



Research article

Spectral recoverability of inverse loads for fractionally damped beams

Huaqing Ma¹, Jichen Zhang^{2,*}, Min Liu³, Chen Gao³ and Zhiwei Yang^{1,*}

¹ School of Qilu Transportation, Shandong University, Jinan 250002, China

² Shandong Inspur Science Research Institute Co., Ltd, Jinan 250101, China

³ OriChip (ShanDong) Semiconductor, Qingdao 266104, China

* **Correspondence:** Email: zhangjichen@inspur.com, zhiweiyang@sdu.edu.cn.

Abstract: This paper investigates the spectral recoverability of a final-time inverse load problem for a Caputo fractionally damped Euler–Bernoulli beam. The unknown source is assumed to be separable in space and time, and the aim is to recover the spatial load profile from terminal displacement data. For a simply supported beam with constant coefficients, the terminal input-output map is diagonalized by the beam eigenfunctions, leading to a modal relation between the unknown load and the observation. The associated singular values are determined by the terminal responses of fractional-order modal equations and depend explicitly on the damping order. A truncated spectral reconstruction method with Tikhonov filtering is used for stable recovery from noisy data. Several effective indicators, including the minimum retained singular value, the effective condition number, and the maximum regularized amplification factor, are introduced to assess reconstruction stability. Numerical experiments show that the fractional damping order can reshape the effective terminal spectrum and influence the stability of the inverse problem. In suitable parameter regimes, fractional memory damping improves recoverable modal responses and reduces noise amplification compared with the classical integer-order damping model.

Keywords: spectral recoverability; final-time inverse load reconstruction; Euler–Bernoulli beam; Tikhonov regularization; fractional damping; Caputo derivative

1. Introduction

Inverse source and inverse load problems form an important class of inverse problems for partial differential equations. They arise when an unknown forcing term or load distribution must be inferred from indirect observations of the system response. Such problems occur in structural health monitoring, nondestructive evaluation, vibration-based diagnostics, dynamic force identification, and material characterization. Mathematically, these problems are usually ill-posed, because the forward

evolution may smooth, attenuate, or partially cancel the information carried by the unknown source before it is observed.

Several classical monographs provide the mathematical foundation for the analysis and regularization of such inverse problems. Engl et al. [1] gave a systematic treatment of regularization methods for ill-posed operator equations, including Tikhonov regularization and spectral filtering techniques. Kirsch [2] presented a comprehensive introduction to the mathematical theory of inverse problems, with emphasis on compact operators, singular systems, and stability issues. Isakov [3] studied inverse problems for partial differential equations from the viewpoint of uniqueness, stability, and reconstruction, and provided many fundamental results for inverse source and coefficient identification problems. These works form the general analytical background for the regularized spectral reconstruction considered in the present paper.

Recent studies have further developed inverse reconstruction theory for partial differential equations under limited or indirect measurements. Diao et al. [4] established a novel unique continuation principle and applied it to inverse conductive scattering, showing how hidden information can be recovered from indirect data under suitable structural assumptions. Diao et al. [5] investigated the determination of anomalies in a semilinear elliptic equation by using a minimal number of measurements, which highlights the role of limited observation data in inverse reconstruction. Diao et al. [6] studied dislocations with corners in elastic bodies and applied their analysis to fault detection, illustrating how geometric singularities can enhance identifiability in elastic inverse problems. These works are related to the present study in the sense that they all address the recovery of hidden information from indirect PDE observations, although the present paper focuses on final-time load reconstruction for a dynamic beam model.

For elastic structures, the Euler–Bernoulli beam equation is a fundamental model for bending vibration of slender beams. Meirovitch [7] presented the classical theory of vibration and modal analysis, including the use of eigenfunction expansions for continuous mechanical systems. Rao [8] provided a detailed treatment of vibration of continuous systems, including beam models, boundary conditions, and modal response analysis. These references justify the use of the Euler–Bernoulli beam framework and the sine eigenfunction basis for simply supported beams. In the inverse load setting, this spectral structure is particularly useful because it allows the terminal input-output map to be diagonalized mode by mode.

Beam vibration models have also been extended to modern multi-field and small-scale structural systems. Fang et al. [9] studied the non-local multi-field coupling response of a piezoelectric semiconductor nanofiber under shear force, demonstrating the importance of nonlocal and coupled effects in advanced beam-type structures. Fang et al. [10] analyzed the size-dependent vibration of laminated functionally graded curved beams covered with piezoelectric layers, showing that material gradation, curvature, and piezoelectric effects may strongly influence beam vibration behavior. These works are not inverse load studies, but they illustrate the continuing importance of accurate beam modeling for modern structural analysis and motivate further investigation of inverse problems for beam-type systems.

A typical final-time inverse load problem aims to recover an unknown spatial load profile from a displacement observation at a prescribed terminal time. For a classically damped Euler–Bernoulli beam, one may consider

$$u_{tt} + \mu u_t + u_{xxxx} = F(x)G(t), \quad (1.1)$$

where $F(x)$ is the unknown spatial load, $G(t)$ is a prescribed temporal excitation, and $\mu > 0$ is the viscous damping coefficient. The final observation $u_T(x) = u(x, T)$ is related to $F(x)$ through a compact terminal observation operator. Its singular values determine how much modal information about the load remains visible at time T . Small singular values cause strong noise amplification in direct inversion, and therefore regularization is necessary.

Computational and data-driven studies also show that reliable modeling of structural responses is essential for engineering diagnosis and interpretation. Meng et al. [11] developed a multi-scale feature fusion network for rapid intensity classification of ground vibration signals, which demonstrates the relevance of vibration-response information in engineering assessment. Zhang et al. [12] proposed a physics-informed neural network framework for stress-field modeling in coal matrix with hard inclusions, showing how physical models and computational learning can be combined to interpret mechanical fields. Although these two works use different methodologies from the present paper, they both emphasize the importance of stable response analysis and interpretable reconstruction in engineering systems.

Classical viscous damping is not always sufficient for describing energy dissipation in materials with memory. In viscoelastic media, the present state may depend on the deformation history, and fractional derivatives provide a useful tool for modeling such hereditary effects. Podlubny [13] introduced the basic theory of fractional differential equations and described the main definitions and analytical tools of fractional calculus. Kilbas et al. [14] provided a detailed mathematical treatment of fractional integrals and derivatives, including Caputo-type derivatives and their applications. Mainardi [15] developed the theory of fractional calculus in linear viscoelasticity and wave propagation, clarifying the physical meaning of memory-dependent constitutive laws. Li and Zeng [16] presented numerical methods for fractional calculus, including discretization techniques that are useful for fractional differential equations. These references provide the theoretical and numerical background for the Caputo damping term used in this paper.

Fractional damping has been increasingly used in structural dynamics to describe memory-dependent dissipation and viscoelastic response. Boukhari et al. [17] investigated fractional derivative boundary control in coupled Euler–Bernoulli beams and analyzed stability and discrete energy decay. Mohamed et al. [18] studied nonlinear dynamics and forced vibrations of simply supported fractional viscoelastic microbeams by using a fractional differential quadrature method, showing the influence of fractional viscoelasticity on microbeam vibration. Abouelregal et al. [19] considered thermomechanical waves in functionally graded viscoelastic nanobeams with an Atangana–Baleanu fractional derivative, illustrating the role of fractional operators in coupled thermoelastic nanobeam models. These works show that fractional derivatives can significantly modify vibration, stability, and wave propagation properties of beam-type structures.

Fractional damping in Euler–Bernoulli beam systems has also been studied from the viewpoint of boundary dissipation and energy decay. Achouri et al. [20] analyzed an Euler–Bernoulli beam equation with boundary dissipation of fractional derivative type and obtained decay properties associated with fractional boundary damping. Akil et al. [21] studied energy decay for boundary coupled wave/Euler–Bernoulli beam systems involving locally singular fractional Kelvin–Voigt damping. Their results further demonstrate that fractional damping mechanisms may strongly influence the long-time behavior and stability of beam systems. In contrast to these studies, the present work does not focus on energy decay or boundary stabilization, but on how the fractional damping order affects the singular spectrum

of a final-time inverse load operator.

Inverse problems for fractional differential equations have also attracted considerable attention. Zhang et al. [22] investigated two regularization methods for identifying a spatial source term in a space-time fractional diffusion-wave equation, emphasizing the stabilizing role of regularization in fractional inverse source recovery. Liu et al. [23] proposed a fractional Tikhonov regularization method for identifying a time-independent source in the fractional Rayleigh–Stokes equation, which is closely related to the use of spectral filtering for source reconstruction. Sun and Zhang [24] developed an exponential Tikhonov regularization method for an inverse source problem in a sub-diffusion equation, showing another way to suppress noise amplification in fractional inverse problems. Chen and Cheng [25] studied conditional stability and regularization for an inverse source problem associated with a distributed-order time-fractional diffusion equation. Hai and Chanh [26] considered regularization of an inverse source problem for fractional diffusion-wave equations under a general noise assumption. These works demonstrate the importance of regularization in fractional inverse source problems, but most of them concern diffusion-type or diffusion-wave-type equations rather than fractionally damped beam equations.

Despite the above developments, the influence of fractional damping on the spectral recoverability of final-time inverse load reconstruction for Euler–Bernoulli beams has not been systematically clarified. Existing fractional beam studies mainly address forward vibration, stability, decay, or coupled field behavior, while existing fractional inverse source studies mainly focus on diffusion-type equations. The present work connects these two directions by studying how a Caputo damping order changes the terminal singular values of a beam inverse load operator.

Motivated by these observations, a Caputo fractionally damped Euler–Bernoulli beam model is considered:

$$u_{tt} + \mu {}^C D_t^\beta u + u_{xxxx} = F(x)G(t), \quad 0 < \beta \leq 1. \quad (1.2)$$

Here ${}^C D_t^\beta$ denotes the Caputo fractional derivative. The case $\beta = 1$ recovers the classical viscously damped beam, whereas $0 < \beta < 1$ introduces long-memory damping. The inverse problem is to recover the unknown spatial load $F(x)$ from the final-time displacement $u(x, T)$. The central question is how the fractional damping order β affects the singular-value structure and hence the spectral recoverability of the inverse problem.

The main contributions of this paper are as follows.

- A final-time inverse spatial load problem is formulated for a Caputo fractionally damped Euler–Bernoulli beam.
- A modal singular-value representation of the terminal observation operator is derived for the simply supported constant-coefficient case.
- A truncated spectral reconstruction method with Tikhonov filtering is used for stable recovery from noisy terminal data.
- Fractional-order-dependent spectral indicators are introduced to quantify recoverability, including the minimum retained singular value, the truncated condition number, and the regularized amplification indicator.
- Numerical experiments show that suitable fractional damping orders can reshape the effective terminal spectrum and reduce regularized noise amplification compared with the classical integer-order damping model in certain parameter regimes.

The novelty of this work is not merely the replacement of classical damping by a Caputo derivative. Rather, the fractional damping order is identified as a spectral tuning parameter that modifies the singular-value structure of the terminal observation operator. This distinguishes the present work from standard inverse load reconstruction studies for classically damped beams, where the damping mechanism is fixed and the terminal spectrum is not analyzed with respect to a memory parameter. By connecting fractional memory damping with singular values, regularized amplification factors, and reconstruction errors, this paper provides a spectral explanation for how the Caputo damping order affects the recoverability and stability of final-time inverse load reconstruction.

The remainder of this paper is organized as follows. The mathematical model and inverse problem are introduced in Section 2. The spectral representation of the terminal observation operator is derived in Section 3. Conditional uniqueness and regularized reconstruction are discussed in Section 4. The numerical approximation of fractional modal responses is described in Section 5. Numerical experiments and spectral recoverability indicators are presented in Section 6. Concluding remarks are given in Section 7.

2. Mathematical model and inverse problem

Throughout this paper, the fractional damping order is denoted by β , the terminal singular values of the observation operator are denoted by σ_n^β , and the Tikhonov regularization parameter is denoted by α . The truncation level used in the spectral reconstruction is denoted by N .

Let $0 < x < \ell$ and $0 < t < T$. A simply supported Euler–Bernoulli beam with fractional damping is considered. The fourth-order spatial term in the model comes from the classical Euler–Bernoulli bending theory for slender beams [8], while the Caputo fractional derivative is introduced to describe memory-dependent viscoelastic dissipation [13, 14]. The governing equation is given by

$$u_{tt}(x, t) + \mu {}^C D_t^\beta u(x, t) + u_{xxxx}(x, t) = F(x)G(t), \quad (2.1)$$

where $u(x, t)$ denotes the transverse displacement, $F(x)$ is the unknown spatial load profile, $G(t)$ is a prescribed temporal excitation, $\mu > 0$ is the damping coefficient, and $0 < \beta \leq 1$ is the fractional damping order. Thus, Eq (2.1) may be viewed as a fractional viscoelastic extension of the classically damped Euler–Bernoulli beam equation. When $\beta = 1$, the damping term reduces to the standard viscous damping term μu_t ; when $0 < \beta < 1$, the damping is nonlocal in time and accounts for hereditary memory effects.

For $0 < \beta < 1$, the Caputo derivative used in (2.1) is defined by the following formula; see the standard references on fractional calculus [13, 14]:

$${}^C D_t^\beta v(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{v'(\tau)}{(t-\tau)^\beta} d\tau. \quad (2.2)$$

For the limiting classical case $\beta = 1$, the convention

$${}^C D_t^1 v(t) = v'(t) \quad (2.3)$$

is adopted. Therefore, the model continuously connects the fractional memory-damped beam for $0 < \beta < 1$ with the classical viscously damped beam for $\beta = 1$.

The beam is assumed to be simply supported:

$$u(0, t) = u(\ell, t) = 0, \quad u_{xx}(0, t) = u_{xx}(\ell, t) = 0. \quad (2.4)$$

These boundary conditions correspond to zero transverse displacement and zero bending moment at both ends of the beam. They are also chosen because they lead to a sine eigenfunction basis, which allows the terminal observation operator to be diagonalized explicitly in the subsequent spectral analysis.

The initial conditions are taken as homogeneous:

$$u(x, 0) = 0, \quad u_t(x, 0) = 0. \quad (2.5)$$

This assumption is imposed in order to isolate the effect of the unknown external load. If unknown initial states were also present, the final-time observation would contain contributions from both the initial data and the source term, and an additional identification problem would have to be considered.

The inverse problem is to reconstruct $F(x)$ from the terminal displacement

$$u_T(x) = u(x, T). \quad (2.6)$$

In practice, the observation may be noisy. We write $u_T^\delta = u_T + \eta$ and assume

$$\|u_T^\delta - u_T\|_{L^2(0, \ell)} \leq \delta. \quad (2.7)$$

The forward map from the spatial load F to the terminal displacement u_T is called the terminal observation operator in this paper. Since the beam dynamics smooth and attenuate modal information before the terminal time T , this operator is compact in the spectral setting considered below. Consequently, direct inversion is unstable with respect to measurement perturbations, and regularization is required for stable recovery from noisy terminal data.

The final-time inverse load problem can therefore be stated as follows: given the known parameters ℓ, T, μ, β , the prescribed temporal excitation $G(t)$, and the noisy terminal observation u_T^δ , recover the unknown spatial load profile $F(x)$ in a stable manner. The spectral representation and the associated regularized reconstruction method are developed in the following sections.

3. Spectral representation of the terminal observation operator

For a simply supported beam with constant coefficients, the eigenfunctions of the fourth-order spatial operator are

$$\psi_n(x) = \sqrt{\frac{2}{\ell}} \sin \frac{n\pi x}{\ell}, \quad n = 1, 2, \dots, \quad (3.1)$$

and the corresponding eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^4. \quad (3.2)$$

The system $\{\psi_n\}_{n=1}^\infty$ forms an orthonormal basis of $L^2(0, \ell)$.

The unknown spatial load and the displacement are expanded as

$$F(x) = \sum_{n=1}^{\infty} F_n \psi_n(x), \quad u(x, t) = \sum_{n=1}^{\infty} u_n(t) \psi_n(x), \quad (3.3)$$

where

$$F_n = \int_0^\ell F(x)\psi_n(x) dx. \quad (3.4)$$

Substituting (3.3) into (2.1) and using the orthogonality of $\{\psi_n\}$ yield the modal equation

$$u_n''(t) + \mu {}^C D_t^\beta u_n(t) + \lambda_n u_n(t) = F_n G(t), \quad u_n(0) = u_n'(0) = 0. \quad (3.5)$$

Let $q_n(t)$ be the unit-load modal response defined by

$$q_n''(t) + \mu {}^C D_t^\beta q_n(t) + \lambda_n q_n(t) = G(t), \quad q_n(0) = q_n'(0) = 0. \quad (3.6)$$

By linearity of (3.5), one has

$$u_n(t) = F_n q_n(t). \quad (3.7)$$

At the terminal time, the terminal modal coefficient is therefore given by

$$\sigma_n^\beta = q_n(T). \quad (3.8)$$

Then

$$u_{T,n} = \sigma_n^\beta F_n, \quad u_{T,n} = \int_0^\ell u_T(x)\psi_n(x) dx. \quad (3.9)$$

Consequently, the terminal observation operator has the spectral representation

$$u_T(x) = \sum_{n=1}^{\infty} \sigma_n^\beta F_n \psi_n(x). \quad (3.10)$$

This representation is the core of the analysis. It shows that final-time load reconstruction is equivalent to recovering F_n from $u_{T,n}$ through the singular values σ_n^β . Since σ_n^β is generated by the fractional-order modal response, the fractional damping order directly affects the stability of the inverse problem.

A further interpretation of σ_n^β can be obtained from the Laplace-domain representation of the modal response. For $0 < \beta < 1$, applying the Laplace transform to (3.6) with zero initial data gives

$$\widehat{q}_n(s) = \frac{\widehat{G}(s)}{s^2 + \mu s^\beta + \lambda_n}. \quad (3.11)$$

For the classical case $\beta = 1$, the denominator becomes $s^2 + \mu s + \lambda_n$. Thus, $q_n(t)$ can be interpreted as the convolution of the prescribed temporal excitation $G(t)$ with the impulse response associated with the fractional modal transfer function

$$H_n^\beta(s) = \frac{1}{s^2 + \mu s^\beta + \lambda_n}. \quad (3.12)$$

Accordingly, the terminal singular value $\sigma_n^\beta = q_n(T)$ depends on the eigenvalue λ_n , the damping coefficient μ , the fractional order β , the terminal time T , and the excitation profile $G(t)$.

This representation also explains the compactness mechanism of the terminal observation operator. Since

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^4 \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad (3.13)$$

high-frequency modes are increasingly dominated by the large stiffness term λ_n . As a result, the terminal response generated by a unit modal load is attenuated for high modes, and the corresponding terminal coefficients $\sigma_n^\beta = q_n(T)$ tend to be small as n increases. Therefore, the terminal observation operator is compact in the spectral sense, and the inverse reconstruction is ill-posed because recovering F_n requires division by σ_n^β .

The dependence on β is embedded in the fractional damping term μs^β in (3.11). Changing β modifies the frequency-dependent memory dissipation and therefore reshapes the terminal singular spectrum $\{\sigma_n^\beta\}_{n \geq 1}$. This is the spectral mechanism by which the fractional damping order affects recoverability. In general, however, this dependence need not be monotone for all modes or all terminal times, because oscillation, damping, terminal observation time, and the excitation function may interact in a nontrivial way.

4. Conditional uniqueness and regularized reconstruction

The spectral relation derived in the previous section shows that the final-time inverse load problem is reduced to the sequence of scalar equations

$$u_{T,n} = \sigma_n^\beta F_n, \quad n = 1, 2, \dots \quad (4.1)$$

A formal inversion gives

$$F_n = \frac{u_{T,n}}{\sigma_n^\beta}. \quad (4.2)$$

However, this formula is meaningful only when $\sigma_n^\beta \neq 0$. Moreover, even if all σ_n^β are nonzero, the inversion is unstable when σ_n^β is small.

The following conditional uniqueness statement follows immediately from the modal representation. Suppose that two loads $F^{(1)}$ and $F^{(2)}$ generate the same terminal displacement. Let $F_n^{(1)}$ and $F_n^{(2)}$ be their Fourier coefficients. Then

$$\sigma_n^\beta (F_n^{(1)} - F_n^{(2)}) = 0, \quad n = 1, 2, \dots \quad (4.3)$$

Therefore, if

$$\sigma_n^\beta \neq 0 \quad \text{for all } n \geq 1, \quad (4.4)$$

then $F_n^{(1)} = F_n^{(2)}$ for every n , and hence $F^{(1)} = F^{(2)}$ in $L^2(0, \ell)$. This gives uniqueness under the non-vanishing modal response condition (4.4).

If condition (4.4) fails, uniqueness must be understood only on the observable subspace. More precisely, define

$$\mathcal{I}_\beta = \{n \in \mathbb{N} : \sigma_n^\beta \neq 0\}. \quad (4.5)$$

Only the coefficients F_n with $n \in \mathcal{I}_\beta$ can be identified from the terminal observation. If $\sigma_n^\beta = 0$ for some n , then the corresponding component $F_n \psi_n$ belongs to the null space of the terminal observation operator and cannot be recovered from u_T . Thus, without additional a priori information, exact uniqueness cannot be guaranteed for the unobservable modes.

In the presence of noisy data u_T^δ , direct inversion by

$$F_n^\delta = \frac{u_{T,n}^\delta}{\sigma_n^\beta} \quad (4.6)$$

is unstable. Indeed, if

$$u_{T,n}^\delta = u_{T,n} + \eta_n, \quad (4.7)$$

then

$$F_n^\delta - F_n = \frac{\eta_n}{\sigma_n^\beta}. \quad (4.8)$$

Thus, small singular values lead to strong amplification of measurement noise. This is the main ill-posedness mechanism of the final-time inverse load problem.

To obtain a stable approximation, a Tikhonov-type spectral filtering strategy is adopted. For a regularization parameter $\alpha > 0$, the filtered coefficient is defined by

$$F_{n,\alpha}^\delta = \frac{\sigma_n^\beta}{(\sigma_n^\beta)^2 + \alpha} u_{T,n}^\delta. \quad (4.9)$$

Equivalently,

$$F_{n,\alpha}^\delta = g_\alpha(\sigma_n^\beta) u_{T,n}^\delta, \quad g_\alpha(\sigma) = \frac{\sigma}{\sigma^2 + \alpha}. \quad (4.10)$$

The factor $g_\alpha(\sigma)$ behaves like $1/\sigma$ for modes with $\sigma^2 \gg \alpha$, while it suppresses modes with $\sigma^2 \ll \alpha$. In this way, the unstable division by very small singular values is avoided.

Since the numerical implementation uses only finitely many modes, a truncated Tikhonov reconstruction is considered:

$$F_{\alpha,N}^\delta(x) = \sum_{n=1}^N \frac{\sigma_n^\beta}{(\sigma_n^\beta)^2 + \alpha} u_{T,n}^\delta \psi_n(x). \quad (4.11)$$

Here N is the truncation level. The parameters α and N balance data fidelity, noise suppression, and spectral approximation error. A small α retains more modal information but may amplify noise, whereas a large α improves stability at the cost of stronger bias. Similarly, increasing N includes more high-frequency information but may also introduce more unstable components.

For exact data, the corresponding truncated regularized approximation is

$$F_{\alpha,N}(x) = \sum_{n=1}^N \frac{(\sigma_n^\beta)^2}{(\sigma_n^\beta)^2 + \alpha} F_n \psi_n(x). \quad (4.12)$$

Thus the regularization error is caused by the filter factor

$$r_\alpha(\sigma_n^\beta) = 1 - \frac{(\sigma_n^\beta)^2}{(\sigma_n^\beta)^2 + \alpha} = \frac{\alpha}{(\sigma_n^\beta)^2 + \alpha}. \quad (4.13)$$

This expression shows that modes with large singular values are well preserved, while modes with small singular values are strongly damped. Hence the reconstruction quality depends essentially on the distribution of $\{\sigma_n^\beta\}$.

The fractional order β enters the reconstruction through the singular values $\sigma_n^\beta = q_n(T)$. Therefore, changing β changes the Tikhonov filter factors, the noise amplification levels, and the amount of retained modal information. This observation is the basis for the fractional-order-dependent spectral indicators introduced later. It should be emphasized, however, that the optimal choice of β is

problem-dependent and may depend on the terminal time, the damping coefficient, the excitation function, the noise level, and the target load profile.

For later use, the regularized amplification factor is defined by

$$A_{n,\alpha}^\beta = \left| \frac{\sigma_n^\beta}{(\sigma_n^\beta)^2 + \alpha} \right|. \quad (4.14)$$

The maximal regularized amplification over the first N modes is defined as

$$A_{\max}^\beta(\alpha, N) = \max_{1 \leq n \leq N} A_{n,\alpha}^\beta. \quad (4.15)$$

This quantity provides a simple measure of the worst-case noise amplification under the chosen regularization parameter and truncation level.

5. Numerical approximation of fractional modal responses

The reconstruction formula in Section 4 requires the terminal singular values

$$\sigma_n^\beta = q_n(T), \quad n = 1, 2, \dots, N, \quad (5.1)$$

where $q_n(t)$ solves the fractional modal problem

$$q_n''(t) + \mu {}^C D_t^\beta q_n(t) + \lambda_n q_n(t) = G(t), \quad q_n(0) = q_n'(0) = 0. \quad (5.2)$$

Thus, the numerical computation of σ_n^β is reduced to solving a family of scalar fractional ordinary differential equations.

Let $0 = t_0 < t_1 < \dots < t_M = T$ be a uniform partition of the time interval with time step $\tau = T/M$. Denote

$$q_n^k \approx q_n(t_k), \quad G^k = G(t_k). \quad (5.3)$$

For $0 < \beta < 1$, the Caputo derivative is approximated by the classical L1 formula:

$${}^C D_t^\beta q_n(t_k) \approx \frac{1}{\tau^\beta \Gamma(2-\beta)} \sum_{j=0}^{k-1} a_j^{(\beta)} (q_n^{k-j} - q_n^{k-j-1}), \quad k \geq 1, \quad (5.4)$$

where

$$a_j^{(\beta)} = (j+1)^{1-\beta} - j^{1-\beta}, \quad j = 0, 1, \dots, k-1. \quad (5.5)$$

The second derivative is approximated by the central difference

$$q_n''(t_k) \approx \frac{q_n^{k+1} - 2q_n^k + q_n^{k-1}}{\tau^2}. \quad (5.6)$$

Combining (5.4) and (5.6), the following time-stepping scheme is used for $0 < \beta < 1$:

$$\frac{q_n^{k+1} - 2q_n^k + q_n^{k-1}}{\tau^2} + \mu \frac{1}{\tau^\beta \Gamma(2-\beta)} \sum_{j=0}^{k-1} a_j^{(\beta)} (q_n^{k-j} - q_n^{k-j-1}) + \lambda_n q_n^k = G^k. \quad (5.7)$$

The initial conditions are discretized by

$$q_n^0 = 0, \quad q_n^1 = 0, \quad (5.8)$$

when the homogeneous initial velocity is used. More generally, q_n^1 may be obtained from a consistent starting procedure if nonzero initial velocity is considered.

Solving (5.7) gives the discrete modal response q_n^k . The numerical approximation of the terminal singular value is then defined by

$$\sigma_{n,M}^\beta = q_n^M. \quad (5.9)$$

In the reconstruction algorithm, σ_n^β is replaced by $\sigma_{n,M}^\beta$.

For the classical endpoint $\beta = 1$, the Caputo derivative is not discretized by the L1 formula (5.4). Instead, the limiting equation is treated as the classical damped modal oscillator

$$q_n''(t) + \mu q_n'(t) + \lambda_n q_n(t) = G(t), \quad q_n(0) = q_n'(0) = 0. \quad (5.10)$$

A standard finite-difference time-stepping method is then applied, for example

$$\frac{q_n^{k+1} - 2q_n^k + q_n^{k-1}}{\tau^2} + \mu \frac{q_n^k - q_n^{k-1}}{\tau} + \lambda_n q_n^k = G^k. \quad (5.11)$$

This separate treatment avoids applying fractional L1 weights at the endpoint $\beta = 1$. Therefore, any comparison between fractional cases $0 < \beta < 1$ and the classical case $\beta = 1$ is based on their corresponding continuous models and consistent numerical discretizations.

The approximation error in $\sigma_{n,M}^\beta$ affects the reconstruction through the Tikhonov filter. If

$$\sigma_{n,M}^\beta = \sigma_n^\beta + e_{n,M}^\beta, \quad (5.12)$$

then the computed filter coefficient differs from the exact one by

$$\frac{\sigma_{n,M}^\beta}{(\sigma_{n,M}^\beta)^2 + \alpha} - \frac{\sigma_n^\beta}{(\sigma_n^\beta)^2 + \alpha}. \quad (5.13)$$

For fixed $\alpha > 0$, this perturbation remains bounded when $|e_{n,M}^\beta|$ is sufficiently small. However, if α is chosen too small, errors in small singular values may have a stronger influence on the reconstructed coefficients. This provides another reason for using a positive regularization parameter.

For sufficiently smooth modal responses, the L1 approximation of the Caputo derivative has the standard consistency order $O(\tau^{2-\beta})$ away from possible initial singularities [16, 27]. In practical computations, the time step is chosen small enough so that the discretization error in $\sigma_{n,M}^\beta$ is dominated by the measurement noise and regularization effects. The numerical experiments below use the same time grid for all modes at a fixed β , which ensures that the computed singular spectrum is obtained consistently across modal indices.

The complete computational procedure is summarized as follows.

- (1). Choose the truncation level N , the regularization parameter α , and the time step τ .
- (2). Compute the eigenvalues $\lambda_n = (n\pi/\ell)^4$ for $1 \leq n \leq N$.

(3). For each n , solve the modal equation (5.2) numerically and obtain $\sigma_{n,M}^\beta = q_n^M$.

(4). Compute the noisy terminal coefficients

$$u_{T,n}^\delta = \int_0^\ell u_T^\delta(x) \psi_n(x) dx. \quad (5.14)$$

(5). Reconstruct the load by

$$F_{\alpha,N}^{\delta,M}(x) = \sum_{n=1}^N \frac{\sigma_{n,M}^\beta}{(\sigma_{n,M}^\beta)^2 + \alpha} u_{T,n}^\delta \psi_n(x). \quad (5.15)$$

The superscript M in (5.15) indicates that the singular values are computed numerically on the time grid. In the following sections, this notation is omitted when no confusion is possible.

6. Numerical experiments

6.1. Numerical setting

We present numerical experiments to examine how the fractional damping order affects final-time inverse load reconstruction. The beam is simply supported on $0 < x < \pi$. The parameters are

$$\ell = \pi, \quad \mu = 0.5, \quad T = 1.5. \quad (6.1)$$

The prescribed temporal excitation is

$$G(t) = \cos(\omega t), \quad \omega = 0.5, \quad (6.2)$$

and the exact spatial load is

$$F(x) = x \sin(5x). \quad (6.3)$$

This load contains oscillatory components and is useful for testing the recovery of modal information from terminal data.

The representative damping orders are

$$\beta = 1.0, \quad \beta = 0.8, \quad \beta = 0.6. \quad (6.4)$$

The case $\beta = 1.0$ corresponds to the classical viscously damped beam model, while $0 < \beta < 1$ corresponds to the fractionally damped beam model with memory-dependent dissipation.

The reconstruction is performed by the truncated Tikhonov formula

$$F_{\alpha,N}^\delta(x) = \sum_{n=1}^N \frac{\sigma_n^\beta}{(\sigma_n^\beta)^2 + \alpha} u_{T,n}^\delta \psi_n(x), \quad (6.5)$$

with truncation level

$$N = 11 \quad (6.6)$$

and regularization parameter

$$\alpha = 3 \times 10^{-9}. \quad (6.7)$$

The truncation level $N = 11$ is chosen to retain the dominant low- and medium-frequency modes that are still effectively visible in the terminal observation, while excluding highly attenuated high-frequency modes that would lead to severe noise amplification. The regularization parameter $\alpha = 3 \times 10^{-9}$ is selected to balance the regularization bias and the amplification of the 5% measurement noise. The same values of N and α are used for all tested damping orders in order to provide a fair comparison of the influence of β .

For noisy data, perturbations are added to the terminal displacement so that

$$\frac{\|u_T^\delta - u_T\|_{L^2(0,\pi)}}{\|u_T\|_{L^2(0,\pi)}} = 0.05. \quad (6.8)$$

Thus, the relative noise level is 5%. The same noise level, truncation level, and regularization parameter are used for all damping orders.

In all experiments, the terminal singular values $\sigma_n^\beta = q_n(T)$ are computed from the modal response equations described in Section 5. For $0 < \beta < 1$, the L1 approximation is used for the Caputo derivative. For $\beta = 1$, the corresponding classical damped modal oscillator is solved directly, rather than by applying the fractional L1 weights at the endpoint. This ensures that the comparison between fractional damping and classical damping is based on consistent numerical discretizations of their respective continuous models.

6.2. Reconstruction results

We first compare the terminal displacement responses generated by the same spatial load under different damping orders. Since the terminal displacement is the only observation used in the inverse problem, these curves reflect the information retained by the forward dynamics at time T . The corresponding terminal displacement responses are displayed in Figure 1.

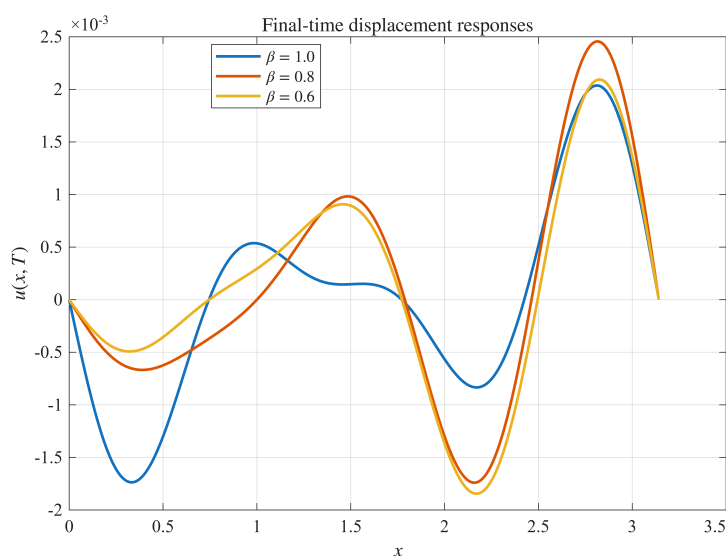


Figure 1. Terminal displacement responses for different fractional damping orders.

In Figure 1, the terminal displacement responses for $\beta = 1.0$, $\beta = 0.8$, and $\beta = 0.6$ are compared. The curves show that changing β modifies both the amplitude and the shape of the terminal response.

This indicates that memory damping changes the terminal information available for load recovery. Since the final-time inverse problem uses only $u(x, T)$, such changes in the terminal response directly affect the singular values of the terminal observation operator and hence the stability of the reconstructed load.

Next, we examine the reconstructed spatial load from exact terminal data and noisy terminal data. The noise-free reconstruction results are shown in Figure 2, while the corresponding reconstructions from noisy terminal data are shown in Figure 3.

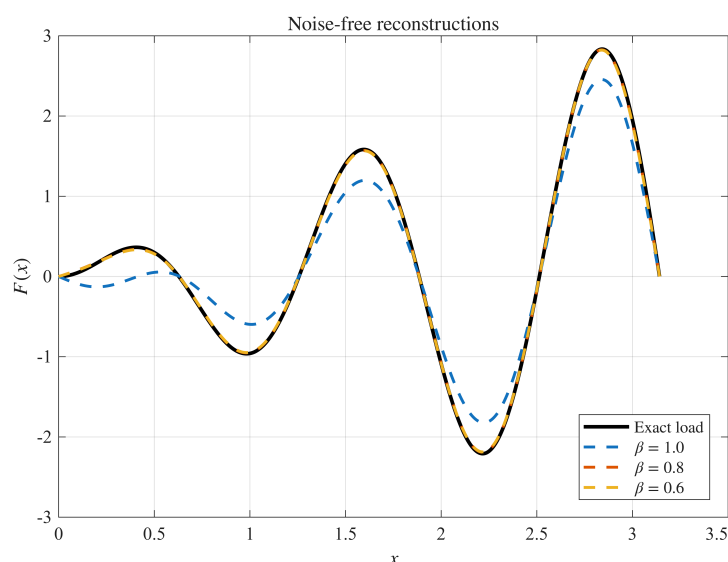


Figure 2. Noise-free reconstructions for different fractional damping orders.

In Figure 2, the reconstructed loads obtained from exact terminal data are compared with the exact load. In this case, the reconstruction error mainly comes from the finite modal truncation and the Tikhonov regularization bias. The results indicate that the damping order affects the accuracy of the recovered spatial profile even when no measurement noise is present.

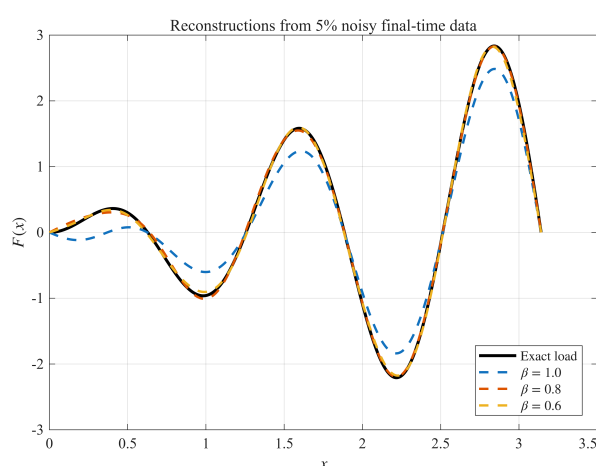


Figure 3. Reconstructions from terminal data with 5% relative noise.

In Figure 3, the reconstructions are obtained from terminal data contaminated by 5% relative noise.

Compared with Figure 2, the reconstructed profiles are additionally affected by noise amplification through the inverse procedure. The results show that the damping order has a visible influence on reconstruction quality. In the tested setting, suitable fractional orders produce reconstructions closer to the exact load, particularly in the presence of noise. This observation does not mean that smaller fractional orders always improve the reconstruction. Rather, it suggests that changing β modifies the effective singular spectrum of the terminal observation operator, and that some values of β may lead to better conditioning for the specific parameters, terminal time, excitation function, and noise level considered here.

To further explain this behavior, we compute the regularized modal amplification factor

$$g_n^\beta(\alpha) = \frac{|\sigma_n^\beta|}{|\sigma_n^\beta|^2 + \alpha}. \quad (6.9)$$

This quantity measures how strongly the Tikhonov filter amplifies the n -th noisy modal coefficient. A large value of $g_n^\beta(\alpha)$ indicates that noise in the corresponding terminal modal coefficient may be amplified more strongly in the reconstruction. The modal amplification factors for different damping orders are plotted in Figure 4.

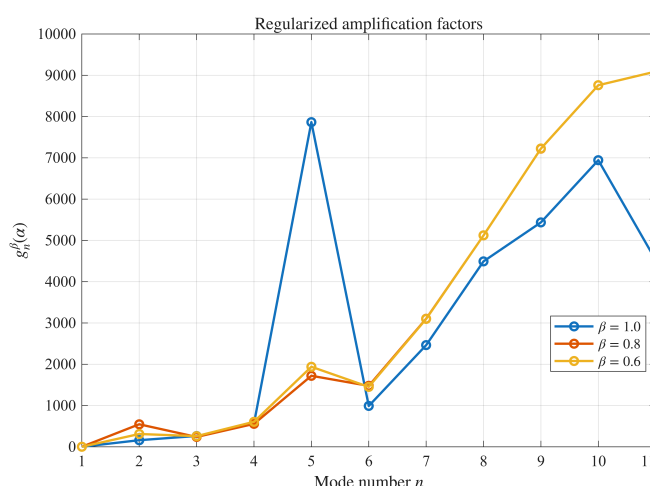


Figure 4. Regularized amplification factors for different fractional damping orders.

In Figure 4, the regularized amplification factors over the retained modal range are compared for different values of β . The figure shows that the fractional damping order changes the modal amplification pattern. Orders with smaller amplification over the retained modes tend to yield more stable reconstructions from noisy data. This supports the interpretation that the improved recovery is related to the effective spectral structure of the terminal observation operator.

It should be noted that $g_n^\beta(\alpha)$ is a regularized amplification factor, not the unregularized inverse factor $1/|\sigma_n^\beta|$. The Tikhonov denominator $|\sigma_n^\beta|^2 + \alpha$ prevents severe amplification of very small singular values. Therefore, the reconstruction quality is determined by the combined effect of the singular spectrum, the regularization parameter α , and the truncation level N .

6.3. Influence of the fractional order

We further scan the fractional order β to examine how reconstruction accuracy varies with the damping order. The relative reconstruction error as a function of β is reported in Figure 5.

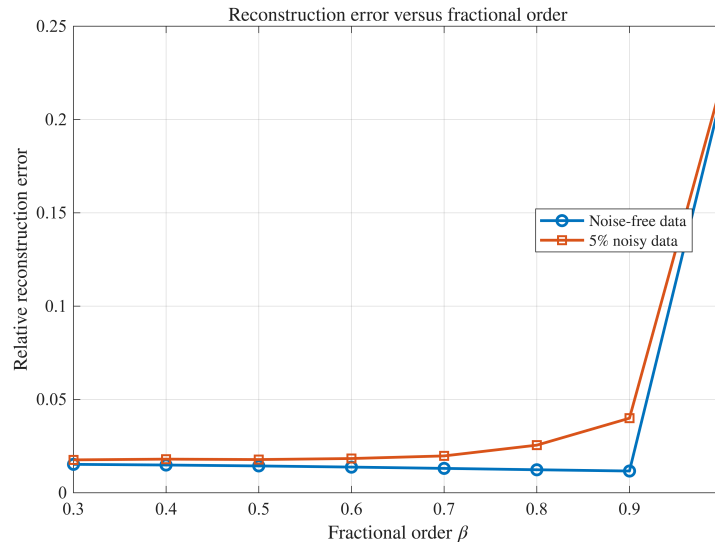


Figure 5. Relative reconstruction error as a function of the fractional damping order β .

In Figure 5, the relative reconstruction error varies with the fractional damping order. The curve shows that the reconstruction error is not necessarily monotone with respect to β . Therefore, fractional damping should not be viewed as a universally improving modification. Rather, the fractional order acts as a tuning parameter that changes the terminal response and the effective stability of the inverse problem.

This non-monotone behavior is consistent with the modal transfer function

$$H_n^\beta(s) = \frac{1}{s^2 + \mu s^\beta + \lambda_n}. \quad (6.10)$$

Changing β modifies the frequency-dependent damping and the phase of the modal response. As a consequence, the terminal values $\sigma_n^\beta = q_n(T)$ may increase for some modes and decrease for others. The final reconstruction error is therefore determined by the interaction between the modal spectrum, the terminal time, the excitation function, and the regularization filter.

The corresponding spectral indicators are shown in Figure 6. These indicators include the minimal retained singular value m_N^β , the truncated condition number κ_N^β , and the global regularized amplification indicator $A_N^\beta(\alpha)$.

For clarity, these indicators are defined as follows. The minimal retained singular value is

$$m_N^\beta = \min_{1 \leq n \leq N} |\sigma_n^\beta|. \quad (6.11)$$

It measures the weakest observable modal response among the first N modes. The truncated spectral condition number is

$$\kappa_N^\beta = \frac{\max_{1 \leq n \leq N} |\sigma_n^\beta|}{\min_{1 \leq n \leq N} |\sigma_n^\beta|}, \quad (6.12)$$

provided that $m_N^\beta > 0$. A smaller value of κ_N^β indicates a more balanced singular spectrum on the retained modal range. Since the full terminal observation operator is compact, this condition number is only a truncated indicator and does not remove the ill-posedness of the infinite-dimensional inverse problem.

The global regularized amplification indicator is defined by the root-mean-square average of the modal amplification factors:

$$A_N^\beta(\alpha) = \left(\frac{1}{N} \sum_{n=1}^N (g_n^\beta(\alpha))^2 \right)^{1/2}. \quad (6.13)$$

Thus, $g_n^\beta(\alpha)$ describes the amplification of a single modal coefficient, whereas $A_N^\beta(\alpha)$ describes the overall regularized amplification level over the first N modes. A smaller $A_N^\beta(\alpha)$ generally indicates weaker average noise amplification after Tikhonov filtering.

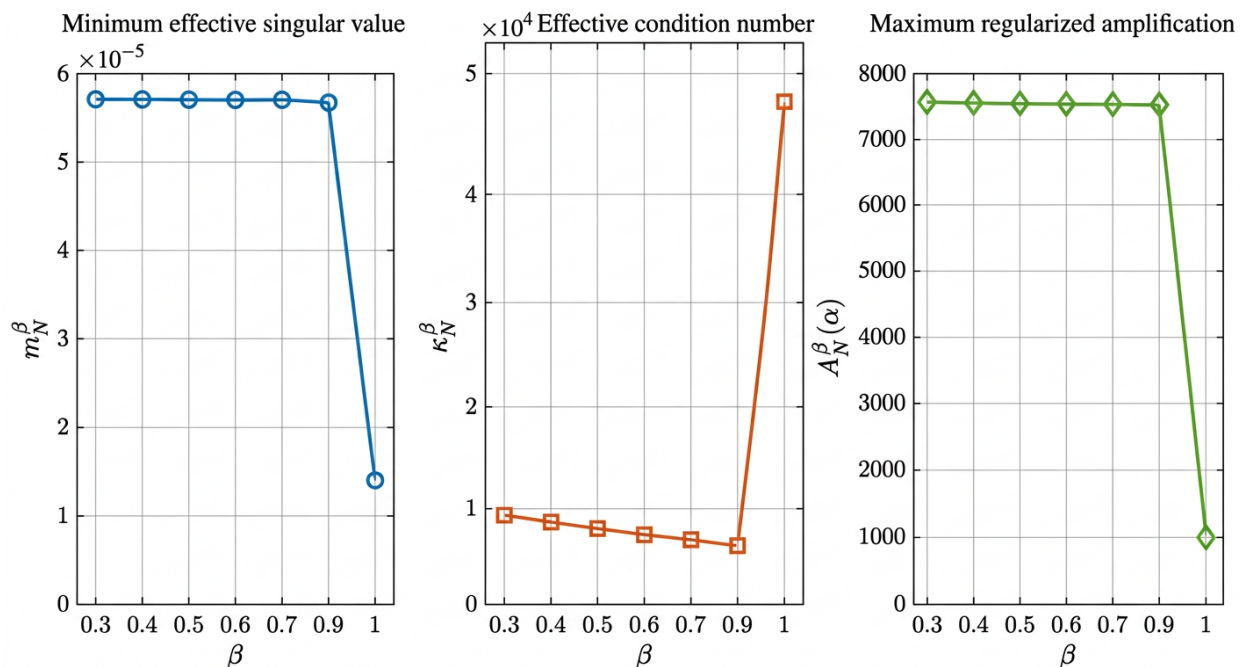


Figure 6. Spectral indicators as functions of the fractional damping order β .

In Figure 6, the dependence of the spectral indicators on β is presented. The results indicate that smaller reconstruction errors are associated with more favorable spectral indicators. In particular, fractional orders that reduce regularized amplification and improve the balance of retained modal responses tend to produce better reconstructions. This supports the conclusion that the reconstruction improvement is caused by a genuine change in the effective spectral structure of the terminal observation operator.

At the same time, Figure 6 confirms that the influence of β is not purely monotone. The fractional damping order should therefore be interpreted as a spectral tuning parameter rather than as a parameter that always improves or always deteriorates reconstruction. For the present example, the best reconstruction is obtained when the singular spectrum and the Tikhonov filter jointly provide a favorable balance between retained modal information and noise suppression.

7. Conclusions

In this paper, a final-time inverse spatial load problem was studied for a Caputo fractionally damped Euler–Bernoulli beam. The unknown spatial load was reconstructed from terminal displacement data by using a spectral representation of the terminal observation operator and a truncated Tikhonov regularization method.

For the simply supported constant-coefficient beam, the problem was diagonalized in the sine eigenfunction basis. This led to the modal relation

$$u_{T,n} = \sigma_n^\beta F_n, \quad (7.1)$$

where $\sigma_n^\beta = q_n(T)$ is the terminal modal response generated by a unit modal load. This representation shows that the recoverability of the inverse load problem is governed by the terminal singular values $\{\sigma_n^\beta\}$. In particular, small singular values cause strong noise amplification, and regularization is therefore required for stable reconstruction.

A conditional uniqueness result was discussed in terms of the modal terminal response. If $\sigma_n^\beta \neq 0$ for all relevant modes, then the corresponding modal coefficients of the load can be uniquely determined. If $\sigma_n^\beta = 0$ for some mode, then that component belongs to the null space of the terminal observation operator and cannot be recovered from a single final-time observation without additional information. Thus, the uniqueness considered in this paper is conditional on modal terminal observability.

The influence of the fractional damping order was analyzed through the modal transfer function

$$H_n^\beta(s) = \frac{1}{s^2 + \mu s^\beta + \lambda_n}. \quad (7.2)$$

This expression shows that changing β modifies the frequency-dependent damping and hence reshapes the singular-value structure of the terminal observation operator. Therefore, the fractional order affects not only the forward response but also the stability of the inverse reconstruction.

The numerical scheme for computing the modal singular values was also clarified. For $0 < \beta < 1$, the Caputo derivative was approximated by the L1 formula. For the classical endpoint $\beta = 1$, the limiting modal equation was treated as a classical damped oscillator and discretized separately. This avoids applying fractional L1 weights at the endpoint and ensures a consistent comparison between the fractionally damped and classically damped models.

Numerical experiments demonstrated that the damping order can have a significant influence on reconstruction accuracy. For the tested parameters, some fractional orders led to smaller reconstruction errors and lower regularized noise amplification than the classical damping case. The spectral indicators, including the minimal retained singular value m_N^β , the truncated condition number κ_N^β , and the global regularized amplification indicator $A_N^\beta(\alpha)$, helped explain these differences in terms of the effective singular spectrum.

It should be emphasized that the results do not imply that smaller fractional orders always improve reconstruction. The dependence of the reconstruction error and spectral indicators on β may be non-monotone, because the terminal singular values depend jointly on the fractional memory effect, modal oscillations, damping coefficient, terminal time, excitation function, and regularization parameters. Therefore, the fractional damping order should be interpreted as a spectral tuning parameter rather than as a universally stabilizing parameter.

The present work has several limitations. The analysis was carried out for a simply supported Euler–Bernoulli beam with constant coefficients and a known temporal excitation. Only a single final-time displacement observation was used. In addition, the reconstruction was based on a finite number of modes and a fixed Tikhonov regularization parameter.

Future work may consider more general boundary conditions, spatially varying beam coefficients, multiple or time-dependent observations, and simultaneous identification of spatial and temporal load components. It would also be useful to develop adaptive strategies for choosing the fractional order β , the truncation level N , and the regularization parameter α . Another possible direction is to extend the present spectral framework to more complex beam, plate, or coupled multi-field structures where fractional damping and memory effects play an important role.

Use of AI tools declaration

The authors declare that no artificial intelligence tools were used in the creation of this article.

Acknowledgments

This research was partially supported by the National Key R&D Program of China (No. 2024YFF0507903), the National Natural Science Foundation of China (No. 12401555), the Shandong Provincial Natural Science Foundation (No. ZR2025LZN010), the Young Talent of Lifting Engineering for Science and Technology in Shandong, China (No. SDAST2025QTB035), and the Taishan Scholars Program of Shandong Province (Nos. tsqn202507039 and tsqn202408001).

Conflict of interest

The authors declare that there are no conflicts of interest.

Data availability

The MATLAB code used to generate the numerical results is available from the corresponding author upon reasonable request.

References

1. H. W. Engl, M. Hanke, A. Neubauer, *Regularization of Inverse Problems*, Kluwer Academic Publishers, Dordrecht, 1996.
2. A. Kirsch, *An Introduction to the Mathematical Theory of Inverse Problems*, Springer, New York, 2011.
3. V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer, New York, 2006.
4. H. Diao, X. Fei, H. Liu, On a novel unique continuation principle result and its application to inverse conductive scattering, *SIAM J. Math. Anal.*, **57** (2025), 3386–3421. <https://doi.org/10.1137/24M1704762>

5. H. Diao, X. Fei, H. Liu, L. Wang, Determining anomalies in a semilinear elliptic equation by a minimal number of measurements, *Inverse Problems*, **41** (2025), 055004. <https://doi.org/10.1088/1361-6420/adc82a>
6. H. Diao, H. Liu, Q. Meng, Dislocations with corners in an elastic body with applications to fault detection, *SIAM J. Appl. Math.*, **85** (2025), 2399–2424. <https://doi.org/10.1137/24M1710267>
7. L. Meirovitch, *Fundamentals of Vibrations*, McGraw-Hill, New York, 2001. <https://doi.org/10.1115/1.1421112>
8. S. S. Rao, *Vibration of Continuous Systems*, Wiley, Hoboken, 2007.
9. X. Q. Fang, H. W. Ma, C. S. Zhu, Non-local multi-fields coupling response of a piezoelectric semiconductor nanofiber under shear force, *Mech. Adv. Mater. Struct.*, **31** (2024), 2452–2459. <https://doi.org/10.1080/15376494.2022.2158503>
10. X. Q. Fang, Y. F. Hu, C. S. Zhu, S. An, L. Q. Chen, Size-dependent vibration of laminated functionally graded curved beams covered with piezoelectric layers, *Mech. Adv. Mater. Struct.*, **30** (2023), 3257–3266.
11. F. C. Meng, X. T. Ai, H. Z. Cai, Y. S. Wang, T. Ren, MSFFCN: A multi-scale feature fusion network for rapid intensity classification of ground vibration signals, *MetaResource*, **2** (2025), 105–114. <https://doi.org/10.23919/METAR.2025.000007>
12. Q. Zhang, W. Liu, Y. Liu, W. Yang, Y. Tian, X. Dong, et al., Physics-informed neural network framework for stress field modeling in coal matrix with hard inclusions, *MetaResource*, **2** (2025), 321–355. <https://doi.org/10.23919/METAR.2025.000019>
13. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
14. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
15. F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, 2010.
16. C. Li, F. Zeng, *Numerical Methods for Fractional Calculus*, Chapman and Hall/CRC, Boca Raton, 2015.
17. B. Boukhari, F. Mtiri, A. Bchatnia, A. Beniani, Fractional derivative boundary control in coupled Euler–Bernoulli beams: stability and discrete energy decay, *AIMS Math.*, **9** (2024), 32102–32123. <https://doi.org/10.3934/math.20241541>
18. S. A. Mohamed, M. A. Eltaher, N. Mohamed, R. M. Abo-bakr, Nonlinear dynamics and forced vibrations of simply-supported fractional viscoelastic microbeams using a fractional differential quadrature method, *Mech. Based Des. Struct. Mach.*, **52** (2024), 10006–10025. <https://doi.org/10.1080/15397734.2024.2353321>
19. A. E. Abouelregal, M. Marin, A. Foul, S. S. Askar, Coupled responses of thermomechanical waves in functionally graded viscoelastic nanobeams via thermoelastic heat conduction model including Atangana–Baleanu fractional derivative, *Sci. Rep.*, **14** (2024), 9122. <https://doi.org/10.1038/s41598-024-58866-2>

20. Z. Achouri, N. E. Amroun, A. Benaissa, The Euler–Bernoulli beam equation with boundary dissipation of fractional derivative type, *Math. Methods Appl. Sci.*, **40** (2017), 3837–3854. <https://doi.org/10.1002/mma.4267>
21. M. Akil, I. Issa, A. Wehbe, Energy decay of some boundary coupled systems involving wave/Euler–Bernoulli beam with one locally singular fractional Kelvin–Voigt damping, *Math. Control Relat. Fields*, **13** (2023), 330–381. <https://doi.org/10.3934/mcrf.2021059>
22. C. Zhang, F. Yang, X. Li, Two regularization methods for identifying the spatial source term problem for a space-time fractional diffusion-wave equation, *Mathematics*, **12** (2024), 231. <https://doi.org/10.3390/math12020231>
23. S. Liu, L. Feng, C. Liu, A fractional Tikhonov regularization method for identifying a time-independent source in the fractional Rayleigh–Stokes equation, *Fractal Fract.*, **8** (2024), 601. <https://doi.org/10.3390/fractalfract8100601>
24. L. Sun, Z. Zhang, Exponential Tikhonov regularization method for an inverse source problem in a sub-diffusion equation, *Taiwan. J. Math.*, **28** (2024), 1111–1136. <https://doi.org/10.11650/tjm/240901>
25. Y. Chen, H. Cheng, Conditional stability and regularization method for inverse source for distributed order time-fractional diffusion equation, *Comput. Appl. Math.*, **43** (2024), 408. <https://doi.org/10.1007/s40314-024-02924-y>
26. D. N. D. Hai, L. V. Chanh, Regularization of an inverse source problem for fractional diffusion-wave equations under a general noise assumption, *Fract. Calc. Appl. Anal.*, **27** (2024), 2544–2576. <https://doi.org/10.1007/s13540-024-00315-y>
27. B. Jin, R. Lazarov, Z. Zhou, An analysis of the L1 scheme for the subdiffusion equation with nonsmooth data, *IMA J. Numer. Anal.*, **36** (2016), 197–221. <https://doi.org/10.1093/imanum/dru063>



AIMS Press

© 2026 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)