



Research article

Accelerated finite-time surrounding control for multiple moving targets

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Abstract: Finite-time surrounding control possesses superior performance in terms of tracking accuracy and rapidity. Nevertheless, most existing finite-time surrounding algorithms are only applicable to integrator systems, restricted to single-target scenarios, and fail to maintain rapid dynamic responses throughout the entire convergence process. To address these limitations, this paper extends the framework to multiple moving targets and generalizes integrators to nonlinear systems with unknown control coefficients. Meanwhile, an accelerated distributed control scheme is developed to improve the convergence performance of each agent throughout the entire process. Specifically, dynamic high-gain distributed observers are constructed for each follower, which achieve target state estimation merely using local relative information. Combined with adaptive techniques, a novel accelerated distributed surrounding protocol based on high-gain is proposed, which realizes finite-time multi-target encirclement and significantly improves global convergence rapidity compared with existing finite-time strategies. The embedded dynamic high-gain mechanism effectively compensates the unknown control coefficients while strictly preserving the fully distributed feature of the protocol, which guarantees strong operational robustness. Numerical simulations further verify the feasibility and effectiveness of the developed control scheme, demonstrating its promising application potential in practical multi-agent cooperative surrounding missions.

Keywords: multi-agent systems; accelerated finite-time surrounding; multiple moving targets; distributed protocol; adaptive technique

1. Introduction

Surrounding control enables multiple agents to construct predefined geometric enclosures for single or multiple targets within finite convergence time. Although existing finite-time surrounding schemes realize accurate encirclement and fast local convergence, they cannot maintain fast

convergence rates over the whole control process [1–5]. These approaches fail to meet the stringent global convergence speed demands of emerging time-critical industrial and military tasks, particularly for full-duration fast interception in cooperative multi-missile scenarios. This motivates the development of an accelerated finite-time surrounding strategy that simultaneously guarantees precise encirclement and global rapid convergence.

Nowadays, most of the finite-time distributed protocols proposed in existing related works [6–11] concern consensus, containment, and microgrid secondary regulation but not surrounding. As a cooperative goal extended from consensus, the investigation of the surrounding in a finite time is complex, let alone the accelerated finite-time surrounding. In addition, the protocols proposed in works [1, 12–14] on finite-time surrounding are only applicable to single-target scenarios, which greatly limits their practical applicability. In engineering and military practices, numerous typical missions such as multi-UAV hunting and important formation protection require the encirclement of multiple dynamic targets [15–17], such as entrapping or protecting important teams, patrolling, and hunting multiple enemy UAVs. Thus, it is meaningful to pursue a powerful distributed control strategy to realize accelerated finite-time surrounding for multiple moving targets.

Notably, multi-target surrounding control faces inherent difficulties in estimating the geometric center of multiple dynamic targets, as follower agents cannot always directly access real-time target state information. To tackle this challenge, works [2, 3, 5, 18, 19] adopted neighbor-interaction estimation strategies based on local error information, which realized geometric center estimation for integrator and parameter uncertain nonlinear multi-agent systems (MASs). However, these techniques still require absolute position information of interactive targets in protocol design. To reduce reliance on direct target information, a distributed observer was developed in [20] to estimate target position for each follower. Nevertheless, this method failed to achieve accelerated finite-time performance with global fast convergence. In practical scenarios, the superposition of accelerated finite-time characteristics and time-varying target motion imposes severe limitations on real-time information acquisition and rapid formation reconstruction. Therefore, sophisticated control protocols and rigorous theoretical analysis are urgently required to address the above unresolved issues.

Motivated by the aforementioned research gaps, this paper constructs a fully distributed control scheme for MASs with unknown control coefficients to achieve an accelerated finite-time surrounding of multiple moving targets. Firstly, an adaptive technique is adopted to design distributed finite-time observers for each follower, which can precisely estimate target states and drive observation errors to zero within finite time. Based on the observers, a distributed protocol based on high gains is further proposed, which guarantees practical accelerated finite-time convergence of surrounding errors irrespective of initial system states. Moreover, the designed dynamic high gains compensate the negative effects brought by unknown control coefficients while preserving the fully distributed property of the protocol. Theoretical analysis demonstrates that the proposed observer-based control approach ensures the boundedness of all closed-loop signals and enables effective accelerated finite-time multi-target surrounding performance.

The primary contributions of this paper over existing surrounding control literature are highlighted as follows:

- Different from the finite-time control strategies in [18, 19, 21, 22], which only achieve fast convergence in the neighborhood of equilibrium while the convergence rate degrades significantly under large initial errors, the developed accelerated finite-time adaptive scheme achieves favorable

global convergence performance and accelerates state evolution even when initial state errors are far away from the equilibrium.

- Unlike the existing results in [1, 12, 23–26], which are restricted to the single-target scenario or targets with zero acceleration, this paper realizes multi-target encirclement for targets with nonlinear acceleration profiles.

- The designed target state estimation approach removes the strict requirement of target absolute position information in control design. Compared with the geometric-center-based estimation methods in [5, 19], this work features a more concise distributed estimation structure. It reduces reliance on global network information and avoids network-wide iterative fusion for target state estimation, thereby lowering topology-dependent computational overhead in the observation stage.

- The proposed protocol is fully distributed and independent of any global topological information of the communication graph. Benefiting from this feature, the protocol requires no redesign when new agents join the network provided that network connectivity is maintained, which demonstrates favorable scalability for practical applications.

Notations: For a vector $x = [x_1, \dots, x_n]^T$, denote $x^\alpha = [x_1^\alpha, \dots, x_n^\alpha]^T$ and $x_i^{[\alpha]} = |x_i|^\alpha \cdot \text{sgn}(x_i)$. Let $\mathbb{R}_{>0}^{\text{odd}}$ represent the set of positive real numbers that possess both odd numerators and odd denominators. For a square matrix M , its transpose, minimal eigenvalue, and maximal eigenvalue are represented by M^T , $\lambda_{\min}(M)$, and $\lambda_{\max}(M)$, respectively. Let $\otimes$ represent the Kronecker product.

2. Preliminaries

For MASs, the interactions among agents can be described by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \dots, n + m\} = \mathcal{V}_F \cup \mathcal{V}_T$ is the node set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set. Note that $(j, i) \in \mathcal{E}$ implies the j th node could transmit the information to the i th node. We use $\mathcal{N}_i = \{j | j \in \mathcal{V}, (j, i) \in \mathcal{E}\}$ to denote the neighbor set of the i th node. Let $\mathcal{A} := [a_{ij}]$ to represent the adjacency matrix between nodes. While, when $(j, i) \in \mathcal{E}$, $a_{ij} > 0$, and otherwise, $a_{ij} = 0$. In this paper, there are no self-loops. A directed path from node i to node j can be represented by a sequence of ordered edges in the form $(i, i + 1), \dots, (j - 1, j)$. The Laplacian matrix is denoted by $\mathcal{L} := [l_{ij}]$, in which, when $i \neq j$, $l_{ij} = -a_{ij}$, and otherwise, $l_{ij} = \sum_{j \neq i}^N a_{ij}$. Note that the Laplacian matrix can be partitioned as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix},$$

where $\mathcal{L}_1 \in \mathbb{R}^{n \times n}$ and $\mathcal{L}_2 \in \mathbb{R}^{n \times m}$.

Next, we introduce several foundational lemmas that will be extensively adopted in analysis.

Lemma 1. [27] For $\dot{x}(t) = g(x(t))$ with $x(t_0) = x_0$ and $g(0) = 0$, $x \in \mathbb{R}^n$, if for any $x \in \mathbb{R}^n \setminus \{0\}$, the continuously differentiable function $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}^+$ satisfies $\dot{V}(x) + a_1 V(x) + \mu V(x)^\alpha \leq 0$ with $a_1 > 0$, $\mu > 0$, and $0 < \alpha < 1$. Then, $V(x(t))$ converges to zero in finite time, and the settling time satisfies $T(x_0) \leq \frac{1}{a_1(1-\alpha)} \ln(1 + \frac{a_1 V^{1-\alpha}(x_0)}{\mu})$.

Lemma 2. [28] Let $\alpha \in \mathbb{R}_{>0}^{\text{odd}}$.

(1) If $0 < \alpha \leq 1$, then $|x^\alpha - y^\alpha| \leq 2^{1-\alpha} |x - y|^\alpha$, $\forall x, y \in \mathbb{R}$.

(2) If $\alpha \geq 1$, then $|x^\alpha - y^\alpha| \geq 2^{1-\alpha} |x - y|^\alpha$, $\forall x, y \in \mathbb{R}$.

Lemma 3. [28] If $a > b \geq 1$, then for any $z_i \geq 0, i = 1, \dots, n$,

$$\left(\sum_{i=1}^n z_i^a \right) \leq \left(\sum_{i=1}^n z_i^b \right)^{\frac{a}{b}} \leq n^{\frac{a}{b}-1} \left(\sum_{i=1}^n z_i^a \right).$$

Definition 1. [29] Consider the nonlinear dynamical system $\dot{x}(t) = f(x(t))$ satisfying $f(0) = 0$. A system is said to achieve **accelerated practical finite-time convergence** if there exists a continuously differentiable positive definite function $V(x)$ satisfying the composite Lyapunov inequality:

$$\dot{V}(x) + aV(x) + bV^\alpha(x) - c \leq 0,$$

where $a > 0, b > 0, c \geq 0$ are constants and $0 < 1$. For arbitrary $0 < \epsilon < a$, the following two properties hold:

- (i) For any $t \geq T_0$, $V(x(t)) \leq \frac{c}{\epsilon}$, which implies that system errors converge to a practical residual set.
- (ii) The settling time is explicitly quantified as

$$T_0 \leq \frac{1}{(a - \epsilon)(1 - \alpha)} \ln \left(\frac{\frac{b}{a - \epsilon} + V^{1-\alpha}(x_0)}{\frac{b}{a - \epsilon} + \left(\frac{c}{\epsilon}\right)^{1-\alpha}} \right).$$

3. Problem formulation

3.1. System models

This work investigates a class of uncertain nonlinear MASs, which consist of follower agents and dynamic moving targets. The dynamical evolution formula for the i th follower agent is formulated as below:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = b_i(t)u_i + g(t, x_i, v_i), \quad i = 1, \dots, n, \end{cases} \quad (3.1)$$

where $x_i, v_i \in \mathbb{R}^N$ represent the follower's states; $u_i \in \mathbb{R}^N$ denotes the control input; unknown nonzero and time-varying continuous function $b_i(t)$ is called the control coefficient, and continuous vector-value function $g(t, x_i, v_i) : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is called the nonlinearity of the system.

The dynamic equation of the k th target is given as below:

$$\begin{cases} \dot{x}_k = v_k, \\ \dot{v}_k = g(t, x_k, v_k), \quad k = n + 1, \dots, n + m, \end{cases} \quad (3.2)$$

where $x_k, v_k \in \mathbb{R}^N$ represent the target's states and $g(\cdot)$ is defined as in (3.1).

Subsequently, three assumptions are given for system (3.1)–(3.2).

Assumption 1. There exist constants $\theta_1 > 0, \theta_2 > 0$ such that for any $x, y, x_k, y_k \in \mathbb{R}^N$ and $t \geq 0$,

$$\|g(t, x, y) - \sum_{k=1}^m a_k g(t, x_k, y_k)\| \leq \theta_1 \|x - \sum_{k=1}^m a_k x_k\| + \theta_2 \|y - \sum_{k=1}^m a_k y_k\|,$$

where $a_k \geq 0$ and $\sum_{k=1}^m a_k = 1$.

Assumption 2. The sign of $b_i(t)$ is priori known. In addition, there exist two unknown nonzero constants $\underline{b}_i, \bar{b}_i$ such that $\underline{b}_i \leq |b_i(t)| \leq \bar{b}_i$ for any $t \geq 0$.

Remark 1. In Assumption 1, this inequality characterizes the bounded residual of nonlinear terms under convex combinations, and it is different from the convex-preserving property of affine mappings. Particularly, if for any $k = 1, \dots, m$, $g(t, x_k, y_k)$ are the same, then the nonlinearities satisfy the Lipschitz condition, which is a common condition on nonlinearity. Assumption 2 indicates that severe uncertainties are allowed in control coefficients. In the context, the considered systems are more generic than existing related works with known control coefficients [30, 31].

Next, two assumptions on the communication graph in this paper are given as follows.

Assumption 3. In the graph, each follower is reachable from at least one target via a directed path. In addition, each target can communicate with at least one follower agent.

Assumption 4. There exist positive constants $q_i > 0$, $i \in \mathcal{V}_F$, obeying $q_j a_{ij} = q_i a_{ji}$ for any, $i, j \in \mathcal{V}_F$, with a_{ij} standing for the (i, j) -th entry of the graph adjacency matrix.

Assumption 5. The maximal offset of each target relative to the group geometric centroid is globally bounded. Specifically, a positive scalar M fulfills

$$M = \sup_{k \in \mathcal{V}_T} \{\|x_k(t) - q(t)\|\},$$

where $q(t) = \frac{1}{m} \sum_{k=n+1}^{n+m} x_k(t)$.

Remark 2. Assumption 3 guarantees that $\mathcal{L}_1$ is a non-singular M -matrix with all eigenvalues having strictly positive real parts. Besides, by Assumption 3, we know the elements of $-\mathcal{L}_1^{-1} \mathcal{L}_2$ are nonnegative and each row sum of $-\mathcal{L}_1^{-1} \mathcal{L}_2$ equals 1 (see [8]). From Assumption 4, we know the interactions between followers are bidirectional and weights on the edges may be different. As such, the adjacency matrix of the followers may be asymmetric, which would make some good properties used in performance analysis vanish and thereby render the existing control strategies invalid [15]. Such topology satisfies the detailed-balance condition $q_j a_{ij} = q_i a_{ji}$, which defines weight-balanced directed graphs. Typical examples include undirected graphs, bidirectional directed graphs with unequal edge weights, and bidirectional ring networks. Physically, this topology corresponds to bidirectional information exchange among multi-agent nodes, where the channel gains of forward and reverse communication links are not identical. This condition ensures that the weighted Laplacian matrix $\mathcal{L}_1 Q$ ($Q = \text{diag}\{q_1, \dots, q_n\}$) is symmetric and positive definite and facilitates the finite-time stability analysis. Note that this topology constraint is stronger than the general leader-reachable directed graph condition and restricts the admissible network structures. In practical systems, Assumption 5 is always satisfied.

3.2. Objective

The paper seeks to design a distributed scheme for dynamics (3.1)–(3.2) subject to Assumptions 1–5, which achieves accelerated practical finite-time surrounding of multiple targets. To facilitate the theoretical analysis, the definition presented below is provided.

Definition 2. For each follower agent i , the topology-induced aggregate target states are defined as

$$\bar{x}_i(t) = \sum_{k=n+1}^{n+m} \omega_{ik} x_k(t), \quad \bar{v}_i(t) = \sum_{k=n+1}^{n+m} \omega_{ik} v_k(t),$$

where $\omega_{ik} \triangleq (-\mathcal{L}_1^{-1} \mathcal{L}_2)_{ik}$ denotes the convex weight of target k for follower i , with $i \in \{1, \dots, n\}$ and $k \in \{n+1, \dots, n+m\}$. These coefficients satisfy $\omega_{ik} \geq 0$ and $\sum_{k=n+1}^{n+m} \omega_{ik} = 1$.

Definition 3. The practical finite-time surrounding is solved if there exists a distributed protocol such that for any initial condition, there hold

$$\begin{cases} \|x_i(t) - \bar{x}_i(t) - \rho_{ix}(t)\| \leq d, & \forall t \geq T_s; \\ \|v_i(t) - \bar{v}_i(t) - \rho_{iv}(t)\| \leq d, & \forall t \geq T_s, \end{cases}$$

where d is a positive adjustable constant and T_s is the limited settling time. Here, $\dot{\rho}_{ix}(t) = \rho_{iv}(t)$, and $\rho_{ix}(t) = \varrho[\rho_{i1x}(t), \dots, \rho_{inx}(t)]^T \in \mathbb{R}^n$, $\rho_{iv}(t) = \varrho[\rho_{i1v}(t), \dots, \rho_{inv}(t)]^T \in \mathbb{R}^n$, $i = 1, \dots, n$ are the desired surrounding configuration for followers, $\varrho > 2M$.

Remark 3. From Definition 2, we know $\bar{x}_i(t) = \sum_{k=n+1}^{n+m} \omega_{ik} x_k(t)$ and $\bar{v}_i(t) = \sum_{k=n+1}^{n+m} \omega_{ik} v_k(t)$ are convex combinations of the target states induced by the communication topology. This notation is distinct from a_{ij} , the (i, j) -th entry of the adjacency matrix. Note that Laplacian matrix $\mathcal{L}$ is constructed from the adjacency matrix whose entries are a_{ij} . After partitioning $\mathcal{L}$ into blocks corresponding to followers and leaders, ω_{ik} is the (i, k) -th entry of $-\mathcal{L}_1^{-1} \mathcal{L}_2$. Therefore, ω_{ik} is implicitly governed by a_{ij} , while no simple explicit closed-form expression directly relating ω_{ik} and a_{ij} is available due to matrix inversion. Moreover, from Definition 3, the compact form of $x_i - \bar{x}_i$ can be written as $x_F - (-\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n)x_T$, where $x_F = [x_1, \dots, x_n]^T$, $x_T = [x_{n+1}, \dots, x_{n+m}]^T$. If $x_F - \rho_F$ with $\rho_F = [\rho_{1x}, \dots, \rho_{nx}]^T$ practical converges to $(-\mathcal{L}_1^{-1} \mathcal{L}_2 \otimes I_n)x_T$, then x_F around the convex hull is formed by targets. As such, the followers could surround the multiple moving targets. Figure 1 is given to exhibit the surrounding scenario for four-target six-follower.

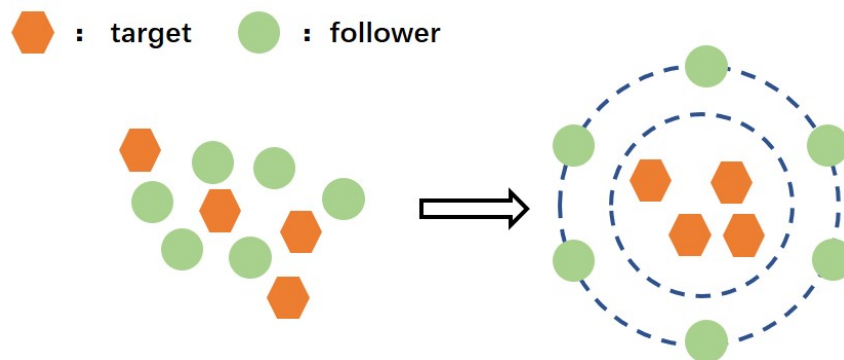


Figure 1. Six followers surrounding four moving targets.

3.3. Distributed accelerated finite-time observers for followers

For each follower ($i \in \mathcal{V}_F$), we design the following distributed accelerated finite-time observer to estimate $x_k, v_k, k \in \mathcal{V}_T$:

$$\begin{cases} \dot{\hat{x}}_{ik} = -\gamma_{ikx}\zeta_{ikx} - \zeta_{ikx}^\alpha + \hat{v}_{ik}, \\ \dot{\gamma}_{ikx} = \zeta_{ikx}^T \zeta_{ikx}, \quad \gamma_{ikx}(0) > 0, \\ \dot{\hat{v}}_{ik} = -\gamma_{ikv}\zeta_{ikv} - \zeta_{ikv}^\alpha + g(t, \hat{x}_{ik}, \hat{v}_{ik}), \\ \dot{\gamma}_{ikv} = \zeta_{ikv}^T \zeta_{ikv}, \quad \gamma_{ikv}(0) > 0, \end{cases} \quad (3.3)$$

where $\zeta_{ikx} = \sum_{j=1}^n a_{ij}(\hat{x}_{ik} - \hat{x}_{jk}) + a_{ik}(\hat{x}_{ik} - x_k)$, $\zeta_{ik} = \sum_{j=1}^n a_{ij}(\hat{v}_{ik} - \hat{v}_{jk}) + a_{ik}(\hat{v}_{ik} - v_k)$, $\alpha \in (0, 1)$.

Let $\tilde{x}_{ik} = \hat{x}_{ik} - x_k$, $\tilde{v}_{ik} = \hat{v}_{ik} - v_k$. Then, by (3.3), we have the following error system:

$$\begin{cases} \dot{\tilde{x}}_{ik} = -\gamma_{ikx}\zeta_{ikx} - \zeta_{ikx}^\alpha + \tilde{v}_{ik}, \\ \dot{\tilde{v}}_{ik} = -\gamma_{ikv}\zeta_{ikv} - \zeta_{ikv}^\alpha + g(t, \hat{x}_{ik}, \hat{v}_{ik}) - g(t, x_k, v_k). \end{cases} \quad (3.4)$$

In what follows, we show that the trajectories of error system (3.4) rapidly and precisely converge to zero in a finite time. Denote $\bar{\mathcal{L}}_k$ as the graph Laplacian matrix associated with the communication topology connecting n follower agents and the k -th target. Additionally, the matrix $\bar{\mathcal{L}}_k$ admits a block decomposition of the form:

$$\bar{\mathcal{L}}_k = \begin{bmatrix} \mathcal{L}_{1k} & \mathcal{L}_{2k} \\ \mathbf{0}_{1 \times n} & 0 \end{bmatrix}.$$

By Assumption 3, we know $\mathcal{L}_{1k}$ is a non-singular M-matrix with all eigenvalues having strictly positive real parts. Let $P_k = \text{diag}\{p_{1k}, \dots, p_{nk}\}$, $p_{ik} > 0$, $i = 1, \dots, n$. Then, from Assumption 4, we know that $\mathcal{L}_{1k}P_k$ is a positive defined and symmetric matrix. We give the following Lyapunov function candidate:

$$\begin{cases} W(\tilde{x}_k, \tilde{v}_k, \gamma_{kx}, \gamma_{kv}) = \sum_{i=1}^4 W_i, \\ W_1(\tilde{x}_k) = \frac{1}{2} \zeta_{kx}^T ((\mathcal{L}_{1k}P_k)^{-1} \otimes I_n) \zeta_{kx}, \quad W_2(\tilde{v}_k) = \frac{1}{2} \zeta_{kv}^T ((\mathcal{L}_{1k}P_k)^{-1} \otimes I_n) \zeta_{kv}, \\ W_3(\gamma_{kx}) = \frac{1}{2} \sum_{i=1}^n \frac{(\gamma_{ikx} - \tau_k p_{ik})^2}{p_{ik}}, \quad W_4(\gamma_{kv}) = \frac{1}{2} \sum_{i=1}^n \frac{(\gamma_{ikv} - \tau_k p_{ik})^2}{p_{ik}}, \end{cases} \quad (3.5)$$

where $\tilde{x}_k = [\tilde{x}_{1k}^T, \dots, \tilde{x}_{nk}^T]^T$, $\tilde{v}_k = [\tilde{v}_{1k}^T, \dots, \tilde{v}_{nk}^T]^T$, $\zeta_{kx} = [\zeta_{1kx}^T, \dots, \zeta_{nkx}^T]^T$, $\zeta_{kv} = [\zeta_{1kv}^T, \dots, \zeta_{nkx}^T]^T$, $\gamma_{kx} = [\gamma_{1kx}, \dots, \gamma_{nkx}]^T$, $\gamma_{kv} = [\gamma_{1kv}, \dots, \gamma_{nkx}]^T$, and τ_k is a positive constant determined later. Based on the proposed distributed observers, a proposition is proposed to describe the behavior of $\tilde{x}_{ik}, \tilde{v}_{ik}$ through the Lyapunov function candidate, and the detailed proof process is presented in Appendix A.

Proposition 1. Under Assumptions 1–5, the states $\tilde{x}_{ik}(t)$, $\tilde{v}_{ik}(t)$, $i \in \mathcal{V}_F$, $k \in \mathcal{V}_T$ of system (3.4) achieve accelerated convergence to zero within a finite settling time T_1 , where $T_1 = \max_{k=1, \dots, m} \{T_{2k}\}$, $T_{2k} = \frac{\lambda_{2k}}{\bar{p}_k(1-\alpha)} \ln(1 + (2 \frac{W_1(\tilde{x}_k(T_{1k})) + W_2(\tilde{v}_k(T_{1k}))}{\lambda_{2k}})^{\frac{1-\alpha}{2}}) + T_{1k}$, $\lambda_{2k} = \lambda_{\max}(P_k^{-1} \mathcal{L}_{1k}^{-1})$, and $\bar{p}_k = \|P_k^{-1}\|$; T_{1k} is a finite time and is given in later.

3.4. Distributed accelerated finite-time surrounding control

Let $e_{ix} = x_i - \bar{x}_i - \rho_{ix}$ and $e_{iv} = v_i - z_i - \rho_{iv}$, $z_i = \sum_{k=n+1}^{n+m} \omega_{ik} \hat{v}_{ik}$. Then, by (3.1)–(3.3), we have the following error system:

$$\begin{cases} \dot{e}_{ix} = v_i - \bar{v}_i - \dot{\rho}_{ix}, \\ \dot{e}_{iv} = b_i(t)u_i + \sum_{k=n+1}^{n+m} \gamma_{ikv} \omega_{ik} \zeta_{ikv} + \sum_{k=n+1}^{n+m} \omega_{ik} \zeta_{ikv}^\alpha + g(t, x_i, v_i) - \sum_{k=n+1}^{n+m} \omega_{ik} g(t, \hat{x}_{ik}, \hat{v}_{ik}) - \dot{\rho}_{iv}. \end{cases} \quad (3.6)$$

By finite-time observer (3.3), a distributed accelerated finite-time surrounding protocol is derived as follows:

$$\begin{cases} u_i = -\text{sign}(b_i(t))h_i(\eta_i^{\frac{\alpha}{2-\alpha}} + \Lambda_{1i}(\eta_i^{\beta(1+\beta)-1} + \eta_i^3) + \Lambda_{2i}D_i) - \text{sign}(b_i(t))\eta_i^\epsilon, \\ \dot{h}_i = \eta_i^T(\eta_i^{\frac{\alpha}{2-\alpha}} + \Lambda_{1i}(\eta_i^{\beta(1+\beta)-1} + \eta_i^3) + \Lambda_{2i}D_i), \\ \eta_i = e_{iv}^{\frac{1}{\beta}} - e_{iv}^{*\frac{1}{\beta}}, \quad e_{iv}^* = -\xi_i^\beta - \xi_i^3, \quad h_i(0) > 0, \end{cases} \quad (3.7)$$

where $0 < \epsilon < \frac{\alpha}{2-\alpha}$, $\beta(1+\beta) > 1$, $\beta < 2(1-\beta)(3-\beta)$, $0 < \beta < 1$, $\beta \in \mathbb{R}_{>0}^{odd}$, $D_i = [|\rho_{ix1}| + |\rho_{iv1}| + |\dot{\rho}_{iv1}|, \dots, |\rho_{ixn}| + |\rho_{ivn}| + |\dot{\rho}_{ivn}|]^T$, $\Lambda_{2i} = \text{diag}\{\text{sign}(\eta_{i1}), \dots, \text{sign}(\eta_{in})\}$, $\xi_i = [\xi_{i1}, \dots, \xi_{in}]^T$, $\xi_{is} = \sum_{j=1}^n a_{ij}(x_{is} - x_{js} - \rho_{ixs} + \rho_{jxs}) + \sum_{k=n+1}^{n+m} a_{ik}(x_{is} - x_{ks} - \rho_{ixs})$, $s = 1, \dots, n$, and $\Lambda_i = \text{diag}\left\{1 + \xi_{i1}^{\frac{4(3-\beta)}{\beta}} + \xi_{i1}^{\frac{(3-\beta)(1-\beta)(1+\beta)}{\beta}}, \dots, 1 + \xi_{in}^{\frac{4(3-\beta)}{\beta}} + \xi_{in}^{\frac{(3-\beta)(1-\beta)(1+\beta)}{\beta}}\right\}$.

Remark 4. Protocol (3.7) merely depends on agent velocity information and relative state information, which is essentially different from the existing works [5, 21–23], where neighboring target information and global topology information are utilized directly. This greatly lowers the topology-induced computational overhead, especially for MASs with a large number of agents. In addition, $-\xi_i^\beta$ and η_i^ϵ are critical in achieving the finite-time surrounding. Moreover, $-\xi_i^3$ in virtual control e_{iv}^* and η_i^3 in u_i play a key role in guaranteeing that the error variables converge to an arbitrarily small bounded region around the origin at an exponential rate. Dynamic gain $h_i(t)$ is introduced to handle the unknown control coefficients and ensure the protocol possesses a fully distributed architecture. In addition, in observer (3.3), the introduction of γ_{ikx} , γ_{ikv} shows that the global graph information is no longer necessary for observer design and reduces the computations. Since if there is no γ_{ikx} , γ_{ikv} , then $\bar{p}_k = \|P_k^{-1}\|$ and $\lambda_{1k} = \lambda_{\min}(\mathcal{L}_{1k})$, $k \in \mathcal{V}_T$ are need to be computed.

Define $Q = \text{diag}\{q_1, \dots, q_n\}$, where $q_i > 0$ for all $i = 1, \dots, n$. Hence, Assumption 4 ensures $\mathcal{L}_1 Q$ is symmetric and positive definite. To confirm the reliability of the devised control law presented beforehand, the below Lyapunov candidate is formulated:

$$\begin{cases} V(e_x, e_v, h) = V_1(e_x) + V_2(e_x, e_v) + V_3(h), \\ V_1(e_x) = \frac{1}{2} \xi^T ((\mathcal{L}_1 Q)^{-1} \otimes I_n) \xi, \quad V_2(e_x, e_v) = \sum_{i=1}^n \int_{e_{iv}^*}^{e_{iv}} (s^{\frac{1}{\beta}} - e_{iv}^{*\frac{1}{\beta}}) ds, \\ V_3(h) = \frac{1}{2} \sum_{i=1}^n |b_i| (h_i - \frac{c}{|b_i|})^2, \end{cases} \quad (3.8)$$

where $e_x = [e_{1x}^T, \dots, e_{nx}^T]^T$, $e_v = [e_{1v}^T, \dots, e_{nv}^T]^T$, $h = [h_1, \dots, h_n]^T$, and $\xi = [\xi_1^T, \dots, \xi_n^T]^T$.

Based on surrounding control protocol (3.7), we present the following proposition (the detailed proof process is presented in Appendix B), which is the preparatory work for the subsequent proof.

Proposition 2. For systems (3.1)–(3.2) satisfying Assumptions 1-5, under the finite-time observer (3.3) and distributed protocol (3.7), the Lyapunov function $V(e_x, e_v, h)$ satisfies

$$\begin{aligned} D^+V \leq & -\mu_1 \sum_{i=1}^n \sum_{s=1}^n (|\xi_{is}|^{1+\beta} + \xi_{is}^4) - \sum_{i=1}^n \sum_{s=1}^n |b_i| |\eta_{is}|^{1+\epsilon} - \mu_2 \sum_{i=1}^n \sum_{s=1}^n (|\eta_{is}|^{\beta(1+\beta)} + \eta_{is}^4) \\ & + \mu_3 \sum_{i=1}^n \sum_{k=n+1}^{n+m} (\zeta_{ik}^T \zeta_{ik} + \omega_{ik}^2 (\tilde{x}_{ik}^T \tilde{x}_{ik} + \tilde{v}_{ik}^T \tilde{v}_{ik})), \end{aligned} \quad (3.9)$$

where μ_1 , μ_2 , and μ_3 are some positive constants to be determined later.

4. Main results

This section focuses on practical accelerated finite-time surrounding for MASs (3.1)–(3.2). First, we analyze the existence of the solutions of the closed-loop system consisting of (3.1), (3.2), and (3.7). Second, Proposition 3 is given based on accelerated finite-time observer (3.3) and distributed protocol (3.7) to exhibit the bounded characteristic for every signal of the closed-loop system and verify their asymptotic convergence of ξ_i and η_i . Finally, based on Proposition 3 and Lemma 1, we show the accelerated finite-time surrounding for multiple moving targets in Theorem 1.

From (3.1), (3.2), (3.3), and (3.7), the closed-loop system includes the sign function. Hence, the vector field of the closed-loop system is not continuous over $\mathbb{R}_+ \times \mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \times \mathbb{R}_+$. Therefore, the solution of the closed-loop system can be understood in the sense of Filippov [32], and the closed-loop system possesses at least one Filippov solution on its maximal interval $[0, t_f)$, with $t_f \leq +\infty$.

Proposition 3. Under Assumptions 1–5 and accelerated finite-time observers (3.3), the signals of the closed-loop system formed by (3.1)–(3.2) and accelerated surrounding control protocol (3.7) are bounded. Moreover, for any $i = 1, \dots, n$, there hold $\lim_{t \rightarrow +\infty} \xi_i(t) = \mathbf{0}_n$ and $\lim_{t \rightarrow +\infty} \eta_i(t) = \mathbf{0}_n$.

Proof: Based on the Filippov differential inclusion theory, we adopt the upper-right Dini derivative D^+V to perform Lyapunov analysis along Filippov solution trajectories. From Proposition 1, we know $\tilde{x}_{ik}(t)$, $\tilde{v}_{ik}(t)$, $\zeta_{ikx}(t)$, and $\zeta_{ikv}(t)$ converge to $\mathbf{0}_n$ in finite time. Thus, by (3.9), we know $V(\cdot)$ is bounded on $[0, t_f)$. Then, by the definition of $V(\cdot)$, we can see $\|\xi_i(t)\|$'s and $|h_i(t) - \frac{c}{|b_i|}|$'s are bounded on $[0, t_f)$, and meanwhile, we know $|h_i(t) - \frac{c}{|b_i|}| \leq \frac{2V(0)}{\underline{b}} := \bar{h}$, where $\underline{b} = \min\{|b_1|, \dots, |b_n|\}$. Noting $e_{iv}^* = -(\xi_i^\beta + \xi_i^3)$, then $\|e_{iv}^*(t)\|$'s are also bounded.

Furthermore, with Lemma 2, the definition and bounded feature of $V(\cdot)$, we confirm that $\|e_{iv}(t) - e_{iv}^*(t)\|$ is bounded for $[0, t_f)$, and $\|e_{iv}(t)\|$'s, $\|\eta_i(t)\|$'s are likewise bounded. From the foregoing derivations and the definition of u_i , each norm $\|u_i\|$ is bounded. Hence, the conclusion holds: Signals of the closed-loop dynamics exhibit global boundedness. Besides, by accelerated finite-time observers (3.3), system (3.2), and the boundeness of the signals, we can obtain that target states x_i 's and v_i 's exist on $[0, t_f)$. Thus, $t_f = +\infty$.

In addition, recalling the fact that $\zeta_{ik}(t)$'s, $\tilde{x}_{ik}(t)$'s, and $\tilde{v}_{ik}(t)$'s converge to $\mathbf{0}_n$ in finite time, one can easily know from (3.9) that $\eta_i(t)$'s and $\xi_i(t)$'s are square integrable. Owing to bounded signals and the differential equations of $\xi_i(t)$'s and $\eta_i(t)$'s, we obtain that $\|\dot{\eta}_i(t)\|$'s, $\|\dot{\xi}_i(t)\|$'s remain bounded. Hence, applying Barbălat's lemma, we reach the conclusion that for all $i = 1, \dots, n$, $\lim_{t \rightarrow +\infty} \xi_i(t) = \mathbf{0}_n$ and $\lim_{t \rightarrow +\infty} \eta_i(t) = \mathbf{0}_n$.

This completes the proof.

Theorem 1. For system (3.1)–(3.2) satisfying Assumptions 1–5, finite-time observer (3.3) and distributed protocol (3.7) guarantee that the practical accelerated finite-time surrounding for multiple moving targets is achieved.

Proof: Let $\bar{V} = V_1 + V_2$. Then, based on (3.9), the definition of $V(\cdot)$, and a series of calculations, we conclude that for any $t \geq T' = \max\{T_1, T_2\}$,

$$D^+ \bar{V} \leq -\mu_4 \bar{V} - \mu_5 \bar{V}^{\frac{1+\beta}{2}} + \varepsilon \mu_6, \quad (4.1)$$

where $\mu_4 = \min\{\frac{\mu_1}{\lambda_3}, \frac{\mu_2}{2^{2-\beta}}\}$, $\mu_5 = \min\{\frac{\mu_1}{4}(\frac{2}{\lambda_3})^{\frac{1+\beta}{2}}, \frac{\mu_2}{2^{\frac{3-\beta^2}{2}}}\}$, $\lambda_3 = \lambda_{\min}((\mathcal{L}_1 P)^{-1})$, $\mu_6 = 3\bar{\rho}\bar{b}\bar{h}n$, $\bar{b} = \max\{|\underline{b}_1|, \dots, |\underline{b}_n|\}$, $c - M_1 - \frac{2-\alpha}{2} - \theta\delta = \mu_2 > 0$, $\tau_k - \theta^2 \bar{p}_k - \frac{3\bar{p}_k}{2\lambda_{1k}^2} - \frac{\bar{p}_k}{2} = \mu_1 > 0$, $\bar{h} = \frac{2V(0)}{\underline{b}}$, $\underline{b} = \min\{|\underline{b}_1|, \dots, |\underline{b}_n|\}$, $\bar{\rho}$ is the bound of $\rho_{ixs}^2 + \rho_{ivs}^2 + \rho_{ivs}^2$, $\underline{b}_i \leq |b_i(t)|$ is the unknown nonzero constant, ε is an arbitrarily small positive constant, and τ_k and c are sufficiently large constants. (The detailed proof process of (4.1) is presented in Appendix D). Therefore, by Definition 1, we can conclude there exists a finite time

$$T_s \leq \frac{2}{(\mu_4 - \mu_8)(1 - \beta)} \cdot \ln \left(\frac{\frac{\mu_5}{\mu_4 - \mu_8} + \bar{V}^{\frac{1-\beta}{2}}(T')}{\frac{\mu_5}{\mu_4 - \mu_8} + (\frac{\varepsilon \mu_6}{\mu_8})^{\frac{1-\beta}{2}}} \right) + T'$$

such that for any $t > T_s$, $\bar{V}(t) \leq \frac{\varepsilon \mu_6}{\mu_8} = \frac{6\varepsilon \bar{\rho} n \bar{b} V(0)}{b \mu_8}$ always holds, where μ_8 is a constant and $0 < \mu_8 < \mu_4$. Note that ε is an arbitrarily small positive constant, thus, for any $t > T_s$, $\bar{V}(t)$ ultimately enters and remains within an arbitrarily small bounded region around the origin, and this region can be shrunk arbitrarily close to the origin by tuning parameter ε . This, together with the definitions of $\bar{V}(\cdot)$ and e_{iv}^* , means that for any $t > T_s$, $\xi_{ix}(t), e_{ix}(t), e_{iv}(t) - e_{iv}^*(t), e_{iv}(t)$ ultimately converge to an arbitrarily small bounded neighborhood of the origin.

In addition, from $\bar{V} = V_1 + V_2$, $V_1 = \frac{1}{2} \xi^T ((\mathcal{L}_1 Q)^{-1} \otimes I_n) \xi$, and the boundness of $\bar{V}$, we can derive

$$0 \leq \frac{1}{2} \lambda_7 \xi^T \xi \leq V_1 \leq \bar{V} \leq \frac{6\varepsilon \bar{\rho} n \bar{b} V(0)}{b \mu_8},$$

where λ_7 denotes the minimum eigenvalue of the positive definite matrix $(\mathcal{L}_1 Q)^{-1} \otimes I_n$. Then, according to $\xi = (\mathcal{L}_1 \otimes I_n) e_x$, we have

$$\|e_x(t)\| \leq \sqrt{\frac{12\lambda_8 \varepsilon \bar{\rho} n \bar{b} V(0)}{b \mu_8 \lambda_7}}, \quad t > T_s,$$

with λ_8 as the minimum eigenvalue of the positive definite matrix $\mathcal{L}_1^T \mathcal{L}_1 \otimes I_n$.

Now, we estimate the bound of e_{iv} . By Lemma 2 and the definition of V_2 , we can obtain that

$$V_2 = \sum_{i=1}^n \int_{e_{iv}^*}^{e_{iv}} (s^{\frac{1}{\beta}} - (e_{iv}^*)^{\frac{1}{\beta}}) ds \geq \sum_{i=1}^n \int_{e_{iv}^*}^{e_{iv}} 2^{1-\frac{1}{\beta}} |s - (e_{iv}^*)|^{\frac{1}{\beta}} ds = \sum_{i=1}^n \frac{\beta}{1+\beta} 2^{\frac{\beta-1}{\beta}} |e_{iv} - e_{iv}^*|^{\frac{\beta+1}{\beta}},$$

which, together with $\bar{V} = V_1 + V_2$ and $\bar{V} \leq \frac{6\varepsilon \bar{\rho} n \bar{b} V(0)}{b \mu_8}$, implies $|e_{iv} - e_{iv}^*| \leq \left(\frac{6(\beta+1)\varepsilon \bar{\rho} n \bar{b} V(0)}{2^{\frac{\beta-1}{\beta}} \beta b \mu_8} \right)^{\frac{\beta}{1+\beta}} := E_1, t > T_s$.

Then, we derive that for any $t > T_s$,

$$|e_{iv}(t)| \leq |e_{iv}^*| + \left(\frac{6(\beta+1)\varepsilon \bar{\rho} n \bar{b} V(0)}{2^{\frac{\beta-1}{\beta}} \beta b \mu_8} \right)^{\frac{\beta}{1+\beta}}$$

$$\begin{aligned} &\leq \left(\sqrt{\frac{12\lambda_8 \varepsilon \bar{\rho} n \bar{b} V(0)}{b \mu_8 \lambda_7}} \right)^\beta + \left(\sqrt{\frac{12\lambda_8 \varepsilon \bar{\rho} n \bar{b} V(0)}{b \mu_8 \lambda_7}} \right)^3 + \left(\frac{6(\beta+1) \varepsilon \bar{\rho} n \bar{b} V(0)}{2^{\frac{\beta-1}{\beta}} \beta b \mu_8} \right)^{\frac{\beta}{1+\beta}} \\ &:= E_2. \end{aligned}$$

According to the above analysis, we have $|e_{iv}^{\frac{1}{\beta}} - (e_{iv}^*)^{\frac{1}{\beta}}| \leq |e_{iv}|^{\frac{1}{\beta}} + |e_{iv}^*|^{\frac{1}{\beta}} \leq 2|e_{iv}|^{\frac{1}{\beta}} + |e_{iv} - e_{iv}^*|^{\frac{1}{\beta}} \leq 2E_1^{\frac{1}{\beta}} + E_2^{\frac{1}{\beta}}$. Then, from the the bounds of e_x , ξ , h and the protocol (3.7), we can obtain a conservative upper bound for protocol u_i , that is, $|u_i| \leq \frac{2V(0)}{b} \left(2 \left(\sqrt{\frac{12\lambda_8 \varepsilon \bar{\rho} n \bar{b} V(0)}{b \mu_8 \lambda_7}} \right)^{Y_1} \times (2E_1^{\frac{1}{\beta}} + E_2^{\frac{1}{\beta}})^{Y_1} + (2E_1^{\frac{1}{\beta}} + E_2^{\frac{1}{\beta}})^{\frac{\alpha}{2-\alpha}} + \bar{\rho}_1 \right) + (2E_1^{\frac{1}{\beta}} + E_2^{\frac{1}{\beta}})^\epsilon$, where $\bar{\rho}_1$ is the bound of $|\rho_{ix}| + |\rho_{iv}| + |\dot{\rho}_{iv}|$.

These bounds are determined by system parameters, control parameters, and topological parameters and can be arbitrarily reduced to the origin by tuning parameter ε . Therefore, we can conclude that the surrounding error between the followers and the multiple moving targets can achieve practical accelerated finite-time convergence.

Corollary 1. For MASs consisting of (3.1)–(3.2), finite-time observer (3.3), and distributed protocol (3.7) under Assumptions 1–5, the settling time of the finite-time surrounding satisfies

$$T_s \leq \frac{2}{(\mu_4 - \mu_8)(1 - \beta)} \cdot \ln \left(\frac{\frac{\mu_5}{\mu_4 - \mu_8} + \bar{V}^{\frac{1-\beta}{2}}(T')}{\frac{\mu_5}{\mu_4 - \mu_8} + (\frac{\varepsilon \mu_6}{\mu_8})^{\frac{1-\beta}{2}}} \right) + T', \quad (4.2)$$

where $T' = \max\{T_1, T_2\}$, $T_1 = \max_{k=n+1, \dots, n+m} \{T_{2,k}\}$, $T_{2,k} = \frac{\lambda_{2k}}{\bar{p}_k(1-\alpha)} \ln(1 + (\frac{2\bar{W}(\tilde{x}_k(T_{1k}), \tilde{v}_k(T_{1k}))}{\lambda_{2k}})^{\frac{1-\alpha}{2}}) + T_{1,k}$, $T_{1,k} = \frac{W(\tilde{x}_k(0), \tilde{v}_k(0), \gamma_{kx}(0), \gamma_{kv}(0))}{\min_i \{p_{ik}^{-1}\} (\frac{\bar{p}_k}{2\tilde{\gamma}_k})^{\frac{1+\alpha}{1-\alpha}}}$, $T_2 = \max\{T_\xi, T_\eta\}$, $T_\xi = \frac{W(\tilde{x}_k(0), \tilde{v}_k(0), \gamma_{kx}(0), \gamma_{kv}(0)) + V(e_x(0), e_v(0), h(0))}{\mu_1 \Gamma_1}$, and $T_\eta = \frac{W(\tilde{x}_k(0), \tilde{v}_k(0), \gamma_{kx}(0), \gamma_{kv}(0)) + V(e_x(0), e_v(0), h(0))}{\min_i \{b_i\} \Gamma_2}$.

Proof: From Definition 1 and (4.1), we see that (4.2) holds. In what follows, we estimate the time T' .

First, we estimate the time T_{1k} . According to the proof of Proposition 1 and (A.6) given in Appendix A, we assume T_{1kx} is the instant when $\psi_1^{\frac{1-\alpha}{1+\alpha}}(t) \leq \frac{\bar{p}_k}{2\tilde{\gamma}_k}$, $\psi_2^{\frac{1-\alpha}{1+\alpha}}(t) \leq \frac{\bar{p}_k}{2\tilde{\gamma}_k}$ for the first time while T_{1kv} is the instant when $\psi_2^{\frac{1-\alpha}{1+\alpha}}(t) \leq \frac{\bar{p}_k}{2\tilde{\gamma}_k}$ for the first time. Let $T_{1,k} = \max\{T_{1kx}, T_{1kv}\}$. Then, $T_{1,k}$ is the instant that (A.6) holds for the first time.

Now, we estimate the times T_{1kx} and T_{1kv} . From (A.6) and $\psi_1(t) = \sum_{i=1}^n \sum_{s=1}^n |\zeta_{ikxs}|^{1+\alpha}(t)$, we know

$$\begin{cases} \psi_1(t) = \sum_{i=1}^n \sum_{s=1}^n |\zeta_{ikxs}|^{1+\alpha}(t) > (\frac{\bar{p}_k}{2\tilde{\gamma}_k})^{\frac{1+\alpha}{1-\alpha}}, & t \in [0, T_{1kx}), \\ \psi_2(t) = \sum_{i=1}^n \sum_{s=1}^n |\zeta_{ikvs}|^{1+\alpha}(t) > (\frac{\bar{p}_k}{2\tilde{\gamma}_k})^{\frac{1+\alpha}{1-\alpha}}, & t \in [0, T_{1kv}), \end{cases}$$

which, together with (A.4), implies

$$\begin{aligned} &W(\tilde{x}_k(0), \tilde{v}_k(0), \gamma_{kx}(0), \gamma_{kv}(0)) - W(\tilde{x}_k(T_{1kx}), \tilde{v}_k(T_{1kx}), \gamma_{kx}(T_{1kx}), \gamma_{kv}(T_{1kx})) \\ &\geq - \int_0^{T_{1kx}} \dot{W}(\tilde{x}_k(s), \tilde{v}_k(s), \gamma_{kx}(s), \gamma_{kv}(s)) ds \geq \int_0^{T_{1kx}} \sum_{i=1}^n \sum_{s=1}^n p_{ik}^{-1} |\zeta_{ikxs}|^{1+\alpha} \\ &\geq T_{1kx} \min_i \{p_{ik}^{-1}\} (\frac{\bar{p}_k}{2\tilde{\gamma}_k})^{\frac{1+\alpha}{1-\alpha}}, \end{aligned}$$

and

$$\begin{aligned} & W(\tilde{x}_k(0), \tilde{v}_k(0), \gamma_{kx}(0), \gamma_{kv}(0)) - W(\tilde{x}_k(T_{1kv}), \tilde{v}_k(T_{1kv}), \gamma_{kx}(T_{1kv}), \gamma_{kv}(T_{1kv})) \\ & \geq - \int_0^{T_{1kv}} \dot{W}(\tilde{x}_k(s), \tilde{v}_k(s), \gamma_{kx}(s), \gamma_{kv}(s)) ds \geq \int_0^{T_{1kv}} \sum_{i=1}^n \sum_{s=1}^n p_{ik}^{-1} |\zeta_{ikvs}|^{1+\alpha} \\ & \geq T_{1kv} \min_i \{p_{ik}^{-1}\} \left(\frac{\tilde{p}_k}{2\gamma_k}\right)^{\frac{1+\alpha}{1-\alpha}}. \end{aligned}$$

Then, we obtain $T_{1kx} \leq \frac{W(\tilde{x}_k(0), \tilde{v}_k(0), \gamma_{kx}(0), \gamma_{kv}(0))}{\min_i \{p_{ik}^{-1}\} \left(\frac{\tilde{p}_k}{2\gamma_k}\right)^{\frac{1+\alpha}{1-\alpha}}}$ and $T_{1kv} \leq \frac{W(\tilde{x}_k(0), \tilde{v}_k(0), \gamma_{kx}(0), \gamma_{kv}(0))}{\min_i \{p_{ik}^{-1}\} \left(\frac{\tilde{p}_k}{2\gamma_k}\right)^{\frac{1+\alpha}{1-\alpha}}}$. Thus, $T_{1,k} = \max\{T_{1kx}, T_{1kv}\} = \frac{W(\tilde{x}_k(0), \tilde{v}_k(0), \gamma_{kx}(0), \gamma_{kv}(0))}{\min_i \{p_{ik}^{-1}\} \left(\frac{\tilde{p}_k}{2\gamma_k}\right)^{\frac{1+\alpha}{1-\alpha}}}$.

Second, we estimate the time T_2 . According to the proof of inequality (4.1) and (B.23) given in Appendix D, we assume T_ξ and T_η are the instants that the following formulations hold for the first time:

$$\begin{cases} \varphi_1(t)^{\frac{8(3-\beta)}{\beta(1+\beta)}-1} + \varphi_1(t)^{\frac{2(3-\beta)(1-\beta)}{\beta}-1} \leq \frac{\mu_1}{8\mu_7}, \\ \varphi_2(t)^{\frac{2}{1+\epsilon}-1} + \varphi_2(t)^{\frac{2}{(2-\alpha)(1+\epsilon)}-1} + \varphi_2(t)^{\frac{2\beta(1+\beta)}{1+\epsilon}-1} + \varphi_2(t)^{\frac{8}{1+\epsilon}-1} \leq \frac{b}{2\mu_7}. \end{cases}$$

Then, we arrive at

$$\begin{cases} \max\{\varphi_1(t)^{\frac{8(3-\beta)}{\beta(1+\beta)}-1}, \varphi_1(t)^{\frac{2(3-\beta)(1-\beta)}{\beta}-1}\} > \frac{\mu_1}{16\mu_7}, & t \in [0, T_\xi), \\ \max\{\varphi_2(t)^{\frac{2}{1+\epsilon}-1}, \varphi_2(t)^{\frac{2}{(2-\alpha)(1+\epsilon)}-1}, \varphi_2(t)^{\frac{2\beta(1+\beta)}{1+\epsilon}-1}, \varphi_2(t)^{\frac{8}{1+\epsilon}-1}\} > \frac{b}{8\mu_7}, & t \in [0, T_\eta). \end{cases}$$

Therefore, for any $t \in [0, T_\xi)$ and $t \in [0, T_\eta)$, there hold $\varphi_1(t) > \bar{\Gamma}_1 = \min\{1, (\frac{\mu_1}{16\mu_7})^{\frac{\beta(1+\beta)}{8(3-\beta)}}, (\frac{\mu_1}{16\mu_7})^{\frac{\beta}{2\beta^2-9\beta+6}}\}$ and $\varphi_2(t) > \bar{\Gamma}_2 = \min\{1, (\frac{b}{8\mu_7})^{\frac{1+\epsilon}{1-\epsilon}}, (\frac{b}{8\mu_7})^{\frac{\alpha(1+\epsilon)-2\epsilon}{(2-\alpha)(1+\epsilon)}}, (\frac{b}{8\mu_7})^{\frac{(1+\epsilon)}{2\beta(1+\beta)-1-\epsilon}}, (\frac{b}{8\mu_7})^{\frac{1+\epsilon}{7-\epsilon}}\}$, respectively. Furthermore, by (3.9), (A.4), $\varphi_1(t) = \sum_{i=1}^n \sum_{s=1}^n |\xi_{is}(t)|^{1+\beta}$, and $\varphi_2(t) = \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}(t)|^{1+\epsilon}$, we can derive that

$$\begin{cases} W(\tilde{x}_k(0), \tilde{v}_k(0), \gamma_{kx}(0), \gamma_{kv}(0)) + V(e_x(0), e_v(0), h(0)) \geq \int_0^{T_\xi} \mu_1 \sum_{i=1}^n \sum_{s=1}^n |\xi_{is}(\tau)|^{1+\beta} d\tau \geq T_\xi \mu_1 \bar{\Gamma}_1, \\ W(\tilde{x}_k(0), \tilde{v}_k(0), \gamma_{kx}(0), \gamma_{kv}(0)) + V(e_x(0), e_v(0), h(0)) \geq \int_0^{T_\eta} \sum_{i=1}^n \sum_{s=1}^n |b_i| |\eta_{is}(\tau)|^{1+\epsilon} d\tau \geq T_\eta \min_i \{|b_i|\} \bar{\Gamma}_2. \end{cases}$$

Then, we have $T_\xi \leq \frac{W(\tilde{x}_k(0), \tilde{v}_k(0), \gamma_{kx}(0), \gamma_{kv}(0)) + V(e_x(0), e_v(0), h(0))}{\mu_1 \bar{\Gamma}_1}$, $T_\eta \leq \frac{W(\tilde{x}_k(0), \tilde{v}_k(0), \gamma_{kx}(0), \gamma_{kv}(0)) + V(e_x(0), e_v(0), h(0))}{\min_i \{|b_i|\} \bar{\Gamma}_2}$. Thus, $T_2 = \max\{T_\xi, T_\eta\}$.

Therefore, we can conclude that the settling time satisfies (4.2).

Remark 5. In this paper, the practical accelerated finite-time surrounding is achieved. Specifically, the surrounding error variables converge into an arbitrarily small neighborhood near the origin within a finite settling time. Compared with conventional finite-time works, the proposed method provides a tighter theoretical upper bound of settling time and superior transient performance under large initial errors. Traditional finite-time algorithms satisfying $\dot{V}(x) \leq -\mu \bar{V}(x)^\alpha$ exhibit non-uniform convergence behavior: They decay rapidly near the origin but converge slowly when the initial error is large and the system state is far from the origin. Specifically, according to (4.1), the Lyapunov

function satisfies $\dot{\bar{V}}(x) \leq -a\bar{V}(x) - \mu\bar{V}(x)^\alpha$ (the small threshold term $\varepsilon\mu_6$ is neglected only for this qualitative theoretical comparison). The corresponding settling-time upper bound is derived as $T_s \leq \frac{1}{a(1-\alpha)} \ln\left(\frac{a\bar{V}^{1-\alpha}(x_0)+\mu}{\mu}\right)$. In contrast, most existing finite-time surrounding methods only rely on the criterion $\dot{\bar{V}}(x) \leq -\mu\bar{V}(x)^\alpha$, whose settling time satisfies $T_s \leq \frac{\bar{V}^{1-\alpha}(x_0)}{\mu(1-\alpha)}$. Based on the mathematical relation $\ln\left(1 + \frac{a\bar{V}^{1-\alpha}(x_0)}{\mu}\right) < \frac{a\bar{V}^{1-\alpha}(x_0)}{\mu}$, it is theoretically verified that the proposed accelerated finite-time strategy possesses a faster convergence speed, especially when the system state is far away from the origin. It should be emphasized that this tighter upper bound improves the worst-case convergence guarantee and constitutes the core accelerated property of the proposed method.

Remark 6. It is worth noting that the term “accelerated control” also appears in [10]. In that paper, acceleration denotes performance improvement over asymptotic convergence, where both finite-time and fixed-time distributed protocols are constructed for microgrid secondary frequency regulation. By contrast, the acceleration mechanism of our scheme operates strictly within the finite-time framework. The specially devised control law induces an additional linear-damping-like term within the Lyapunov derivative, which delivers theoretically faster error-decay rates especially under large initial errors and produces a tighter settling-time upper bound for improved worst-case convergence performance. This induced damping effect constitutes the source of the core accelerated property of our scheme. Nevertheless, the settling-time upper bound remains dependent on initial conditions. This characteristic intrinsically differentiates our design from fixed-time control [33].

Remark 7. Since the proposed protocol contains the sign function with respect to relative state information, the closed-loop system may suffer from the chattering phenomenon in theory. To mitigate chattering, boundary-layer approximation is commonly adopted in simulation and practical implementation to approximate the sign function, such as the boundary-layer function $\frac{\delta}{|\delta|+\epsilon}$ and saturation function $\text{sat}(\frac{\delta}{\epsilon})$, where ϵ is a small positive constant. As $\epsilon \rightarrow 0$, these approximate functions point-wise converge to the ideal sign function. It should be noted that with a fixed positive ϵ , the tracking error cannot converge exactly to zero but will stay within a bounded neighborhood of the origin. This property is compatible with the practical finite-time surrounding objective considered in this paper.

5. Simulation example

In this section, we give an example for the uncertain nonlinear MASs to illustrate that the developed distributed adaptive protocol has the ability to achieve the accelerated finite-time surrounding for multiple moving targets. Consider nonlinear MASs (3.1)–(3.2) with six followers labeled by $1, \dots, 6$ and four targets labeled by $7, \dots, 10$. Moreover, in (3.1)–(3.2), where the nonlinearities $g(t, x_i, v_i) = 0.5 \sin(t)(x_i + v_i)$ satisfy Assumption 1 with $\theta = 0.5$, the control coefficients are selected as

$$\begin{cases} b_1(t) = 2 + 0.5 \sin(t), & b_2(t) = 2 - 0.5 \cos(t), & b_3(t) = 2 + \sin(t), \\ b_4(t) = 2 - 0.2 \sin(t), & b_5(t) = 1 + 0.3 \cos(t), & b_6(t) = 2 + \sin(t). \end{cases}$$

The communication graph $\mathcal{G}$ is described by Figure 2, and the weights are labeled on the edges. Then, the corresponding Laplacian matrix is given as

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ 0_{4 \times 6} & 0_{4 \times 4} \end{bmatrix},$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are given as follows:

$$\mathcal{L}_1 = \begin{bmatrix} 4.5 & -0.5 & 0 & -2 & 0 & 0 \\ -1 & 3.3 & -0.3 & 0 & 0 & 0 \\ 0 & -0.3 & 1.3 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1.3 & -0.3 & 0 \\ 0 & 0 & 0 & -0.6 & 2.6 & -1 \\ 0 & 0 & 0 & 0 & -0.2 & 0.2 \end{bmatrix}, \mathcal{L}_2 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $P = \text{diag}\{1, 2, 2, 0.5, 1, 0.2\}$. Then, we can obtain

$$\mathcal{L}_1 P = \begin{bmatrix} 4.5 & -1 & 0 & -1 & 0 & 0 \\ -1 & 6.6 & -0.6 & 0 & 0 & 0 \\ 0 & -0.6 & 2.6 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0.65 & -0.3 & 0 \\ 0 & 0 & 0 & -0.3 & 2.6 & -0.2 \\ 0 & 0 & 0 & 0 & -0.2 & 0.04 \end{bmatrix}.$$

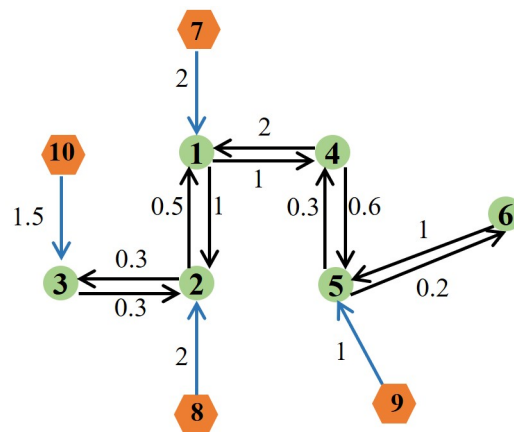


Figure 2. Graph with six followers and four targets.

From this, we know $\mathcal{L}_1 P$ is a positive defined and symmetric matrix. Thus, the weights of the graph satisfy Assumption 4. In what follows, we implement the proposed observer (3.3) and control protocol (3.7). Select the parameters as $\alpha = \frac{2}{3}$, $\beta = \frac{11}{13}$, $\epsilon = \frac{3}{7}$. Moreover, the formation is given as follows:

$$\begin{cases} \rho_{ix} = \varrho[\cos(ct + \frac{2i\pi}{6}), \sin(ct + \frac{2i\pi}{6}) + e^{-2t}]^T, \\ \rho_{iv} = \varrho[-c \sin(ct + \frac{2i\pi}{6}), c \cos(ct + \frac{2i\pi}{6}) - 2e^{-2t}]^T, \end{cases}$$

with $\varrho = 0.18$, $c = 0.00005$, $i = 1, \dots, 6$. Set the initial values of the six followers as

$$\begin{cases} x_1(0) = [0.05, 0.2]^T, x_2(0) = [0.04, 0.1]^T, x_3(0) = [0.01, 0.3]^T, \\ x_4(0) = [0.08, 0.1]^T, x_5(0) = [0.04, 0.2]^T, x_6(0) = [0.09, 0.6]^T, \\ v_1(0) = [1.2, 1.1]^T, v_2(0) = [0.5, 0.1]^T, v_3(0) = [0.1, 0.6]^T, \\ v_4(0) = [0.2, 1.1]^T, v_5(0) = [0.1, 0.1]^T, v_6(0) = [0.1, 0.2]^T. \end{cases}$$

Choose the initial values of the four targets as

$$\begin{cases} x_7(0) = [-0.04, 0.2]^T, x_8(0) = [0.02, 0.14]^T, x_9(0) = [-0.04, 0.4]^T, \\ x_{10}(0) = [0.01, 0.2]^T, v_7(0) = [0.01, 0.01]^T, v_8(0) = [0.01, 0.01]^T, \\ v_9(0) = [0.01, 0.01]^T, v_{10}(0) = [0.01, 0.01]^T. \end{cases}$$

Randomly select the values of $\hat{x}_{ik1}(0)$'s, $\hat{x}_{ik2}(0)$'s, $\hat{v}_{ik1}(0)$'s, and $\hat{v}_{ik2}(0)$'s in $[-1, 1]$ and the values of $\gamma_{ikx}(0)$'s, $\gamma_{ikv}(0)$'s, and $h_i(0)$'s in $(0, 2]$. Using MATLAB, we obtain Figures 3–12, which reveal the trajectories of the closed-loop system signals. Specifically, from Figure 3, one can see that the followers form the expected time-varying formation and surround the moving targets. Therefore, under observers and protocols, the MASs can accomplish the finite-time surrounding with expected time-varying formation. We see from Figures 4–7 that observer errors $(\hat{v}_{ik1}(t) - v_{k1}(t))$'s, $(\hat{v}_{ik2}(t) - v_{k2}(t))$'s, $(\hat{x}_{ik1}(t) - x_{k1}(t))$'s, and $(\hat{x}_{ik2}(t) - x_{k2}(t))$'s converge to zero. Moreover, from Figures 8–12, we see error variables $e_{iv1}(t)$'s and $e_{iv2}(t)$'s converge to zero; besides, the trajectories of control protocols $u_i(t)$'s and dynamic gains $\gamma_{ikx}(t)$'s, $\gamma_{ikv}(t)$'s, and $h_i(t)$'s are bounded. In addition, from the fourth subplot in Figure 11, we know $\varrho - 2M(t) > 0$ for all $t > 65$ s (let $M(t) = \max_k \{\|x_k(t) - q(t)\|\}$), indicating that the geometric condition for successful encirclement, $\varrho > 2M(t)$, holds after $t = 65$ s.

Simulation results verify the convergence of observer errors, surrounding errors, dynamic gains, and protocols. Different from conventional finite-time methods that converge slowly at large state errors, the proposed method introduces a linear acceleration term in the Lyapunov function and achieves faster convergence for states far from the origin, thus improving the transient performance. Further quantitative benchmark comparisons under identical conditions will be conducted in future work.

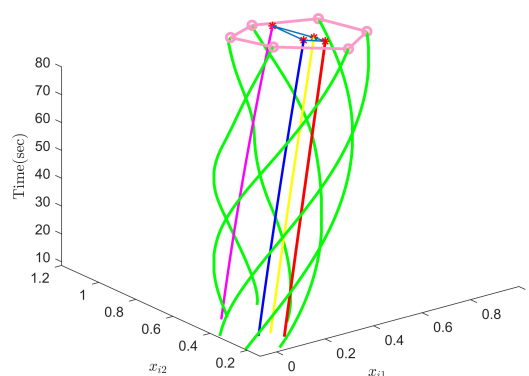


Figure 3. Encirclement configuration with six followers encircling four moving targets.

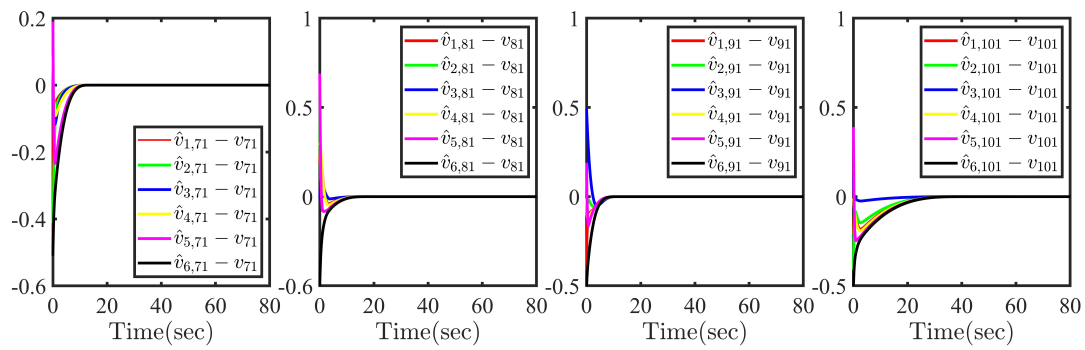


Figure 4. The trajectories of observer errors $\hat{v}_{ik1} - v_{k1}$, $i = 1, \dots, 6$, $k = 7, \dots, 10$.

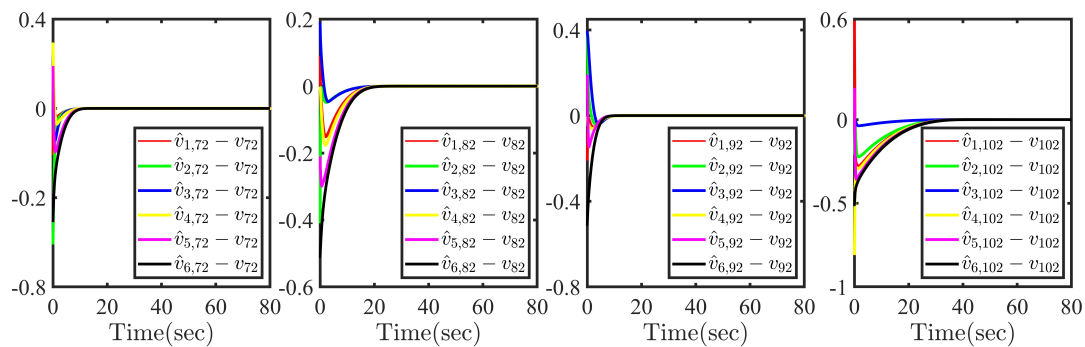


Figure 5. The trajectories of observer errors $\hat{v}_{ik2} - v_{k2}$, $i = 1, \dots, 6$, $k = 7, \dots, 10$.

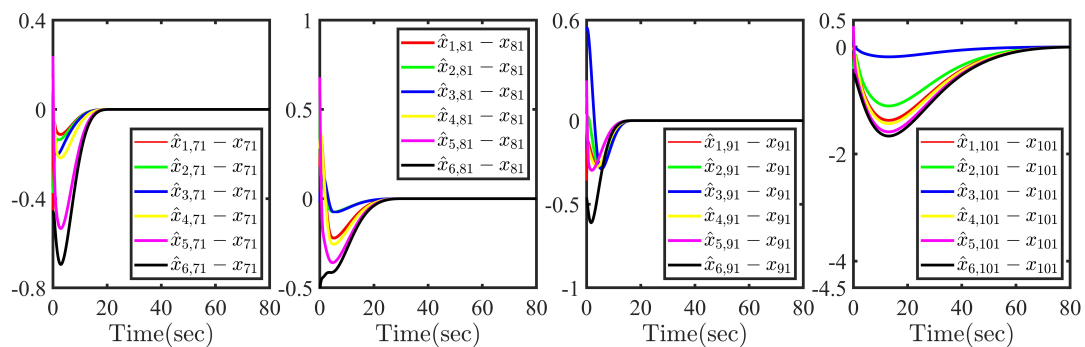


Figure 6. The trajectories of observer errors $\hat{x}_{ik1} - x_{k1}$, $i = 1, \dots, 6$, $k = 7, \dots, 10$.

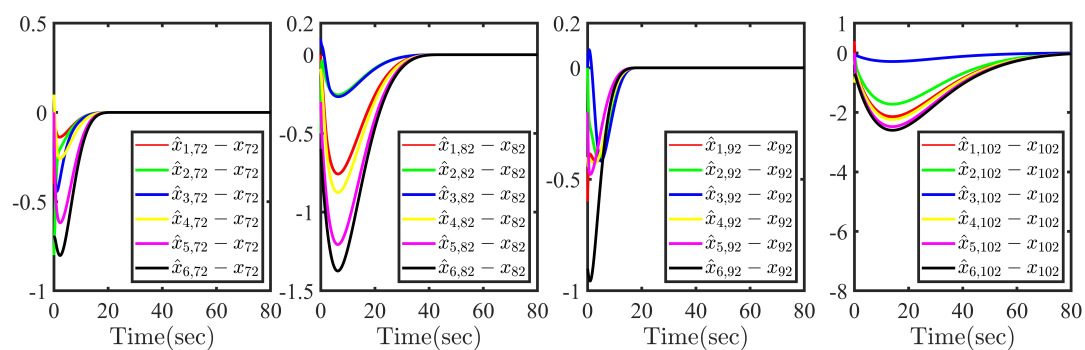


Figure 7. The trajectories of observer errors $\hat{x}_{ik2} - x_{k2}$, $i = 1, \dots, 6$, $k = 7, \dots, 10$.

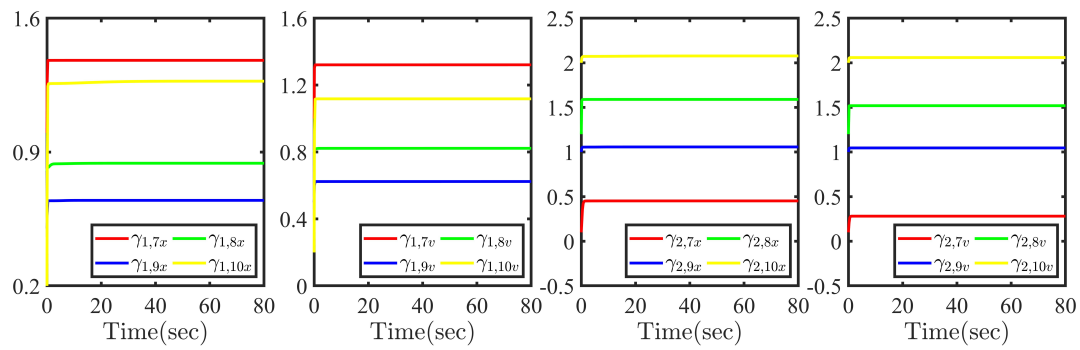


Figure 8. The trajectories of high gains $\gamma_{ikx}, \gamma_{ikv}, i = 1, 2, k = 7, \dots, 10$.

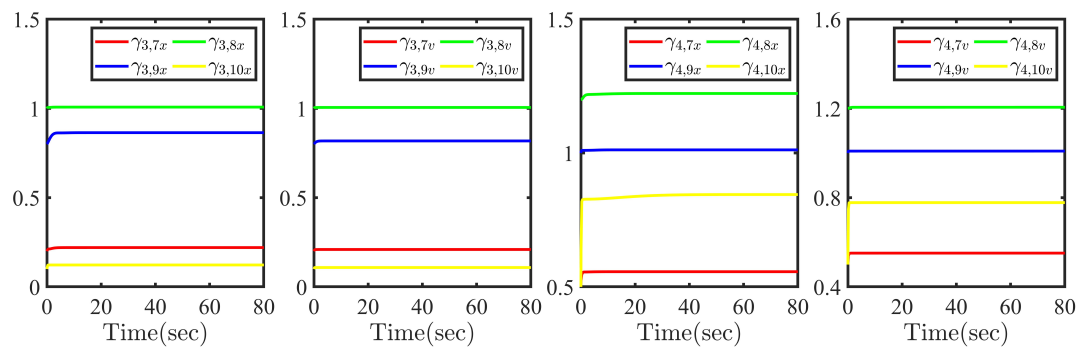


Figure 9. The trajectories of high gains $\gamma_{ikx}, \gamma_{ikv}, i = 3, 4, k = 7, \dots, 10$.

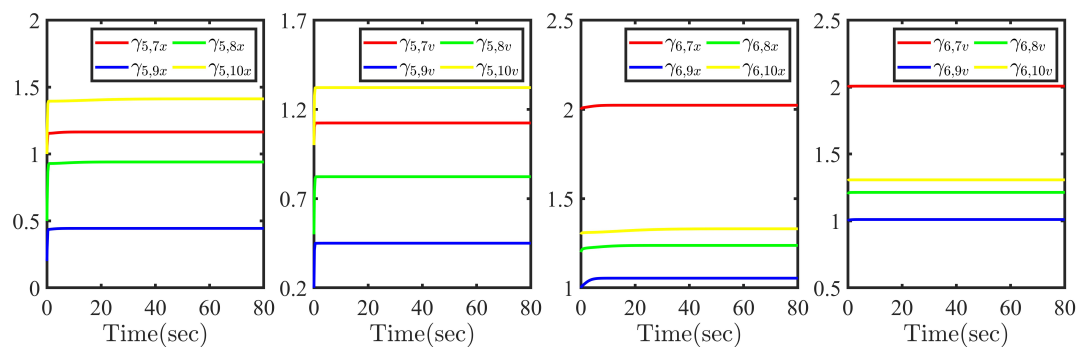


Figure 10. The trajectories of high gains $\gamma_{ikx}, \gamma_{ikv}, i = 5, 6, k = 7, \dots, 10$.

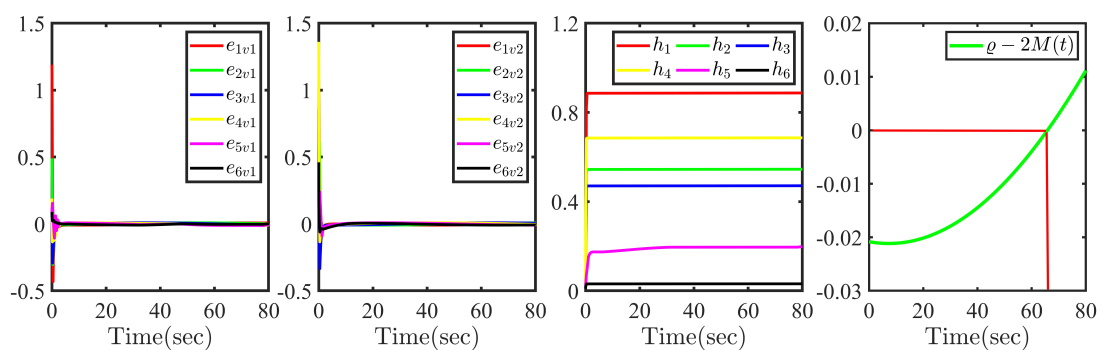


Figure 11. The trajectories of errors e_{ivs} , high gains h_i , and $\varrho - 2M(t), i = 1, \dots, 6, s = 1, 2$.

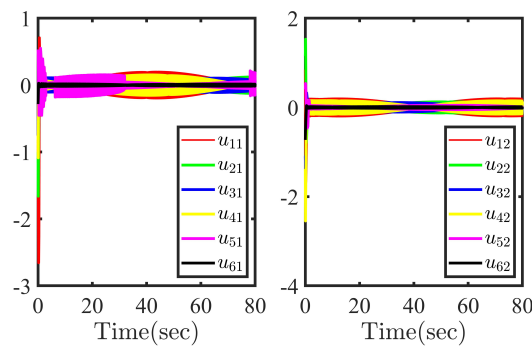


Figure 12. The trajectories of controls u_{is} , $i = 1, \dots, 6$, $s = 1, 2$.

6. Conclusions

In this paper, we have solved the practical accelerated finite-time surrounding problem for multiple moving targets, in which the nonlinear MASs with time-varying unknown control coefficients were concerned. Because only a portion of followers could obtain target information directly, by adaptive technique and merely utilizing the relative information, distributed observers for each follower have been given to estimate targets states, and then we have obtained the convex hull formed by the estimated states. This is different from the related works [25, 26] where the target information was applied directly. Based on the observers, we have given an adaptive distributed protocol with dynamic high gain. It proved that the multi-target surrounding errors achieve practical accelerated finite-time convergence. Compared with conventional finite-time surrounding strategies in [5, 19, 20, 22], the proposed scheme has obtained shorter settling time and superior transient convergence performance. This work has provided an effective distributed control solution for multi-target encirclement under complex unknown system conditions. Future research will extend the established framework to more practical scenarios, including systems with unknown nonlinear disturbances, fully unmeasurable target velocities, and time-varying communication topologies.

Use of AI tools declaration

The author declares that no Artificial Intelligence (AI) tools were used in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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Appendices

A. Appendix A

Proof of Proposition 1:

First, by error system (3.4), $\zeta_{kx} = (\mathcal{L}_{1k} \otimes I_n)\tilde{x}_k$, and $\zeta_{kv} = (\mathcal{L}_{1k} \otimes I_n)\tilde{v}_k$, we obtain

$$\begin{aligned}
 \dot{\tilde{W}} &= \zeta_{kx}^T((\mathcal{L}_{1k}P_k)^{-1} \otimes I_n)\dot{\zeta}_{kx} + \zeta_{kv}^T((\mathcal{L}_{1k}P_k)^{-1} \otimes I_n)\dot{\zeta}_{kv} \\
 &= \zeta_{kx}^T(P_k^{-1} \otimes I_n)\dot{\tilde{x}}_k + \zeta_{kv}^T(P_k^{-1} \otimes I_n)\dot{\tilde{v}}_k \\
 &\leq -\sum_{i=1}^n \gamma_{ikx} \zeta_{ikx}^T P_k^{-1} \zeta_{ikx} - \sum_{i=1}^n \zeta_{ikx}^T P_k^{-1} \zeta_{ikx}^\alpha - \sum_{i=1}^n \zeta_{ikx}^T P_k^{-1} \tilde{v}_k - \sum_{i=1}^n \gamma_{ikv} \zeta_{ikv}^T P_k^{-1} \zeta_{ikv} \\
 &\quad - \sum_{i=1}^n \zeta_{ikv}^T P_k^{-1} \zeta_{ikv}^\alpha + \sum_{i=1}^n \zeta_{ikv}^T P_k^{-1} (g(t, \hat{x}_{ik}, \hat{v}_{ik}) - g(t, x_k, v_k)),
 \end{aligned} \tag{A.1}$$

where $\dot{\tilde{W}} = \dot{\tilde{W}}_1 + \dot{\tilde{W}}_2$.

By Young's inequality and $\zeta_{kv} = (\mathcal{L}_{1k} \otimes I_n)\tilde{v}_k$, we obtain that

$$\begin{aligned}
 -\sum_{i=1}^n \zeta_{ikx}^T P_k^{-1} \tilde{v}_k &\leq \sum_{i=1}^n \|P_k^{-1}\| \cdot \|\zeta_{ikx}\| \cdot \|\tilde{v}_k\| \leq \frac{\bar{p}_k}{2} \sum_{i=1}^n \|\zeta_{ikx}\|^2 + \frac{\bar{p}_k}{2} \sum_{i=1}^n \|\tilde{v}_k\|^2 \\
 &\leq \frac{\bar{p}_k}{2} \sum_{i=1}^n \sum_{s=1}^n \zeta_{iks}^2 + \frac{\bar{p}_k}{2\lambda_{1k}^2} \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikvs}^2.
 \end{aligned} \tag{A.2}$$

From Assumption 1, Young's inequality, and $\zeta_{kx} = (\mathcal{L}_{1k} \otimes I_n)\tilde{x}_k$, $\zeta_{kv} = (\mathcal{L}_{1k} \otimes I_n)\tilde{v}_k$, it follows that

$$\begin{aligned}
 & \sum_{i=1}^n \zeta_{ikv}^T P_k^{-1} (g(t, \hat{x}_{ik}, \hat{v}_{ik}) - g(t, x_k, v_k)) \\
 & \leq \sum_{i=1}^n \theta_1 \|P_k^{-1}\| \cdot \|\zeta_{ikv}\| \cdot \|\hat{x}_{ik} - x_k\| + \sum_{i=1}^n \theta_2 \|P_k^{-1}\| \cdot \|\zeta_{ikv}\| \cdot \|\hat{v}_{ik} - v_k\| \\
 & \leq \sum_{i=1}^n \bar{p}_k \theta \|\zeta_{ikv}\| \cdot \|\tilde{x}_{ik}\| + \sum_{i=1}^n \bar{p}_k \theta \|\zeta_{ikv}\| \cdot \|\tilde{v}_{ik}\| \\
 & \leq \theta^2 \bar{p}_k \sum_{i=1}^n \|\zeta_{ikv}\|^2 + \frac{\bar{p}_k}{2} \sum_{i=1}^n \tilde{x}_{ik}^T \tilde{x}_{ik} + \frac{\bar{p}_k}{2} \sum_{i=1}^n \tilde{v}_{ik}^T \tilde{v}_{ik} \\
 & \leq \theta^2 \bar{p}_k \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikvs}^2 + \frac{\bar{p}_k}{2\lambda_{1k}^2} \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikxs}^2 + \frac{\bar{p}_k}{2\lambda_{1k}^2} \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikvs}^2, \tag{A.3}
 \end{aligned}$$

where $\theta = \max\{\theta_1, \theta_2\}$, $\lambda_{1k} = \lambda_{\min}(\mathcal{L}_{1k})$. Substituting (A.2)–(A.3) into (A.1), we derive

$$\begin{aligned}
 \dot{\bar{W}} & \leq - \sum_{i=1}^n \sum_{s=1}^n \gamma_{ikx} p_{ik}^{-1} \zeta_{ikxs}^2 - \sum_{i=1}^n \sum_{s=1}^n p_{ik}^{-1} |\zeta_{ikxs}|^{1+\alpha} - \sum_{i=1}^n \sum_{s=1}^n \gamma_{ikv} p_{ik}^{-1} \zeta_{ikvs}^2 \\
 & \quad - \sum_{i=1}^n \sum_{s=1}^n p_{ik}^{-1} |\zeta_{ikvs}|^{1+\alpha} + \left(\frac{\bar{p}_k}{2} + \frac{\bar{p}_k}{2\lambda_{1k}^2} \right) \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikxs}^2 + \left(\frac{\bar{p}_k}{\lambda_{1k}^2} + \theta^2 \bar{p}_k \right) \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikvs}^2,
 \end{aligned}$$

which, together with the formulation of $W(\cdot)$ and (3.3), implies

$$\begin{aligned}
 \dot{W} & \leq - \sum_{i=1}^n \sum_{s=1}^n \gamma_{ikx} p_{ik}^{-1} \zeta_{ikxs}^2 - \sum_{i=1}^n \sum_{s=1}^n p_{ik}^{-1} |\zeta_{ikxs}|^{1+\alpha} - \sum_{i=1}^n \sum_{s=1}^n \gamma_{ikv} p_{ik}^{-1} \zeta_{ikvs}^2 \\
 & \quad - \sum_{i=1}^n \sum_{s=1}^n p_{ik}^{-1} |\zeta_{ikvs}|^{1+\alpha} + \left(\frac{\bar{p}_k}{2} + \frac{\bar{p}_k}{2\lambda_{1k}^2} \right) \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikxs}^2 + \left(\frac{\bar{p}_k}{\lambda_{1k}^2} + \theta^2 \bar{p}_k \right) \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikvs}^2 \\
 & \quad + \sum_{i=1}^n \sum_{s=1}^n \frac{(\gamma_{ikx} - \tau_k p_{ik})}{p_{ik}} \zeta_{ikxs}^2 + \sum_{i=1}^n \sum_{s=1}^n \frac{(\gamma_{ikv} - \tau_k p_{ik})}{p_{ik}} \zeta_{ikvs}^2.
 \end{aligned}$$

Choose a sufficiently large constant τ_k such that $\tau_k - \theta^2 \bar{p}_k - \frac{3\bar{p}_k}{2\lambda_{1k}^2} - \frac{\bar{p}_k}{2} = \mu_1 > 0$, then there holds

$$\dot{W} = -\mu_1 \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikxs}^2 - \sum_{i=1}^n \sum_{s=1}^n p_{ik}^{-1} |\zeta_{ikxs}|^{1+\alpha} - \mu_1 \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikvs}^2 - \sum_{i=1}^n \sum_{s=1}^n p_{ik}^{-1} |\zeta_{ikvs}|^{1+\alpha}. \tag{A.4}$$

By this and from (3.5), we know $W(\tilde{x}_k(t), \tilde{v}_k(t), \gamma_{kx}(t), \gamma_{kv}(t))$ is bounded on $[0, +\infty)$, and furthermore, $|\gamma_{ikx}(t) - \tau_k p_{ik}|$'s, $|\gamma_{ikv}(t) - \tau_k p_{ik}|$'s, $\|\zeta_{ikx}(t)\|$'s, $\|\zeta_{ikv}(t)\|$'s, $\|\dot{\zeta}_{ikx}(t)\|$'s, and $\|\dot{\zeta}_{ikv}(t)\|$'s are also bounded. In addition, by (A.4), it's easy to verify that $\zeta_{ikxs}(t)$'s, $\zeta_{ikvs}(t)$'s are square integrable. Therefore, from Barbălat's lemma, it follows that $\lim_{t \rightarrow +\infty} \zeta_{ikxs}(t) = 0$, $\lim_{t \rightarrow +\infty} \zeta_{ikvs}(t) = 0$, $i \in \mathcal{V}_F$, $k \in \mathcal{V}_T$, $s = 1, \dots, n$. In what follows, we prove that $\zeta_{ikxs}(t)$'s, $\zeta_{ikvs}(t)$'s converge to 0 in finite time. By the definition of $\bar{W}$, (3.3), and (A.4), we can obtain

$$\dot{\bar{W}} \leq -\mu_1 \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikxs}^2 - \sum_{i=1}^n \sum_{s=1}^n p_{ik}^{-1} |\zeta_{ikxs}|^{1+\alpha} - \sum_{i=1}^n \sum_{s=1}^n p_{ik}^{-1} |\zeta_{ikvs}|^{1+\alpha}$$

$$\begin{aligned}
& -\mu_1 \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikvs}^2 - \sum_{i=1}^n \sum_{s=1}^n \frac{(\gamma_{ikx} - \tau_k p_{ik})}{p_{ik}} \zeta_{ikxs}^2 - \sum_{i=1}^n \sum_{s=1}^n \frac{(\gamma_{ikv} - \tau_k p_{ik})}{p_{ik}} \zeta_{ikvs}^2 \\
& \leq -\frac{2\mu_1}{\lambda_{2k}} \bar{W} - \frac{\tilde{p}_k}{2} \left(\frac{2}{\lambda_{2k}}\right)^{\frac{1+\alpha}{2}} W_1^{\frac{1+\alpha}{2}} - \frac{\tilde{p}_k}{2} \left(\frac{2}{\lambda_{2k}}\right)^{\frac{1+\alpha}{2}} W_2^{\frac{1+\alpha}{2}} + \bar{\gamma}_k \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikvs}^2 \\
& \quad + \bar{\gamma}_k \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikxs}^2 - \frac{\tilde{p}_k}{2} \sum_{i=1}^n \sum_{s=1}^n |\zeta_{ikvs}|^{1+\alpha} - \frac{\tilde{p}_k}{2} \sum_{i=1}^n \sum_{s=1}^n |\zeta_{ikxs}|^{1+\alpha} \\
& \leq -\frac{2\mu_1}{\lambda_{2k}} \bar{W} - \frac{\tilde{p}_k}{2} \left(\frac{2}{\lambda_{2k}}\right)^{\frac{1+\alpha}{2}} \bar{W}^{\frac{1+\alpha}{2}} - \frac{\tilde{p}_k}{2} \sum_{i=1}^n \sum_{s=1}^n |\zeta_{ikxs}|^{1+\alpha} \\
& \quad + \bar{\gamma}_k \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikxs}^2 - \frac{\tilde{p}_k}{2} \sum_{i=1}^n \sum_{s=1}^n |\zeta_{ikvs}|^{1+\alpha} + \bar{\gamma}_k \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikvs}^2,
\end{aligned} \tag{A.5}$$

where $\tilde{p}_k = \min_{i=1, \dots, n} \{p_{ik}^{-1}\}$, $|\frac{(\gamma_{ikx} - \tau_k p_{ik})}{p_{ik}}| \leq \bar{\gamma}_k$, $|\frac{(\gamma_{ikv} - \tau_k p_{ik})}{p_{ik}}| \leq \bar{\gamma}_k$.

Let $\varphi_1(t) = \sum_{i=1}^n \sum_{s=1}^n |\zeta_{ikxs}|^{1+\alpha}(t)$, $\varphi_2(t) = \sum_{i=1}^n \sum_{s=1}^n |\zeta_{ikvs}|^{1+\alpha}(t)$. Then, we know

$$\begin{cases} -\frac{\tilde{p}_k}{2} \sum_{i=1}^n \sum_{s=1}^n |\zeta_{ikxs}|^{1+\alpha} + \bar{\gamma}_k \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikxs}^2 \leq -\varphi_1(t) \left(\frac{\tilde{p}_k}{2} - \bar{\gamma}_k \varphi_1^{\frac{1-\alpha}{1+\alpha}}(t)\right), \\ -\frac{\tilde{p}_k}{2} \sum_{i=1}^n \sum_{s=1}^n |\zeta_{ikvs}|^{1+\alpha} + \bar{\gamma}_k \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikvs}^2 \leq -\varphi_2(t) \left(\frac{\tilde{p}_k}{2} - \bar{\gamma}_k \varphi_2^{\frac{1-\alpha}{1+\alpha}}(t)\right). \end{cases} \tag{A.6}$$

Noting that $\lim_{t \rightarrow +\infty} \zeta_{ikxs}(t) = 0$, $\lim_{t \rightarrow +\infty} \zeta_{ikvs}(t) = 0$, $i \in \mathcal{V}_F$, $k \in \mathcal{V}_T$, $s = 1, \dots, n$, thus $\varphi_1^{\frac{1-\alpha}{1+\alpha}}(t) \leq \frac{\tilde{p}_k}{2\bar{\gamma}_k}$, $\varphi_2^{\frac{1-\alpha}{1+\alpha}}(t) \leq \frac{\tilde{p}_k}{2\bar{\gamma}_k}$, $\forall t \geq T_{1,k}$, $T_{1,k}$ is a finite time. Therefore, for any $t \geq T_{1,k}$, we arrive at

$$-\frac{\tilde{p}_k}{2} \sum_{i=1}^n \sum_{s=1}^n |\zeta_{ikxs}|^{1+\alpha} + \bar{\gamma}_k \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikxs}^2 - \frac{\tilde{p}_k}{2} \sum_{i=1}^n \sum_{s=1}^n |\zeta_{ikvs}|^{1+\alpha} + \bar{\gamma}_k \sum_{i=1}^n \sum_{s=1}^n \zeta_{ikvs}^2 \leq 0.$$

Substituting this into (A.5), we deduce that, for any $t \geq T_{1,k}$,

$$\dot{\bar{W}} \leq -\frac{2\mu_1}{\lambda_{2k}} \bar{W} - \frac{\tilde{p}_k}{2} \left(\frac{2}{\lambda_{2k}}\right)^{\frac{1+\alpha}{2}} \bar{W}^{\frac{1+\alpha}{2}}. \tag{A.7}$$

Invoking Lemma 1, we can conclude that $\bar{W}(\cdot)$ accelerated converges to 0 at an exponential rate in finite time, that is,

$$\begin{cases} \lim_{t \rightarrow T_{2k}} \bar{W}(\tilde{x}_k(t), \tilde{v}_k(t)) = 0, \\ \bar{W}(\tilde{x}_k(t), \tilde{v}_k(t)) = 0, \quad \forall t \geq T_{2k}, \end{cases}$$

where $T_{2,k} = \frac{\lambda_{2k}}{\tilde{p}_k(1-\alpha)} \ln(1 + (\frac{2\bar{W}(\tilde{x}_k(t), \tilde{v}_k(t))}{\lambda_{2k}})^{\frac{1-\alpha}{2}}) + T_{1,k}$. Then, we have for any $k \in \mathcal{V}_T$,

$$\begin{cases} \lim_{t \rightarrow T_1} (\zeta_{kx}(t), \zeta_{kv}(t)) = (\mathbf{0}_{n^2}, \mathbf{0}_{n^2}), \\ (\zeta_{kx}(t), \zeta_{kv}(t)) = (\mathbf{0}_{n^2}, \mathbf{0}_{n^2}), \quad t \geq T_1, \end{cases}$$

where $T_1 = \max\{T_{2,n+1}, \dots, T_{2,n+m}\}$. Furthermore, we can obtain that for any $i \in \mathcal{V}_F$, $k \in \mathcal{V}_T$,

$$\begin{cases} \lim_{t \rightarrow T_1} (\tilde{x}_{ik}(t), \tilde{v}_{ik}(t)) = (\mathbf{0}_n, \mathbf{0}_n), \\ (\tilde{x}_{ik}(t), \tilde{v}_{ik}(t)) = (\mathbf{0}_n, \mathbf{0}_n), \quad t \geq T_1. \end{cases}$$

B. Appendix B

Proof of Proposition 2:

First, by error system (3.6), $\xi = (\mathcal{L}_1 \otimes I_n)e_x$, and $\dot{\rho}_{ix}(t) = \rho_{iv}(t)$, we know

$$\begin{aligned} D^+V_1 &= \xi^T(Q^{-1}\mathcal{L}_1^{-1} \otimes I_n)(\mathcal{L}_1 \otimes I_n)\dot{e}_x = \xi^T(Q^{-1} \otimes I_n)(v - \bar{v} - \dot{\rho}_x) \\ &= \sum_{i=1}^n \xi_i^T Q^{-1}(v_i - z_i - \rho_{iv} + z_i - \bar{v}_i + \rho_{iv} - \dot{\rho}_{ix}) \\ &= \sum_{i=1}^n \xi_i^T Q^{-1}(e_{iv} - e_{iv}^*) + \sum_{i=1}^n \xi_i^T Q^{-1}e_{iv}^* + \sum_{i=1}^n \xi_i^T Q^{-1} \sum_{k=n+1}^{n+m} \omega_{ik} \tilde{v}_{ik} \\ &= : \textcircled{1} + \textcircled{2} + \textcircled{3}, \end{aligned} \quad (\text{B.1})$$

where $\bar{v} = [\bar{v}_1, \dots, \bar{v}_n]^T$.

In what follows, we handle the three items, respectively. From Young's inequality and Lemma 2, we derive

$$\begin{aligned} \textcircled{1} &\leq \frac{\bar{q}\delta}{2} \sum_{i=1}^n \xi_i^T \xi_i + \frac{\bar{q}}{2\delta} \sum_{i=1}^n (e_{iv} - e_{iv}^*)^T (e_{iv} - e_{iv}^*) \\ &\leq \frac{\bar{q}\delta}{2} \sum_{i=1}^n \sum_{s=1}^n \xi_{is}^2 + \frac{\bar{q}2^{2(1-\beta)}}{2\delta} \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{2\beta}, \end{aligned} \quad (\text{B.2})$$

where $\bar{q} = \|Q^{-1}\|$. In addition, noting that $e_{iv}^* = -\xi_i^\beta - \xi_i^3$, then we have

$$\textcircled{2} = -\sum_{i=1}^n \xi_i^T Q^{-1} \xi_i^\beta - \sum_{i=1}^n \xi_i^T Q^{-1} \xi_i^3 \leq -\underline{q} \sum_{i=1}^n \sum_{s=1}^n (|\xi_{is}|^{1+\beta} + \xi_{is}^4), \quad (\text{B.3})$$

with $\underline{q} = \min_{i=1, \dots, n} \{q_i^{-1}\}$. Utilizing Young's inequality and Lemma 3, we deduce

$$\textcircled{3} \leq \frac{\bar{q}\sqrt{m}}{2\delta} \sum_{i=1}^n \sum_{s=1}^n \sum_{k=n+1}^{n+m} \omega_{ik}^2 \tilde{v}_{iks}^2 + \delta \bar{q} \sum_{i=1}^n \sum_{s=1}^n \xi_{is}^2,$$

which, together with $\omega_{ik} \leq 1$, implies

$$\textcircled{3} \leq \frac{\bar{q}\sqrt{m}}{2\delta} \sum_{i=1}^n \sum_{k=n+1}^{n+m} \tilde{v}_{ik}^T \tilde{v}_{ik} + \delta \bar{q} \sum_{i=1}^n \sum_{s=1}^n \xi_{is}^2. \quad (\text{B.4})$$

Substituting (B.2)–(B.4) into (B.1), we derive that

$$\begin{aligned} D^+V_1 &\leq -\underline{q} \sum_{i=1}^n \sum_{s=1}^n (|\xi_{is}|^{1+\beta} + \xi_{is}^4) + \frac{3\delta\bar{q}}{2} \sum_{i=1}^n \sum_{s=1}^n \xi_{is}^2 \\ &\quad + \frac{\bar{q}2^{2(1-\beta)}}{2\delta} \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{2\beta} + \frac{\bar{q}\sqrt{m}}{2\delta} \sum_{i=1}^n \sum_{k=n+1}^{n+m} \tilde{v}_{ik}^T \tilde{v}_{ik}. \end{aligned} \quad (\text{B.5})$$

Next, we estimate D^+V_2 . According to the definition of $V_2(\cdot)$, we have

$$D^+V_2 = \sum_{i=1}^n \eta_i^T \dot{e}_{iv} + \int_{e_{iv}^*}^{e_{iv}} ds \cdot \frac{\partial e_{iv}^{*\frac{1}{\beta}}}{\partial \xi_i} \dot{\xi}_i =: \textcircled{4} + \textcircled{5}. \quad (\text{B.6})$$

In what follows, we estimate $\textcircled{4}$ and $\textcircled{5}$, respectively.

First, by (3.6), we derive that

$$\textcircled{4} = \sum_{i=1}^n \eta_i^T (b_i(t)u_i + \sum_{k=n+1}^{n+m} \gamma_{ikv} \omega_{ik} \zeta_{ikv} + \sum_{k=n+1}^{n+m} \omega_{ik} \zeta_{ikv}^\alpha + g(t, x_i, v_i) - \sum_{k=n+1}^{n+m} \omega_{ik} g(t, \hat{x}_{ik}, \hat{v}_{ik}) - \dot{\rho}_{iv}). \quad (\text{B.7})$$

Noting the boundedness of γ_{ikv} , one can assume $|\gamma_{ikv}| \leq l_k$. Then, by Young's inequality, it's easy to obtain

$$\begin{aligned} \eta_i^T \left(\sum_{k=n+1}^{n+m} \gamma_{ikv} \omega_{ik} \zeta_{ikv} + \sum_{k=n+1}^{n+m} \omega_{ik} \zeta_{ikv}^\alpha \right) &\leq \sum_{k=n+1}^{n+m} \omega_{ik} (\gamma_{ikv} \eta_i^T \zeta_{ikv} + \sum_{s=1}^n |\eta_{is}| \cdot |\zeta_{ikvs}|^\alpha) \\ &\leq \sum_{k=n+1}^{n+m} \omega_{ik} \left(\frac{1}{2} \zeta_{ikv}^T \zeta_{ikv} + \frac{l_k^2}{2} \sum_{s=1}^n \eta_{is}^2 + \frac{\alpha}{2} \sum_{s=1}^n \zeta_{ikvs}^2 + \frac{2-\alpha}{2} \sum_{s=1}^n \eta_{is}^{\frac{2}{2-\alpha}} \right), \end{aligned}$$

which, together with $\sum_{k=n+1}^{n+m} \omega_{ik} = 1$, means

$$\eta_i^T \left(\sum_{k=n+1}^{n+m} \gamma_{ikv} \omega_{ik} \zeta_{ikv} + \sum_{k=n+1}^{n+m} \omega_{ik} \zeta_{ikv}^\alpha \right) \leq \frac{1+\alpha}{2} \sum_{k=n+1}^{n+m} \zeta_{ik}^T \zeta_{ik} + \frac{l^2}{2} \sum_{s=1}^n \eta_{is}^2 + \frac{2-\alpha}{2} \sum_{s=1}^n \eta_{is}^{\frac{2}{2-\alpha}}, \quad (\text{B.8})$$

where $l = \max\{l_1, \dots, l_m\}$.

In addition, according to Assumption 1 and the definitions of e_{ix} , e_{iv} , $\tilde{x}_{ik}$, $\tilde{v}_{ik}$, and e_{iv}^* , we know

$$\begin{aligned} &\sum_{i=1}^n \eta_i^T (g(t, x_i, v_i) - \sum_{k=n+1}^{n+m} \omega_{ik} g(t, \hat{x}_{ik}, \hat{v}_{ik})) \\ &\leq \sum_{i=1}^n \theta_1 \|\eta_i\| \cdot \|x_i - \bar{x}_i - \rho_{ix} + \rho_{ix} + \bar{x}_i - \sum_{k=n+1}^{n+m} \omega_{ik} \hat{x}_{ik}\| + \sum_{i=1}^n \theta_2 \|\eta_i\| \cdot \|v_i - \sum_{k=n+1}^{n+m} \omega_{ik} \hat{v}_{ik}\| \\ &= \sum_{i=1}^n \theta_1 \|\eta_i\| \cdot \|x_i - \bar{x}_i - \rho_{ix} + \rho_{ix} + \bar{x}_i - \sum_{k=n+1}^{n+m} \omega_{ik} \hat{x}_{ik}\| + \sum_{i=1}^n \theta_2 \|\eta_i\| \cdot \|v_i - \sum_{k=n+1}^{n+m} \omega_{ik} \hat{v}_{ik} - \rho_{iv} + \rho_{iv}\| \\ &\leq \sum_{i=1}^n \theta \|e_{ix}\| \cdot \|\eta_i\| + \sum_{i=1}^n \theta \|e_{iv} - e_{iv}^*\| \cdot \|\eta_i\| + \sum_{i=1}^n \theta \|\eta_i\| \cdot \|\xi_i^\beta + \xi_i^3\| \\ &\quad + \sum_{i=1}^n \theta \|\eta_i\| \cdot \left\| \sum_{k=n+1}^{n+m} \omega_{ik} \tilde{x}_{ik} \right\| + \sum_{i=1}^n \theta \|\eta_i\| \cdot \|\rho_{ix} + \rho_{iv}\|, \end{aligned}$$

which, together with Lemma 2, Lemma 3, Young's inequality, $\zeta_{kx} = (\mathcal{L}_{1k} \otimes I_n) \tilde{x}_k$, $\zeta_{kv} = (\mathcal{L}_{1k} \otimes I_n) \tilde{v}_k$, and $\xi = (\mathcal{L}_{1k} \otimes I_n) e_x$, implies

$$\left\{ \begin{array}{l} \sum_{i=1}^n \theta \|e_{ix}\| \cdot \|\eta_i\| \leq \frac{\theta}{2} \sum_{i=1}^n \eta_i^T \eta_i + \frac{\theta}{2} \sum_{i=1}^n e_{ix}^T e_{ix} \leq \frac{\theta}{2} \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^2 + \frac{\theta}{2\lambda_{1k}^2} \sum_{i=1}^n \sum_{s=1}^n \xi_{is}^2, \\ \sum_{i=1}^n \theta \|\eta_i\| \cdot \|\rho_{ix} + \rho_{ix}\| \leq \theta \delta \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}| \cdot |\rho_{ixs} + \rho_{ivs}|, \\ \sum_{i=1}^n \theta \|\eta_i\| \cdot \left\| \sum_{k=n+1}^{n+m} \omega_{ik} \tilde{x}_{ik} \right\| \leq \frac{\theta}{2} \sum_{i=1}^n \eta_i^T \eta_i + \frac{\theta}{2} \sum_{i=1}^n \left\| \sum_{k=n+1}^{n+m} \omega_{ik} \tilde{x}_{ik} \right\|^2 \\ \leq \frac{\theta}{2} \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^2 + \frac{n\theta}{2} \sum_{i=1}^n \sum_{k=n+1}^{n+m} \omega_{ik}^2 \tilde{x}_{ik}^T \tilde{x}_{ik}, \\ \sum_{i=1}^n \theta \|e_{iv} - e_{iv}^*\| \cdot \|\eta_i\| + \sum_{i=1}^n \theta \|\eta_i\| \cdot \|\xi_i^\beta + \xi_i^3\| \leq \frac{\beta\delta}{1+\beta} \sum_{i=1}^n \sum_{s=1}^n |\xi_{is}|^{1+\beta} + \frac{3\delta}{4} \sum_{i=1}^n \sum_{s=1}^n \xi_{is}^4 \\ + \tau_1 \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}|^{1+\beta} + \frac{\theta^4}{4\delta^{\frac{1}{3}}} \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^4. \end{array} \right.$$

Then, we derive that

$$\begin{aligned} & \sum_{i=1}^n \eta_i^T (g(t, x_i, v_i) - \sum_{k=n+1}^{n+m} \omega_{ik} g(t, \hat{x}_{ik}, \hat{v}_{ik})) \\ & \leq \frac{\beta\delta}{1+\beta} \sum_{i=1}^n \sum_{s=1}^n |\xi_{is}|^{1+\beta} + \frac{3\delta}{4} \sum_{i=1}^n \sum_{s=1}^n \xi_{is}^4 + \frac{\theta}{2\lambda_{1k}^2} \sum_{i=1}^n \sum_{s=1}^n \xi_{is}^2 + \tau_1 \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}|^{1+\beta} \\ & \quad + \frac{\theta^4}{4\delta^{\frac{1}{3}}} \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^4 + \theta \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^2 + \theta \delta \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}| \cdot |\rho_{ixs} + \rho_{ivs}| + \frac{n\theta}{2} \sum_{i=1}^n \sum_{k=n+1}^{n+m} \omega_{ik}^2 \tilde{x}_{ik}^T \tilde{x}_{ik}, \quad (\text{B.9}) \end{aligned}$$

where $\tau_1 = 2^{1-\beta}\theta + \frac{\theta^{1+\beta}}{(1+\beta)\delta^{\frac{1}{\beta}}}$. Substituting (B.8)–(B.9) into (B.7), we can obtain the following estimation:

$$\begin{aligned} \textcircled{4} & \leq \sum_{i=1}^n b_i(t) \eta_i^T u_i + \frac{1+\alpha}{2} \sum_{i=1}^n \sum_{k=n+1}^{n+m} \zeta_{ik}^T \zeta_{ik} + \frac{l^2 + 2\theta}{2} \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^2 + \frac{2-\alpha}{2} \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{\frac{2}{2-\alpha}} \\ & \quad + \tau_1 \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}|^{1+\beta} + \frac{\theta^4}{4\delta^{\frac{1}{3}}} \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^4 + \frac{\beta\delta}{1+\beta} \sum_{i=1}^n \sum_{s=1}^n |\xi_{is}|^{1+\beta} + \frac{3\delta}{4} \sum_{i=1}^n \sum_{s=1}^n \xi_{is}^4 \\ & \quad + \frac{\theta}{2\lambda_{1k}^2} \sum_{i=1}^n \sum_{s=1}^n \xi_{is}^2 + \theta \delta \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}| \cdot |\rho_{ixs} + \rho_{ivs} + \dot{\rho}_{ivs}| \\ & \quad + \frac{n\theta}{2} \sum_{i=1}^n \sum_{k=n+1}^{n+m} \omega_{ik}^2 \tilde{x}_{ik}^T \tilde{x}_{ik}. \quad (\text{B.10}) \end{aligned}$$

Now, we estimate $\textcircled{5}$ in (B.6). By $e_{iv}^* = -(\xi_i^\beta + \xi_i^3)$, we arrive at $-e_{iv}^{*\frac{1}{\beta}} = A_{i1} B_{i1}$, where $A_{i1} = \text{diag}\{\xi_{i1}, \dots, \xi_{in}\}$, $B_{i1} = [(1 + \xi_{i1}^{3-\beta})^{\frac{1}{\beta}}, \dots, (1 + \xi_{in}^{3-\beta})^{\frac{1}{\beta}}]^T$. Then, we have $\frac{\partial e_{iv}^{*\frac{1}{\beta}}}{\partial \xi_i} \dot{\xi}_i = A_{i2} B_{i2} \dot{\xi}_i$, where $A_{i2} = \text{diag}\{1 + \frac{3}{\beta} \xi_{i1}^{3-\beta}, \dots, 1 + \frac{3}{\beta} \xi_{in}^{3-\beta}\}$, $B_{i2} = \text{diag}\{(1 + \xi_{i1}^{3-\beta})^{\frac{1}{\beta}-1}, \dots, (1 + \xi_{in}^{3-\beta})^{\frac{1}{\beta}-1}\}$. By this and noting $\dot{\rho}_x = \rho_v$, we derive that

$$\begin{aligned} \textcircled{5} & \leq \sum_{i=1}^n (e_{iv} - e_{iv}^*)^T A_{i2} B_{i2} \dot{\xi}_i = (e_v - e_v^*)^T \Lambda (\mathcal{L}_1 \otimes I_n) (v - \bar{v} - \dot{\rho}_x) \\ & = (e_v - e_v^*)^T \Lambda \mathcal{L}_1 (v - z - \rho_v + \rho_v + z - \bar{v} - \dot{\rho}_x) \end{aligned}$$

$$\begin{aligned}
&= (e_v - e_v^*)^T \Lambda(\mathcal{L}_1 \otimes I_n)(e_v - e_v^*) + (e_v - e_v^*)^T \Lambda(\mathcal{L}_1 \otimes I_n) e_v^* \\
&\quad + (e_v - e_v^*)^T \Lambda(\mathcal{L}_1 \otimes I_n)(z - \bar{v}) \\
&=: \textcircled{6} + \textcircled{7} + \textcircled{8},
\end{aligned} \tag{B.11}$$

where $\Lambda = \text{diag}\{A_{12}B_{12}, \dots, A_{n2}B_{n2}\}$. By a series of computations, we can obtain the following estimations for the three items $\textcircled{6}$, $\textcircled{7}$, and $\textcircled{8}$ (the detailed proof is given in Appendix C):

$$\textcircled{6} \leq 2^{1-2\beta} \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{2\beta} + c_1 \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{2\beta} (1 + \xi_{is}^{\frac{(2-2\beta)(3-\beta)}{\beta}} + \xi_{is}^{\frac{6-2\beta}{\beta}} + \xi_{is}^{6-2\beta}), \tag{B.12}$$

where $c_1 = \lambda_4 2^{1-2\beta} 2^{\frac{2-3\beta}{2(1-\beta)}} \cdot \frac{9\sqrt{2}}{\beta^2}$, $\lambda_4 = \lambda_{\max}(\mathcal{L}_1 \mathcal{L}_1^T)$.

$$\begin{aligned}
\textcircled{7} \leq & c_2 \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{\beta(1+\beta)} (\xi_{is}^{\frac{(1-\beta^2)(3-\beta)}{\beta}} + \xi_{is}^{(3-\beta)(1+\beta)} + \xi_{is}^{\frac{(3-\beta)(1+\beta)}{\beta}} + 1) \\
& + c_3 \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{4\beta} (\xi_{is}^{\frac{(4-4\beta)(3-\beta)}{\beta}} + \xi_{is}^{12-4\beta} + \xi_{is}^{\frac{12-4\beta}{\beta}} + 1) + \delta \sum_{i=1}^n \sum_{s=1}^n (|\xi_{is}|^{1+\beta} + \xi_{is}^4),
\end{aligned} \tag{B.13}$$

where $c_2 = \frac{\lambda_5^{1+\beta} n^{\frac{\beta}{1-\beta}}}{\delta^\beta} 2^{2-\beta^2} \cdot 2^{\frac{\beta^2}{1-\beta^2}} \cdot (\frac{3}{\beta})^{1+\beta}$, $c_3 = \frac{\lambda_5^4 n^{\frac{3}{4}}}{\delta^3} 2^{4-4\beta} \cdot 2^{\frac{7-8\beta^2}{4(1-\beta)}} \cdot (\frac{3}{\beta})^4$, $\lambda_5 = \lambda_{\max}(\mathcal{L}_1)$.

$$\textcircled{8} \leq c_1 \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{2\beta} (1 + \xi_{is}^{\frac{(2-2\beta)(3-\beta)}{\beta}} + \xi_{is}^{\frac{6-2\beta}{\beta}} + \xi_{is}^{6-2\beta}) + \frac{\sqrt{m}}{2\delta} \sum_{i=1}^n \sum_{k=n+1}^{n+m} \omega_{ik}^2 \tilde{v}_{ik}^T \tilde{v}_{ik}. \tag{B.14}$$

Substituting (B.12)–(B.14) into (B.11) and noting $\beta(1+\beta) < 2\beta < 4\beta$ and the other power exponents of ξ_{is} in $\textcircled{6}$, $\textcircled{7}$, and $\textcircled{8}$ belong to the interval $(\frac{(1-\beta^2)(3-\beta)}{\beta}, \frac{12-4\beta}{\beta})$, we can obtain the estimation of $\textcircled{5}$:

$$\begin{aligned}
\textcircled{5} \leq & M_0 \sum_{i=1}^n \sum_{s=1}^n (\eta_{is}^{\beta(1+\beta)} + \eta_{is}^{4\beta}) (1 + \xi_{is}^{\frac{(1-\beta^2)(3-\beta)}{\beta}} + \xi_{is}^{\frac{12-4\beta}{\beta}}) \\
& + \delta \sum_{i=1}^n \sum_{s=1}^n (\xi_{is}^{1+\beta} + \xi_{is}^4) + \frac{\sqrt{m}}{2\delta} \sum_{i=1}^n \sum_{k=n+1}^{n+m} \omega_{ik}^2 \tilde{v}_{ik}^T \tilde{v}_{ik},
\end{aligned} \tag{B.15}$$

where M_0 is a positive constant. Thus, by (B.5), (B.6), (B.10), (B.15), and the formulation of $V(\cdot)$, we know

$$\begin{aligned}
D^+V \leq & \sum_{i=1}^n b_i(t) \eta_i^T u_i + \sum_{i=1}^n \sum_{s=1}^n (-q(|\xi_{is}|^{1+\beta} + \xi_{is}^4) + (\frac{3\delta\bar{q}}{2} + \frac{\theta}{2\lambda_{1k}^2}) \xi_{is}^2 + 2\delta(\xi_{is}^{1+\beta} + \xi_{is}^4)) \\
& + \sum_{i=1}^n \sum_{s=1}^n (\frac{\bar{q}2^{2(1-\beta)}}{2\delta} \eta_{is}^{2\beta} + \frac{l^2 + 2\theta}{2} \eta_{is}^2 + \frac{2-\alpha}{2} \eta_{is}^{\frac{2}{2-\alpha}} + \tau_1 |\eta_{is}|^{1+\beta} + \frac{\theta^4}{4\delta^{\frac{1}{3}}} \eta_{is}^4) \\
& + \sum_{i=1}^n \sum_{s=1}^n M_0 (\eta_{is}^{\beta(1+\beta)} + \eta_{is}^{4\beta}) (1 + \xi_{is}^{\frac{(1-\beta^2)(3-\beta)}{\beta}} + \xi_{is}^{\frac{12-4\beta}{\beta}}) + \theta\delta \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}| \cdot |\rho_{ixs} + \rho_{ivs} + \dot{\rho}_{ivs}| \\
& + \frac{1+\alpha}{2} \sum_{i=1}^n \sum_{k=n+1}^{n+m} \zeta_{ik}^T \zeta_{ik} + \sum_{i=1}^n |b_i| (h_i - \frac{c}{|b_i|}) \dot{h}_i + (\frac{\sqrt{m}}{2\delta} + \frac{n\theta}{2}) \sum_{i=1}^n \sum_{k=n+1}^{n+m} \omega_{ik}^2 (\tilde{x}_{ik}^T \tilde{x}_{ik} + \tilde{v}_{ik}^T \tilde{v}_{ik}).
\end{aligned} \tag{B.16}$$

Noting $\xi_{is}^2 < \xi_{is}^{1+\beta} + \xi_{is}^4$, $\eta_{is}^{\beta(1+\beta)} < \eta_{is}^{2\beta} < \eta_{is}^{1+\beta} < \eta_{is}^{\beta(1+\beta)} < \eta_{is}^{4\beta} < \eta_{is}^4$ and choosing a sufficiently small constant δ such that $\underline{q} - \frac{3\delta\bar{q}}{2} - \frac{\theta}{2\lambda_{ik}^2} - 2\delta > \mu_1 > 0$, then we can obtain

$$\begin{aligned} D^+V &\leq \sum_{i=1}^n b_i(t) \eta_i^T u_i + \sum_{i=1}^n \sum_{s=1}^n (-\mu_1(|\xi_{is}|^{1+\beta} + \xi_{is}^4) + \frac{2-\alpha}{2} \eta_{is}^{\frac{2}{2-\alpha}} + \theta\delta \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}| \cdot |\rho_{ixs} + \rho_{ivs} + \dot{\rho}_{ivs}| \\ &\quad + M_1(|\eta_{is}|^{\beta(1+\beta)} + \eta_{is}^4)(1 + \xi_{is}^{\frac{12-4\beta}{\beta}} + \xi_{is}^{\frac{(1-\beta^2)(3-\beta)}{\beta}})) + \frac{1+\alpha}{2} \sum_{i=1}^n \sum_{k=n+1}^{n+m} \zeta_{ik}^T \zeta_{ik} + \sum_{i=1}^n |\underline{b}_i| (h_i - \frac{c}{|\underline{b}_i|}) \dot{h}_i \\ &\quad + (\frac{\sqrt{m}}{2\delta} + \frac{n\theta}{2}) \sum_{i=1}^n \sum_{k=n+1}^{n+m} \omega_{ik}^2 (\tilde{x}_{ik}^T \tilde{x}_{ik} + \tilde{v}_{ik}^T \tilde{v}_{ik}), \end{aligned} \quad (B.17)$$

where M_1 is a positive constant. Substituting protocol (3.7) into (B.18) and choosing a large enough constant c such that $c - M_1 - \frac{2-\alpha}{2} - \theta\delta > \mu_2 > 0$, moreover, by Assumption 2, we have

$$\begin{aligned} D^+V &\leq -\mu_1 \sum_{i=1}^n \sum_{s=1}^n (|\xi_{is}|^{1+\beta} + \xi_{is}^4) - \sum_{i=1}^n \sum_{s=1}^n |\underline{b}_i| |\eta_{is}|^{1+\epsilon} - \mu_2 \sum_{i=1}^n \sum_{s=1}^n (|\eta_{is}|^{\beta(1+\beta)} + \eta_{is}^4) \\ &\quad + \mu_3 \sum_{i=1}^n \sum_{k=n+1}^{n+m} (\zeta_{ik}^T \zeta_{ik} + \omega_{ik}^2 (\tilde{x}_{ik}^T \tilde{x}_{ik} + \tilde{v}_{ik}^T \tilde{v}_{ik})), \end{aligned} \quad (B.18)$$

where $\mu_3 = \max\{\frac{1+\alpha}{2}, \frac{\sqrt{m}}{2\delta} + \frac{n\theta}{2}\}$.

C. Appendix C

Proof of Inequality (B.12):

By Young's inequality and Lemma 2, we derive

$$\begin{aligned} &(e_v - e_v^*)^T \Lambda(\mathcal{L}_1 \otimes I_n)(e_v - e_v^*) \\ &\leq \frac{1}{2}(e_v - e_v^*)^T \Lambda(\mathcal{L}_1 \otimes I_n)(\mathcal{L}_1^T \otimes I_n) \Lambda(e_v - e_v^*) + \frac{1}{2}(e_v - e_v^*)^T (e_v - e_v^*) \\ &\leq \frac{\lambda_4}{2} \sum_{i=1}^n \sum_{s=1}^n (e_{ivs} - e_{ivs}^*)^2 (1 + \frac{3}{\beta} \xi_{is}^{3-\beta})^2 (1 + \xi_{is}^{3-\beta})^{\frac{2-2\beta}{\beta}} + 2^{1-2\beta} \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{2\beta} \\ &\leq c_1 \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{2\beta} (1 + \xi_{is}^{\frac{(2-2\beta)(3-\beta)}{\beta}} + \xi_{is}^{\frac{6-2\beta}{\beta}} + \xi_{is}^{6-2\beta}) + 2^{1-2\beta} \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{2\beta}, \end{aligned} \quad (C.1)$$

where $c_1 = \lambda_4 2^{1-2\beta} 2^{\frac{2-3\beta}{2(1-\beta)}} \cdot \frac{9\sqrt{2}}{\beta^2}$, $\lambda_4 = \lambda_{\max}(\mathcal{L}_1 \mathcal{L}_1^T)$.

Proof of Inequality (B.13):

According to the formulation of $\bar{v}_i^*$ and Young's inequality, we can obtain

$$\begin{aligned} &(e_v - e_v^*)^T \Lambda(\mathcal{L}_1 \otimes I_n) e_v^* = -(e_v - e_v^*)^T \Lambda(\mathcal{L}_1 \otimes I_n) (\xi^\beta + \xi^3) \\ &\leq \delta \sum_{i=1}^n \sum_{s=1}^n (\xi_{is}^{1+\beta} + \xi_{is}^4) + \frac{\lambda_5^4 n^{\frac{3}{4}}}{\delta^\beta} \sum_{i=1}^n \sum_{s=1}^n (|e_{ivs} - e_{ivs}^*| \cdot |1 + \frac{3}{\beta} \xi_{is}^{3-\beta}| \cdot |1 + \xi_{is}^{3-\beta}|^{\frac{1-\beta}{\beta}})^4 \end{aligned}$$

$$+ \frac{\lambda_5^{1+\beta} n^{\frac{\beta}{1-\beta}}}{\delta^\beta} \sum_{i=1}^n \sum_{s=1}^n (|e_{ivs} - e_{ivs}^*| \cdot |1 + \frac{3}{\beta} \xi_{is}^{3-\beta}| \cdot |1 + \xi_{is}^{3-\beta}|^{\frac{1-\beta}{\beta}})^{1+\beta},$$

which, together with Lemmas 2 and 3, implies

$$\begin{aligned} & (e_v - e_v^*)^T \Lambda(\mathcal{L}_1 \otimes I_n) e_v^* \\ & \leq c_2 \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{\beta(1+\beta)} (\xi_{is}^{\frac{(1-\beta^2)(3-\beta)}{\beta}} + \xi_{is}^{(3-\beta)(1+\beta)} + \xi_{is}^{\frac{(3-\beta)(1+\beta)}{\beta}} + 1) \\ & \quad + \delta \sum_{i=1}^n \sum_{s=1}^n (\xi_{is}^{1+\beta} + \xi_{is}^4) + c_3 \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{4\beta} (\xi_{is}^{\frac{(4-4\beta)(3-\beta)}{\beta}} + \xi_{is}^{12-4\beta} + \xi_{is}^{\frac{12-4\beta}{\beta}} + 1), \end{aligned}$$

where $c_2 = \frac{\lambda_5^{1+\beta} n^{\frac{\beta}{1-\beta}}}{\delta^\beta} 2^{2-\beta^2} \cdot 2^{\frac{\beta^2}{1-\beta^2}} \cdot (\frac{3}{\beta})^{1+\beta}$, $c_3 = \frac{\lambda_5^4 n^{\frac{3}{4}}}{\delta^3} 2^{4-4\beta} \cdot 2^{\frac{7-8\beta^2}{4(1-\beta)}} \cdot (\frac{3}{\beta})^4$, $\lambda_5 = \lambda_{\max}(\mathcal{L}_1)$.

Proof of Inequality (B.14):

By Young's inequality and Lemma 3, we deduce

$$(e_v - e_v^*)^T \Lambda(\mathcal{L}_1 \otimes I_n) (z - \bar{v}) \leq \delta (e_v - e_v^*)^T \Lambda(\mathcal{L}_1 \otimes I_n) (\mathcal{L}_1^T \otimes I_n) \Lambda(e_v - e_v^*) + \frac{\sqrt{m}}{2\delta} \sum_{i=1}^n \sum_{k=n+1}^{n+m} \omega_{ik}^2 \tilde{v}_{ik}^T \tilde{v}_{ik},$$

which, together with (B.19), implies

$$\begin{aligned} (e_v - e_v^*)^T \Lambda(\mathcal{L}_1 \otimes I_n) (z - \bar{v}) & \leq c_1 \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{2\beta} (1 + \xi_{is}^{\frac{(2-2\beta)(3-\beta)}{\beta}} + \xi_{is}^{\frac{6-2\beta}{\beta}} + \xi_{is}^{6-2\beta}) \\ & \quad + \frac{\sqrt{m}}{2\delta} \sum_{i=1}^n \sum_{k=n+1}^{n+m} \omega_{ik}^2 \tilde{v}_{ik}^T \tilde{v}_{ik}. \end{aligned}$$

Appendix D

Proof of Inequality (4.1):

Based on (3.9) and the formulation of $\bar{V}$, we derive

$$\begin{aligned} D^+ \bar{V} & \leq -\mu_1 \sum_{i=1}^n \sum_{s=1}^n (|\xi_{is}|^{1+\beta} + \xi_{is}^4) - \sum_{i=1}^n |b_i| (h_i - \frac{c}{|b_i|}) \dot{h}_i - \sum_{i=1}^n \sum_{s=1}^n |b_i| |\eta_{is}|^{1+\epsilon} \\ & \quad - \mu_2 \sum_{i=1}^n \sum_{s=1}^n (|\eta_{is}|^{\beta(1+\beta)} + \eta_{is}^4) + \mu_3 \sum_{i=1}^n \sum_{k=n+1}^{n+m} (\zeta_{ik}^T \zeta_{ik} + \omega_{ik}^2 (\tilde{x}_{ik}^T \tilde{x}_{ik} + \tilde{v}_{ik}^T \tilde{v}_{ik})). \end{aligned} \quad (\text{C.2})$$

By $|b_i| \geq \underline{b}$ and noting $|h_i(t) - \frac{c}{|b_i|}| \leq \frac{2V(0)}{\underline{b}} := \bar{h}$, $\xi_{is}^2 < \xi_{is}^{1+\beta} + \xi_{is}^4$, and $|\eta_{is}|^{\beta(1+\beta)} < |\eta_{is}|^{\frac{(1+\beta)^2}{2}} < |\eta_{is}|^{1+\beta} < \eta_{is}^4$, we can obtain that

$$D^+ \bar{V} \leq -\frac{\mu_1}{2} \sum_{i=1}^n \sum_{s=1}^n \xi_{is}^2 - \frac{\mu_1}{4} \sum_{i=1}^n \sum_{s=1}^n (\xi_{is}^{1+\beta} + \xi_{is}^4) - \frac{\mu_2}{2} \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}|^{\frac{(1+\beta)^2}{2}}$$

$$\begin{aligned}
& + \bar{b}\bar{h} \sum_{i=1}^n \sum_{s=1}^n (\eta_{is}^{\frac{2}{2-\alpha}} + (\eta_{is}^{\beta(1+\beta)} + \eta_{is}^4)(1 + \xi_{is}^{\frac{(12-4\beta)}{\beta}} + \xi_{is}^{\frac{(3-\beta)(1-\beta^2)}{\beta}})) \\
& + |\eta_{is}| \cdot |\rho_{ixs} + \rho_{ivs} + \dot{\rho}_{ivs}| - \underline{b} \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}|^{1+\epsilon} - \frac{\mu_1}{4} \sum_{i=1}^n \sum_{s=1}^n (|\xi_{is}|^{1+\beta} + \xi_{is}^4) \\
& + \mu_3 \sum_{i=1}^n \sum_{k=n+1}^{n+m} (\zeta_{ik}^T \zeta_{ik} + \omega_{ik}^2 (\tilde{x}_{ik}^T \tilde{x}_{ik} + \tilde{v}_{ik}^T \tilde{v}_{ik})) - \frac{\mu_2}{2} \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}|^{1+\beta},
\end{aligned}$$

where $\bar{b} = \max\{|\underline{b}_1|, \dots, |\underline{b}_n|\}$. Moreover, from Young's inequality and noting $\rho_{ixs}, \rho_{ivs}, \dot{\rho}_{ivs}$ are known and bounded, we arrive at

$$\begin{aligned}
|\eta_{is}| \cdot |\rho_{ixs} + \rho_{ivs} + \dot{\rho}_{ivs}| & \leq \frac{1}{2\epsilon} \eta_{is}^2 + \frac{\epsilon}{2} (\rho_{ixs} + \rho_{ivs} + \dot{\rho}_{ivs})^2 \\
& \leq \frac{1}{2\epsilon} \eta_{is}^2 + \frac{3\epsilon}{2} (\rho_{ixs}^2 + \rho_{ivs}^2 + \dot{\rho}_{ivs}^2) \leq \frac{1}{2\epsilon} \eta_{is}^2 + \frac{3\epsilon}{2} \bar{\rho},
\end{aligned}$$

where $\bar{\rho}$ is the bound of $\rho_{ixs}^2 + \rho_{ivs}^2 + \dot{\rho}_{ivs}^2$. Then, we conclude that

$$\begin{aligned}
D^+ \bar{V} & \leq -\frac{\mu_1}{2} \sum_{i=1}^n \sum_{s=1}^n \xi_{is}^2 - \frac{\mu_1}{4} \sum_{i=1}^n \sum_{s=1}^n (\xi_{is}^{1+\beta} + \xi_{is}^4) - \frac{\mu_2}{2} \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}|^{\frac{(1+\beta)^2}{2}} - \underline{b} \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}|^{1+\epsilon} \\
& - \frac{\mu_2}{2} \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}|^{1+\beta} + \mu_7 \sum_{i=1}^n \sum_{s=1}^n (\eta_{is}^{\frac{2}{2-\alpha}} + \eta_{is}^2 + (\eta_{is}^{\beta(1+\beta)} + \eta_{is}^4)(1 + \xi_{is}^{\frac{(12-4\beta)}{\beta}} + \xi_{is}^{\frac{(3-\beta)(1-\beta^2)}{\beta}})) \\
& - \frac{\mu_1}{4} \sum_{i=1}^n \sum_{s=1}^n (|\xi_{is}|^{1+\beta} + \xi_{is}^4) + \mu_3 \sum_{i=1}^n \sum_{k=n+1}^{n+m} (\zeta_{ik}^T \zeta_{ik} + \omega_{ik}^2 (\tilde{x}_{ik}^T \tilde{x}_{ik} + \tilde{v}_{ik}^T \tilde{v}_{ik})) + \epsilon \mu_6, \tag{C.3}
\end{aligned}$$

where $\mu_6 = 3\bar{\rho}\bar{b}\bar{h}n$, $\mu_7 = \bar{b}\bar{h} + \frac{1}{2\epsilon}$, and ϵ is an arbitrarily small positive constant. In addition, from the definition of $V(\cdot)$, it follows that

$$V_1 \leq \frac{\lambda_3}{2} \sum_{i=1}^n \sum_{s=1}^n \xi_{is}^2, \quad V_2 \leq 2^{1-\beta} \sum_{i=1}^n \sum_{s=1}^n \eta_{is}^{1+\beta}.$$

Then, by (B.21) and the first inequality in Lemma 3, we have

$$\begin{aligned}
D^+ \bar{V} & \leq -\frac{\mu_1}{\lambda_3} V_1 - \frac{\mu_1}{4} \left(\frac{2}{\lambda_3}\right)^{\frac{1+\beta}{2}} V_1^{\frac{1+\beta}{2}} + \mu_3 \sum_{i=1}^n \sum_{k=n+1}^{n+m} (\zeta_{ik}^T \zeta_{ik} + \omega_{ik}^2 (\tilde{x}_{ik}^T \tilde{x}_{ik} + \tilde{v}_{ik}^T \tilde{v}_{ik})) \\
& - \frac{\mu_2}{2^{2-\beta}} V_2 - \frac{\mu_2}{2^{\frac{3-\beta^2}{2}}} V_2^{\frac{1+\beta}{2}} + \epsilon \mu_6 - \underline{b} \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}|^{1+\epsilon} - \frac{\mu_1}{4} \sum_{i=1}^n \sum_{s=1}^n (|\xi_{is}|^{1+\beta} + \xi_{is}^4) \\
& + \mu_7 \sum_{i=1}^n \sum_{s=1}^n (\eta_{is}^{\frac{2}{2-\alpha}} + \eta_{is}^2 + (\eta_{is}^{\beta(1+\beta)} + \eta_{is}^4)(1 + \xi_{is}^{\frac{(12-4\beta)}{\beta}} + \xi_{is}^{\frac{(3-\beta)(1-\beta^2)}{\beta}})). \tag{C.4}
\end{aligned}$$

In what follows, we handle the last three items (denoted by ③). First, by Young's inequality, we deduce that

$$\mu_7 \sum_{i=1}^n \sum_{s=1}^n (\eta_{is}^{\frac{2}{2-\alpha}} + \eta_{is}^2 + (\eta_{is}^{\beta(1+\beta)} + \eta_{is}^4)(1 + \xi_{is}^{\frac{(12-4\beta)}{\beta}} + \xi_{is}^{\frac{(3-\beta)(1-\beta^2)}{\beta}}))$$

$$\leq 2\mu_7 \sum_{i=1}^n \sum_{s=1}^n (\eta_{is}^{\frac{2}{2-\alpha}} + \eta_{is}^2 + \eta_{is}^{2\beta(1+\beta)} + \eta_{is}^8 + \xi_{is}^{\frac{8(3-\beta)}{\beta}} + \xi_{is}^{\frac{(6-2\beta)(1-\beta^2)}{\beta}}).$$

Let $\varphi_1(t) = \sum_{i=1}^n \sum_{s=1}^n |\xi_{is}(t)|^{1+\beta}$ and $\varphi_2(t) = \sum_{i=1}^n \sum_{s=1}^n |\eta_{is}(t)|^{1+\epsilon}$. Then, by Lemma 3, we know ③ can be estimated as follows:

$$\begin{aligned} \textcircled{3} &\leq 2\mu_7(\varphi_1(t)^{\frac{8(3-\beta)}{\beta(1+\beta)}} + \varphi_1(t)^{\frac{2(3-\beta)(1-\beta)}{\beta}} + \varphi_2(t)^{\frac{2}{1+\epsilon}} + \varphi_2(t)^{\frac{2}{(2-\alpha)(1+\epsilon)}} + \varphi_2(t)^{\frac{2\beta(1+\beta)}{1+\epsilon}} + \varphi_2(t)^{\frac{8}{1+\epsilon}}) - \frac{\mu_1}{4}\varphi_1(t) - \underline{b}\varphi_2(t) \\ &= -\varphi_1(t)(-2\mu_7(\varphi_1(t)^{\frac{8(3-\beta)}{\beta(1+\beta)}-1} + \varphi_1(t)^{\frac{2(3-\beta)(1-\beta)}{\beta}-1}) + \frac{\mu_1}{4}) - \varphi_2(t)(\underline{b} - 2\mu_7(\varphi_2(t)^{\frac{2}{1+\epsilon}-1} \\ &\quad + \varphi_2(t)^{\frac{2}{(2-\alpha)(1+\epsilon)}-1} + \varphi_2(t)^{\frac{2\beta(1+\beta)}{1+\epsilon}-1} + \varphi_2(t)^{\frac{8}{1+\epsilon}-1})), \end{aligned}$$

which, together with the facts $\lim_{t \rightarrow +\infty} \xi_{ix}(t) = \mathbf{0}$ and $\lim_{t \rightarrow +\infty} \eta_i(t) = \mathbf{0}$, implies that for any $t \geq T_2$, T_2 is a finite time

$$\begin{cases} \varphi_1(t)^{\frac{8(3-\beta)}{\beta(1+\beta)}-1} + \varphi_1(t)^{\frac{2(3-\beta)(1-\beta)}{\beta}-1} \leq \frac{\mu_1}{8\mu_7}, \\ \varphi_2(t)^{\frac{2}{1+\epsilon}-1} + \varphi_2(t)^{\frac{2}{(2-\alpha)(1+\epsilon)}-1} + \varphi_2(t)^{\frac{2\beta(1+\beta)}{1+\epsilon}-1} + \varphi_2(t)^{\frac{8}{1+\epsilon}-1} \leq \frac{\underline{b}}{2\mu_7}. \end{cases} \quad (\text{C.5})$$

Thus, by (B.22), we know, for any $t \geq T_2$,

$$\begin{aligned} D^+ \bar{V} &\leq -\frac{\mu_1}{\lambda_3} V_1 - \frac{\mu_1}{4} \left(\frac{2}{\lambda_3}\right)^{\frac{1+\beta}{2}} V_1^{\frac{1+\beta}{2}} - \frac{\mu_2}{2^{\frac{3-\beta^2}{2}}} V_2^{\frac{1+\beta}{2}} - \frac{\mu_2}{2^{2-\beta}} V_2 \\ &\quad + \mu_3 \sum_{i=1}^n \sum_{k=n+1}^{n+m} (\zeta_{ik}^T \zeta_{ik} + \omega_{ik}^2 (\tilde{x}_{ik}^T \tilde{x}_{ik} + \tilde{v}_{ik}^T \tilde{v}_{ik})) + \varepsilon \mu_6. \end{aligned} \quad (\text{C.6})$$

Then, according to Proposition 1, we have, for any $t \geq T' = \max\{T_1, T_2\}$,

$$\begin{aligned} D^+ \bar{V} &\leq -\frac{\mu_1}{\lambda_3} V_1 - \frac{\mu_1}{4} \left(\frac{2}{\lambda_3}\right)^{\frac{1+\beta}{2}} V_1^{\frac{1+\beta}{2}} - \frac{\mu_2}{2^{\frac{3-\beta^2}{2}}} V_2^{\frac{1+\beta}{2}} - \frac{\mu_2}{2^{2-\beta}} V_2 + \varepsilon \mu_6 \\ &\leq -\mu_4 \bar{V} - \mu_5 \bar{V}^{\frac{1+\beta}{2}} + \varepsilon \mu_6, \end{aligned}$$

where $\mu_4 = \min\{\frac{\mu_1}{\lambda_3}, \frac{\mu_2}{2^{2-\beta}}\}$, $\mu_5 = \min\{\frac{\mu_1}{4} \left(\frac{2}{\lambda_3}\right)^{\frac{1+\beta}{2}}, \frac{\mu_2}{2^{\frac{3-\beta^2}{2}}}\}$.



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