



Research article

Adaptive memory switching for chaos synchronization in variable-order fractional dynamical systems

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Abstract: We study the synchronization of piecewise-constant-order Caputo systems whose lower terminal is reset at prescribed times. Under linear feedback, a computable dissipativity margin yields an error bound consisting of one Mittag-Leffler factor per completed reset interval. The argument combines standard local Caputo inequalities with continuous endpoint matching; periodic schedules give an explicit cycle factor and a physical-time decay envelope. We also quantify the difference between resetting and retaining memory: A total-variation estimate controls the integrated omitted history and the resulting trajectory discrepancy. An exact scalar example shows that refining the reset mesh need not approximate a fixed-origin equation. A conditional bounded-region Lorenz result gives explicit sufficient gains. Matched-grid computations with identical gains and order schedules compare reset and fixed-origin models and include step refinement and discarded-history diagnostics. In the alternating 0.94/0.86 benchmark with unit dwell, the reset model reaches a 10^{-5} error near $t = 2.04$, whereas the fixed-origin error is approximately 2.15×10^{-3} at $t = 8$. The reset master approaches an equilibrium over this horizon. These results concern synchronization under prescribed memory reinitialization; they establish neither chaotic dynamics nor a general performance advantage for a fixed-memory plant.

Keywords: Caputo memory reset; piecewise fractional order; synchronization; Mittag-Leffler bounds

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1. Introduction

Fractional differential equations encode hereditary response through convolution kernels whose weights extend over the preceding evolution. Caputo models are widely used in nonlinear dynamics because initial data retain the familiar integer-order form while the state evolution carries a power-law

history [1–3].

Synchronization asks whether a controlled response approaches a prescribed master trajectory. The classical construction of Pecora and Carroll concerns chaotic systems [4]; fractional extensions involve Mittag-Leffler decay and fractional Lyapunov estimates [5–8]. Synchronization itself does not imply that the master is chaotic. Here, the objective is convergence of the master–slave error, independently of the asymptotic dynamical regime of the master.

Variable-order operators allow the memory exponent to change [9–11]. The exponent and the lower terminal are separate modeling choices. A fixed-origin variable-order Caputo derivative retains the entire history from t_0 . A fixed-origin piecewise-order derivative uses an exponent α_k on each interval while retaining that same history. We study instead

$$\alpha(t) = \alpha_k, \quad t \in I_k = [t_k, t_{k+1}), \quad (1.1)$$

with

$${}^C D_{t_k}^{\alpha_k} x(t) = \frac{1}{\Gamma(1 - \alpha_k)} \int_{t_k}^t (t - s)^{-\alpha_k} x'(s) ds. \quad (1.2)$$

The state is continuous at t_k , but the active memory begins again there. We call this a *switched-reset piecewise-order Caputo model*. Time and state are nondimensionalized using fixed reference scales, so comparisons across different orders use the same numerical coefficients and time unit.

There is substantial prior work on variable-order dynamics and control. Xu and He studied active synchronization of a variable-order financial model [12]; Pérez-Díaz et al. studied variable-order hyperjerk synchronization [13]; Chen et al. derived linear matrix inequality (LMI) conditions with gain fluctuations [14]; and Kiruthika and Manivannan treated sampled-data control with delays [15]. Predefined-time sliding-mode control is a distinct objective [16]; the bounds below are asymptotic and do not guarantee exact synchronization at a prescribed finite time. Analytical solutions of variable-order Liu systems [17], comparisons of constant- and variable-order dynamics [18], and discrete state-dependent order maps [19] illustrate other questions that should be distinguished from a synchronization estimate. Existence and stability for variable-order initial-value problems have also been considered using fixed-point and noncompactness methods [20, 21].

The closest modeling antecedent is the short-memory construction of Wu et al., which changes the initial point of the Caputo derivative and uses piecewise-constant orders [22]. Short-memory variable-order network synchronization with event-triggered control has also been studied by Li et al. [23]. Thus, changing the memory origin and obtaining exponential synchronization in short-memory settings are established ideas. Table 1 identifies the role of these related studies. The comparison in Table 1 places the present analysis within that existing literature: The focus is a shared prescribed reset clock, continuous state matching, a directly computable feedback margin, and a quantitative comparison with the corresponding history-preserving equation.

A reset can represent deliberate reinitialization of an internal digital memory buffer: retain the current state, discard the stored increments, and load the next kernel exponent. This is a modeling interpretation, rather than a claim that a physical hereditary material can erase its past. A history-preserving piecewise-order operator satisfies the exact identity

$${}^C D_{t_0}^{\alpha_k} x(t) = H_k[x](t) + {}^C D_{t_k}^{\alpha_k} x(t), \quad (1.3)$$

Table 1. Related settings and their relationship to the present analysis. The entries describe the cited contributions, without inferring unreported operator properties.

Study	Setting or result	Relationship to this work
Wu et al. [22]	Variable Caputo initial point and piecewise orders	Direct antecedent for the short-memory model
Li et al. [23]	Event-triggered synchronization of short-memory networks	Prior synchronization results with a different control architecture
Chen et al. [14]	Variable-order one-sided Lipschitz systems and gain fluctuations	LMI-based control context
Kiruthika and Manivannan [15]	Sampled-data synchronization with delays	Control and delay context
Present analysis	Continuous state matching, prescribed resets, and linear feedback	Explicit interval product, omitted-history estimate, and matched-model numerical comparison

where

$$H_k[x](t) = \frac{1}{\Gamma(1 - \alpha_k)} \int_{t_0}^{t_k} (t - s)^{-\alpha_k} x'(s) ds. \quad (1.4)$$

For $t \geq t_k + \delta$, $\delta > 0$, and $x' \in L^1([t_0, t_k])$,

$$\|H_k[x](t)\| \leq \frac{\delta^{-\alpha_k}}{\Gamma(1 - \alpha_k)} \int_{t_0}^{t_k} \|x'(s)\| ds. \quad (1.5)$$

This bound concerns an operator evaluated on one path. It is not, by itself, a bound on the difference of solutions of two different equations. Section 5 supplies that additional step, with an integrated-history estimate that remains meaningful at a reset.

The analytical task is to connect the local damping inequality to the global schedule without discarding the distinction between state matching and history matching. Under $u = -Ke$, a feedback margin $\eta_K > 0$ gives

$$\|e(t)\|^2 \leq \|e(0)\|^2 \left(\prod_{j=0}^{k-1} E_{\alpha_j}(-2\eta_K h_j^{\alpha_j}) \right) E_{\alpha_k}(-2\eta_K (t - t_k)^{\alpha_k}), \quad t \in I_k. \quad (1.6)$$

The controller determines η_K ; the orders and dwell lengths determine how that margin accumulates. For two alternating orders and dwell T , the squared-error cycle factor and its rate are

$$\Theta(T) = E_{\alpha_1}(-2\eta_K T^{\alpha_1}) E_{\alpha_2}(-2\eta_K T^{\alpha_2}), \quad r_\Theta(T) = -\frac{\log \Theta(T)}{2T}. \quad (1.7)$$

The corresponding error-norm rate is $r_\Theta/2$.

The specific results are as follows. First, an intervalwise existence construction and standard Caputo comparison tools yield the product estimate, an explicit gain condition, and a physical-time envelope under dwell constraints. The local inequalities are established results; the schedule calculation is an application of them. Second, a total-variation estimate quantifies the state discrepancy caused by deleting the preswitch history. An exactly solved scalar example shows that refining the reset mesh need

not approximate a fixed-origin equation, even when the fractional order is constant. Third, a conditional Lorenz calculation makes the gain thresholds explicit, and matched-grid experiments compare both memory conventions. For the 0.94/0.86, $T = 1$ benchmark, the fixed-origin error remains approximately 2.15×10^{-3} at $t = 8$, whereas the reset error reaches 10^{-5} near $t = 2.04$. This difference is attributable to the changed memory model and is not a general controller advantage. The reset master approaches an equilibrium over this experiment; no chaotic regime is asserted.

The schedule is fixed offline. No state-dependent adaptation law or convergence theorem for approximation of a continuously varying order is assumed. Section 2 gives the solution framework; Sections 3 and 4 derive the synchronization and cycle bounds; Section 5 compares memory conventions; and Sections 6 and 7 present the Lorenz application and computations.

2. Preliminaries and well-posedness

We work with a strictly increasing switching sequence

$$0 = t_0 < t_1 < t_2 < \cdots, \quad t_k \longrightarrow \infty, \quad (2.1)$$

and define

$$I_k = [t_k, t_{k+1}), \quad \bar{I}_k = [t_k, t_{k+1}], \quad h_k = t_{k+1} - t_k. \quad (2.2)$$

The condition $t_k \rightarrow \infty$ excludes finite-time accumulation of switching instants. Every finite horizon therefore intersects finitely many reset intervals. The admissible orders satisfy

$$0 < \underline{\alpha} \leq \alpha_k \leq \bar{\alpha} < 1, \quad k \geq 0. \quad (2.3)$$

Here, $\underline{\alpha}$ and $\bar{\alpha}$ denote the lower and upper bounds, respectively.

We write $\text{sym } B = (B + B^\top)/2$ and

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(1 + \alpha m)}. \quad (2.4)$$

For $0 < \alpha < 1$, $E_\alpha(-s)$ is positive and completely monotone for $s \geq 0$ [24]. The active order is exactly constant on each interval; no variable-order differentiation rule is used inside a local proof.

Definition 1 (Switched-reset mild and regular solutions). *On I_k , consider*

$${}^C D_{t_k}^{\alpha_k} x(t) = f_k(t, x(t)), \quad x(t_k) = x_k. \quad (2.5)$$

A function $x \in C(\bar{I}_k; \mathbb{R}^n)$ is called a mild solution on $\bar{I}_k$ when

$$x(t) = x_k + \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^t (t-s)^{\alpha_k-1} f_k(s, x(s)) ds, \quad t \in \bar{I}_k. \quad (2.6)$$

A mild solution is called regular when it belongs to $AC(\bar{I}_k; \mathbb{R}^n) \cap C^1(I_k^\circ; \mathbb{R}^n)$, and its derivative is locally bounded on $I_k^\circ = (t_k, t_{k+1})$. A global switched-reset mild solution is continuous on $[0, \infty)$, satisfies (2.6) on every $\bar{I}_k$, and uses the endpoint convention

$$x_{k+1} = x(t_{k+1}^-) = x(t_{k+1}^+). \quad (2.7)$$

This solution concept separates existence from the additional pointwise regularity required by the Lyapunov calculation. Standard fractional initial-value theory gives the following equivalence under the stated regularity [2, 3].

Lemma 1. Fix $k \geq 0$. Every regular solution of the Caputo initial-value problem (2.5) satisfies the Volterra equation (2.6). Conversely, a mild solution that belongs to the regularity class above satisfies (2.5) for $t \in I_k^\circ$ and attains $x(t_k) = x_k$ continuously.

Proof. For

$$I_{t_k}^{\alpha_k} g(t) = \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^t (t-s)^{\alpha_k-1} g(s) ds, \quad (2.8)$$

the standard Caputo identity gives

$$I_{t_k}^{\alpha_k C} D_{t_k}^{\alpha_k} x(t) = x(t) - x(t_k) \quad (2.9)$$

for regular x . Applying $I_{t_k}^{\alpha_k}$ yields (2.6). The converse follows from the corresponding inverse identity on the regularity class. This is the usual Caputo–Volterra equivalence applied with the interval-dependent lower terminal t_k . $\square$

Theorem 1 (Global mild well-posedness). Assume that, for every $k \geq 0$, $f_k : \bar{I}_k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous and globally Lipschitz in the state with constant L_k :

$$\|f_k(t, x) - f_k(t, y)\| \leq L_k \|x - y\|. \quad (2.10)$$

Then, every $x_0 \in \mathbb{R}^n$ generates a unique global switched-reset mild solution on $[0, \infty)$. The construction is valid on every finite horizon because (2.1) excludes Zeno accumulation.

Proof. Fix k and an endpoint value x_k . On $C(\bar{I}_k; \mathbb{R}^n)$, define the Bielecki norm

$$\|x\|_{\beta, k} = \sup_{t \in \bar{I}_k} e^{-\beta(t-t_k)} \|x(t)\|, \quad \beta > 0. \quad (2.11)$$

On the finite interval $\bar{I}_k$, this norm is equivalent to the uniform norm:

$$e^{-\beta h_k} \|x\|_{\infty, k} \leq \|x\|_{\beta, k} \leq \|x\|_{\infty, k}. \quad (2.12)$$

Thus, the topology, completeness, and convergent sequences are the same as for the usual sup norm. The exponential weight is used only to obtain a convenient contraction constant.

Define

$$(T_k x)(t) = x_k + \frac{1}{\Gamma(\alpha_k)} \int_{t_k}^t (t-s)^{\alpha_k-1} f_k(s, x(s)) ds. \quad (2.13)$$

For $x, y \in C(\bar{I}_k; \mathbb{R}^n)$,

$$e^{-\beta(t-t_k)} \|(T_k x)(t) - (T_k y)(t)\| \leq \frac{L_k \|x - y\|_{\beta, k}}{\Gamma(\alpha_k)} \int_0^{t-t_k} r^{\alpha_k-1} e^{-\beta r} dr \quad (2.14)$$

$$\leq L_k \beta^{-\alpha_k} \|x - y\|_{\beta, k}. \quad (2.15)$$

Choosing $\beta > L_k^{1/\alpha_k}$ makes T_k a contraction. Banach's fixed-point theorem therefore yields a unique mild solution on $\bar{I}_k$. Its right endpoint defines x_{k+1} through (2.7). Repetition produces a unique concatenated solution on $[0, t_m]$ for each m . Because $t_m \rightarrow \infty$, these finite-horizon solutions define the unique global mild solution. $\square$

The next tools are standard, not new variable-order lemmas. Lemma 2 is the quadratic Caputo inequality of Aguila-Camacho et al. [8]. Proposition 1 uses the Caputo extremum principle underlying Luchko's maximum principle [25]; Lemma 3 is its standard scalar-comparison consequence, consistent with fractional Lyapunov theory [2, 7]. The proofs are included to specify regularity and the lower terminal. They claim no distinction in mathematical content from these established tools. Their role for a piecewise-order schedule is to be applied separately with $(a, \alpha) = (t_k, \alpha_k)$, after which endpoint continuity concatenates the estimates.

Lemma 2 (Standard quadratic Caputo inequality). *Let $a < b$, $\alpha \in (0, 1)$, and*

$$e \in AC([a, b]; \mathbb{R}^n) \cap C^1((a, b]; \mathbb{R}^n), \quad (2.16)$$

with e' locally bounded on $(a, b]$. Then, for every $t \in (a, b]$,

$${}^C D_a^\alpha \left(\frac{1}{2} \|e(t)\|^2 \right) \leq e(t)^\top {}^C D_a^\alpha e(t). \quad (2.17)$$

Proof. For a scalar $m \in AC([a, b]) \cap C^1((a, b])$ with a locally bounded derivative, integration by parts gives, at each fixed $t > a$,

$${}^C D_a^\alpha m(t) = \frac{m(t) - m(a)}{\Gamma(1 - \alpha)(t - a)^\alpha} + \frac{\alpha}{\Gamma(1 - \alpha)} \int_a^t \frac{m(t) - m(s)}{(t - s)^{\alpha+1}} ds. \quad (2.18)$$

Local boundedness of m' near the fixed endpoint t gives $m(s) - m(t) = O(t - s)$ as $s \uparrow t$, which justifies the boundary limit. Set $V(z) = (1/2)\|z\|^2$. Convexity gives

$$V(e(t)) - V(e(s)) \leq e(t)^\top (e(t) - e(s)). \quad (2.19)$$

Applying (2.18) to $V(e)$ and componentwise to e , followed by the convexity inequality for $s \in [a, t]$ and $s = a$, yields (2.17). $\square$

Proposition 1 (Caputo extremum principle). *Let $w \in AC([a, b]) \cap C^1((a, b])$ have a locally bounded derivative on $(a, b]$ and satisfy $w(a) = 0$. If*

$$w(t_*) = \max_{s \in [a, t_*]} w(s) > 0, \quad (2.20)$$

then

$${}^C D_a^\alpha w(t_*) \geq \frac{w(t_*)}{\Gamma(1 - \alpha)(t_* - a)^\alpha} > 0. \quad (2.21)$$

Proof. Equation (2.18) applies at t_* . Both $w(t_*) - w(s)$ and $w(t_*) - w(a) = w(t_*)$ are non-negative, which gives (2.21). $\square$

Lemma 3 (Standard scalar comparison). *Let $c > 0$, and let $z \in AC([a, b]) \cap C^1((a, b])$ have a locally bounded derivative. If $z(a) = z_a \geq 0$, and*

$${}^C D_a^\alpha z(t) \leq -cz(t), \quad t \in (a, b], \quad (2.22)$$

then

$$z(t) \leq z_a E_\alpha(-c(t - a)^\alpha), \quad t \in [a, b]. \quad (2.23)$$

Proof. Let $v(t) = z_a E_\alpha(-c(t-a)^\alpha)$, so ${}^C D_a^\alpha v = -cv$, and $v(a) = z_a$. With $w = z - v$,

$${}^C D_a^\alpha w(t) + cw(t) \leq 0, \quad w(a) = 0. \quad (2.24)$$

A positive maximum of w would give a strictly positive Caputo derivative by Proposition 1, together with $cw > 0$, which conflicts with the displayed inequality. Hence, $w \leq 0$, and (2.23) follows. $\square$

The preliminary results serve two distinct roles. Theorem 1 provides interval-by-interval mild well-posedness in a Banach space. Lemmas 2 and 3 apply to regular intervalwise solutions. In the Lorenz application, the vector field is smooth, and the synchronization theorem is stated directly for regular trajectories on the bounded working region used in the estimate.

3. Synchronization of switched-reset piecewise-order systems

Throughout this section, x and y are regular intervalwise solutions in the sense of Section 2. The master and slave share the same prescribed sequence $\{t_k, \alpha_k\}$. Thus, every reset is applied at the same physical time in both systems. A timing mismatch would create different lower terminals in the two Caputo operators and hence, a different error equation; that robustness problem lies outside the theorem below.

On I_k , the master system is

$${}^C D_{t_k}^{\alpha_k} x(t) = Ax(t) + \phi(x(t)), \quad (3.1)$$

and the slave system is

$${}^C D_{t_k}^{\alpha_k} y(t) = Ay(t) + \phi(y(t)) + u(t), \quad (3.2)$$

where $A \in \mathbb{R}^{n \times n}$. We use linear state feedback

$$u(t) = -K(y(t) - x(t)), \quad (3.3)$$

with $K \in \mathbb{R}^{n \times n}$. For $e = y - x$,

$${}^C D_{t_k}^{\alpha_k} e(t) = (A - K)e(t) + \phi(y(t)) - \phi(x(t)). \quad (3.4)$$

Definition 2. *Identical synchronization means*

$$\lim_{t \rightarrow \infty} \|e(t)\| = 0. \quad (3.5)$$

For a tolerance $\varepsilon > 0$, the synchronization time is

$$t_\varepsilon = \inf\{t \geq 0 : \|e(s)\| \leq \varepsilon \text{ for every } s \geq t\}. \quad (3.6)$$

The fractional order and synchronization speed refer to different quantities. On I_k , α_k controls the kernel weighting

$$K_{\alpha_k}(t, s) = \frac{(t-s)^{-\alpha_k}}{\Gamma(1-\alpha_k)}, \quad (3.7)$$

and t_ε , the interval damping factors, and the cycle rate r_Θ quantify synchronization performance.

Theorem 2 (Feedback-dependent product estimate). *Let the master (3.1) and the slave (3.2) share the same prescribed reset times and orders, and let both trajectories be regular on every interval. Assume*

$$\|\phi(\xi) - \phi(\zeta)\| \leq L\|\xi - \zeta\|, \quad \xi, \zeta \in \mathbb{R}^n, \quad (3.8)$$

for $L \geq 0$. Apply the controller $u(t) = -K(y(t) - x(t))$ with a constant matrix K , and define its damping margin by

$$\eta_K = \lambda_{\min}(\text{sym } K - \text{sym } A) - L > 0. \quad (3.9)$$

For any $0 < \eta \leq \eta_K$, equivalently

$$\lambda_{\max}(\text{sym}(A - K)) + L \leq -\eta, \quad (3.10)$$

the error $e = y - x$ satisfies, for $t \in I_k$,

$$\|e(t)\|^2 \leq \|e(0)\|^2 \left(\prod_{j=0}^{k-1} E_{\alpha_j}(-2\eta h_j^{\alpha_j}) \right) E_{\alpha_k}(-2\eta(t - t_k)^{\alpha_k}). \quad (3.11)$$

A uniform positive lower dwell bound

$$h_k \geq h_* > 0 \quad (k \geq 0) \quad (3.12)$$

implies identical synchronization. In particular, $K = \kappa I$ with $\kappa \geq 0$ and $\kappa > \lambda_{\max}(\text{sym } A) + L$ are sufficient, with $\eta_K = \kappa - \lambda_{\max}(\text{sym } A) - L$.

Proof. Subtracting the controlled slave and master equations produces the term $-Ke$ in (3.4). Thus, the controller changes the symmetric linear part from $\text{sym } A$ to $\text{sym}(A - K)$; Eq (3.9) is precisely the amount of damping remaining after allowing for L . Set $V(t) = (1/2)\|e(t)\|^2$. Lemma 2 and (3.4) give

$${}^C D_{t_k}^{\alpha_k} V(t) \leq e(t)^\top (A - K)e(t) + e(t)^\top (\phi(y(t)) - \phi(x(t))) \quad (3.13)$$

$$\leq \left[\lambda_{\max} \left(\frac{(A - K) + (A - K)^\top}{2} \right) + L \right] \|e(t)\|^2 \quad (3.14)$$

$$\leq -2\eta V(t). \quad (3.15)$$

Lemma 3, applied on the active reset interval, yields

$$V(t) \leq V(t_k) E_{\alpha_k}(-2\eta(t - t_k)^{\alpha_k}). \quad (3.16)$$

Taking $t \uparrow t_{k+1}$ and using endpoint continuity gives

$$V(t_{k+1}) \leq V(t_k) E_{\alpha_k}(-2\eta h_k^{\alpha_k}). \quad (3.17)$$

Iteration over the completed reset intervals gives

$$V(t_k) \leq V(0) \prod_{j=0}^{k-1} E_{\alpha_j}(-2\eta h_j^{\alpha_j}), \quad (3.18)$$

which combined with (3.16) yields (3.11).

For the asymptotic statement, the function $x \mapsto E_\alpha(-x)$ is decreasing on $x > 0$ for $0 < \alpha < 1$. The map

$$\alpha \longmapsto E_\alpha(-2\eta h_*^\alpha) \quad (3.19)$$

is continuous on the compact interval $[\underline{\alpha}, \bar{\alpha}]$ and takes values in $(0, 1)$. Hence there exists

$$q = \max_{\alpha \in [\underline{\alpha}, \bar{\alpha}]} E_\alpha(-2\eta h_*^\alpha) < 1. \quad (3.20)$$

Using $h_k \geq h_*$,

$$E_{\alpha_k}(-2\eta h_k^{\alpha_k}) \leq q, \quad (3.21)$$

so $V(t_k) \leq q^k V(0)$. Equation (3.16) also gives $V(t) \leq V(t_k)$ on I_k . Because $t_k \rightarrow \infty$, the interval index tends to infinity with time, and $V(t) \rightarrow 0$. $\square$

Remark 1 (Role of interval lengths). *The factor generated by interval k is*

$$q_k = E_{\alpha_k}(-2\eta h_k^{\alpha_k}). \quad (3.22)$$

For a constant order $\alpha_k \equiv \alpha$, the completed-interval product is

$$\prod_{j=0}^{k-1} E_\alpha(-2\eta h_j^\alpha). \quad (3.23)$$

A repeated power $[E_\alpha(-2\eta h^\alpha)]^k$ arises when the dwell lengths also satisfy $h_j \equiv h$. Thus, order and dwell time enter each factor separately.

Corollary 1 (Reset-induced exponential envelope). *Under the hypotheses of Theorem 2, suppose*

$$0 < h_* \leq h_k \leq h^* < \infty. \quad (3.24)$$

Then, constants $M \geq 1$ and $\sigma_* > 0$ exist such that

$$\|e(t)\| \leq M e^{-\sigma_* t} \|e(0)\|, \quad t \geq 0. \quad (3.25)$$

Proof. Let $N(t)$ satisfy $t \in I_{N(t)}$. The upper dwell bound gives

$$t \leq (N(t) + 1)h^*, \quad N(t) \geq \frac{t}{h^*} - 1. \quad (3.26)$$

With q from (3.20), Theorem 2 gives

$$\|e(t)\|^2 \leq q^{N(t)} \|e(0)\|^2 \leq q^{-1} \exp\left(\frac{\log q}{h^*} t\right) \|e(0)\|^2. \quad (3.27)$$

Thus, one may take

$$M = q^{-1/2}, \quad \sigma_* = -\frac{\log q}{2h^*}. \quad (3.28)$$

$\square$

Theorem 2 gives a sufficient feedback condition; failure of (3.9) is not a proof of instability. A larger scalar gain increases the margin, but actuator limits and control cost are absent from the theorem.

The exponential envelope in Corollary 1 is generated by repeated reset-to-reset contractions. Its mechanism is the linear growth of the number of completed reset intervals with physical time. Fixed-origin long-memory fractional dynamics possess a different cross-interval structure and require a separate decay analysis.

The lower dwell condition also has a physical interpretation. If $h_k \downarrow 0$ along a subsequence, then individual factors may approach one, and the uniform contraction constant in (3.20) can be lost. A positive h_* represents a minimum time during which a selected kernel remains active before the next memory reinitialization.

4. Periodic schedules and explicit cycle factors

Consider a two-level prescribed schedule with equal half-cycle length $T > 0$,

$$\alpha_{2m} = \alpha_1, \quad \alpha_{2m+1} = \alpha_2, \quad h_k = T, \quad m = 0, 1, 2, \dots, \quad (4.1)$$

where $0 < \alpha_1, \alpha_2 < 1$.

Theorem 3. *Under the hypotheses of Theorem 2, the cycle endpoints satisfy*

$$\|e(2mT)\|^2 \leq \Theta(T)^m \|e(0)\|^2, \quad (4.2)$$

with

$$\Theta(T) = E_{\alpha_1}(-2\eta T^{\alpha_1}) E_{\alpha_2}(-2\eta T^{\alpha_2}), \quad 0 < \Theta(T) < 1. \quad (4.3)$$

The corresponding squared-error contraction rate per unit physical time is

$$r_{\Theta}(T) = -\frac{\log \Theta(T)}{2T} > 0. \quad (4.4)$$

Proof. For one complete cycle, Eq (3.17) gives

$$\|e(2(m+1)T)\|^2 \leq E_{\alpha_2}(-2\eta T^{\alpha_2}) E_{\alpha_1}(-2\eta T^{\alpha_1}) \|e(2mT)\|^2 \quad (4.5)$$

$$= \Theta(T) \|e(2mT)\|^2. \quad (4.6)$$

Iteration gives (4.2). Each Mittag-Leffler factor lies in $(0, 1)$ for the present positive argument, yielding $0 < \Theta(T) < 1$. Taking logarithms of the cycle inequality motivates (4.4). $\square$

The pair (Θ, r_{Θ}) separates squared-error contraction per cycle from contraction per unit time; the corresponding error-norm rate is $r_{\Theta}/2$. Increasing T can reduce the one-cycle factor because each active interval lasts longer, and the number of resets per unit time simultaneously decreases. The scan in Section 7 compares distinct memory-reset models, and Section 5 explains why the resulting rates are not performance guarantees for a fixed-origin model.

5. Discrepancy caused by a memory reset

The piecewise order in (1.1) is an exact model specification. It does not identify resetting the terminal with approximation of a fixed-origin variable-order operator. The next estimate separates the two operations at the level of solution trajectories.

Proposition 2 (Integrated-history and trajectory bounds). *Fix a reset time $a > 0$, an order $\alpha \in (0, 1)$, and $h > 0$. Let $v \in AC([0, a + h]; \mathbb{R}^n)$ satisfy the fixed-origin equation*

$${}^C D_0^\alpha v(t) = f(t, v(t)), \quad a < t \leq a + h,$$

and let z satisfy ${}^C D_a^\alpha z = f(t, z)$ on this interval. Assume, the Caputo–Volterra identities hold, both paths are continuous, and $\|f(t, z(t)) - f(t, v(t))\| \leq L\|z(t) - v(t)\|$ for some $L \geq 0$. Define, for $0 < \tau \leq h$,

$$J_a(\tau) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^a \|v'(r)\| \int_0^\tau (\tau-s)^{\alpha-1} (a+s-r)^{-\alpha} ds dr, \quad (5.1)$$

$$J_a(0) = 0, \quad d_a = \|z(a) - v(a)\|. \quad (5.2)$$

Then, J_a is nondecreasing, $J_a(\tau) \rightarrow 0$ as $\tau \downarrow 0$, and

$$0 \leq J_a(\tau) \leq \int_0^a \|v'(r)\| dr. \quad (5.3)$$

Moreover,

$$\|z(a+\tau) - v(a+\tau)\| \leq (d_a + J_a(\tau))E_\alpha(L\tau^\alpha). \quad (5.4)$$

Proof. Splitting the fixed-origin integral gives ${}^C D_a^\alpha v = f(t, v) - H_a[v]$, where $H_a[v]$ is (1.4) with $t_k = a$. The two local integral equations therefore imply, with $d(\tau) = \|z(a+\tau) - v(a+\tau)\|$,

$$d(\tau) \leq d_a + L I_0^\alpha d(\tau) + J_a(\tau). \quad (5.5)$$

Tonelli's theorem justifies the non-negative double integral defining J_a . For $b = a - r > 0$, the substitution $u = (b + s)/(b + \tau)$ yields

$$\int_0^\tau (\tau-s)^{\alpha-1} (b+s)^{-\alpha} ds = \int_{b/(b+\tau)}^1 (1-u)^{\alpha-1} u^{-\alpha} du \leq \Gamma(\alpha)\Gamma(1-\alpha). \quad (5.6)$$

The lower limit decreases as τ increases, which proves monotonicity. The integrand integrated over this shrinking interval tends to zero for each $r < a$ as $\tau \downarrow 0$. Dominated convergence with the dominating function $\|v'(r)\|$ proves the endpoint limit and (5.3).

Set $b(\tau) = d_a + J_a(\tau)$. Iterating (5.5) m times and using the semigroup identity for constant-order fractional integrals gives

$$d(\tau) \leq \sum_{j=0}^{m-1} L^j I_0^{j\alpha} b(\tau) + L^m I_0^{m\alpha} d(\tau).$$

For $j \geq 0$, monotonicity gives $I_0^{j\alpha} b(\tau) \leq b(\tau)\tau^{j\alpha}/\Gamma(j\alpha + 1)$. The remainder is at most $\|d\|_{C[0,h]} L^m h^{m\alpha}/\Gamma(m\alpha + 1)$, which tends to zero. Letting $m \rightarrow \infty$ proves (5.4). $\square$

Apply Proposition 2 on each interval with the fixed-origin path in the history term. If D_k bounds the state discrepancy at t_k , then

$$D_{k+1} = (D_k + J_{t_k}(h_k))E_{\alpha_k}(L_k h_k^{\alpha_k}), \quad D_0 = 0, \quad (5.7)$$

is a valid endpoint majorant whenever the hypotheses hold. This is an a posteriori comparison: It uses the variation of the history-preserving path. It need not be small, and the positive-argument Mittag-Leffler factor may make it conservative. A claim of accurate memory truncation requires quantitative control of these terms over the entire horizon.

Proposition 3 (Reset refinement need not preserve the model). *Let $0 < \alpha < 1$, and $\lambda > 0$. The fixed-origin relaxation equation ${}^C D_0^\alpha q = -\lambda q$, $q(0) = 1$ has the solution $q_0(t) = E_\alpha(-\lambda t^\alpha) > 0$. Reset the terminal every $h > 0$ while keeping the state continuous, and denote the resulting solution by q_h . Then, for every fixed $t > 0$,*

$$q_h(t) = [E_\alpha(-\lambda h^\alpha)]^{\lfloor t/h \rfloor} E_\alpha(-\lambda(t - h\lfloor t/h \rfloor)^\alpha) \longrightarrow 0 \quad (h \downarrow 0). \quad (5.8)$$

Consequently, $q_h(t)$ does not converge to $q_0(t)$.

Proof. Solving on each reset interval and matching its endpoint gives the product in (5.8). The Mittag-Leffler series yields

$$\log E_\alpha(-\lambda h^\alpha) = -\frac{\lambda h^\alpha}{\Gamma(1 + \alpha)} + O(h^{2\alpha}).$$

Thus, the logarithm of the completed-interval product is $-\lambda t h^{\alpha-1}(1 + o(1))/\Gamma(1 + \alpha) \rightarrow -\infty$. The remaining factor tends to one. Positivity of $q_0(t)$ follows from complete monotonicity [24]. $\square$

Proposition 3 already uses a constant order, so making a piecewise representation of a variable order more accurate cannot remove this terminal-reset obstruction. The same small-dwell expansion gives, for (4.3),

$$r_\Theta(T) = \eta \sum_{i=1}^2 \frac{T^{\alpha_i-1}}{\Gamma(1 + \alpha_i)} (1 + O(T^{\alpha_i})) \longrightarrow \infty. \quad (5.9)$$

This divergence describes the family of increasingly frequent reset models. It is not an unbounded performance improvement for a fixed hereditary plant. Any implementation would need a separate justification for erasing memory and for the cost and admissible frequency of that operation.

6. Conditional bounded-region Lorenz synchronization

We now specialize the reset model to a controlled Lorenz pair. The Lorenz vector field is polynomial and locally Lipschitz. The argument below is derived directly from the Lorenz error equations on a bounded working region. It provides a local/bounded-region result alongside the global-Lipschitz framework of Theorem 2.

On every reset interval I_k , the master is

$$\begin{aligned} {}^C D_{t_k}^{\alpha_k} x_1 &= \sigma(x_2 - x_1), \\ {}^C D_{t_k}^{\alpha_k} x_2 &= \rho x_1 - x_2 - x_1 x_3, \\ {}^C D_{t_k}^{\alpha_k} x_3 &= x_1 x_2 - \beta x_3, \end{aligned} \quad (6.1)$$

and the slave is

$$\begin{aligned} {}^C D_{t_k}^{\alpha_k} y_1 &= \sigma(y_2 - y_1) - k_1(y_1 - x_1), \\ {}^C D_{t_k}^{\alpha_k} y_2 &= \rho y_1 - y_2 - y_1 y_3 - k_2(y_2 - x_2), \\ {}^C D_{t_k}^{\alpha_k} y_3 &= y_1 y_2 - \beta y_3 - k_3(y_3 - x_3), \end{aligned} \quad (6.2)$$

with $k_1, k_2, k_3 > 0$. The master and slave use the same reset times and the same order sequence.

Let $e_i = y_i - x_i$. Direct subtraction gives

$$\begin{aligned} {}^C D_{t_k}^{\alpha_k} e_1 &= -(\sigma + k_1)e_1 + \sigma e_2, \\ {}^C D_{t_k}^{\alpha_k} e_2 &= (\rho - x_3)e_1 - (1 + k_2)e_2 - y_1 e_3, \\ {}^C D_{t_k}^{\alpha_k} e_3 &= x_2 e_1 + y_1 e_2 - (\beta + k_3)e_3. \end{aligned} \quad (6.3)$$

For $R > 0$, define the bounded working region

$$\Omega_R = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x_i| \leq R, |y_i| \leq R, i = 1, 2, 3\}. \quad (6.4)$$

The theorem is conditional on the trajectories remaining in this region. An all-time certificate additionally requires an independent positive-invariance or absorbing-set argument for Ω_R .

Theorem 4. Assume that the regular trajectories of (6.1) and (6.2) remain in Ω_R for all $t \geq 0$. Let $\eta > 0$. If

$$\sigma + k_1 \geq \eta + \frac{\sigma + \rho + 2R}{2}, \quad (6.5)$$

$$1 + k_2 \geq \eta + \frac{\sigma + \rho + R}{2}, \quad (6.6)$$

and

$$\beta + k_3 \geq \eta + \frac{R}{2}, \quad (6.7)$$

then

$$\|e(t)\|^2 \leq \|e(0)\|^2 \left(\prod_{j=0}^{k-1} E_{\alpha_j}(-2\eta h_j^{\alpha_j}) \right) E_{\alpha_k}(-2\eta(t - t_k)^{\alpha_k}), \quad t \in I_k. \quad (6.8)$$

A uniform lower dwell bound $h_k \geq h_* > 0$ yields identical synchronization.

Proof. Let

$$V(t) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2). \quad (6.9)$$

Lemma 2 and (6.3) give

$${}^C D_{t_k}^{\alpha_k} V \leq -(\sigma + k_1)e_1^2 - (1 + k_2)e_2^2 - (\beta + k_3)e_3^2 \quad (6.10)$$

$$+ (\sigma + \rho - x_3)e_1 e_2 + x_2 e_1 e_3. \quad (6.11)$$

The terms involving $y_1 e_2 e_3$ cancel exactly. On Ω_R ,

$$|\sigma + \rho - x_3| \leq \sigma + \rho + R, \quad |x_2| \leq R. \quad (6.12)$$

Using $|ab| \leq (1/2)(a^2 + b^2)$,

$$(\sigma + \rho - x_3)e_1e_2 \leq \frac{\sigma + \rho + R}{2}(e_1^2 + e_2^2), \quad (6.13)$$

$$x_2e_1e_3 \leq \frac{R}{2}(e_1^2 + e_3^2). \quad (6.14)$$

Hence,

$${}^c D_{t_k}^{\alpha_k} V \leq \left[-(\sigma + k_1) + \frac{\sigma + \rho + 2R}{2} \right] e_1^2 \quad (6.15)$$

$$+ \left[-(1 + k_2) + \frac{\sigma + \rho + R}{2} \right] e_2^2 \quad (6.16)$$

$$+ \left[-(\beta + k_3) + \frac{R}{2} \right] e_3^2 \quad (6.17)$$

$$\leq -\eta(e_1^2 + e_2^2 + e_3^2) = -2\eta V. \quad (6.18)$$

The interval comparison and reset concatenation from Lemma 3 then give (6.8). The positive-dwell conclusion follows from the compact-order argument in Theorem 2. $\square$

Corollary 2. For $\sigma = 10$, $\rho = 28$, $\beta = 8/3$, $R = 30$, and $\eta = 1$, the integer gains

$$k_1 = 41, \quad k_2 = 34, \quad k_3 = 14 \quad (6.19)$$

satisfy (6.5)–(6.7).

Proof. The three threshold inequalities reduce to

$$k_1 \geq 40, \quad k_2 \geq 34, \quad k_3 \geq \frac{40}{3}, \quad (6.20)$$

and the stated integer values satisfy them. $\square$

The gain conditions are deliberately transparent and correspond to the Euclidean quadratic Lyapunov function $V = e^\top e/2$ together with scalar Young bounds. This choice produces conservative sufficient gains. A weighted function $V = e^\top P e/2$ with $P > 0$ would expose additional degrees of freedom and can be coupled to an LMI search for smaller gains or a tighter decay certificate. The present algebraic form is retained because it makes the dependence on R , η , and the Lorenz parameters explicit.

For the numerical working region $R = 35$ and the value $\eta = 4$ used in Section 7, Conditions (6.5)–(6.7) require

$$k_1 \geq 48, \quad k_2 \geq 39.5, \quad k_3 \geq 18.833 \dots \quad (6.21)$$

The simulation therefore uses $(k_1, k_2, k_3) = (50, 40, 20)$. These gains provide a simple integer choice compatible with the sharper theorem parameter $\eta = 4$ on Ω_{35} . Minimum-energy tuning is a separate controller-design task.

A representative prescribed schedule is

$$\alpha_k = \begin{cases} 0.94, & k \text{ even,} \\ 0.86, & k \text{ odd,} \end{cases} \quad h_k = T. \quad (6.22)$$

Its cycle factor and cycle rate are

$$\Theta_L(T) = E_{0.94}(-2\eta T^{0.94}) E_{0.86}(-2\eta T^{0.86}), \quad r_{\Theta,L}(T) = -\frac{\log \Theta_L(T)}{2T}. \quad (6.23)$$

These quantities support an offline schedule-design calculation once the admissible operating modes and reset cadence are specified.

7. Numerical simulations

This section tests the switched-reset Lorenz model with a numerical method that follows the lower-terminal reset used in the analysis. All computations were carried out in IEEE-754 binary64 arithmetic. The supplied scripts regenerate the figures, numerical summaries, and memory-model comparisons. Mittag-Leffler factors are evaluated using a non-negative spectral integral, with a Taylor series only for small arguments. Comparisons at 25 parameter points against 60-digit arithmetic give a maximum relative difference below 5×10^{-15} ; this avoids cancellation in a large-argument alternating series.

7.1. Reset-consistent PECE discretization

On one reset interval $I_k = [t_k, t_{k+1}]$, let $h = (t_{k+1} - t_k)/N_k$, $t_n = t_k + nh$, and $z_n \approx z(t_n)$ for the six-dimensional master-slave state $z = (x, y)$. For a local fractional order $\alpha = \alpha_k$, the Diethelm–Ford–Freed Adams–Bashforth–Moulton predictor–corrector method [26] uses the predictor

$$z_{n+1}^P = z_0 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^n b_{j,n+1} f(t_j, z_j), \quad b_{j,n+1} = (n + 1 - j)^\alpha - (n - j)^\alpha, \quad (7.1)$$

and the corrector

$$z_{n+1} = z_0 + \frac{h^\alpha}{\Gamma(\alpha + 2)} \left[f(t_{n+1}, z_{n+1}^P) + \sum_{j=0}^n a_{j,n+1} f(t_j, z_j) \right], \quad (7.2)$$

where

$$a_{0,n+1} = n^{\alpha+1} - (n - \alpha)(n + 1)^\alpha, \quad (7.3)$$

$$a_{j,n+1} = (n - j + 2)^{\alpha+1} + (n - j)^{\alpha+1} - 2(n - j + 1)^{\alpha+1}, \quad 1 \leq j \leq n. \quad (7.4)$$

At t_{k+1} , the computed endpoint becomes the initial state of the next interval and a new local history starts there. This intervalwise restart is the discrete realization of the switched-reset Caputo law in Sections 2–6.

The baseline parameters are

$$\sigma = 10, \quad \rho = 28, \quad \beta = \frac{8}{3}, \quad (k_1, k_2, k_3) = (50, 40, 20), \quad (7.5)$$

with initial data

$$x(0) = (-8, 8, 27)^\top, \quad y(0) = (-6, 7, 25)^\top. \quad (7.6)$$

The two-level schedule alternates $\alpha_H = 0.94$ and $\alpha_L = 0.86$, with baseline half-cycle length $T = 1$. The production step is $h = 1.25 \times 10^{-3}$ on $0 \leq t \leq 8$, and the theorem parameter is $\eta = 4$. For $R = 35$, Theorem 4 requires

$$k_1 \geq 48, \quad k_2 \geq 39.5, \quad k_3 \geq 18.833 \dots, \quad (7.7)$$

so the gains in (7.5) satisfy the stated conditional bounded-region criterion.

Over the computed horizon, the componentwise maxima are

$$\max_t |x(t)| = (12.350, 15.317, 33.818), \quad \max_t |y(t)| = (12.356, 15.317, 33.805), \quad (7.8)$$

component by component. Hence, the simulated trajectories remain inside Ω_{35} on $[0, 8]$. This finite-horizon observation supplies an a posteriori diagnostic for the bounded-region hypothesis. An all-time Lorenz certificate additionally requires a positive-invariance or absorbing-set analysis.

7.2. Quantitative synchronization and theorem envelope

The numerical tolerance time is the finite-grid statistic

$$\widehat{t}_{\varepsilon,8} = \min\{t_n : \|e(t_j)\|_2 \leq \varepsilon \text{ for all grid nodes } t_j \in [t_n, 8]\}, \quad \varepsilon = 10^{-5}, \quad (7.9)$$

with “not attained” if this set is empty. It is different from the all-time quantity (3.6). For the baseline switched schedule, $\widehat{t}_{\varepsilon,8} = 2.040$. Figure 1 displays the first coordinates and reset times. Their overlap demonstrates error decay over the computed horizon, not chaotic dynamics.

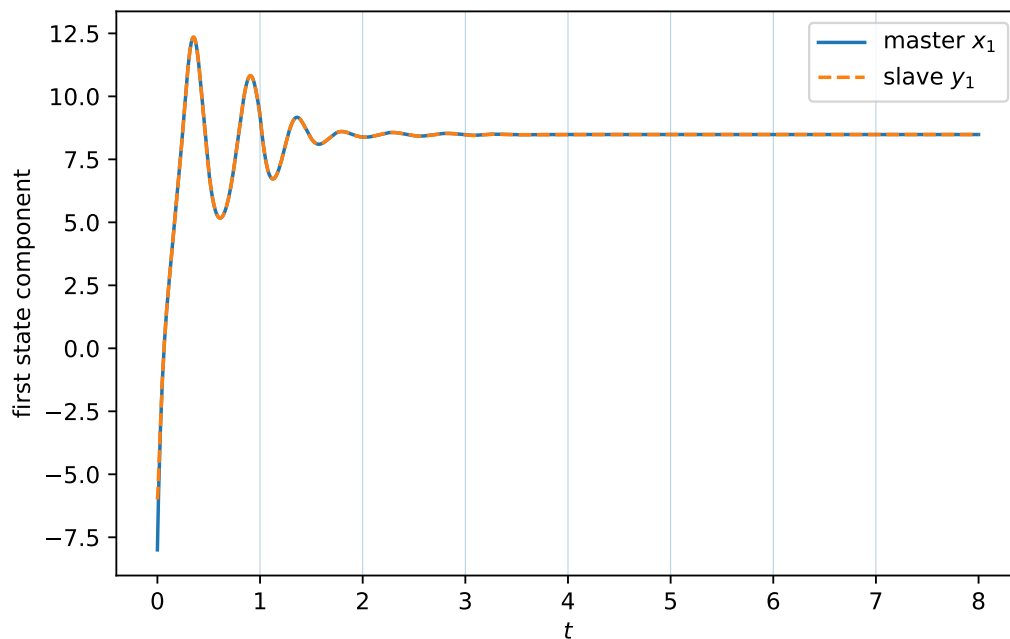


Figure 1. First master and slave coordinates for the baseline switched-reset schedule 0.94/0.86 with $T = 1$. Thin vertical markers indicate reset times. The finite-grid tolerance time (7.9) is $\widehat{t}_{\varepsilon,8} = 2.040$. The traces settle toward an equilibrium.

Figure 2 supplies a phase projection and the distance to the positive equilibrium

$$x_* = (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho-1)^\top = (\sqrt{72}, \sqrt{72}, 27)^\top.$$

The distance is approximately 8.26×10^{-7} at $t = 8$ and at most 5.26×10^{-5} on $[6, 8]$. Thus, this run displays a decaying transient toward the equilibrium. A phase projection is not a Lyapunov-exponent

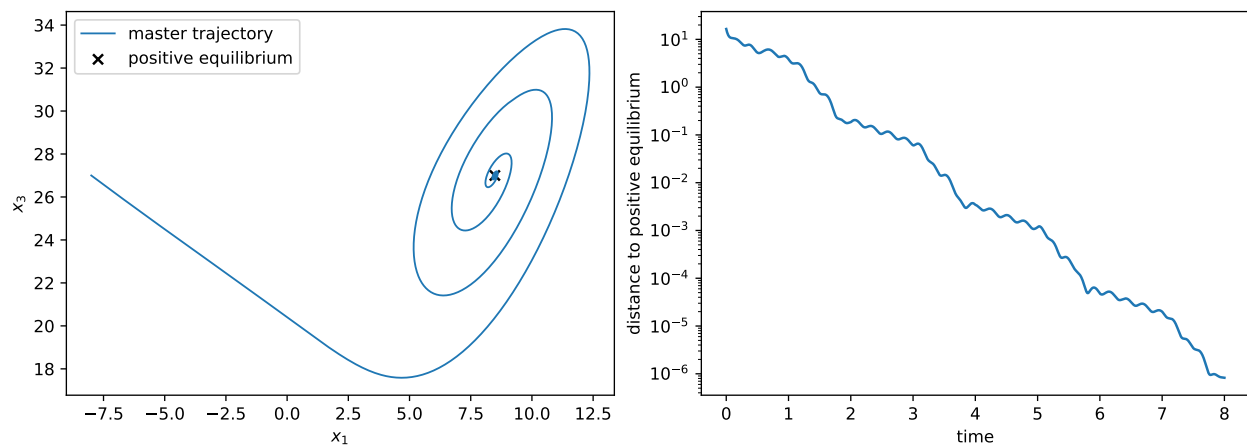


Figure 2. Baseline reset master: Phase projection and distance to its positive equilibrium. The experiment illustrates transient synchronization followed by near-equilibrium behavior.

calculation, and no chaotic attractor is claimed. Theorem 2 concerns the synchronization error and makes no assertion that resetting preserves any dynamical regime of the master.

Theorem 4 yields the squared-error envelope with $\eta = 4$. Figure 3 compares this envelope with the computed $\|e(t)\|_2^2$. Table 2 gives reset-time values through $t = 4$, where the state-space error remains well above its eventual binary64 roundoff floor.

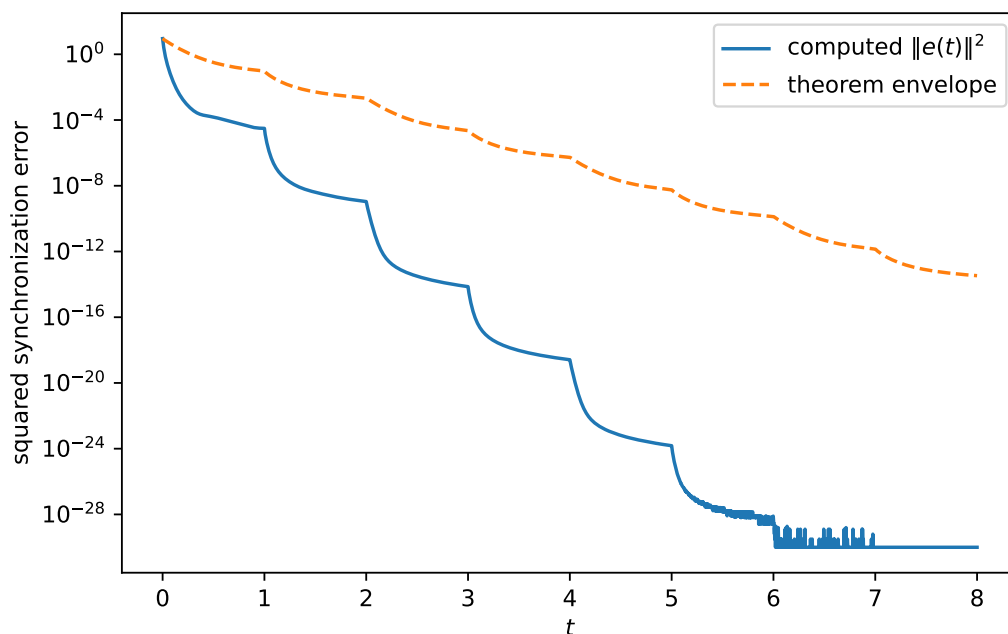


Figure 3. Computed squared synchronization error and the product-type theorem envelope for the baseline switched-reset Lorenz pair.

The separation of the curves in Figure 3 and the large ratios in Table 2 quantify the conservatism of the scalar Lyapunov estimate. The proof uses $P = I$ and scalar Young bounds to dominate the cross terms uniformly over Ω_R . A weighted quadratic form $V = e^\top P e$ together with an LMI or direct matrix

Table 2. Computed squared error, theorem envelope, and conservatism ratio at selected reset times.

t	$\ e(t)\ _2^2$	Theorem envelope	Envelope/ $\ e(t)\ _2^2$
1	3.103×10^{-5}	9.537×10^{-2}	3.074×10^3
2	1.089×10^{-9}	2.226×10^{-3}	2.043×10^6
3	7.242×10^{-15}	2.359×10^{-5}	3.257×10^9
4	2.581×10^{-19}	5.507×10^{-7}	2.134×10^{12}

optimization can tighten the gain and decay estimates. The present form is retained because it gives closed algebraic gain conditions and an explicit schedule factor.

7.3. Schedule dependence and cycle-factor maps

For equal reset cadence $T = 1$, Figure 4 compares the switched schedule with constant-order reset references. Table 3 reports the tolerance-based synchronization time and feedback-energy integral

$$\mathcal{E}_u = \int_0^8 \|u(t)\|_2^2 dt. \quad (7.10)$$

The reference cases keep the same reset times, so this comparison isolates the fractional-order sequence within the same reset architecture.

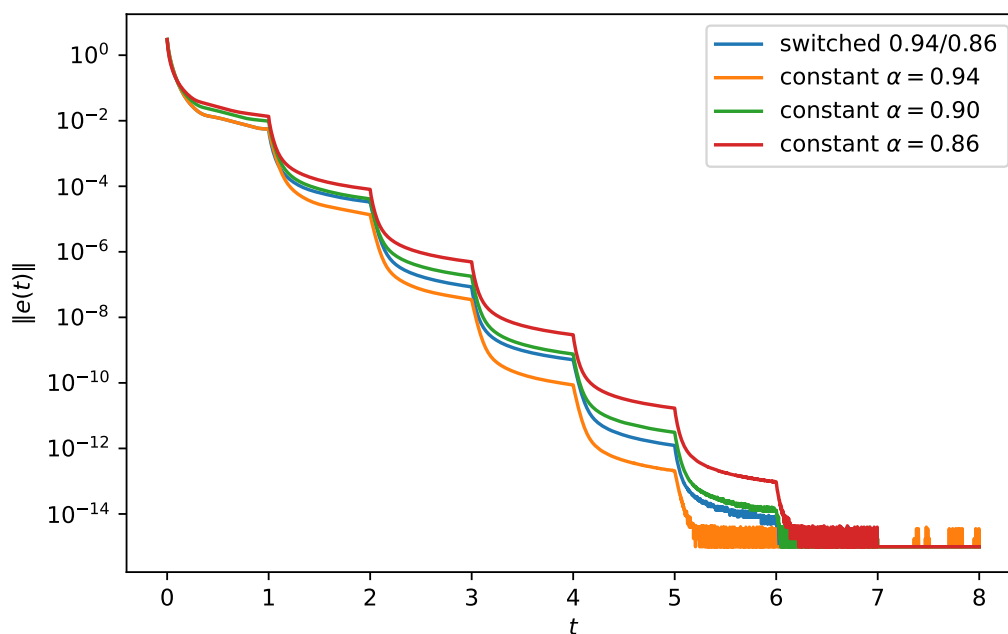


Figure 4. Synchronization-error norm for the switched schedule and three constant-order reset references with common half-cycle length $T = 1$.

Figure 4 and Table 3 compare order sequences within the reset architecture. At $T = 1$, constant $\alpha = 0.94$ reaches the tolerance slightly earlier than the alternating schedule (2.0100 versus 2.0400). These rows neither show an advantage for order switching nor supply a history-preserving baseline.

Table 3. Schedule comparison for common reset cadence $T = 1$ on $0 \leq t \leq 8$.

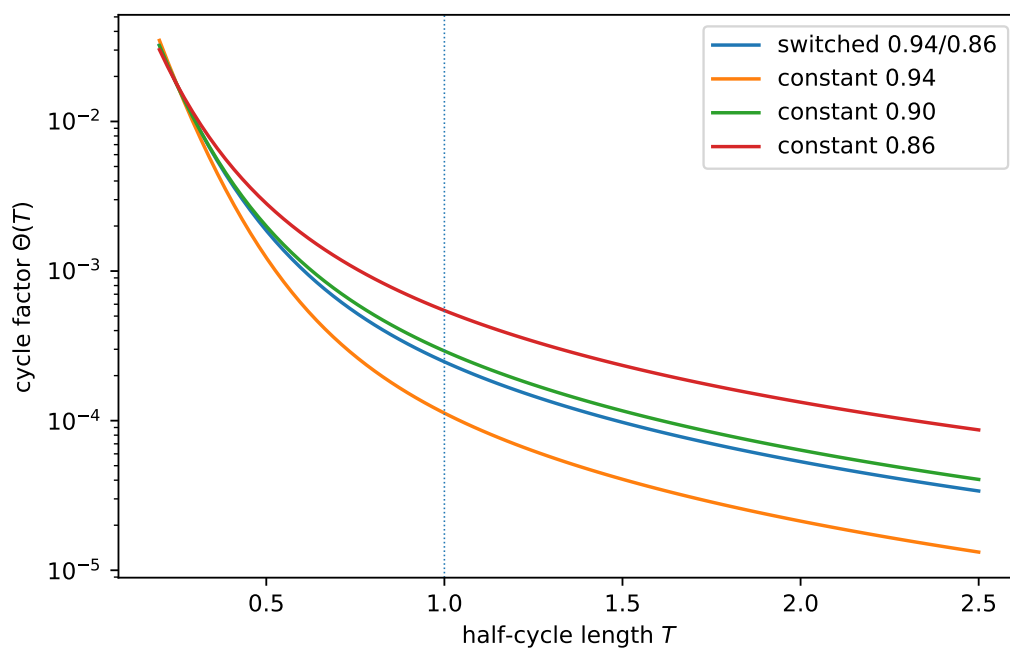
Schedule	$\widehat{t}_{\varepsilon,8}$	$\mathcal{E}_u$
constant $\alpha = 0.94$	2.0100	99.2998
switched 0.94/0.86	2.0400	99.2998
constant $\alpha = 0.90$	2.0438	84.2281
constant $\alpha = 0.86$	2.0675	71.2207

That separate comparison is provided below using a matched numerical method and the same order function in each pair.

At $T = 1$ and $\eta = 4$, the corresponding theorem-level cycle factors are

$$\Theta_{0.94} = 1.123 \times 10^{-4}, \quad \Theta_{0.94/0.86} = 2.474 \times 10^{-4}, \quad \Theta_{0.86} = 5.449 \times 10^{-4}. \quad (7.11)$$

Figure 5 shows their dependence on the half-cycle length. The per-cycle contraction becomes stronger as T increases over the plotted range, whereas the number of completed cycles per unit physical time decreases. The rate r_Θ in (4.4) incorporates both effects.

**Figure 5.** Cycle factor $\Theta(T)$ for switched and constant-order reset schedules at $\eta = 4$. The vertical marker identifies the baseline $T = 1$.

The two-dimensional maps in Figure 6 extend this calculation to order pairs $(\alpha_1, \alpha_2) \in [0.82, 0.98]^2$. They are theoretical design maps: A chosen reset cadence fixes T , and every point gives the contraction banked over one complete two-level cycle. The symmetry in Figure 6 reflects the product structure of (4.3) for equal half-cycle lengths. Like the scan in Figure 5, these maps describe the analytical squared-error bound, rather than a measured synchronization time or a diagnosis of the master dynamics.

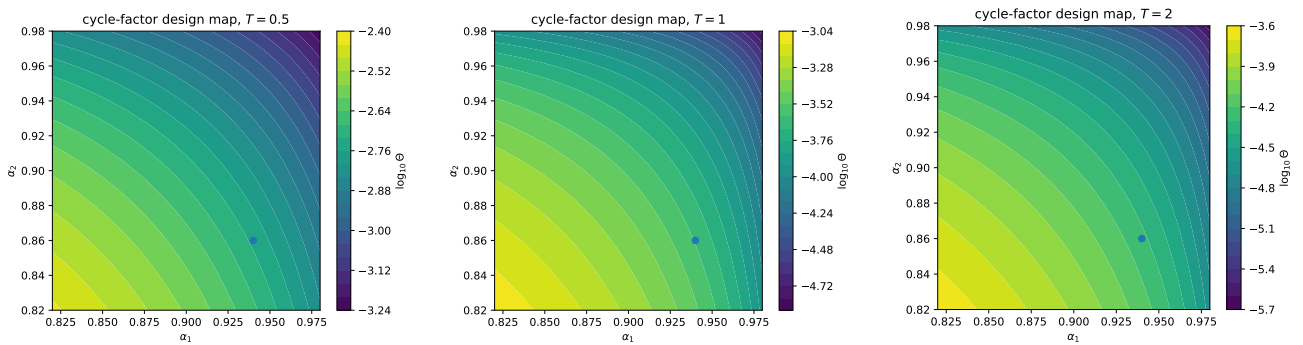


Figure 6. Two-dimensional maps of $\log_{10} \Theta(\alpha_1, \alpha_2; T)$ at $\eta = 4$ for $T = 0.5$ (left), $T = 1$ (center), and $T = 2$ (right).

7.4. Dependence on the chosen reset model

The reset cadence changes both the theorem factor and the actual transient. To quantify this effect, we scan $T \in [0.25, 2]$ while keeping the alternating orders, gains, initial states, and final time fixed. Figure 7 and Table 4 report representative points. The reset count is the number of interior switching times on $0 < t < 8$.

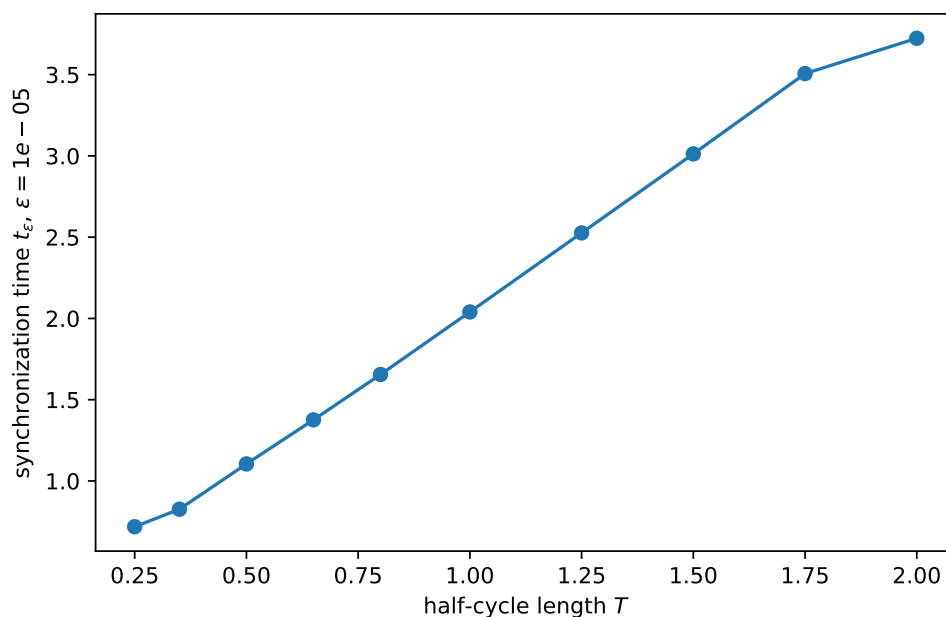


Figure 7. Computed synchronization time versus the half-cycle/reset dwell T for the alternating 0.94/0.86 schedule. The same scan also records the feedback-energy integral and reset count in Table 4.

Within this benchmark, changing T from 1 to 0.25 reduces the finite-grid tolerance time from 2.040 to 0.7188, whereas the reported feedback-energy integral stays near 99.3, and the interior reset count increases from 7 to 31 (Figure 7 and Table 4). This is a comparison of different memory laws: Both the reset master and the controlled slave change with T . It does not demonstrate faster control of one fixed hereditary system. The apparent acceleration is a consequence of modifying memory, as Proposition 3

Table 4. Reset-cadence scan for the alternating 0.94/0.86 schedule. Here, $r_\Theta = -\log \Theta/(2T)$.

T	$\widehat{t}_{\varepsilon,8}$	$\mathcal{E}_u$	Interior resets	r_Θ
0.25	0.7188	99.2343	31	8.094
0.50	1.1050	99.2776	15	6.285
1.00	2.0400	99.2998	7	4.152
1.50	3.0125	99.3071	5	3.079
2.00	3.7238	99.3108	3	2.460

shows already for scalar relaxation. The energy functional counts only u and contains no implementation or physical cost of a reset.

7.5. Step refinement, roundoff, and parameter mismatch

A three-level refinement study checks sensitivity to the predict–evaluate–correct–evaluate (PECE) step. Table 5 reports $\widehat{t}_{\varepsilon,8}$, $\mathcal{E}_u$, and the error norm at $t = 2$. The production step 1.25×10^{-3} and the finer step 6.25×10^{-4} differ by 6.25×10^{-4} in the reported synchronization time, by about 0.37% in $\mathcal{E}_u$ and by about 1.65% in $\|e(2)\|_2$. These values give a numerical resolution check for the quantities used in the figures and tables.

Table 5. PECE step-refinement diagnostics for the baseline switched schedule.

h	$\widehat{t}_{\varepsilon,8}$	$\mathcal{E}_u$	$\ e(2)\ _2$
2.50×10^{-3}	2.0375	100.7438	3.067×10^{-5}
1.25×10^{-3}	2.0400	99.2998	3.301×10^{-5}
6.25×10^{-4}	2.0406	98.9371	3.356×10^{-5}

The baseline binary64 trajectory enters the 10^{-14} error range near $t = 5.81$, and the tail for $t \geq 6$ reaches a maximum error norm of about 5.4×10^{-15} . Subsequent coordinate differences are therefore governed by floating-point resolution. Quantitative error tables stop at $t = 4$, where the values remain suitable for meaningful decimal reporting.

The synchronization theorem assumes matched Lorenz parameters. A simple robustness diagnostic perturbs the slave value of ρ by $\pm 2\%$ while keeping the controller and switched schedule fixed. Figure 8 and Table 6 show the resulting residual errors. The root-mean-square error is computed over $6 \leq t \leq 8$.

Table 6. Tail synchronization diagnostics under slave-parameter mismatch.

Slave ρ mismatch	RMS $\ e\ _2, t \in [6, 8]$	Max $\ e\ _2, t \in [6, 8]$	$\mathcal{E}_u$
−2%	0.117527	0.117527	252.1549
0	7.25×10^{-16}	5.33×10^{-15}	99.2998
+2%	0.118010	0.118011	256.6460

The residual plateaus in Figure 8 and Table 6 illustrate a robustness boundary for the present theorem: Exact identical synchronization follows from the matched-parameter error dynamics, and parameter uncertainty produces a practical residual with the fixed gain. Robust or adaptive compensation for parametric uncertainty is therefore a natural extension of the controller architecture.

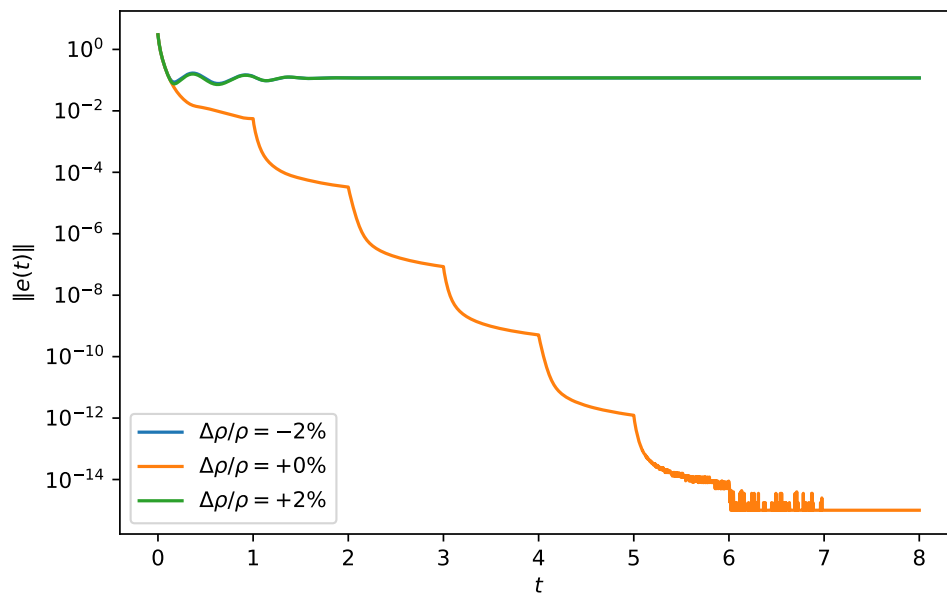


Figure 8. Synchronization-error norm under ρ mismatch in the slave system for the baseline switched schedule. Parameter mismatch produces a persistent residual under the fixed linear controller.

7.6. History-preserving baseline and matched numerical method

The fixed-origin comparator uses the identical vector field, gains, initial data, and order schedule, but retains the lower terminal 0 in both master and slave equations. On a grid $t_n = n\Delta$, direct product integration of the Caputo derivative gives the implicit L1 formula [27]

$$\frac{\Delta^{-\alpha_n}}{\Gamma(2 - \alpha_n)} \sum_{j=m_n}^{n-1} b_{n-1-j}^{(\alpha_n)} (z_{j+1} - z_j) = F(z_n), \quad b_\ell^{(\alpha)} = (\ell + 1)^{1-\alpha} - \ell^{1-\alpha}. \quad (7.12)$$

Here, $z = (x, y)$, F is the coupled vector field, and $m_n = 0$ for the fixed-origin model. For the reset model, m_n is the grid index of the latest reset strictly preceding t_n . The active order at an arriving switching endpoint is that of the interval just completed; the new order is used at the next node. This convention implements endpoint continuity and avoids imposing two derivative laws at the same instant. In particular, changing α_n in the fixed-origin run reweights every retained increment: It never restarts a Volterra formula as though the earlier history were zero.

Both memory conventions are solved with (7.12), the same grid, and Newton iteration with the analytic Jacobian. The production comparison uses $\Delta = 0.00125$, and all reset times are grid nodes. The largest nonlinear algebraic residual over the comparison runs is below 3×10^{-12} . Before the first reset, the two numerical trajectories agree exactly. A scalar check against $E_\alpha(-2t^\alpha)$ gives decreasing errors under step halving; the code and machine-readable data record those errors. This check and the refinement below are numerical validation, not a uniform convergence theorem for a nonsmooth variable-order problem.

Let x^r and x^f denote the reset and fixed-origin masters. We measure

$$D_x = \max_{t_n \in [0, 8]} \|x^r(t_n) - x^f(t_n)\|_2 \quad (7.13)$$

as well as each model's own master–slave error. These are different quantities: A small synchronization error within either model says nothing about agreement of the two master trajectories. Figure 9 displays both comparisons for $T = 1$. Table 7 repeats the calculation for several dwell lengths and for constant-order controls, which isolate memory resetting without changing the order function.

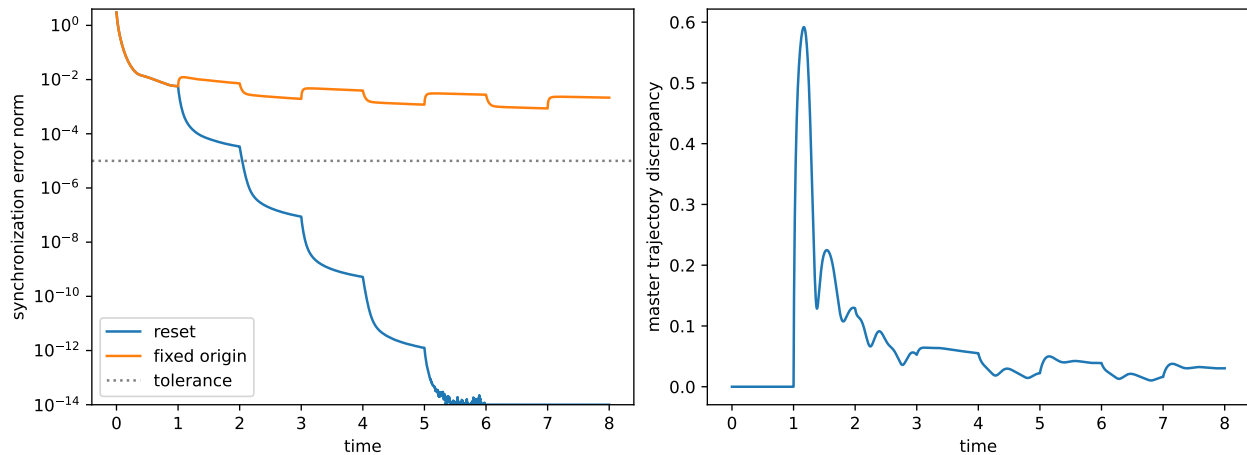


Figure 9. Matched-grid L1 comparison for the alternating 0.94/0.86 schedule with $T = 1$. Left: Synchronization errors within the two memory models. Right: Discrepancy between their master trajectories. Values below 10^{-14} are clipped for display.

Table 7. Matched-grid comparisons on $[0, 8]$, $\Delta = 0.00125$. “> 8” means that the tolerance 10^{-5} is not reached and maintained on the computed grid; it is not an assertion about infinite-time behavior.

Order schedule	T	$\hat{t}_{\varepsilon,8}^r$	$\hat{t}_{\varepsilon,8}^f$	$\ e^f(8)\ _2$	D_x
0.94/0.86	0.25	0.7200	> 8	2.1349×10^{-3}	1.772
0.94/0.86	0.5	1.1062	> 8	2.1461×10^{-3}	1.104
0.94/0.86	1	2.0425	> 8	2.1510×10^{-3}	0.592
0.94/0.86	2	3.7550	> 8	2.1534×10^{-3}	0.244
0.94	1	2.0112	> 8	0.7570×10^{-3}	0.318
0.90	1	2.0450	> 8	1.3961×10^{-3}	0.109
0.86	1	2.0688	> 8	2.1574×10^{-3}	0.235

For $T = 1$, Table 7 gives a fixed-origin error of approximately 2.15×10^{-3} at $t = 8$, well above the tolerance reached by the reset model near $t = 2.04$. The nonzero values of D_x demonstrate that the reset intervention changes the master trajectory. This persists for constant orders, so it cannot be attributed solely to alternating α . Across the alternating rows, changing T changes both the order timing and the terminal-reset schedule; only the within-row memory comparison holds the entire order function fixed.

Table 8 shows that halving the two finest steps changes the fixed-origin final error by less than 0.002%, the reset tolerance time by 0.0003125, and D_x by about 2.2%. Thus, the qualitative model discrepancy is stable, although its reported magnitude should be read as approximately 0.6, not as a certified high-precision value. The energy columns in Table 8 are more step-sensitive: The finest reset value is about 100.14, compared with 98.94 for the finest PECE result in Table 5. The near

Table 8. L1 step refinement for the alternating schedule with $T = 1$. Each discrepancy is computed between reset and fixed-origin solutions on the same grid. The energy columns expose the slower resolution of the initial transient.

Δ	$\widehat{t}_{\varepsilon,8}^r$	$\ e^f(8)\ _2$	D_x	$\mathcal{E}_u^r$	$\mathcal{E}_u^f$
0.0025000	2.042500	2.151130×10^{-3}	0.5402	109.240	109.334
0.0012500	2.042500	2.150979×10^{-3}	0.5919	103.999	104.092
0.0006250	2.041250	2.150903×10^{-3}	0.6193	101.428	101.521
0.0003125	2.040938	2.150866×10^{-3}	0.6330	100.140	100.234

equality of reset and fixed-origin energies within each L1 row does not include any cost of deleting memory. Numerical zeros in the reset tail reflect floating-point resolution and do not prove finite-time synchronization.

7.7. Measured discarded history

We evaluate Eq (1.4) along the fixed-origin master, using its piecewise-linear interpolant. At $t_n > t_k$, this gives

$$H_k^\Delta[x^f](t_n) = \frac{\Delta^{-\alpha_k}}{\Gamma(2 - \alpha_k)} \sum_{j=0}^{m_k-1} b_{n-1-j}^{(\alpha_k)} (x_{j+1}^f - x_j^f), \quad t_{m_k} = t_k. \quad (7.14)$$

The total variation of that interpolant on $[0, t_k]$ is $V_k^\Delta = \sum_{j=0}^{m_k-1} \|x_{j+1}^f - x_j^f\|_2$. Consequently, (1.5) gives the explicit bound $V_k^\Delta / [\Gamma(1 - \alpha_k)(t_n - t_k)^{\alpha_k}]$ on this computed history term. These are quadrature diagnostics of the interpolant, not certified bounds on the unknown exact trajectory.

Figure 10 and Table 9 quantify the omitted contribution during the first interval after resetting. It is appreciable: At $t = 1.1$, the computed norm is about 4.17. The total-variation estimate is conservative, particularly close to t_k , where its distance factor is singular. Proposition 2 explains how to integrate the history contribution before estimating the resulting state discrepancy; it avoids interpreting the pointwise singular bound as an instantaneous state jump.

Table 9. Discarded-history diagnostics in the first post-reset interval for the alternating $T = 1$ case.

t	$\ H_1^\Delta[x^f](t)\ _2$	Total-variation upper bound
1.01	7.988	593.627
1.10	4.172	81.943
1.50	2.274	20.531
2.00	1.630	11.311

7.8. Numerical scope

The reset computations share an exact master–slave clock and matched parameters except in the explicitly labeled mismatch experiment. The fixed-origin comparisons retain all history increments. The bounded working region is checked numerically on $[0, 8]$ only; no all-time invariance proof for Ω_{35} follows from these samples. Figures 1 and 2 describe transient synchronization and near-equilibrium

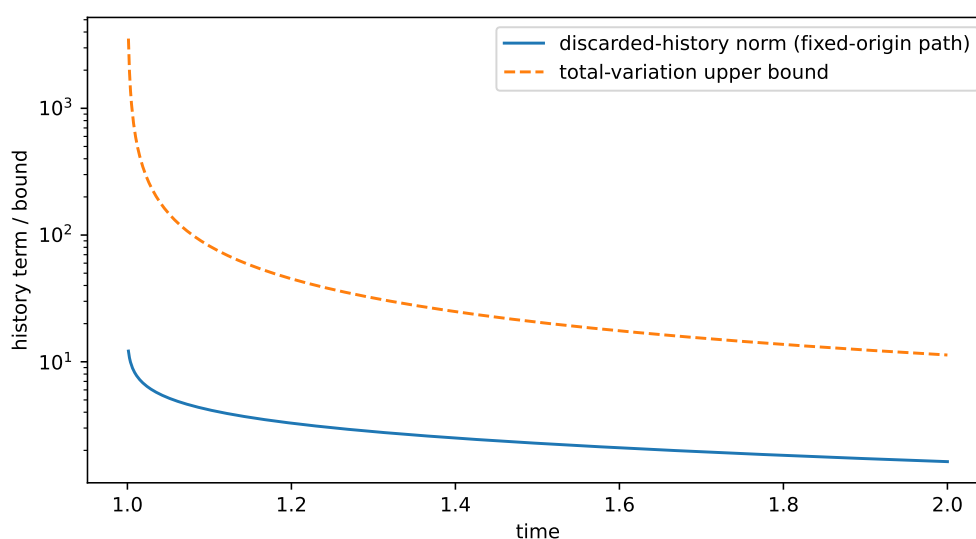


Figure 10. Discarded pre- $t = 1$ history along the fixed-origin master for $T = 1$ and its total-variation upper bound. Both quantities use the same piecewise-linear L1 trajectory.

behavior. Figures 9 and 10, together with Tables 7–9, establish the numerical scale of the model discrepancy and its step sensitivity. Continuous-order approximation, uncertain reset clocks, and robust compensation require separate analyses.

8. Conclusions

Continuous state matching and terminal resetting lead to a product of intervalwise Mittag-Leffler synchronization factors. The feedback gain enters explicitly through the dissipativity margin η_K , and the prescribed order and dwell sequences determine the accumulated contraction. Standard local Caputo inequalities provide the proof; the resulting estimate belongs to the shared-clock reset model. A bounded-region Lorenz calculation gives transparent sufficient gains but remains conditional on the stated all-time trajectory hypothesis.

The history-preserving comparator has an additional pre-switch integral. An integrated-history estimate bounds its effect on the state discrepancy, and the scalar relaxation example proves that increasingly frequent resets need not approach the fixed-origin dynamics. The numerical comparisons show this distinction with unchanged gains and, within each pair, the same complete order schedule. The reset model attains the specified tolerance much earlier in the tested examples, but its master trajectory also changes. Such acceleration is a consequence of modified memory and cannot be transferred to a fixed hereditary plant as a general control advantage.

The Lorenz benchmark approaches an equilibrium over the computed horizon and provides evidence of transient synchronization without a claim of chaos. Neither phase projections nor finite-horizon tolerances establish an asymptotic dynamical classification. The computations include step refinement, explicit retained-history weights, a scalar solver check, roundoff diagnostics, and parameter mismatch. Remaining questions include a positive-invariance certificate for the Lorenz region, tighter weighted Lyapunov estimates, controller and reset-clock uncertainty, and a costed implementation of memory reinitialization. Approximation of a genuinely variable-order fixed-origin equation requires a separate

history-preserving analysis.

Use of AI tools declaration

During the preparation of this work, the authors independently developed the mathematical content, wrote the manuscript, and developed the computational routines. Generative AI, specifically Anthropic's Claude Opus, was subsequently used solely for minor language proofreading and cosmetic code refactoring to improve the presentation of simulation visualizations implemented in Python 3.14.

All mathematical formulations, analytical derivations, numerical models, and computational results were independently reviewed and validated by the authors. The authors take full responsibility for the accuracy, integrity, and final content of the manuscript.

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Conflict of interest

The authors declare there are no conflicts of interest.

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