}^n dg_k = g_n - g_{k_0}.$$

By the definition of the stopping time  $\sigma$ , we get

$$M(a) \leq \|g_n\| + \|g_{k_0}\| \leq 2\delta\lambda.$$

On the set  $\{\mu = \infty\}$ , for each  $n$ , we have  $\{\mu < n \leq \min\{\nu, \sigma\}\} = \emptyset$ , thus  $u_n = 0$ . Immediately,  $a_n = 0$  for all  $n$ , so  $M(a) = \sup_n \|a_n\| = 0$ . Therefore,

$$\|M(a)\|_1 = \int_{\{\mu < \infty\}} M(a) d\mathbb{P} \leq \int_{\{\mu < \infty\}} 2\delta\lambda d\mathbb{P} = 2\delta\lambda \mathbb{P}\{M(Tg) > \lambda\}.$$

Choose appropriate numbers  $\beta > 0$  and  $\delta > 0$  such that  $\beta > \delta + 1$ . It follows from the condition (i) that

$$\mathbb{P}\{M(Ta) > (\beta - \delta - 1)\lambda\} \leq \frac{C\|M(a)\|_1}{(\beta - \delta - 1)\lambda} \leq \frac{2C\delta}{\beta - \delta - 1} \mathbb{P}\{M(Tg) > \lambda\}.$$

It's clear that if  $\omega \in \{\mu < n \leq \min(\nu, \sigma)\}$ , then  $(Ta)_n = (Tg)_n$ .

By the definitions of the stopping time  $\nu$  and  $\sigma$ , the set  $\{M(Tg) > \beta\lambda, M(W) \leq \delta\lambda\} = \{\nu < \infty, \sigma = \infty\}$ , and  $M(g) \leq \delta\lambda$ ,  $M(W) \leq \delta\lambda$ , for the  $k_0$  mentioned above, we have  $k_0 - 1 < \mu$ , therefore

$$\begin{aligned} M(Ta) &= \sum_{k=1}^n \|u_k(Tg)_k\| \\ &= \sum_{k=1}^n \|(Tg)_k\| - \|(Tg)_{k_0}\| - \sum_{k=1}^{k_0} \|(Tg)_k\| \geq (\beta - \delta - 1)\lambda. \end{aligned}$$

Hence,

$$\mathbb{P}\{M(Tg) > \beta\lambda, M(W) \leq \delta\lambda\} \leq \mathbb{P}\{M(Ta) > (\beta - \delta - 1)\lambda\}.$$

Therefore,

$$\begin{aligned} \mathbb{P}\{M(Tg) > \beta\lambda\} &\leq \mathbb{P}\{M(Tg) > \beta\lambda, M(W) \leq \delta\lambda\} + \mathbb{P}\{M(W) > \delta\lambda\} \\ &\leq \frac{2C\delta}{\beta - \delta - 1} \mathbb{P}\{M(Tg) > \lambda\} + \mathbb{P}\{M(W) > \delta\lambda\}. \end{aligned}$$

Let  $\rho = 2C\delta/(\beta - \delta - 1)$ . According to the definition of  $L_{p,q,b}$  norm, we obtain

$$\begin{aligned} \beta^{-1} \|M(Tg)\|_{p,q,b} &\leq \rho^{1/p} \|M(Tg)\|_{p,q,b} + \delta^{-1} \|M(W)\|_{p,q,b} \\ &\leq \rho^{1/p} \|M(Tg)\|_{p,q,b} + 8\delta^{-1} \|M(W)\|_{p,q,b}. \end{aligned}$$

Now choose a proper  $\delta$  such that  $1 - \beta\rho^{1/p} > 0$ , then  $\|M(Tg)\|_{p,q,b} \leq C\|M(f)\|_{p,q,b}$ . Combine the above derivation, and we get the desired conclusion (ii), namely

$$\|M(Tf)\|_{p,q,b} \leq C\|M(f)\|_{p,q,b}.$$

Since the Lorentz-Karamata spaces are rearrangement invariant spaces, the proof of (ii)  $\Rightarrow$  (i) is referred to Theorem 3.1 in [24]. Then the proof is complete.  $\square$

### 3. Embedding results of Lorentz-Karamata martingale spaces

In this section, the embedding results of  $\mathbf{B}$ -valued Lorentz-Karamata martingale spaces are established by the tool of martingale transform. In fact, we need to consider  $M(f)$  and  $S^{(q)}(f)$  as martingale transform operators under these circumstances.

**Theorem 3.1.** *Let  $\mathbf{B}$  be a Banach space and  $b$  be a slowly varying function. For  $2 \leq q < \infty$ , the following conditions are equivalent:*

- (i)  $\mathbf{B}$  is isomorphic to a  $q$ -uniformly convex space.
- (ii) There exists a positive constant  $C$  such that for every  $\mathbf{B}$ -valued martingale  $f = (f_n)_{n \geq 0}$ , we have

$$\lambda \mathbb{P}\{S^{(q)}(f) > \lambda\} \leq C \|M(f)\|_1, \quad \forall \lambda > 0.$$

- (iii) For any (or equivalently, some)  $r$  and  $s$ ,  $1 < r < \infty$ ,  $1 \leq s \leq \infty$ , and we have

$$H_{r,s,b}^* \hookrightarrow {}_q H_{r,s,b}^S.$$

*Proof.* (i)  $\Leftrightarrow$  (ii) First, we prove the sufficiency. Fix  $\lambda > 0$  and a small parameter  $\theta > 0$  to be chosen later. Define two stopping times (with convention  $\inf \emptyset = \infty$ ):

$$\tau_1 = \inf\{n \geq 0 : S_n^{(q)}(f) > \lambda\}, \quad \tau_2 = \inf\{n \geq 0 : \|f_n\| > \theta\lambda\}.$$

On the set  $\{S^{(q)}(f) > 2\lambda, M(f) \leq \theta\lambda\}$ , we have  $\tau_1 < \infty$  and  $\tau_2 = \infty$ . Since  $S_{\tau_1-1}^{(q)}(f) \leq \lambda$ , we get

$$S_{\tau_1}^{(q)}(f)^q = S_{\tau_1-1}^{(q)}(f)^q + \|df_{\tau_1}\|^q \leq \lambda^q + \|df_{\tau_1}\|^q.$$

On the set  $\{\tau_2 = \infty\}$ ,  $\|f_{\tau_1}\| \leq \theta\lambda$ ,  $\|f_{\tau_1-1}\| \leq \theta\lambda$ , so  $\|df_{\tau_1}\| = \|f_{\tau_1} - f_{\tau_1-1}\| \leq \|f_{\tau_1}\| + \|f_{\tau_1-1}\| \leq 2\theta\lambda$ , thus

$$S_{\tau_1}^{(q)}(f)^q \leq \lambda^q + (2\theta\lambda)^q = [1 + (2\theta)^q]\lambda^q.$$

Define the stopped martingale  $g_n = f_{\tau_1+n} - f_{\tau_1}$  ( $n \geq 0$ ) after  $\tau_1$ , then  $g = (g_n)$  is a  $\mathbf{B}$ -valued martingale with respect to the filtration  $(\mathcal{F}_{\tau_1+n})_{n \geq 0}$ , and

$$S^{(q)}(g)^q = \sum_{n > \tau_1} \|dg_n\|^q = S^{(q)}(f)^q - S_{\tau_1}^{(q)}(f)^q.$$

On  $\{S^{(q)}(f) > 2\lambda, \tau_2 = \infty\}$ ,

$$S^{(q)}(g)^q > (2\lambda)^q - [1 + (2\theta)^q]\lambda^q = (2^q - 1 - (2\theta)^q)\lambda^q,$$

where we choose a proper  $\theta$  such that  $2^q - 1 - (2\theta)^q > 0$ . Let  $\eta^q = 2^q - 1 - (2\theta)^q$ . Then

$$S^{(q)}(g) > \eta\lambda.$$

On the set  $\{\tau_2 = \infty\}$ , the maximal function of  $g$  satisfies

$$M(g) = \sup_{n \geq 0} \|f_{\tau_1+n} - f_{\tau_1}\| \leq M(f) + \|f_{\tau_1}\| \leq \theta\lambda + \theta\lambda = 2\theta\lambda.$$

Condition (ii) applies to the martingale  $g$ . Conditioning on  $\mathcal{F}_{\tau_1}$  and on  $\{\tau_1 < \infty, \tau_2 = \infty\}$ ,  $\mathbb{E}[M(g) | \mathcal{F}_{\tau_1}] \leq 2\theta\lambda$ , hence

$$\mathbb{P}\{S^{(q)}(g) > \eta\lambda | \mathcal{F}_{\tau_1}\} \leq \frac{C \mathbb{E}[M(g) | \mathcal{F}_{\tau_1}]}{\eta\lambda} \leq \frac{2C\theta}{\eta} = \varepsilon_\theta.$$

Thus,

$$\begin{aligned} \mathbb{P}\{S^{(q)}(f) > 2\lambda, M(f) \leq \theta\lambda\} &\leq \mathbb{P}\{\tau_1 < \infty, \tau_2 = \infty, S^{(q)}(g) > \eta^{1/q}\lambda\} \\ &= \mathbb{E}\left[\chi_{\{\tau_1 < \infty, \tau_2 = \infty\}} \mathbb{P}\{S^{(q)}(g) > \eta^{1/q}\lambda | \mathcal{F}_{\tau_1}\}\right] \\ &\leq \varepsilon_\theta \mathbb{P}\{\tau_1 < \infty\} = \varepsilon_\theta \mathbb{P}\{S^{(q)}(f) > \lambda\}. \end{aligned}$$

Immediately,

$$\mathbb{P}\{S^{(q)}(f) > 2\lambda, M(f) \leq \theta\lambda\} \leq \varepsilon_\theta \mathbb{P}\{S^{(q)}(f) > \lambda\}, \quad \forall \lambda > 0,$$

where  $\varepsilon_\theta = \frac{2C\theta}{\eta} \rightarrow 0$  as  $\theta \rightarrow 0$ . For any  $\lambda > 0$ ,

$$\begin{aligned} \mathbb{P}\{S^{(q)}(f) > \lambda\} &= \mathbb{P}\{S^{(q)}(f) > \lambda, M(f) > \theta\lambda/2\} + \mathbb{P}\{S^{(q)}(f) > \lambda, M(f) \leq \theta\lambda/2\} \\ &\leq \mathbb{P}\{M(f) > \theta\lambda/2\} + \varepsilon_\theta \mathbb{P}\{S^{(q)}(f) > \lambda/2\}. \end{aligned}$$

For any non-negative random variable  $X \geq 0$ , we have

$$\mathbb{E}[X^p] = p \int_0^\infty \lambda^{p-1} \mathbb{P}\{X > \lambda\} d\lambda, \quad p > 0.$$

Let  $X = S^{(q)}(f)$ ,  $p = q$ , then

$$\begin{aligned} \mathbb{E}[S^{(q)}(f)^q] &= q \int_0^\infty \lambda^{q-1} \mathbb{P}\{S^{(q)}(f) > \lambda\} d\lambda \\ &\leq q \int_0^\infty \lambda^{q-1} \mathbb{P}\{M(f) > \theta\lambda/2\} d\lambda + \varepsilon_\theta q \int_0^\infty \lambda^{q-1} \mathbb{P}\{S^{(q)}(f) > \lambda/2\} d\lambda. \end{aligned}$$

For the first integral substitute  $t = \theta\lambda/2$ ,  $\lambda = \frac{2t}{\theta}$ ,  $d\lambda = \frac{2}{\theta} dt$ , we obtain

$$q \int_0^\infty \lambda^{q-1} \mathbb{P}\{M(f) > \theta\lambda/2\} d\lambda = \left(\frac{2}{\theta}\right)^q q \int_0^\infty t^{q-1} \mathbb{P}\{M(f) > t\} dt = \left(\frac{2}{\theta}\right)^q \mathbb{E}[M(f)^q].$$

For the second integral substitute  $t = \lambda/2$ ,  $\lambda = 2t$ ,  $d\lambda = 2dt$ , it yields

$$\varepsilon_\theta q \int_0^\infty \lambda^{q-1} \mathbb{P}\{S^{(q)}(f) > \lambda/2\} d\lambda = \varepsilon_\theta 2^q q \int_0^\infty t^{q-1} \mathbb{P}\{S^{(q)}(f) > t\} dt = 2^q \varepsilon_\theta \mathbb{E}[S^{(q)}(f)^q].$$

Hence,

$$\mathbb{E}[S^{(q)}(f)^q] \leq \left(\frac{2}{\theta}\right)^q \mathbb{E}[M(f)^q] + 2^q \varepsilon_\theta \mathbb{E}[S^{(q)}(f)^q].$$

Rearranging gives

$$\mathbb{E}[S^{(q)}(f)^q] \leq \frac{(2/\theta)^q}{1 - 2^q \varepsilon_\theta} \mathbb{E}[M(f)^q],$$

where we choose a proper  $\theta$  satisfying  $1 - 2^q \varepsilon_\theta > 0$ , then there exists a constant  $C_q > 0$  such that

$$\mathbb{E}[S^{(q)}(f)^q] \leq C_q \mathbb{E}[M(f)^q].$$

It follows from Doob's maximal inequality that

$$\mathbb{E}[M(f)^q] \leq \left(\frac{q}{q-1}\right)^q \sup_{n \geq 0} \mathbb{E}\|f_n\|^q.$$

Hence,

$$\sum_{n=1}^{\infty} \mathbb{E}\|df_n\|^q \leq C \sup_{n \geq 0} \mathbb{E}\|f_n\|^q,$$

where  $C = C_q \left(\frac{q}{q-1}\right)^q$ , which implies  $\mathbf{B}$  is isomorphic to a  $q$ -uniformly convex space by using Lemma 2.7(ii).

Now we prove the necessity. For each  $\mathbf{B}$ -valued martingale  $f = (f_n)_{n \geq 0}$ , set  $W = (W_n)$  as any predictable dominating function sequence for  $(\|df_n\|)$  ( $\|df_n\| \leq W_n$ , where  $W_n$  is increasing and  $\mathcal{F}_{n-1}$  is measurable). Fix a  $\lambda > 0$ . For  $\beta > 0$  and  $\delta > 0$  such that  $\beta^q > \delta^q + 1$ , define stopping times below:

$$\begin{aligned} \mu &= \inf\{n : S_n^{(q)}(f) > \lambda\}, \\ \nu &= \inf\{n : S_n^{(q)}(f) > \beta\lambda\}, \\ \sigma &= \inf\{n : \|f_n\| \vee W_{n+1} > \delta\lambda\}. \end{aligned}$$

Denote  $A_k = \{\omega : \mu < k \leq \nu \wedge \sigma\}$ ,  $u_k = \chi_{A_k}$ , then  $(u_k)$  is a predictable sequence. Let

$$\hat{f}_n = \sum_{k=1}^n u_k df_k, \quad \hat{f} = (\hat{f}_n).$$

Note that when  $u_n = 1$ ,  $\omega \in \{\mu < n \leq \nu \wedge \sigma\} = \{\mu < n\} \cap \{n \leq \nu \wedge \sigma\}$ . At this time there exists  $k_0$ ,  $1 \leq k_0 < n$ ,  $\omega \notin A_{k_0}$ , and  $\omega \in A_{k_0+1}$ , thus

$$\hat{f}_n = \sum_{k=k_0+1}^n df_k = f_n - f_{k_0} = f_{n-1} + df_n - f_{k_0}.$$

From the definition of  $\sigma$  and  $n \leq \sigma$ , it follows that

$$\|\hat{f}_n\| \leq \|f_{n-1}\| + \|df_n\| + \|f_{k_0}\| \leq 3\delta\lambda.$$

Hence,

$$\mathbb{E}\|\hat{f}_n\|^q \leq (3\delta\lambda)^q \mathbb{P}\{\mu < \infty\} = (3\delta\lambda)^q \mathbb{P}\{S^{(q)}(f) > \lambda\}.$$

On the set  $\{\nu = n, \sigma = \infty\}$ , we have  $S_n^{(q)}(\hat{f})^q \geq (\beta^q - \delta^q - 1)\lambda^q$ , thus

$$\mathbb{P}\{S^{(q)}(f) > \beta\lambda, M(f) \vee M(W) \leq \delta\lambda\} \leq \mathbb{P}\{S^{(q)}(\hat{f})^q \geq (\beta^q - \delta^q - 1)\lambda^q\}.$$

Since  $\mathbf{B}$  is isomorphic to a  $q$ -uniformly convex space, it follows from Lemma 2.7 that

$$C \sup_{n \geq 0} \mathbb{E}\|f_n\|^q \geq \sum_{n=0}^{\infty} \mathbb{E}\|df_n\|^q,$$

then

$$\|S^{(q)}(f)\|_q \leq C_q \|f\|_q.$$

Furthermore,

$$\lambda^q \mathbb{P}\{S^{(q)}(f) > \lambda\} \leq \|S^{(q)}(f)\|_q^q \leq C_q^q \|f\|_q^q.$$

Applying  $(\hat{f}_n)$  to the above inequality, we have

$$\begin{aligned} \mathbb{P}\{S^{(q)}(f) > \beta\lambda, M(f) \vee M(W) \leq \delta\lambda\} &\leq C_q^q (\beta^q - \delta^q - 1)^{-1} \lambda^{-q} \|\hat{f}\|_q^q \\ &\leq C_q^q (3\delta)^q (\beta^q - \delta^q - 1)^{-1} \mathbb{P}\{S^{(q)}(f) > \lambda\}, \end{aligned}$$

therefore

$$\begin{aligned} \mathbb{P}\{S^{(q)}(f) > \beta\lambda\} &\leq \mathbb{P}\{S^{(q)}(f) > \beta\lambda, M(f) \vee M(W) \leq \delta\lambda\} \\ &\quad + \mathbb{P}\{M(f) \vee M(W) > \delta\lambda\} \\ &\leq C_q^q (3\delta)^q (\beta^q - \delta^q - 1)^{-1} \mathbb{P}\{S^{(q)}(f) > \lambda\} \\ &\quad + \mathbb{P}\{M(f) \vee M(W) > \delta\lambda\}. \end{aligned}$$

This is a good- $\lambda$  inequality for the pairwise  $(S^{(q)}(f), M(f) \vee M(W))$ . Equipping with integration yields

$$\|S^{(q)}(f)\|_1 \leq C \|M(f) \vee M(W)\|_1.$$

By using Lemma 2.8,  $f$  can be decomposed as  $f = g + h$ , where  $f$  and  $g$  are martingales, respectively. First for  $g = (g_n)_{n \geq 0}$ , by the sublinearity of the operator  $M$ , we have

$$\begin{aligned} \|S^{(q)}(g)\|_1 &\leq C \|M(g) \vee 4M(df)\|_1 \\ &\leq 8C \|M(f)\|_1 + C \|M(g)\|_1 \\ &\leq 9C \|M(f)\|_1 + C \|M(h)\|_1. \end{aligned}$$

Then for  $h = (h_n)_{n \geq 0}$ , obviously

$$\begin{aligned} \|S^{(q)}(h)\|_1 &\leq \mathbb{E} \left( \sum_{i=1}^{\infty} \|dh_i\| \right) \leq 4\mathbb{E}[M(df)] \leq 8 \|M(f)\|_1, \\ \|M(h)\|_1 &\leq \mathbb{E} \left( \sum_{i=1}^{\infty} \|dh_i\| \right) \leq 8 \|M(f)\|_1. \end{aligned}$$

In summary, we obtain

$$\|S^{(q)}(f)\|_1 \leq \|S^{(q)}(g)\|_1 + \|S^{(q)}(h)\|_1 \leq C \|M(f)\|_1.$$

Using Fubini's theorem, we have

$$\|S^{(q)}(f)\|_1 = \int_0^\infty \mathbb{P}\{S^{(q)}(f) > \lambda\} d\lambda,$$

and

$$\lambda \mathbb{P}\{S^{(q)}(f) > \lambda\} \leq \int_0^\infty \mathbb{P}\{S^{(q)}(f) > t\} dt \leq C \|S^{(q)}(f)\|_1.$$

Thus,

$$\lambda \mathbb{P}\{S^{(q)}(f) > \lambda\} \leq C\|M(f)\|_1,$$

which implies (i)  $\Rightarrow$  (ii) holds. Then the equivalence of (i) and (ii) is finished.

Now we show (ii)  $\Leftrightarrow$  (iii).

We first deal with (ii)  $\Rightarrow$  (iii). Consider the following operator  $T$ , which maps every family of  $\mathbf{B}$ -valued martingales to an  $\ell^q(\mathbf{B})$ -valued martingale family.

For each  $k \in \mathbb{N}$ , let  $v_k \in L(\mathbf{B}, \ell^q(\mathbf{B}))$  be the operator defined by  $v_k x = \{x_j\}_{j=1}^\infty$  for  $x \in \mathbf{B}$ , where  $x_j = x$  when  $j = k$ ; otherwise  $x_j = 0$ . Set

$$(Tf)_n = \sum_{k=1}^n v_k df_k = (df_1, df_2, \dots, df_n, 0, \dots) \in \ell^q(\mathbf{B}).$$

Then it is obvious to know that

$$M(Tf) = \sup_{n \geq 0} \|(Tf)_n\|_{\ell^q(\mathbf{B})} = \sup_{n \geq 0} \left( \sum_{k=1}^n \|df_k\|^q \right)^{1/q} = S^{(q)}(f).$$

If the condition (ii) holds, i.e.,

$$\lambda \mathbb{P}\{M(Tf) > \lambda\} \leq \|M(f)\|_1,$$

which guarantees the condition (i) in Lemma 2.9 holds, consequently the condition (ii) in Lemma 2.9 also holds, i.e., for any  $1 < r < \infty$ ,  $1 \leq s \leq \infty$ ,

$$\|M(Tf)\|_{r,s,b} \leq C\|M(f)\|_{r,s,b}.$$

Replacing  $M(Tf)$  by  $S^{(q)}(f)$  in the above inequality, we immediately obtain

$$\|S^{(q)}(f)\|_{r,s,b} \leq C\|M(f)\|_{r,s,b},$$

which is exactly the embedding  $H_{r,s,b}^* \hookrightarrow {}_q H_{r,s,b}^S$ .

Conversely, if the condition (iii) holds, i.e., for any  $1 < r < \infty$ ,  $1 \leq s \leq \infty$  such that the embedding  $H_{r,s,b}^* \hookrightarrow {}_q H_{r,s,b}^S$  holds, that is, for each  $\mathbf{B}$ -valued martingale  $f = (f_n)_{n \geq 0}$ , we have  $\|S^{(q)}(f)\|_{r,s,b} \leq C\|M(f)\|_{r,s,b}$ , then

$$\|M(Tf)\|_{r,s,b} = \|S^{(q)}(f)\|_{r,s,b} \leq C\|M(f)\|_{r,s,b},$$

i.e., the condition (ii) in Lemma 2.9 holds. Then the condition (i) in Lemma 2.9 also holds by equivalence, so there exists a constant  $C' > 0$  such that

$$\lambda \mathbb{P}\{M(Tf) > \lambda\} \leq C'\|M(f)\|_1,$$

i.e.,  $\lambda \mathbb{P}\{S^{(q)}(f) > \lambda\} \leq C'\|M(f)\|_1$ , which completes the proof of (iii)  $\Rightarrow$  (ii). At the same time, the proof of the equivalence of (ii) and (iii) is finished.

Combining the above derivation, conditions (i), (ii), and (iii) in Theorem 3.1 are pairwise equivalent. The proof is complete.  $\square$

**Theorem 3.2.** Let  $\mathbf{B}$  be a Banach space and  $b$  be a slowly varying function. For  $1 < q \leq 2$ , the following conditions are equivalent:

- (i)  $\mathbf{B}$  is isomorphic to a  $q$ -uniformly smooth space.
- (ii) There exists a positive constant  $C$  such that for every  $\mathbf{B}$ -valued martingale  $f = (f_n)_{n \geq 0}$ , we have

$$\lambda \mathbb{P}\{M(f) > \lambda\} \leq C \|S^{(q)}(f)\|_1, \quad \forall \lambda > 0.$$

- (iii) For any (or equivalently, some)  $r$  and  $s$ ,  $1 < r < \infty$ ,  $1 \leq s \leq \infty$ , we have

$${}_q H_{r,s,b}^S \hookrightarrow H_{r,s,b}^*.$$

*Proof.* (i)  $\Leftrightarrow$  (ii) First, we prove the sufficiency. Define the stopping time

$$\tau = \inf\{n \geq 0 : \|f_n\| > \lambda\}, \quad \lambda > 0.$$

Consider the stopped martingale  $f^\tau = (f_{n \wedge \tau})$ . Since  $\{M(f) > \lambda\} = \{\tau < \infty\}$ , it follows that  $f^\tau$  remains a  $\mathbf{B}$ -valued martingale. According to the condition (ii), we have

$$\lambda \mathbb{P}\{M(f^\tau) > \lambda\} \leq C \mathbb{E}[S^{(q)}(f^\tau)].$$

Note that on the set  $\{\tau = \infty\} = \{M(f) \leq \lambda\}$ , we can eliminate this region by multiplying by the indicator function  $\chi_{\{M(f) > \lambda\}}$ . Therefore, the expectation on the right-hand side can be written as

$$\mathbb{E}[S^{(q)}(f^\tau)] = \mathbb{E}[S^{(q)}(f^\tau) \cdot \chi_{\{M(f) > \lambda\}}].$$

Since  $f^\tau$  is the truncation of  $f$  at the stopping time  $\tau$ , by the monotonicity of the martingale square function, on the set  $\{M(f) > \lambda\}$ , we have

$$S^{(q)}(f^\tau) \leq S^{(q)}(f).$$

Thus we obtain

$$\lambda \mathbb{P}\{M(f) > \lambda\} \leq C \mathbb{E}[S^{(q)}(f) \cdot \chi_{\{M(f) > \lambda\}}],$$

then according to Fubini's theorem, we have

$$\begin{aligned} \mathbb{E}[M(f)^q] &= q \int_0^\infty \lambda^{q-1} \mathbb{P}\{M(f) > \lambda\} d\lambda \\ &\leq qC \int_0^\infty \lambda^{q-2} \mathbb{E}[S^{(q)}(f) \chi_{\{M(f) > \lambda\}}] d\lambda \\ &= qC \mathbb{E}[S^{(q)}(f) \int_0^{M(f)} \lambda^{q-2} d\lambda] \\ &= qC \mathbb{E}[S^{(q)}(f) \frac{M(f)^{q-1}}{q-1}] \\ &= \frac{qC}{q-1} \mathbb{E}[S^{(q)}(f) M(f)^{q-1}]. \end{aligned}$$

Applying the Hölder inequality to the last expression above yields

$$\begin{aligned}\mathbb{E}[S^{(q)}(f)M(f)^{q-1}] &\leq \left(\mathbb{E}[S^{(q)}(f)]^q\right)^{\frac{1}{q}}\left(\mathbb{E}[M(f)^{q-1}]^{q'}\right)^{\frac{1}{q'}} \\ &\leq \left(\mathbb{E}[S^{(q)}(f)]^q\right)^{\frac{1}{q}}\left(\mathbb{E}[M(f)^q]\right)^{\frac{1}{q'}}.\end{aligned}$$

Furthermore,

$$\left(\mathbb{E}[M(f)^q]\right)^{1-\frac{1}{q'}} \leq \frac{qC}{q-1}\left(\mathbb{E}[S^{(q)}(f)]^q\right)^{\frac{1}{q}}.$$

Therefore,

$$\mathbb{E}[M(f)^q] \leq \left(\frac{qC}{q-1}\right)^q \left(\mathbb{E}[S^{(q)}(f)]^q\right) = \left(\frac{qC}{q-1}\right)^q \sum_{n=0}^{\infty} \mathbb{E}\|df_n\|^q.$$

For every  $n \in \mathbb{N}$ , we have

$$\mathbb{E}\|f_n\|^q \leq \mathbb{E}\left[\sup_{k \geq 0} \|f_k\|^q\right] = \mathbb{E}[M(f)^q].$$

Then we obtain

$$\sup_{n \geq 0} \mathbb{E}\|f_n\|^q \leq C \sum_{n=0}^{\infty} \mathbb{E}\|df_n\|^q,$$

which implies  $\mathbf{B}$  is isomorphic to a  $q$ -uniformly smooth space by Lemma 2.7.

Now we prove the necessity. Let  $f = (f_n)_{n \geq 0} \in H_{p,q,b}^*(\mathbf{B})$ . Set  $W = (W_n)$  as any predictable dominating function sequence for  $(\|df_n\|)$  ( $\|df_n\| \leq W_n$ , where  $W_n$  is increasing and  $W_n$  is  $\mathcal{F}_{n-1}$  measurable). Fix a  $\lambda > 0$ . For  $\beta > 0$  and  $\delta > 0$  such that  $\beta^q > \delta^q + 1$ , define

$$\begin{aligned}\mu &= \inf\{n : M_n(f) > \lambda\}, \\ \nu &= \inf\{n : M_n(f) > \beta\lambda\}, \\ \sigma &= \inf\{n : S_n^{(q)}(f) \vee W_{n+1} > \delta\lambda\}.\end{aligned}$$

Denote  $A_k = \{\mu < k \leq \nu \wedge \sigma\}$ ,  $u_k = \chi_{A_k}$ , then  $(u_k)$  is a predictable sequence. Let

$$\hat{f}_n = \sum_{k=1}^n u_k df_k, \quad \hat{f} = (\hat{f}_n)_{n \geq 0}.$$

Note that when  $u_n = 1$ ,  $\omega \in \{\mu < n \leq \nu \wedge \sigma\} = \{\mu < n\} \cap \{n \leq \nu \wedge \sigma\}$ , so there exists a  $k_0 \in \mathbb{N}$  such that  $1 \leq k_0 < n$  and  $\omega \notin A_{k_0}$ ,  $\omega \in A_{k_0+1}$ , thus

$$\hat{f}_n = \sum_{k=k_0+1}^n df_k = f_n - f_{k_0} = f_{n-1} + df_n - f_{k_0}.$$

By the definition of  $\sigma$  and noting that  $n \leq \sigma$ , it follows that

$$\|\hat{f}_n\| \leq \|f_{n-1}\| + \|df_n\| + \|f_{k_0}\| \leq 3\delta\lambda.$$

Hence,

$$\mathbb{E}\|\hat{f}_n\|^q \leq (3\delta\lambda)^q \mathbb{P}\{\mu < \infty\} = (3\delta\lambda)^q \mathbb{P}\{M(f) > \lambda\}.$$

On the set  $\{\nu = n, \sigma = \infty\}$ , we have  $M_n(\hat{f})^q \geq (\beta^q - \delta^q - 1)\lambda^q$ , thus

$$\mathbb{P}\{M(f) > \beta\lambda, S^{(q)}(f) \vee M(W) \leq \delta\lambda\} \leq \mathbb{P}\{M(\hat{f})^q \geq (\beta^q - \delta^q - 1)\lambda^q\}.$$

Since  $\mathbf{B}$  is isomorphic to a  $q$ -uniformly smooth space, by applying Lemma 2.7, we have

$$\sup_{n \geq 0} \mathbb{E}\|f_n\|^q \leq C \sum_{n=0}^{\infty} \mathbb{E}\|df_n\|^q.$$

According to the above derivation, we obtain

$$\lambda^q \mathbb{P}\{M(f) > \lambda\} \leq C_q \|S^{(q)}(f)\|_q^q.$$

Then,

$$\begin{aligned} \mathbb{P}\{M(f) > \beta\lambda, S^{(q)}(f) \vee M(W) \leq \delta\lambda\} &\leq C_q (\beta^q - \delta^q - 1)^{-1} \lambda^{-q} \|\hat{f}\|_q^q \\ &\leq C_q (3\delta)^q (\beta^q - \delta^q - 1)^{-1} \mathbb{P}\{M(f) > \lambda\}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathbb{P}\{M(f) > \beta\lambda\} &\leq \mathbb{P}\{M(f) > \beta\lambda, S^{(q)}(f) \vee M(W) \leq \delta\lambda\} \\ &\quad + \mathbb{P}\{S^{(q)}(f) \vee M(W) > \delta\lambda\} \\ &\leq C_q (3\delta)^q (\beta^q - \delta^q - 1)^{-1} \mathbb{P}\{M(f) > \lambda\} \\ &\quad + \mathbb{P}\{S^{(q)}(f) \vee M(W) > \delta\lambda\}. \end{aligned}$$

This is a good- $\lambda$  inequality for the pairwise  $(M(f), S^{(q)}(f) \vee M(W))$ . Integration yields

$$\|M(f)\|_1 \leq C \|S^{(q)}(f) \vee M(W)\|_1.$$

Applying Lemma 2.8 again,  $f$  can be written as  $f = g + h$ . For  $g = (g_n)_{n \geq 0}$ , we get

$$\begin{aligned} \|M(g)\|_1 &\leq C \|S^{(q)}(g) \vee 4M(df)\|_1 \\ &\leq 8C \|S^{(q)}(f)\|_1 + C \|S^{(q)}(g)\|_1 \\ &\leq 9C \|S^{(q)}(f)\|_1 + C \|S^{(q)}(h)\|_1, \end{aligned}$$

in which we utilize the sublinearity of the operator  $S^{(q)}$ .

Then for  $h = (h_n)_{n \geq 0}$ , note that

$$\begin{aligned} \|S^{(q)}(h)\|_1 &\leq \mathbb{E} \left( \sum_{i=1}^{\infty} \|dh_i\| \right) \leq 4\mathbb{E}[M(df)] \leq C \|S^{(q)}(f)\|_1, \\ \|M(h)\|_1 &\leq \mathbb{E} \left( \sum_{i=1}^{\infty} \|dh_i\| \right) \leq C \|S^{(q)}(f)\|_1. \end{aligned}$$

In summary, we obtain

$$\|M(f)\|_1 \leq \|M(g)\|_1 + \|M(h)\|_1 \leq C \|S^{(q)}(f)\|_1.$$

By the maximal lemma of the martingale, we get

$$\mathbb{P}\{M(f) > \lambda\} \leq \frac{1}{\lambda} \|f\|_1, \quad \lambda > 0.$$

Furthermore,

$$\lambda \mathbb{P}\{M(f) > \lambda\} \leq \|f\|_1 \leq \|M(f)\|_1.$$

Thus,

$$\lambda \mathbb{P}\{M(f) > \lambda\} \leq C \|S^{(q)}(f)\|_1.$$

Now we show the equivalence of (ii) and (iii).

(ii)  $\Rightarrow$  (iii). Consider an arbitrary  $\ell^q(\mathbf{B})$ -valued martingale  $G = (G_n)_{n \geq 1}$ , where  $G_n = \sum_{k=1}^n J_k$  and  $J_k = (J_j^k)_{j \geq 1}$ , and the corresponding martingale transform operator  $T$ , which maps a family of  $\ell^q(\mathbf{B})$ -valued martingales to a family of  $\mathbf{B}$ -valued martingales. Let  $v_k \in L(\ell^q(\mathbf{B}), \mathbf{B})$  be the operator defined by the formula  $v_k x = x^k$ , where  $x = \{x^j\}_{j \geq 1} \in \ell^q(\mathbf{B})$ , and let

$$(TG)_n = \sum_{k=1}^n v_k J_k = \sum_{k=1}^n v_k J_k^k.$$

Now, for each  $\mathbf{B}$ -valued martingale  $f$ , where  $f_n = \sum_{k=1}^n df_k$ , we take an  $\ell^q(\mathbf{B})$ -valued martingale  $G = (G_n)_{n \geq 1}$  such that  $J_j^k = df_k$  when  $j = k$  and  $J_j^k = 0$  when  $j \neq k$ . Then

$$(TG)_n = \sum_{k=1}^n J_k^k = \sum_{k=1}^n df_k = f_n, \quad M(TG) = M(f),$$

and

$$\|G_n\|_{\ell^q(\mathbf{B})} = \|(df_1, df_2, \dots, df_n, 0, \dots)\|_{\ell^q(\mathbf{B})} = S_n^{(q)}(f), \quad M(G) = S^{(q)}(f).$$

If the condition (ii) holds, i.e.,

$$\lambda \mathbb{P}\{M(TG) > \lambda\} \leq \|M(G)\|_1,$$

which means the condition (i) of Lemma 2.9 holds, due to the equivalence, condition (ii) in Lemma 2.9 also holds, i.e., for any  $1 < r < \infty$ ,  $1 \leq s \leq \infty$ , we have

$$\|M(TG)\|_{r,s,b} \leq C_{r,s,b} \|M(G)\|_{r,s,b}.$$

Substituting  $M(f)$  for  $M(TG)$  and  $S^{(q)}(f)$  for  $M(G)$ , we get

$$\|M(f)\|_{r,s,b} \leq C_{r,s,b} \|S^{(q)}(f)\|_{r,s,b},$$

which is the desired result of (ii)  $\Rightarrow$  (iii).

Conversely, according to the condition (iii) for each  $\mathbf{B}$ -valued martingale  $f = (f_n)$ , we have

$$\|M(TG)\|_{r,s,b} = \|M(f)\|_{r,s,b} \leq C \|S^{(q)}(f)\|_{r,s,b},$$

which implies that the condition (ii) in Lemma 2.9 is valid. Consequently, the condition (i) in Lemma 2.9 also holds. Then there exists a constant  $C' > 0$  such that

$$\lambda \mathbb{P}\{M(TG) > \lambda\} \leq C' \|M(G)\|_1,$$

i.e.,  $\lambda \mathbb{P}\{M(f) > \lambda\} \leq C' \|S^{(q)}(f)\|_1$ , which is the condition (ii) of Theorem 3.2. Thus (iii)  $\Rightarrow$  (ii) holds.

In summary, (ii) and (iii) are equivalent.

Combining the first and the second step, conditions (i)–(iii) in Theorem 3.2 are pairwise equivalent. The proof is complete.  $\square$

**Remark 3.3.** If the slowly varying function  $b \equiv 1$  in Theorems 3.1 and 3.2, respectively, the results return to the corresponding conclusions in [21].

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there are no conflicts of interest.

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