



Research article

On the existence of solutions for first-order nonlinear conformable impulsive dynamic equations on time scales

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Abstract: Using a fixed point theorem due to Krasnosel'skii, we showed that the first-order nonlinear conformable impulsive dynamic equations on time scales (3.1) has a solution. Using the idea of contraction, we demonstrated that the solution is unique under a somewhat stricter inequality.

Keywords: conformable derivative; time scales; impulsive dynamic equations

1. Introduction

The conformable derivative, first introduced in [1, 2], is a new mathematical tool. This approach has been integrated into Newtonian mechanics [3], cobweb models [4], and logistic frameworks [5], with additional foundations in physics [6] and mathematics [7–10]. The analytical solution of Van der Pol and Lienard differential equations within the conformable fractional operator by retarded integral inequalities is presented in [11–14]. Several noteworthy contributions on the various methods of solving differential equations may be found in [15–18].

See the following references [19–22] for more literature reviews that demonstrate how the idea has developed. The main tool for analyzing these problems is impulsive differential equations, which are well-documented in the literature [23–26]. In fact, many scholars [27–30] study these equations in relation to conformable derivatives. Which is defined as follows:

$$\begin{aligned}\mathfrak{D}_{\varpi}^{\alpha}c(w) &= g(w, c(w)), \quad w \in K := [a, b] \setminus \{w_1, \dots, w_m\}, \quad 0 < \varpi < 1, \\ \Delta c(w_l) &= K_l(c(w_l^-)), \quad l = 1, 2, \dots, m,\end{aligned}$$

In this context, $\mathfrak{D}_{\varpi}^{\alpha}$ is the conformable derivative having order α with a lower bound of ϖ . Consider a mapping $g : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ which maintains continuity, while $K_l : \mathbb{R} \rightarrow \mathbb{R}$ represents the (instantaneous)

impulsive function $a = w_0 < w_1 < \dots < w_m < w_{m+1} = b$, $b > 0$, $c(w_l^-) = \lim_{\epsilon \rightarrow 0^-} c(w_l + \epsilon)$, $c(w_l^+) = \lim_{\epsilon \rightarrow 0^+} c(w_l + \epsilon)$, and $\Delta c(w_l) = w_l^+ - w_l^-$.

In [31], the authors proved the existence of solutions to the equation of nonlinear pulse dynamics.

$$\begin{aligned} c^\Delta(w) &= -a(w)c^\sigma(w) + \delta(w, c(w)), w \in (0, T], \\ c(0) &= 0, \\ c(w_i^+) &= c(w_i^-) + I(w_i, c(w_i)), i = 1, 2, \dots, n. \end{aligned} \quad (1.1)$$

where $c(w^{\pm i}) = \lim_{w \rightarrow w^{\pm i}} c(w)$, $c(w_i) = c(w_i^-)$, and $w \in \mathbb{W} : 0 \leq w \leq T$, and $\mathbb{W}$ is a Time Scale.

Stefan Hilger developed the notion of time scales in 1988 as a mathematical framework to unite and generalize discrete and continuous analysis. This was accomplished by defining a “time scale,” which can be any closed set of real numbers, and creating a new calculus that applies to these arbitrary sets. Intriguing research on the conformable calculus on time scales is viewed in [32–35]. This has implications in disciplines, including biology, economics, and physics, and enables the unified treatment of differential equations (continuous time) and difference equations (discrete time). Recent advancements of the presented concepts about Laplace transform are found in [36, 37].

The rest of this paper is structured as follows: In Section 2, we introduce the basic definitions and properties of the conformable derivative on time scales. In Section 3, we prove some results of nonlinear impulsive dynamic equations formulated on time scales as a generalization of the results of the equation above.

2. Preliminaries

Definition 2.1. Let $\mathbb{W}$ be a time scale.

1) For $w \in \mathbb{W}$, we define the forward jump operator $\sigma : \mathbb{W} \rightarrow \mathbb{W}$ by

$$\sigma(w) := \inf\{s \in \mathbb{W} : s > w\},$$

2) The graininess function $\mu : \mathbb{W} \rightarrow [0, \infty)$ is defined by

$$\mu(w) := \sigma(w) - w.$$

3) w is right-scattered, if $\sigma(w) > w$.

4) w is right-dense, if $\sigma(w) = w$ and $w < \sup(\mathbb{W})$.

Given an increasing about the first slot, let $\nu \in (0, 1]$ and $\delta : \mathbb{W} \times (0, 1] \rightarrow (0, \infty]$; for example, $\delta(w, \nu) = 3\nu^2 w^2 + 2\nu w + 1$ for all $w \in \mathbb{W} \cap [0, +\infty)$.

Definition 2.2. Assume that $\Upsilon : \mathbb{W} \rightarrow \mathbb{R}$, $w \in \mathbb{W}^k$. The delta N_δ^ν -derivative of Υ at w noted by $\Upsilon_\delta^{(\nu)}(w)$ is defined by

$$\left| (\Upsilon(\varsigma(w)) - \Upsilon(s))\delta(w, \nu) - \Upsilon_\delta^{(\nu)}(w)(\varsigma(w) - s) \right| \leq \epsilon |\varsigma(w) - s|.$$

Which is in the neighbourhood $U \subset \mathbb{W}$ of w .

Using the above definition, the following property can be proven:

Theorem 2.3. [36] Let $w \in \mathbb{W}^k$ and assume that $\Upsilon : \mathbb{W} \rightarrow \mathbb{R}$ is a function. Υ is continuous at t if it is N_δ^ν -differentiable at w .

Theorem 2.4. [36] Let $\nu \in]0, 1]$ and $\mathbb{W}$ be a time scale. Assume that $\Upsilon : \mathbb{W} \rightarrow \mathbb{R}$ and let $w \in \mathbb{W}^k$. The following properties hold.

1) If Υ is continuous at w and w is right-scattered, then Υ is conformable fractional differentiable of order ν at w with

$$\Upsilon_{\delta}^{(\nu)}(w) = \frac{\Upsilon(\sigma(w)) - \Upsilon(w)}{\mu(w)} \Upsilon(w, \nu).$$

2) If w is right-dense, then Υ is conformable fractional differentiable of order α at w , if and only if, the limit $\lim_{s \rightarrow w} \frac{\Upsilon(s) - \Upsilon(w)}{s - w} \Upsilon(w, \alpha)$ exists as a finite number. In this case,

$$\Upsilon_{\delta}^{(\nu)}(w) = \lim_{s \rightarrow w} \frac{\Upsilon(s) - \Upsilon(w)}{s - w} \Upsilon(w, \nu).$$

Theorem 2.5. [36] Assume $\Upsilon_1, \Upsilon_2 : \mathbb{W} \rightarrow \mathbb{R}$ are differentiable at $w \in \mathbb{W}^k$. Then:

- 1) $\Upsilon_1 + \Upsilon_2$ is conformable differentiable with $(\Upsilon_1 + \Upsilon_2)_{\delta}^{(\nu)}(w) = (\Upsilon_1)_{\delta}^{(\nu)}(w) + (\Upsilon_2)_{\delta}^{(\nu)}(w)$.
- 2) $\Upsilon_1 \Upsilon_2$ is conformable differentiable with $(\Upsilon_1 \Upsilon_2)_{\delta}^{(\nu)}(w) = (\Upsilon_1)_{\delta}^{(\nu)}(w) \Upsilon_2(w) + \Upsilon_1(w) (\Upsilon_2)_{\delta}^{(\nu)}(w)$.
- 3) $\frac{1}{\Upsilon_1}$ is conformable differentiable with $(\frac{1}{\Upsilon_1})_{\delta}^{(\nu)}(w) = -\frac{(\Upsilon_1)_{\delta}^{(\nu)}(w)}{\Upsilon_1(w) \Upsilon_1^{\sigma}(w)}$ for which $\Upsilon_1 \Upsilon_1^{\sigma} \neq 0$.
- 4) $\frac{\Upsilon_1}{\Upsilon_2}$ is conformable differentiable with $(\frac{\Upsilon_1}{\Upsilon_2})_{\delta}^{(\nu)}(w) = \frac{(\Upsilon_1)_{\delta}^{(\nu)}(w) \Upsilon_2(w) - \Upsilon_1(w) (\Upsilon_2)_{\delta}^{(\nu)}(w)}{\Upsilon_2(w) \Upsilon_2^{\sigma}(w)}$, $\Upsilon_2(w) \Upsilon_2^{\sigma}(w) \neq 0$.

Theorem 2.6. [36] We have

- 1) $(w^2)_{\delta}^{(\nu)} = (\sigma(w) + w) \delta(w, \nu)$;
- 2) $(w^n)_{\delta}^{(\nu)} = (w^{n-1} + \sigma^{n-1}(w) + \sum_{j=1}^{n-2} \sigma^{n-1-j}(w) t^j) \delta(w, \nu), n \geq 3$.

Definition 2.7. Consider that $\Upsilon : \mathbb{W} \rightarrow \mathbb{R}$ is a regulated function. Then, the delta N_{δ}^{ν} -integral of Υ , for $\nu \in (0, 1]$, is defined by

$$\int \Upsilon(w) \Delta_{\delta}^{\nu} w = \int \Upsilon(w) \delta(w, \nu) \Delta w.$$

Theorem 2.8. [36] If $w \in C_{rd}(\mathbb{W})$ and $w \in \mathbb{W}^k$, then

$$\int_w^{\sigma(w)} \Upsilon(\tau) \delta(\tau, \nu) \Delta \tau = \mu(w) \Upsilon(w) \delta(w, \nu).$$

With $\mathcal{R}_{\delta}^{\nu}$, we represent the set of all functions Υ on $\mathbb{W}$ for which

$$1 + \mu(w) \Upsilon(w) \delta(w, \nu) \neq 0, \quad w \in \mathbb{W}.$$

With $\mathcal{R}_{\delta+}^{\nu}$, we present the set of all functions Υ on $\mathbb{W}$ for which

$$1 + \mu(w) \Upsilon(w) \delta(w, \nu) > 0, \quad w \in \mathbb{W}.$$

Definition 2.9. In $\mathcal{R}_{\delta}^{\nu}$ 'N $_{\delta}^{\nu}$ -circle plus' addition, $\oplus_{\delta, \nu}$ is defined as below

$$(\Upsilon_1 \oplus_{\delta, \nu} \Upsilon_2)(w) = \Upsilon_1(w) + \Upsilon_2(w) + \mu(w) \Upsilon_1(w) \Upsilon_2(w) \delta(w, \nu), \text{ for all } w \in \mathbb{W}^k.$$

The set $\mathcal{R}_{\delta}^{\nu}$ forms an abelian group under $\oplus_{\delta, \nu}$.

For $h \in \mathcal{R}_{\delta}^{\nu}$, the inverse of h is given as

$$\ominus_{\delta, \nu} \Upsilon = \frac{-\Upsilon(w)}{1 + \mu(w) \Upsilon(w) \delta(w, \nu)}, \text{ for all } w \in \mathbb{W}^k.$$

Definition 2.10. For $\mathcal{R}_\delta^\nu$, N_δ^ν -circle minus' subtraction, $\Theta_{\delta,\nu}$ is defined as

$$(\mathfrak{T}_1 \ominus_{\delta,\nu} \mathfrak{T}_2)(w) = \frac{\mathfrak{T}_1(w) - \mathfrak{T}_2(w)}{1 + \mu(w)\mathfrak{T}_2(w)\delta(w,\nu)}, \text{ for all } w \in \mathbb{W}^k.$$

Definition 2.11. Let $g \in \mathcal{R}_\delta^\nu$, then the N_δ^ν -exponential function is defined by

$$\mathbb{E}_{\delta,g}(w, s) = \exp\left(\int_s^w \xi_{\mu(\varrho)}(g(\varrho)\delta(\varrho,\nu))\Delta\varrho\right), \text{ for all } w, s \in \mathbb{W}. \quad (2.1)$$

where the cylindrical transformation $\xi_h : \mathbb{C} \setminus \{-\frac{1}{h}\} \rightarrow \{z \in \mathbb{C} : -\frac{\pi}{h} < \text{Im}(z) \leq \frac{\pi}{h}\}$ is defined as

$$\xi_h(z) := \begin{cases} \frac{1}{h} \text{Log}(1 + zh), & \text{if } h > 0 \\ z, & \text{if } h = 0. \end{cases} \quad (2.2)$$

Using the concept of the cylindrical transformation, Eq (2.1) can be rewritten as

$$\mathbb{E}_{\delta,g}(w, s) = \exp\left(\int_s^w \frac{1}{\mu(\varrho)} \text{Log}(1 + \mu(\varrho)g(\varrho)\delta(\varrho,\nu))\Delta\varrho\right), \text{ for all } w, s \in \mathbb{W}. \quad (2.3)$$

Theorem 2.12. [36] Let $g \in \mathcal{R}_\delta^\nu$, then for all $s, w, r \in \mathbb{W}$

- 1) $\mathbb{E}_{\delta,g}(w, s)\mathbb{E}_{\delta,g}(s, r) = \mathbb{E}_{\delta,g}(w, r)$.
- 2) $\mathbb{E}_{\delta,g}(\sigma(w), s) = (1 + \mu(w)g(w)\delta(w,\nu))\mathbb{E}_{\delta,g}(w, s)$.
- 3) $\left(\mathbb{E}_{\delta,g}\right)_\delta^{(\nu)}(w, s) = \delta^2(w, \nu)g(w)\mathbb{E}_{\delta,g}(w, s)$, with $\delta^2(w, \nu) = \delta^2(w, \nu) \times \delta^2(w, \nu)$.
- 4) $\mathbb{E}_{\delta \ominus_{\delta,\nu} g}(w, s) = \frac{1}{\mathbb{E}_{\delta,g}(w, s)}$.

Let $\mathbb{W}$ be the time scale having forward jump operator and delta differentiation operator ς and Δ , respectively. Suppose that $0, w_i, T \in \mathbb{W}$, $i \in \{1, \dots, n\}$, $0 < w_i < w_{i+1}$, $i \in \{1, \dots, n-1\}$, and w_i is dense in $\mathbb{W}$, $i \in \{1, \dots, n\}$.

Theorem 2.13 (Krasnosel'skii Fixed Point Theorem). Suppose that M is a closed convex nonempty subset of a Banach space $(\mathcal{B}, \|\cdot\|)$. If

- 1) $A : M \rightarrow \mathcal{B}$ is completely continuous.
- 2) $B : M \rightarrow \mathcal{B}$ is a contraction mapping.
- 3) $x, \wp \in M$ entails $Ax + B\wp \in M$.

Then, the mapping $A + B$ has a fixed point in M .

3. Existence of solutions for first order nonlinear conformable impulsive dynamic equations

Consider the following conformable impulsive dynamic equation

$$\begin{cases} \wp_\delta^{(\nu)}(w) &= p(w)\wp(w) + \mathfrak{T}(w, \wp(w)), & w \in [0, T]_{\mathbb{W}}^k \setminus \{t_1, \dots, t_n\}, \\ \wp(0) &= \wp_0, \\ \wp(w_i^+) &= \wp(w_i^-) + K(w_i, \wp(w_i)), & i \in \{1, \dots, n\}, \end{cases} \quad (3.1)$$

where

$$\begin{cases} \wp(w_i^\pm) &= \lim_{w \rightarrow w_i^\pm} \wp(w), \\ \wp(w_i) &= \wp(w_i^-), \quad i \in \{1, \dots, n\}, \end{cases} \quad (3.2)$$

We will assume that the following conditions hold.

(Fact 1) $p \in \mathcal{R}_\delta^\nu$, $\wp_0 \in \mathbb{R}$.

(Fact 2) $\Upsilon \in C(\mathbb{W} \times \mathbb{R})$.

(Fact 3) There occur two positive r and q such that

$$\begin{aligned} \xi &= \max_{w \in [0, T]} \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \frac{q(\varrho)}{(\delta(\varrho, \nu))^2} \Delta_\delta^\nu \varrho < \infty, \\ \eta &= \max_{w \in [0, T]} \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \frac{r(\varrho)}{(\delta(\varrho, \nu))^2} \Delta_\delta^\nu \varrho < \infty, \end{aligned}$$

$$|\Upsilon(w, x) - \Upsilon(w, \wp)| \leq r(w)|x - \wp|,$$

$$|\Upsilon(w, \wp)| \leq q(w) + r(w)|\wp|, \quad w \in \mathbb{W}, \quad c \in \mathbb{R}.$$

(Fact 4) There occurs a positive constant A such that

$$|K(w, x) - K(w, \wp)| \leq A|x - \wp|, \quad x, \wp \in \mathbb{R}.$$

Define

$$w_{n+1} = T,$$

$$J_0 = [0, w_1],$$

$$J_l = (w_l, w_{l+1}], \quad l \in \{1, \dots, n\},$$

$$PC = \left\{ \wp : [0, T] \rightarrow \mathbb{R}, \quad \wp \in C(J_l), \quad \exists \wp(w_l^\pm), \right.$$

$$\left. \wp(w_l^-) = \wp(w_l), \quad l \in \{1, \dots, n\} \right\},$$

$$PC^1 = \left\{ \wp : [0, T] \rightarrow \mathbb{R}, \quad \wp \in C^1(J_l), \quad l \in \{1, \dots, n\} \right\}.$$

The Banach space with the following supremum norm is the set PC

$$\|u\| = \max_{l \in \{0, \dots, n\}} \|u\|_l,$$

where

$$\|u\|_l = \sup_{w \in J_l} |u(w)|, \quad l \in \{0, \dots, n\}.$$

Theorem 3.1. Equation (3.1) will be solved by the function $\wp \in PC^1$ if and only if $\wp \in PC$ satisfies the integral equation that follows.

$$\begin{aligned} \wp(w) &= \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho + \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0) \wp_0 \\ &+ \sum_{\{l: w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) K(w_l, \wp(w_l)). \end{aligned} \quad (3.3)$$

Proof. 1) Let $\wp \in PC^1$ be a solution of Eq (3.1). For $w \in J_0$, we have

$$\begin{aligned} \wp_{\delta}^{(\nu)}(w) &= p(w) \wp(w) + \Upsilon(w, \wp(w)), \quad w \in J_0, \\ \wp(0) &= \wp_0. \end{aligned} \quad (3.4)$$

For its solution, we have

$$\wp(w) = \mathbb{E}_{\delta, \frac{p}{F^2}}(w, 0) \wp_0 + \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho, \quad w \in J_0,$$

and

$$\wp(w_1) = \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1, 0) \wp_0 + \int_0^{w_1} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho.$$

For $w \in J_1$, we get the following IVP

$$\begin{aligned} \wp_{\delta}^{(\nu)}(w) &= p(w) \wp(w) + \Upsilon(w, \wp(w)), \quad w \in J_1, \\ \wp(w_1^+) &= \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1, 0) \wp_0 + \int_0^{w_1} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \\ &+ K(w_1, \wp(w_1)). \end{aligned}$$

We have the following depiction of its solution.

$$\begin{aligned} \wp(w) &= \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_1) \left(\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1, 0) \wp_0 + \int_0^{w_1} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \right. \\ &\quad \left. + K(w_1, \wp(w_1)) \right) + \int_{w_1}^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \\ &= \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0) \wp_0 + \int_0^{w_1} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \\ &\quad + \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_1) K(w_1, \wp(w_1)) + \int_{w_1}^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)\varphi_0 + \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \varphi(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \\
&\quad + \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_1)K(w_1, \varphi(w_1)).
\end{aligned}$$

Applying the similar process, we obtain (3.3).

2) Let $\varphi \in PC$ satisfy (3.3). For $w \in J_0$, we have

$$\varphi(w) = \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)\varphi_0 + \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \varphi(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho,$$

whereupon we get the IVP (3.4) and

$$\begin{aligned}
\varphi(w_1^-) &= \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1^-, 0)\varphi_0 + \int_0^{w_1^-} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1^-, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \varphi(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \\
&= \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1^+, 0)\varphi_0 + \int_0^{w_1^+} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1^+, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \varphi(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho.
\end{aligned}$$

For $w \in J_1$, we have

$$\begin{aligned}
\varphi(w) &= \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)\varphi_0 + \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \varphi(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \\
&\quad + \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_1)K(w_1, \varphi(w_1)).
\end{aligned}$$

Hence,

$$\begin{aligned}
\varphi_{\delta}^{(\nu)}(w) &= p(w)\varphi(w) + \Upsilon(w, \varphi(w)), \quad w \in J_1, \\
\varphi(w_1^+) &= \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1^+, 0)\varphi_0 + \int_0^{w_1^+} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1^+, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \varphi(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \\
&\quad + K(w_1, \varphi(w_1)) \\
&= \varphi(w_1^-) + K(w_1, \varphi(w_1)).
\end{aligned}$$

Applying the same process, we find $\varphi \in PC^1$ and it fulfills (3.1). We get the required conclusion. $\square$

Define operators $G, H, Y : PC \rightarrow PC$ as follows

$$\begin{aligned}
G\varphi(w) &= \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \varphi(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho, \\
H\varphi(w) &= \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)\varphi_0 + \sum_{\{l: w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)K(w_l, \varphi(w_l)), \\
Y\varphi(w) &= G\varphi(w) + H\varphi(w), \quad w \in [0, T].
\end{aligned}$$

Lemma 3.2. Assume (Fact 1)–(Fact 3). Then $G : PC \rightarrow PC$ is continuous.

Proof. As $G : PC \rightarrow PC$ and $c \in PC$ arbitrarily. Then, we have the following

$$\begin{aligned}
 |G\wp(w)| &= \left| \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\mathbb{T}(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \right| \\
 &\leq \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \left| \frac{\mathbb{T}(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \right| \Delta_{\delta}^{\nu} \varrho \\
 &\leq \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \left(\frac{q(\varrho)}{(\delta(\varrho, \nu))^2} + \frac{r(\varrho)}{(\delta(\varrho, \nu))^2} |\wp(\varrho)| \right) \Delta_{\delta}^{\nu} \varrho \\
 &= \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \frac{q(\varrho)}{(\delta(\varrho, \nu))^2} |\nabla_{\delta}^{\nu} \varrho| + \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \frac{r(\varrho)}{(\delta(\varrho, \nu))^2} |\wp(\varrho)| \Delta_{\delta}^{\nu} \varrho \\
 &\leq \xi + \left(\int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \frac{r(\varrho)}{(\delta(\varrho, \nu))^2} |\nabla_{\delta}^{\nu} \varrho| \right) \|\wp\| \\
 &\leq \xi + \eta \|\wp\|, \quad w \in [0, T].
 \end{aligned}$$

Hence,

$$\|G\wp\| \leq \xi + \eta \|\wp\|, \quad w \in [0, T].$$

Let $\{\wp_n\}_{n \in \mathbb{N}} \subset PC$ be such that

$$\wp_n \rightarrow \wp \quad \text{as } n \rightarrow \infty,$$

where $c \in PC$. Then

$$\begin{aligned}
 |G\wp_n(w) - G\wp(w)| &= \left| \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\mathbb{T}(\varrho, \wp_n(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho - \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\mathbb{T}(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \right| \\
 &= \left| \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \left(\frac{\mathbb{T}(\varrho, \wp_n(\varrho))}{(\delta(\varrho, \nu))^2} - \frac{\mathbb{T}(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \right) \Delta_{\delta}^{\nu} \varrho \right| \\
 &\leq \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \left| \frac{\mathbb{T}(\varrho, \wp_n(\varrho))}{(\delta(\varrho, \nu))^2} - \frac{\mathbb{T}(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \right| \Delta_{\delta}^{\nu} \varrho \\
 &\leq \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \frac{r(\varrho)}{(\delta(\varrho, \nu))^2} |\wp_n(\varrho) - \wp(\varrho)| \Delta_{\delta}^{\nu} \varrho \\
 &\leq \left(\int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{r(\varrho)}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \right) \|\wp_n - \wp\| \\
 &\leq \eta \|\wp_n - \wp\|, \quad w \in [0, T],
 \end{aligned}$$

and

$$\|G\wp_n - G\wp\| \leq \eta \|\wp_n - \wp\|.$$

Therefore, $G : PC \rightarrow PC$ is continuous. Now, we take $\wp \in PC$ arbitrarily and $w_1, w_2 \in [0, T]$. Without loss of generality, suppose that $w_1 < w_2$. Then

$$\begin{aligned}
 |G\wp(w_2) - G\wp(w_1)| &= \left| \int_0^{w_2} \mathbb{E}_{\delta, \frac{p}{F^2}}(w_2, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \right. \\
 &\quad \left. - \int_0^{w_1} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \right| \\
 &= \left| \int_0^{w_1} \left(\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_2, \varsigma(\varrho)) - \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1, \varsigma(\varrho)) \right) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \right. \\
 &\quad \left. + \int_{w_1}^{w_2} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_2, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \right| \\
 &\leq |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_2, w_1) - 1| \int_0^{w_1} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_1, \varsigma(\varrho))| \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} |\Delta_{\delta}^{\nu} \varrho| \\
 &\quad + \int_{w_1}^{w_2} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w_2, \varsigma(\varrho))| \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} |\Delta_{\delta}^{\nu} \varrho| \\
 &\rightarrow 0 \quad \text{as } w_1 \rightarrow w_2.
 \end{aligned}$$

We now determine that $G : PC \rightarrow PC$ is continuous by applying the Arzela-Ascoli theorem. $\square$

Lemma 3.3. Assume (Fact 1)–(Fact 4) hold and

$$A \max_{w \in [0, T]} \sum_{l=1}^n |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| \leq \zeta < 1. \quad (3.5)$$

Then $H : PC \rightarrow PC$ is a contraction.

Proof. Note that $\mathbb{E}_{\delta, \frac{p}{F^2}}(w, w_l)$, $l \in \{1, \dots, n\}$, is a continuous. Then, $H : PC \rightarrow PC$. Let now, $\wp, z \in PC$ be arbitrarily chosen. Then

$$\begin{aligned}
 |H\wp(w) - Hz(w)| &= \left| \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)\wp_0 + \sum_{\{l: w_l < w\}} K(w_l, \wp(w_l)) \right. \\
 &\quad \left. - \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)\wp_0 - \sum_{\{l: w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) K(w_l, z(w_l)) \right| \\
 &= \left| \sum_{\{l: w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) (K(w_l, \wp(w_l)) - K(w_l, z(w_l))) \right| \\
 &\leq \sum_{\{l: w_l < w\}} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |K(w_l, \wp(w_l)) - K(w_l, z(w_l))| \\
 &\leq A \sum_{\{l: w_l < w\}} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |\wp(w_l) - z(w_l)|
 \end{aligned}$$

$$\begin{aligned} &\leq A \sum_{\{l:w_l < w\}} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| \|\varphi - z\| \\ &\leq \zeta \|\varphi - z\|, \quad w \in [0, T], \end{aligned}$$

whereupon

$$\|H\varphi - Hz\| \leq \zeta \|\varphi - z\|.$$

Consequently, $H : PC \rightarrow PC$ is a contraction. $\square$

Theorem 3.4. Assume that

$$\gamma = \max_{w \in [0, T]} \sum_{l=1}^n |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |K(w_l, 0)| < \infty. \quad (3.6)$$

Additionally, let (3.5) and (Fact 1)–(Fact 4) hold, and let ξ , γ , η , A , and S be positive constants that meet the inequality

$$\xi + \gamma + \max_{w \in [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)| \|\varphi_0\| + S \left(\eta + A + \max_{w \in [0, T]} \sum_{l=1}^n |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| \right) \leq S. \quad (3.7)$$

Then Eq (3.1) has a solution φ such that

$$\|\varphi\| \leq S.$$

Proof. Define a set

$$C = \{\varphi \in PC : \|\varphi\| \leq S\}.$$

According to Lemma 3.2, $G : PC \rightarrow PC$ is continuous. Lemma 3.3 implies that $H : PC \rightarrow PC$ is a contraction. Let us now select $\varphi, z \in C$ at random. Next,

$$\begin{aligned} \|\varphi\| &\leq S, \\ |G\varphi(w) + Hz(w)| &= \left| \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \varphi(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho + \sum_{l=1}^n \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) K(w_l, z(w_l)) \right. \\ &\quad \left. + \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0) y_0 \right| \\ &\leq \left| \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \varphi(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \right| + \left| \sum_{l=1}^n \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) K(w_l, z(w_l)) \right| \\ &\quad + |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)| \|\varphi_0\| \\ &\leq \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \frac{\Upsilon(\varrho, \varphi(\varrho))}{(\delta(\varrho, \nu))^2} |\Delta_{\delta}^{\nu} \varrho| + \sum_{l=1}^n |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |K(w_l, z(w_l))| \\ &\quad + |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)| \|\varphi_0\| \end{aligned}$$

$$\begin{aligned}
&= \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} |\Delta_{\delta}^{\nu} \varrho| \\
&\quad + \sum_{l=1}^n |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |K(w_l, z(w_l)) - K(w_l, 0) + K(w_l, 0)| + |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)| |\wp_0| \\
&\leq \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \left(\frac{q(\varrho)}{(\delta(\varrho, \nu))^2} + \frac{r(\varrho)}{(\delta(\varrho, \nu))^2} |\wp(\varrho)| \right) \Delta_{\delta}^{\nu} \varrho \\
&\quad + \sum_{l=1}^n |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |K(w_l, z(w_l)) - K(w_l, 0)| \\
&\quad + \sum_{l=1}^n |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |K(w_l, 0)| + |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)| |\wp_0| \\
&\leq \xi + \eta \|\wp\| + A \sum_{l=1}^n |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |z(w_l)| + \gamma + |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)| \|y_0\| \\
&\leq \xi + \gamma + \eta S + |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)| \|\wp_0\| + \left(A \sum_{l=1}^n |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| \right) S \\
&= \xi + \gamma + |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)| \|\wp_0\| + S \left(\eta + A \sum_{l=1}^n |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| \right) \\
&\leq \xi + \gamma + \max_{w \in [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)| \|\wp_0\| + S \left(\eta + A \max_{w \in [0, T]} \sum_{l=1}^n |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| \right) \\
&\leq S, \quad w \in [0, T].
\end{aligned}$$

Hence,

$$\|G\wp + Hz\| \leq S, \quad w \in [0, T],$$

i.e., $G\wp + Hz \in C$. Hence, by Krasnosel'skii theorem, there is a fixed point in C for the operator $G + H$. Then Eq (3.1) has a solution in the set C , according to Theorem 3.1. The evidence is now complete. $\square$

Theorem 3.5. Consider (Fact 1)–(Fact 4), (3.5)–(3.7), and

$$\eta + \zeta < 1.$$

Then, Eq (3.1) has a unique solution.

Proof. Equation (3.1) has at least one solution in the set C , according to Theorem 3.4. Assume that $\wp$ and z are the two solutions to the Eq (3.1). Next,

$$|Y\wp(w) - Yz(w)| = \left| \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho + \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0) \wp_0 \right|$$

$$\begin{aligned}
& + \sum_{\{l:w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) K(w_l, c(w_l)) \\
& - \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\mathfrak{T}(\varrho, z(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho - \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0) \wp_0 \\
& - \sum_{\{l:w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) K(w_l, z(w_l)) \Big| \\
= & \left| \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \left(\frac{\mathfrak{T}(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} - \frac{\mathfrak{T}(\varrho, z(\varrho))}{(\delta(\varrho, \nu))^2} \right) \Delta_{\delta}^{\nu} \varrho \right. \\
& \left. + \sum_{\{l:w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) (K(w_l, \wp(w_l)) - K(w_l, z(w_l))) \right| \\
\leq & \left| \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \left(\frac{\mathfrak{T}(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} - \frac{\mathfrak{T}(\varrho, z(\varrho))}{(\delta(\varrho, \nu))^2} \right) \Delta_{\delta}^{\nu} \varrho \right| \\
& + \left| \sum_{\{l:w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) (K(w_l, \wp(w_l)) - K(w_l, z(w_l))) \right| \\
\leq & \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \left| \frac{\mathfrak{T}(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} - \frac{\mathfrak{T}(\varrho, z(\varrho))}{(\delta(\varrho, \nu))^2} \right| \Delta_{\delta}^{\nu} \varrho \\
& + \sum_{\{l:w_l < w\}} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |K(w_l, \wp(w_l)) - K(w_l, z(w_l))| \\
\leq & \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \frac{r(\varrho)}{(\delta(\varrho, \nu))^2} |\wp(\varrho) - z(\varrho)| \Delta_{\delta}^{\nu} \varrho \\
& + A \sum_{\{l:w_l < w\}} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |\wp(w_l) - z(w_l)| \\
\leq & \eta \|\wp - z\| + \zeta \|\wp - z\| \\
= & (\eta + \zeta) \|\wp - z\| \\
< & \|\wp - z\|, \quad w \in [0, T].
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\wp - z\| &= \|Y\wp - Yz\| \\
&< \|\wp - z\|,
\end{aligned}$$

that contradicts. □

Suppose

(Fact 5) There exist nondecreasing functions $l_l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$|K(w_l, \varphi(w_l))| \leq l_l(|\varphi(w_l)|), \quad l \in \{0, \dots, n\},$$

and

$$\lim_{r \rightarrow \infty} \frac{l_l(r)}{r} = 0, \quad l \in \{0, \dots, n\},$$

(Fact 6) For each $r > 0$, there exists a continuous function $h_r : [0, T] \rightarrow (0, \infty)$ such that

$$|\Upsilon(w, \varphi)| \leq h_r(w) \quad \text{for each } (w, \varphi) \in [0, T] \times \mathbb{R}, \quad |\varphi| \leq r,$$

and

$$\liminf_{r \rightarrow \infty} \frac{1}{r} \int_0^T \frac{h_r(\varrho)}{\delta(\varrho, \nu)^2} \Delta_{\delta\varrho}^\nu = \varpi_1 < \infty.$$

Theorem 3.6. Suppose (Fact 1), (Fact 2), (Fact 4)–(Fact 6), and

$$\varpi_1 \max_{(w, \varrho) \in [0, T] \times [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| < 1. \quad (3.8)$$

Then Eq (3.1) has at least one solution.

Proof. We have $Y : PC \rightarrow PC$.

1) We will prove that Y is continuous. Let $\{\varphi_m\}_{m \in \mathbb{W}}$ be a sequence of elements of PC such that

$$\varphi_m \rightarrow \varphi, \quad \text{as } m \rightarrow \infty, \quad \varphi \in PC.$$

Then

$$\begin{aligned} |Y\varphi_m(w) - Y\varphi(w)| &= \left| \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \varphi_m(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta\varrho}^\nu + \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)\varphi_0 \right. \\ &\quad + \sum_{\{l: w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) K(w_l, \varphi_m(w_l)) \\ &\quad - \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \varphi(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta\varrho}^\nu - \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)\varphi_0 \\ &\quad \left. - \sum_{\{l: w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) K(w_l, \varphi(w_l)) \right| \\ &= \left| \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \left(\frac{\Upsilon(\varrho, \varphi_m(\varrho))}{(\delta(\varrho, \nu))^2} - \frac{\Upsilon(\varrho, \varphi(\varrho))}{(\delta(\varrho, \nu))^2} \right) \Delta_{\delta\varrho}^\nu \right. \\ &\quad \left. + \sum_{\{l: w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) (K(w_l, \varphi_m(w_l)) - K(w_l, \varphi(w_l))) \right| \\ &\leq \left| \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \Delta_{\delta\varrho}^\nu \left(\frac{\Upsilon(\varrho, \varphi_m(\varrho))}{(\delta(\varrho, \nu))^2} - \frac{\Upsilon(\varrho, \varphi(\varrho))}{(\delta(\varrho, \nu))^2} \right) \Delta_{\delta\varrho}^\nu \right| \end{aligned}$$

$$\begin{aligned}
& + \left| \sum_{\{l: w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) (K(w_l, \wp_m(w_l)) - K(w_l, \wp(w_l))) \right| \\
& \leq \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \frac{\Upsilon(\varrho, \wp_m(\varrho))}{(\delta(\varrho, \nu))^2} - \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} |\Delta_{\delta}^{\nu} \varrho| \\
& \quad + \sum_{\{l: w_l < w\}} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |K(w_l, \wp_m(w_l)) - K(w_l, \wp(w_l))| \\
& \leq \max_{(w, \varrho) \in [0, T] \times [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \int_0^w \left| \frac{\Upsilon(\varrho, \wp_m(\varrho))}{(\delta(\varrho, \nu))^2} - \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \right| |\Delta_{\delta}^{\nu} \varrho| \\
& \quad + A \sum_{\{l: w_l < w\}} \max_{w \in [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |\wp_m(w_l) - \wp(w_l)| \\
& \rightarrow 0 \quad \text{as } m \rightarrow \infty, \quad w \in [0, T].
\end{aligned}$$

Hence,

$$\|Y\wp_m - Y\wp\| \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

2) For each constant $r > 0$, denote

$$B_r = \{\wp \in PC : \|\wp\| \leq r\}.$$

Note that B_r is bounded, closed, and convex set in PC . We will prove that there exists a number $r > 0$ such that $Y(B_r) \subseteq B_r$. Suppose the contrary, i.e., for each $r > 0$ there exists a function $\wp_r \in B_r$, such that

$$|Y(\wp_r(w))| > r \quad \text{for some } w \in [0, T].$$

Then

$$\begin{aligned}
r & < |Y(\wp_r(w))| \\
& = \left| \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp_r(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho + \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0) \wp_0 \right. \\
& \quad \left. + \sum_{\{l: w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) K(w_l, \wp_r(w_l)) \right| \\
& \leq \left| \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp_r(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \right| + |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0) \wp_0| \\
& \quad + \left| \sum_{\{l: w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) K(w_l, \wp_r(w_l)) \right| \\
& \leq \max_{w \in [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)| |\wp_0|
\end{aligned}$$

$$\begin{aligned}
& + \int_0^w |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \frac{\Upsilon(\varrho, \wp_r(\varrho))}{(\delta(\varrho, \nu))^2} |\Delta_{\delta}^{\nu} \varrho| \\
& + \sum_{\{l: w_l < w\}} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |K(w_l, \wp_r(w_l))| \\
& \leq \max_{w \in [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)| |\wp_0| \\
& + \max_{(w, \varrho) \in [0, T] \times [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \int_0^w \frac{h_r(\varrho)}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \\
& + \sum_{\{l: w_l < w\}} \max_{w \in [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| l_l(r),
\end{aligned}$$

whereupon

$$\begin{aligned}
1 & < \max_{w \in [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)| \frac{|\wp_0|}{r} \\
& + \max_{(w, \varrho) \in [0, T] \times [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho))| \frac{1}{r} \int_0^w \frac{h_r(\varrho)}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \\
& + \sum_{\{l: w_l < w\}} \max_{w \in [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| \frac{l_l(r)}{r}
\end{aligned}$$

and

$$1 \leq \varpi_1 \max_{(w, \varrho) \in [0, T] \times [0, T]} |\mathbb{E}_{\delta, \frac{p}{F^2}}(w, \varsigma(\varrho))|,$$

which is a contradiction. Consequently, there is a $r_1 > 0$, such that

$$Y(B_{r_1}) \subset B_{r_1}.$$

- 3) We will show that $Y(B_{r_1})$ is an equicontinuous family of functions. Let $w^1, w^2 \in [0, T]$. Without loss of generality, suppose that $w^1 < w^2$. Then

$$\begin{aligned}
|Y(\wp)(w^2) - Y(\wp)(w^1)| & = \left| \int_0^{w^2} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^2, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho + \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^2, 0) \wp_0 \right. \\
& + \sum_{\{l: w_l < w^2\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^2, w_l) K(w_l, \wp(w_l)) \\
& - \int_0^{w^1} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^1, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho - \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^1, 0) \wp_0 \\
& \left. - \sum_{\{l: w_l < w^1\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^1, w_l) K(w_l, \wp(w_l)) \right| \\
& = \left| \int_0^{w^1} \left(\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^2, \varsigma(\varrho)) - \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^1, \varsigma(\varrho)) \right) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \right.
\end{aligned}$$

$$\begin{aligned}
& + \int_{w^1}^{w^2} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^2, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho \\
& + \left(\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^2, 0) - \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^1, 0) \right) \wp_0 \\
& + \sum_{\{l: w_l < w\}} \left(\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^2, w_l) - \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^1, w_l) \right) K(w_l, \wp(w_l)) \\
& + \sum_{\{l: w_l < w\}} \left| \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^2, w_l) K(w_l, \wp(w_l)) \right| \\
\leq & \int_0^{w^1} \left| \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^2, \varsigma(\varrho)) - \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^1, \varsigma(\varrho)) \right| \left| \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \right| \Delta_{\delta}^{\nu} \varrho \\
& + \int_{w^1}^{w^2} \left| \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^2, \varsigma(\varrho)) \right| \left| \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \right| \Delta_{\delta}^{\nu} \varrho \\
& + \left| \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^2, 0) - \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^1, 0) \right| |\wp_0| \\
& + \sum_{\{l: w_l < w\}} \left| \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^2, w_l) - \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^1, w_l) \right| |K(w_l, \wp(w_l))| \\
& + \sum_{\{l: w_l < w\}} \left| \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w^2, w_l) \right| |K(w_l, \wp(w_l))| \\
\rightarrow & 0 \quad \text{as } w^1 \rightarrow w^2.
\end{aligned}$$

Because of this and the Arzela-Ascoli theorem, we may deduce that $Y : PC \rightarrow PC$ is condensing and fully continuous.

Then, from Sadovskii's fixed point theorem, Let G be a condensing operator on a Banach space X . If $G(D) \subset D$ for a convex, closed and bounded set D of X , then G has a fixed point in D . Consequently, Y has a fixed point in G , a solution to Eq (3.1). The evidence is complete. $\square$

Theorem 3.7. Suppose (Fact 1), (Fact 2), (Fact 4), (Fact 6), (3.8), and

$$A \sum_{\{l: w_l < w\}} \max_{w \in [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| < 1.$$

Then Eq (3.1) has at least one solution.

Proof. On PC , define the operators

$$\begin{aligned}
Y_1 \wp(w) &= \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0) \wp_0 + \sum_{\{l: w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l) K(w_l, \wp(w_l)), \\
Y_2 \wp(w) &= \int_0^w \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, \varsigma(\varrho)) \frac{\Upsilon(\varrho, \wp(\varrho))}{(\delta(\varrho, \nu))^2} \Delta_{\delta}^{\nu} \varrho, \quad w \in [0, T].
\end{aligned}$$

As demonstrated in Theorem 3.6, there is a positive number r such that $Y(B_r) \subseteq B_r$, and the operator Y_2 is a compact operator on B_r . Now, let $c, z \in PC$. Next,

$$\begin{aligned}
 |Y_1\varphi(w) - Y_1z(w)| &= \left| \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)\varphi_0 + \sum_{\{l:w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)K(w_l, \varphi(w_l)) \right. \\
 &\quad \left. - \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, 0)z_0 - \sum_{\{l:w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)K(w_l, z(w_l)) \right| \\
 &= \left| \sum_{\{l:w_l < w\}} \mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)(K(w_l, \varphi(w_l)) - K(w_l, z(w_l))) \right| \\
 &\leq \sum_{\{l:w_l < w\}} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |K(w_l, \varphi(w_l)) - K(w_l, z(w_l))| \\
 &\leq A \sum_{\{l:w_l < w\}} \max_{w \in [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| |\varphi(w_l) - z(w_l)| \\
 &\leq A \sum_{\{l:w_l < w\}} \max_{w \in [0, T]} |\mathbb{E}_{\delta, \frac{p}{\delta^2}}(w, w_l)| \|\varphi - z\|, \quad w \in [0, T].
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \|Y_1\varphi - Y_1z\| &\leq A \sum_{\{l:w_l < w\}} \max_{w \in [0, T]} |\mathbb{E}_{F, \frac{p}{F^2}}(w, w_l)| \|\varphi - z\| \\
 &< \|\varphi - z\|.
 \end{aligned}$$

As a result, $Y = Y_1 + Y_2$ is a condensing operator on B_r . Equation (3.1) has a solution in PC as a result of Sadovskii's fixed point theorem. The evidence is now complete. $\square$

4. Conclusions

Throughout this paper, we extended the concept of first-order nonlinear conformable impulsive dynamic equations to time scales. We obtained certain results concerning the existence and uniqueness of solutions. Moreover, we believe that, in the context of general time scales, the results presented here are promising in the simultaneous modeling of both continuous and discrete processes. Further, it is possible to extend the results of this paper to systems of equations and other systems. Additionally, as a continuation of this work, other aspects, like the approximation of solutions and upper and lower solutions and studying models and practical examples, can be studied in the near future.

Use of AI tools declaration

The authors declare that there are no Artificial Intelligence (AI) tools used in the creation of this article.

Acknowledgments

The authors are thankful to the reviewers for their suggestions and useful recommendations. This scientific paper is derived from a research grant funded by Taibah University, Madinah, Kingdom of Saudi Arabia, with grant number (447-16-1181).

Conflict of interest

The authors declare that there are no conflicts of interest.

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