



Research article

Well-posedness of the Lord–Shulman thermoelastic model in general anisotropic materials

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Abstract: In this research, we investigated the well-posedness of the Lord–Shulman model of generalized thermoelasticity for inhomogeneous anisotropic materials with variable coefficients. The presence of the differential thermoelastic coupling operator $(1 + \tau_0 \partial_t)$ prevents the use of the standard energy method due to the lack of cancellation of the coupling terms in the weak formulation. To overcome this difficulty, we introduced a suitable transformation that absorbs the time derivative in the coupling term into new variables, which is an algebraic device to facilitate cancellation in energy estimates. The transformed problem was analyzed using the Faedo–Galerkin method and energy estimates, leading to the existence and uniqueness of weak solutions. The solution of the original system was then recovered through the inverse transformation.

Keywords: existence and uniqueness; Lord–Shulman thermoelastic model; differential coupling operator; higher-order estimates; Galerkin method

1. Introduction

Lord and Shulman [1] introduced the “one-relaxation-time” theory of generalized thermoelasticity for homogeneous isotropic materials. In this model, the governing equations have constant coefficients, and the classical thermoelastic system, consisting of a hyperbolic equation of motion coupled with a parabolic heat equation, is transformed into a fully hyperbolic system in which both the mechanical and thermal equations are of hyperbolic type. This modification resolves the paradox of infinite speed of heat propagation inherent in the classical theory. The uniqueness of solutions for the Lord–Shulman model has been established by several authors; see, for example, [2].

Subsequently, Dhaliwal and Sherief [3] extended the Lord–Shulman theory to anisotropic thermoelastic materials and proved the uniqueness of solutions for the corresponding model. In recent years, considerable attention has been devoted to the mathematical analysis of generalized

thermoelastic systems of Lord–Shulman type, particularly in the presence of nonuniform material properties, anisotropy, nonlinear effects, and additional physical couplings. For example, Fernández et al. [4] investigated a two-dimensional Lord–Shulman model for a homogeneous thermoelastic material incorporating four dissipative mechanisms and established well-posedness using the theory of linear semigroups. Apalara et al. [5] analyzed the well-posedness and asymptotic stability of a model describing double-wall carbon nanotubes with Lord–Shulman heat conduction. It is worth emphasizing that all of these works deal with hyperbolic systems of partial differential equations with constant material coefficients. Feng et al. [6] investigated a nonuniform laminated beam within the Lord–Shulman framework and established well-posedness using semigroup theory. The three-dimensional anisotropic formulation of Lord–Shulman thermoelasticity has also been examined from a variational perspective by Giorgio and Placidi [7], highlighting the role of thermal inertia in anisotropic thermoelastic bodies. More recently, well-posedness and stability of Lord–Shulman thermoelastic systems have been studied using semigroup and energy methods, including porous thermoelastic models with microtemperature effects [8] and swelling porous thermoelastic systems [9]. Related recent work has further addressed stability properties of Lord–Shulman systems with additional time-delay and microtemperature effects [10], while very recent developments have considered coupled thermoelastic structures with Lord–Shulman heat conduction [11]. These studies demonstrate the continuing development of modern analytical approaches, particularly semigroup, variational, and energy methods, for establishing well-posedness and stability of generalized thermoelastic systems. Motivated by these developments, the present work considers a Lord–Shulman thermoelastic model with spatially variable and anisotropic coefficients, and develops a Galerkin-based framework for establishing the existence and uniqueness of strong and weak solutions under the stated assumptions. Recent studies on nonlinear beam-type models have also investigated the qualitative behavior of their solutions, particularly the derivation of upper and lower bounds for the blow-up time in nonlinear extensible beam equations [12, 13].

We also mention the work of Stelmashchuk et al. [14], who investigated the well-posedness of the Lord–Shulman model for thermopiezoelectricity in general inhomogeneous anisotropic materials. In their formulation, the governing system consists of a hyperbolic equation of motion coupled with parabolic equations describing the electric potential, heat flux, and temperature variation. Consequently, the analytical treatment is considerably simpler than that of a fully coupled hyperbolic thermoelastic system. In contrast, the model considered in the present work consists of two coupled hyperbolic equations, where the thermal equation contains the coupling term $T_0 \gamma_{ij}(x) (1 + \tau_0 \partial_t) \varepsilon_{ij}(\partial_t \mathbf{u})$. The presence of the time differential operator in the coupling term prevents the standard cancellation of the coupling terms in the weak formulation, making the derivation of suitable energy estimates significantly more challenging. To overcome this difficulty, we introduce the transformed variables $\widetilde{v} = v + \tau_0 \partial_t v$, following the idea of Sherief [15]. This transformation absorbs the time derivative in the coupling term into new variable, for which suitable a priori estimates can be established. Finally, the estimates obtained for the transformed variables are transferred back to the original variables by means of the inverse transformation $v(t) = e^{-t/\tau_0} v(0) + \frac{1}{\tau_0} \int_0^t e^{-(t-s)/\tau_0} \widetilde{v}(s) ds$, which enables us to recover the well-posedness of the original Lord–Shulman model.

2. Problem formulation

We consider a general inhomogeneous, anisotropic material represented by an open bonded domain Ω in $\mathbb{R}^n$ with two disjoint boundaries S_1 and S_2 such that $\partial\Omega = S_1 \cup S_2$. More specifically, we consider the model described by the following equations. The equations of state relating the stress tensor, the strain tensor, and the temperature variation are given by

$$\sigma_{ij} = \alpha_{ijkl} \varepsilon_{kl}(\mathbf{u}) - \gamma_{ij} \theta, \quad (2.1)$$

where σ_{ij} denotes the components of the stress tensor, $\varepsilon_{kl}(\mathbf{u})$ the components of the infinitesimal strain tensor, $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement, α_{ijkl} the elastic moduli tensor, γ_{ij} the thermoelastic coupling tensor, and θ the temperature increment. The energy equation is given by

$$\partial_{x_i} q_i = -\rho c_E \partial_t \theta - T_0 \gamma_{ij} \varepsilon_{ij}(\partial_t \mathbf{u}), \quad (2.2)$$

where q_i is the heat conduction vector, ρ is the density assumed independent of time, c_E is the specific heat for constant deformation, and T_0 is the initial temperature. q_i satisfies the generalized heat equation:

$$q_i + \tau_0 \partial_t q_i = -\kappa_{ij} \partial_{x_j} \theta, \quad (2.3)$$

where κ_{ij} is the thermal conductivity tensor. The displacement equations is:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho F_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad (2.4)$$

where F_i represents the body force and

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} (\partial_{x_j} u_i + \partial_{x_i} u_j), \quad (2.5)$$

$$\rho(x) \partial_{t^2}^2 u_i - \partial_{x_j} (\alpha_{ijmn} \varepsilon_{nm}(\mathbf{u})) - \partial_{x_j} (\gamma_{ij} \theta) = \rho F_i, \text{ in } \Omega \times (0, T), \quad (2.6)$$

$$\rho(x) c_E (1 + \tau_0 \partial_t) \partial_t \theta + T_0 \gamma_{ij}(x) (1 + \tau_0 \partial_t) \varepsilon_{ij}(\partial_t \mathbf{u}) - \partial_{x_i} (\kappa_{ij} \partial_{x_j} \theta) = 0, \text{ in } \Omega \times (0, T). \quad (2.7)$$

The boundary conditions are

$$u_i = 0 \quad \text{on } S_1 \times (0, T), \quad \alpha_{ijmn} \varepsilon_{nm}(\mathbf{u}) \nu_j = 0 \quad \text{on } S_2 \times (0, T), \quad (2.8)$$

$$\theta = 0 \quad \text{on } \partial\Omega \times (0, T). \quad (2.9)$$

The initial conditions are

$$u_i(x, 0) = u_i^0, \quad \partial_t u_i(x, 0) = u_i^1, \quad \theta(x, 0) = \theta^0, \quad \partial_t \theta(x, 0) = \theta^1, \quad x \in \Omega. \quad (2.10)$$

Let us state the following functional spaces: for an integer k , $H^k(\Omega) = \{u \in L^2(\Omega) \text{ such that } \partial^\alpha u \in L^2(\Omega) \text{ with } |\alpha| \leq k\}$, $\mathcal{V}_0(\Omega) = \{u : u \in H^1(\Omega) \text{ and } u = 0 \text{ on } S_1\}$ and $\mathcal{V}(\Omega) = \{u : u \in H^1(\Omega) \text{ and } u = 0 \text{ on } \partial\Omega\}$.

Assumptions on the data

- (i) There exist constants $c_\rho, C_\rho > 0$ such that $c_\rho \leq \rho(x) \leq C_\rho$ for all $x \in \Omega$, c_E, τ_0 , and T_0 are positive constants.
- (ii) $\alpha_{ijmn}, \gamma_{ij}$, and κ_{ij} are in $W^{1,\infty}(\Omega)$ and satisfy the following:
- (a) $\alpha_{ijmn}(x) = \alpha_{mni j}(x) = \alpha_{jimn}(x)$ for all $x \in \Omega$ and all $i, j, m, n = 1, 2, 3$. Furthermore, there exist constants $c_\alpha, C_\alpha > 0$ such that for all $\eta_{ij} = \eta_{ji}$ and all $x \in \Omega$,

$$c_\alpha |\eta_{ij}|^2 \leq \alpha_{ijmn} \eta_{ij} \eta_{mn} \leq C_\alpha |\eta_{ij}|^2. \quad (2.11)$$

- (b) $\gamma_{ij}(x) = \gamma_{ji}(x)$ and $\kappa_{ij}(x) = \kappa_{ji}(x)$ for all $x \in \Omega$ and all $i, j = 1, 2, 3$. Furthermore, there exist constants $c_\kappa, C_\kappa > 0$ such that for all $\eta_i \in \mathbb{R}$ and all $x \in \Omega$,

$$c_\kappa |\eta_i|^2 \leq \kappa_{ij} \eta_i \eta_j \leq C_\kappa |\eta_i|^2. \quad (2.12)$$

(iii) $F_i, \partial_t F_i$ and $\partial_{t^2}^2 F_i$ are in $L^2(0, T; L^2(\Omega))$.

(iv) $u_i^0 \in H^2(\Omega) \cap \mathcal{V}_0(\Omega)$, $\theta^0 \in H^2(\Omega) \cap \mathcal{V}(\Omega)$, $u_i^1 \in \mathcal{V}_0(\Omega)$, and $\theta^1 \in \mathcal{V}(\Omega)$.

In order to obtain the weak formulation of the systems (2.6)–(2.10), we multiply Eq (2.6) by a function $\omega_i \in \mathcal{V}_0(\Omega)$ and Eq (2.7) by a function $\varphi \in \mathcal{V}(\Omega)$, we have

$$\int_{\Omega} \rho(x) \partial_{t^2}^2 u_i \omega_i dx + \int_{\Omega} (\alpha_{ijmn} \varepsilon_{mn}(\mathbf{u}) + \gamma_{ij}(x) \theta) \partial_{x_j} \omega_i dx = \int_{\Omega} \rho(x) F_i \omega_i dx \quad (2.13)$$

and

$$\begin{aligned} & \int_{\Omega} \rho(x) c_E (1 + \tau_0 \partial_t) \partial_t \theta \varphi dx + T_0 \int_{\Omega} \gamma_{ij}(x) \varepsilon_{ij}(\partial_t \mathbf{u}) \varphi dx \\ & + T_0 \tau_0 \int_{\Omega} \gamma_{ij}(x) \varepsilon_{ij}(\partial_t \mathbf{u}) \varphi dx + \int_{\Omega} \kappa_{ij}(x) \partial_{x_j} \theta \partial_{x_i} \varphi dx = 0. \end{aligned} \quad (2.14)$$

Using the symmetry of α_{ijmn} and γ_{ij} , and the boundary conditions (2.8) and (2.9), we have the weak formulation as:

$$(\rho(x) \partial_{t^2}^2 u_i, \omega_i) + (\alpha_{ijmn} \varepsilon_{mn}(\mathbf{u}), \varepsilon_{ij}(\omega)) + (\gamma_{ij}(x) \theta, \varepsilon_{ij}(\omega)) = (\rho(x) F_i, \omega_i) \quad (2.15)$$

and

$$\begin{aligned} & (\rho(x) c_E (1 + \tau_0 \partial_t) \partial_t \theta, \varphi) + (T_0 \gamma_{ij}(x) \varepsilon_{ij}(\partial_t \mathbf{u}), \varphi) + (T_0 \tau_0 \gamma_{ij}(x) \partial_{t^2}^2 \varepsilon_{ij}(\mathbf{u}), \varphi) \\ & + (\kappa_{ij}(x) \partial_{x_j} \theta, \partial_{x_i} \varphi) = 0, \end{aligned} \quad (2.16)$$

with the initial and boundary conditions (2.8)–(2.10), where $(\cdot, \cdot)$ denotes the inner product on the $L^2(\Omega)$.

Theorem 2.1. *Under assumptions (i)–(iv), the system (2.6)–(2.10) admits a unique solution*

$$(u_i, \partial_t u_i, \theta, \partial_t \theta) \in C([0, T]; \mathcal{V}_0(\Omega)) \times C([0, T]; L^2(\Omega)) \times C([0, T]; \mathcal{V}(\Omega)) \times C([0, T]; L^2(\Omega)),$$

such that the following energy equality holds

$$E(t) + \frac{1}{T_0} \int_0^t \int_{\Omega} \kappa_{ij} \partial_{x_j} \theta \partial_{x_i} \theta dx ds = E(0) + M(\rho, \tau_0, \tilde{u}_i, F_i, \partial_t F_i)(t) \quad (2.17)$$

where

$$E(t) = \frac{1}{2} \int_{\Omega} \left\{ \rho(x) (\partial_t \tilde{u}_i)^2 + \alpha_{ijmn} \varepsilon_{ij}(\tilde{\mathbf{u}}) \varepsilon_{mn}(\tilde{\mathbf{u}}) + \frac{\rho(x) c_E}{T_0} \tilde{\theta}^2 + \frac{\tau_0}{T_0} \kappa_{ij} (\partial_{x_j} \theta \partial_{x_i} \theta) \right\} dx, \quad (2.18)$$

and

$$\begin{aligned} M(\rho, \tau_0, u_i, F_i, \partial_t F_i)(t) &= \int_0^t \int_{\Omega} \rho(x) F_i(s, x) \partial_t \tilde{u}_i(s, x) dx ds \\ &\quad + \tau_0 \int_0^t \int_{\Omega} \rho(x) \partial_t F_i(s, x) \partial_t \tilde{u}_i(s, x) dx ds. \end{aligned} \quad (2.19)$$

Moreover, there exist positive constants c_1, c_2 , and c_3 dependent on $T_0, \tau_0, c_\rho, c_\alpha, c_\kappa, T$, and Ω such that

$$\|\partial_t u_i\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\partial_{x_j} u_i\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\theta\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq c_1, \quad (2.20)$$

$$\begin{aligned} &\|\partial_t \tilde{u}_i\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\partial_{x_j} \tilde{u}_i\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\tilde{\theta}\|_{L^\infty(0,T;L^2(\Omega))}^2 \\ &\quad + \|\partial_{x_j} \theta\|_{L^\infty(0,T;L^2(\Omega))}^2 + \int_0^T \|\partial_{x_j} \theta(s)\|_{L^2(\Omega)}^2 ds \leq c_2, \end{aligned} \quad (2.21)$$

and

$$\begin{aligned} &\|\partial_t^2 \tilde{u}_i\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\partial_t \partial_{x_j} \tilde{u}_i\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\partial_t \tilde{\theta}\|_{L^\infty(0,T;L^2(\Omega))}^2 \\ &\quad + \|\partial_t \partial_{x_j} \theta\|_{L^\infty(0,T;L^2(\Omega))}^2 + \int_0^T \|\partial_t \partial_{x_j} \theta(s)\|_{L^2(\Omega)}^2 ds \leq c_3, \end{aligned} \quad (2.22)$$

where $\tilde{f} := (1 + \tau_0 \partial_t) f$.

3. Preliminaries

The following preliminaries will be useful in our analysis.

Lemma 3.1. *The following inequality holds true:*

$$\int_{\Omega} \theta \left[\rho c_E \tau_0 \partial_t^2 \theta + T_0 \tau_0 \gamma_{ij} (\varepsilon_{ij}(\partial_t^2 \mathbf{u})) - \partial_{x_i} (\kappa_{ij} \partial_{x_j} \theta) \right] dx \geq 0. \quad (3.1)$$

Proof. As in [3], we use the second law of thermodynamics, i.e.,

$$-q_i \partial_{x_i} \theta \geq 0. \quad (3.2)$$

From (2.2) and (2.3), one easily obtains

$$\partial_{x_i} q_i = \rho c_E \tau_0 \partial_t^2 \theta + T_0 \tau_0 \gamma_{ij} (\varepsilon_{ij}(\partial_t^2 \mathbf{u})) - \partial_{x_i} (\kappa_{ij} \partial_{x_j} \theta). \quad (3.3)$$

From (3.2) and (3.3), and integration by parts, we get (3.1). $\square$

Lemma 3.2. [16, P. 192, Proposition 10.5] *The seminorm*

$$u \mapsto \|\varepsilon(u)\|_{L^2(\Omega)}$$

is equivalent to the norm of $(H_0^1(\Omega))^n$. More precisely, there exist positive constants c_1 and c_2 such that

$$c_p \|u\|_{H_0^1(\Omega)} \leq \|\varepsilon(u)\|_{L^2(\Omega)} \leq C_p \|u\|_{H_0^1(\Omega)}, \quad \forall u \in (H_0^1(\Omega))^n.$$

Lemma 3.3. *Let $a > 0$ and let $\{v^n\}$ be a sequence of functions satisfying that there exists a constant $C > 0$, independent of n , such that $\|v^n + a\partial_t v^n\|_{L^2(0,T;L^2(\Omega))} \leq C$ and $\|v^n(0)\|_{L^2(\Omega)} \leq C$. Then there exists a constant $M > 0$, independent of n , such that*

$$\|v^n\|_{L^2(0,T;L^2(\Omega))} \leq M$$

and

$$\|\partial_t v^n\|_{L^2(0,T;L^2(\Omega))} \leq M.$$

Proof. Setting

$$v^n + a\partial_t v^n = \widetilde{v}^n,$$

the function v^n satisfies the first-order evolution equation

$$\partial_t v^n + \frac{1}{a} v^n = \frac{1}{a} \widetilde{v}^n.$$

Multiplying by the integrating factor $e^{t/a}$ and integrating from 0 to t yields

$$v^n(t) = e^{-t/a} v^n(0) + \frac{1}{a} \int_0^t e^{-(t-s)/a} \widetilde{v}^n(s) ds.$$

Therefore,

$$\|v^n(t)\|_{L^2(\Omega)} \leq e^{-t/a} \|v^n(0)\|_{L^2(\Omega)} + \frac{1}{a} \int_0^t e^{-(t-s)/a} \|\widetilde{v}^n(s)\|_{L^2(\Omega)} ds.$$

Taking the $L^2(0, T)$ -norm and applying Young's inequality for convolutions, we obtain

$$\|v^n\|_{L^2(0,T;L^2(\Omega))} \leq \|e^{-t/a}\|_{L^2(0,T)} \|v^n(0)\|_{L^2(\Omega)} + \left\| \frac{1}{a} e^{-t/a} \right\|_{L^1(0,T)} \|\widetilde{v}^n\|_{L^2(0,T;L^2(\Omega))}.$$

Since

$$\|e^{-t/a}\|_{L^2(0,T)} = \left(\frac{a}{2} (1 - e^{-2T/a}) \right)^{1/2}$$

and

$$\left\| \frac{1}{a} e^{-t/a} \right\|_{L^1(0,T)} = 1 - e^{-T/a} \leq 1,$$

it follows that

$$\|v^n\|_{L^2(0,T;L^2(\Omega))} \leq \left(\frac{a}{2} \right)^{1/2} \|v^n(0)\|_{L^2(\Omega)} + \|\widetilde{v}^n\|_{L^2(0,T;L^2(\Omega))} \leq M,$$

for some constant $M > 0$ independent of n . Finally,

$$a\partial_t v^n = \widetilde{v}^n - v^n,$$

and therefore

$$\|\partial_t v^n\|_{L^2(0,T;L^2(\Omega))} \leq \frac{1}{a} \left(\|\widetilde{v}^n\|_{L^2(0,T;L^2(\Omega))} + \|v^n\|_{L^2(0,T;L^2(\Omega))} \right) \leq M,$$

possibly after enlarging the constant M . This completes the proof. $\square$

Lemma 3.4. [17] Let H be a Hilbert space, $\psi \in C^1[0, T]$ such that $\psi(0) \neq 0$ and $\psi(T) = 0$, and $v_n, \partial_t v^n \in L^2(0, T; H)$, and $\phi^n \in H$. Then

$$\int_0^T (\partial_t v^n(t), \phi^n \psi(t)) dt = -(v^n(0), \phi^n \psi(0)) - \int_0^T (v^n(t), \phi^n \dot{\psi}(t)) dt.$$

Proof of Theorem 2.1

The proof is done in several steps.

Construction of approximation

The idea of this step is to obtain an appropriate approximation to the solution (u_i, θ) of (2.6)–(2.10) and to the initial data $(u_i^0, u_i^1, \theta^0, \theta^1)$. For this, we consider the following eigenvalues problems:

$$\begin{aligned} (v_i, w_i^j)_{\mathcal{V}_0(\Omega)} &= \mu^j (v_i, w_i^j)_{L^2(\Omega)} \quad \text{for all } v_i \in \mathcal{V}_0(\Omega), k = 1, 2, 3, \dots, \\ (\varphi, \phi^j)_{\mathcal{V}(\Omega)} &= \lambda^j (\varphi, \phi^j)_{L^2(\Omega)} \quad \text{for all } \varphi \in \mathcal{V}(\Omega), k = 1, 2, 3, \dots, \end{aligned}$$

where the sequences $(\{w_i^k\}_{k=1}^\infty, \{\phi^k\}_{k=1}^\infty)$ and $(\{\mu^k\}_{k=1}^\infty, \{\lambda^k\}_{k=1}^\infty)$ are the corresponding eigenfunctions and the non-decreasing eigenvalues. Assume that $\{w_i^k\}_{k=1}^\infty$ is an orthonormal set of $L^2(\Omega)$ and the orthogonal basis of $\mathcal{V}_0(\Omega)$, and $\{\phi^k\}_{k=1}^\infty$ is an orthonormal set of $L^2(\Omega)$ and the orthogonal basis of $\mathcal{V}(\Omega)$ and define the spaces $\mathcal{V}_0^N = \text{span}\{w_i^1, w_i^2, \dots, w_i^N\}$ and $\mathcal{V}^N = \text{span}\{\phi^1, \phi^2, \dots, \phi^N\}$. We search for the sequence (u_i^N, θ^N) such that

$$\begin{aligned} u_i^N(\cdot, t) &= \sum_{k=1}^N d_k^N(t) w_i^k \quad \text{for all } w_i^k \in \mathcal{V}_0^N, \\ \theta^N(\cdot, t) &= \sum_{k=1}^N e_k^N(t) \phi^k \quad \text{for all } \phi^k \in \mathcal{V}^N, \end{aligned}$$

for all $k = 1, 2, \dots, N$, where $d_k^N(t) = (u_i^N, w_i^k)$ and $e_k^N(t) = (\theta^N, \phi^k)$. Furthermore, assume the following for the initial data:

$$\begin{aligned} u_i^{N0} &:= u_i^N(\cdot, 0) = \sum_{k=1}^n d_k^N(0) w_i^k, \quad \text{where } d_k^N(0) = (u_i^0, w_i^k) \text{ for all } w_i^k \in \mathcal{V}_0^N, \\ u_i^{N1} &:= \partial_t u_i^N(\cdot, 0) = \sum_{k=1}^n \dot{d}_k^N(0) w_i^k, \quad \text{where } \dot{d}_k^N(0) = (u_i^1, w_i^k) \text{ for all } w_i^k \in \mathcal{V}_0^N, \\ \theta^{N0} &:= \theta^N(\cdot, 0) = \sum_{k=1}^n e_k^N(0) \phi^k, \quad \text{where } e_k^N(0) = (\theta^0, \phi^k) \text{ for all } \phi^k \in \mathcal{V}^N, \\ \theta^{N1} &:= \partial_t \theta^N(\cdot, 0) = \sum_{k=1}^n \dot{e}_k^N(0) \phi^k, \quad \text{where } \dot{e}_k^N(0) = (\theta^1, \phi^k) \text{ for all } \phi^k \in \mathcal{V}^N. \end{aligned} \tag{3.4}$$

Now we introduce the projections $\mathcal{P} : L^2(\Omega) \rightarrow \mathcal{V}_0^N$ and $\mathcal{Q} : L^2(\Omega) \rightarrow \mathcal{V}^N$ as:

$$\mathcal{P}(u) = \sum_{k=1}^N (u, w_i^k) w_i^k \text{ for all } u \in L^2(\Omega)$$

and

$$\mathcal{Q}(\psi) = \sum_{k=1}^N (\psi, \phi^k) \phi^k \text{ for all } \psi \in L^2(\Omega). \quad (3.5)$$

By the classical argument, one sees that at $n \rightarrow \infty$,

$$\begin{aligned} u_i^{N0} &= \mathcal{P}(u_i^0) \rightarrow u_i^0 \text{ strongly in } \mathcal{V}_0(\Omega), \\ u_i^{N1} &= \mathcal{P}(u_i^1) \rightarrow u_i^1 \text{ strongly in } L^2(\Omega), \\ \theta^{N0} &= \mathcal{Q}(\theta_i^0) \rightarrow \theta^0 \text{ strongly in } \mathcal{V}(\Omega), \\ \theta^{N1} &= \mathcal{Q}(\theta_i^1) \rightarrow \theta^1 \text{ strongly in } L^2(\Omega). \end{aligned} \quad (3.6)$$

We now require that the Fourier coefficients $d_k^N(t) = (u_i^N, w_i^k)$ and $e_k^K(t) = (\theta^N, \phi^k)$ satisfy the following system of ordinary differential equations:

$$(\rho \partial_t^2 u_i^N, w_i^k) + (\alpha_{ijmn} \varepsilon_{nm}(\mathbf{u}^N), \varepsilon_{ij}(\mathbf{w}^k)) - (\gamma_{ij} \theta^N, \varepsilon_{ij}(\mathbf{w}^k)) = (\rho F_i, w_i^k), \quad (3.7)$$

for $k = 1, \dots, N$, together with

$$(\rho c_E (1 + \tau_0 \partial_t) \partial_t \theta^N, \phi^k) + (T_0 \gamma_{ij} (1 + \tau_0 \partial_t) \varepsilon_{ij}(\partial_t \mathbf{u}^N), \phi^k) + (\kappa_{ij} \partial_{x_j} \theta^N, \partial_{x_i} \phi^k) = 0, \quad (3.8)$$

for $k = 1, \dots, N$. This yields a finite-dimensional system of ordinary differential equations for the coefficients $\{d_k^N(t)\}_{k=1}^N$ and $\{e_k^K(t)\}_{k=1}^N$.

First a priori estimates

Multiplying Eq (3.7) by $\dot{d}_k^N$ and summing from $k = 1$ to N , we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left\{ \rho (\partial_t u_i^N)^2 + \alpha_{ijmn} \varepsilon_{nm}(\mathbf{u}^N) \varepsilon_{ij}(\mathbf{u}^N) \right\} dx - \int_{\Omega} \theta^N \gamma_{ij} \varepsilon_{ij}(\partial_t \mathbf{u}^N) dx = \int_{\Omega} \rho F_i \partial_t u_i^N dx. \quad (3.9)$$

From Eq (3.8), we see that

$$\gamma_{ij} \varepsilon_{ij}(\partial_t \mathbf{u}^N) = \frac{1}{T_0} \left[\frac{\partial}{\partial x_i} \left(\kappa_{ij} \frac{\partial \theta^N}{\partial x_j} \right) - \rho c_E (1 + \tau_0 \partial_t) \frac{\partial \theta^N}{\partial t} - \tau_0 T_0 \gamma_{ij} \varepsilon_{ij}(\partial_t^2 \mathbf{u}^N) \right]. \quad (3.10)$$

Substituting into (3.9), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left\{ \rho (\partial_t u_i^N)^2 + \alpha_{ijmn} \varepsilon_{nm}(\mathbf{u}^N) \varepsilon_{ij}(\mathbf{u}^N) + \frac{\rho c_E}{T_0} (\theta^N)^2 \right\} dx \\ & + \frac{1}{T_0} \int_{\Omega} \left\{ \tau_0 \rho c_E (\partial_t \theta^N) \theta^N + \tau_0 T_0 \gamma_{ij} (\varepsilon_{ij}(\partial_t^2 \mathbf{u}^N)) \theta^N + \kappa_{ij} \frac{\partial \theta^N}{\partial x_i} \frac{\partial \theta^N}{\partial x_j} \right\} dx \\ & = \int_{\Omega} \rho F_i \partial_t u_i^N dx. \end{aligned} \quad (3.11)$$

From Lemma 3.1, assumptions (i)–(iv), Cauchy's inequality, and integration over $(0, t)$, we get

$$\begin{aligned} & \int_{\Omega} \left\{ c_{\rho} (\partial_t u_i^N(t))^2 + c_{\alpha} (\varepsilon_{ij}(\mathbf{u}(t))^N)^2 + \frac{c_{\rho} c_E}{T_0} (\theta^N(t))^2 \right\} dx \\ & \leq \int_{\Omega} \left\{ C_{\rho} (\partial_t u_i^N(0))^2 + C_{\alpha} (\varepsilon_{ij}(\mathbf{u}(0))^N)^2 + \frac{C_{\rho} c_E}{T_0} (\theta^N(0))^2 \right\} dx \\ & + C_{\rho} \int_0^t \|F_i\|_{L^2(\Omega)}^2 + \|\partial_t u_i^N\|_{L^2(\Omega)}^2 dt. \end{aligned} \quad (3.12)$$

By Korn's and Gronwall's inequalities, we get

$$\|\partial_t u_i^N(t)\|_{L^2(\Omega)}^2 + \|\partial_{x_j} u_i^N(t)\|_{L^2(\Omega)}^2 + \|\theta^N(t)\|_{L^2(\Omega)}^2 \leq C. \quad (3.13)$$

Thus, $\partial_t u_i^N(t)$ is bounded in $L^{\infty}(0, T; L^2(\Omega))$, $u_i^N(t)$ is bounded in $L^{\infty}(0, T; \mathcal{V}_0(\Omega))$, and $\theta^N(t)$ is bounded in $L^{\infty}(0, T; L^2(\Omega))$.

Higher estimates

The main difficulty in obtaining higher estimates is caused by the differential thermoelastic coupling operator $1 + \tau_0 \partial_t$. Indeed, the thermal equation contains the term $T_0 \gamma_{ij}(x)(1 + \tau_0 \partial_t) \varepsilon_{ij}(\partial_t u)$, which can be written as

$$T_0 \gamma_{ij}(x) \{ \varepsilon_{ij}(\partial_t u) + \tau_0 \varepsilon_{ij}(\partial_{tt} u) \}.$$

Consequently, if the mechanical equation is tested by $\partial_t u$ and the thermal equation by $\partial_t \theta$, the thermoelastic coupling terms do not cancel in the usual way. This prevents the direct application of the standard energy method. The choice of the transformation used below is dictated by the structure of the Lord–Shulman constitutive law. For any sufficiently regular function f , define $\widetilde{f} = (1 + \tau_0 \partial_t)f = f + \tau_0 \partial_t f$. In particular, we introduce

$$\widetilde{u}_i = u_i + \tau_0 \partial_t u_i, \quad \widetilde{\theta} = \theta + \tau_0 \partial_t \theta, \quad \widetilde{F}_i = F_i + \tau_0 \partial_t F_i.$$

The crucial identities are

$$\partial_t \widetilde{u}_i = \partial_t u_i + \tau_0 \partial_{tt} u_i = (1 + \tau_0 \partial_t) \partial_t u_i$$

and

$$\partial_t \widetilde{\theta} = \partial_t \theta + \tau_0 \partial_{tt} \theta = (1 + \tau_0 \partial_t) \partial_t \theta.$$

Moreover, since the strain operator is linear and independent of time,

$$(1 + \tau_0 \partial_t) \varepsilon_{ij}(\partial_t u) = \varepsilon_{ij}(\partial_t \widetilde{u}).$$

Applying $1 + \tau_0 \partial_t$ to the equation of motion gives

$$\rho(x) \partial_{tt} \widetilde{u}_i - \partial_{x_j} (\alpha_{ijmn} \varepsilon_{mn}(\widetilde{u})) - \partial_{x_j} (\gamma_{ij}(x) \widetilde{\theta}) = \rho(x) \widetilde{F}_i.$$

On the other hand, using the above identities directly in the heat equation gives

$$\rho(x) c_E \partial_t \widetilde{\theta} + T_0 \gamma_{ij}(x) \varepsilon_{ij}(\partial_t \widetilde{u}) - \partial_{x_i} (\kappa_{ij} \partial_{x_j} \theta) = 0.$$

Thus, the transformed problem is

$$\rho(x) \frac{\partial^2 \widetilde{u}_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left(\alpha_{ijmn} \varepsilon_{mn}(\widetilde{\mathbf{u}}) \right) - \frac{\partial}{\partial x_j} (\gamma_{ij} \widetilde{\theta}) = \rho(x) \widetilde{F}_i, \quad \text{in } \Omega \times (0, T), \quad (3.14)$$

$$\rho(x) c_E \frac{\partial \widetilde{\theta}}{\partial t} + T_0 \gamma_{ij} \varepsilon_{ij}(\partial_t \widetilde{\mathbf{u}}) - \frac{\partial}{\partial x_i} \left(\kappa_{ij} \frac{\partial \theta}{\partial x_j} \right) = 0, \quad \text{in } \Omega \times (0, T). \quad (3.15)$$

The boundary conditions are

$$\widetilde{u}_i = 0 \text{ on } S_1 \times (0, T), \quad \alpha_{ijmn} \varepsilon_{mn}(\widetilde{\mathbf{u}}) \nu_j = 0 \text{ on } S_2 \times (0, T), \quad (3.16)$$

and

$$\widetilde{\theta} = 0 \text{ on } \partial\Omega \times (0, T). \quad (3.17)$$

The initial conditions are

$$\widetilde{u}_i(x, 0) = \widetilde{u}_i^0, \quad \partial_t \widetilde{u}_i(x, 0) = \widetilde{u}_i^1, \quad x \in \Omega, \quad (3.18)$$

and

$$\theta(x, 0) = \theta^0, \quad \widetilde{\theta}(x, 0) = \widetilde{\theta}^0, \quad x \in \Omega, \quad (3.19)$$

where $\widetilde{u}_i^0 = u_i^0 + \tau_0 u_i^1$, $\widetilde{\theta}^0 = \theta^0 + \tau_0 \theta^1$, and $\widetilde{u}_i^1 = u_i^1 + \tau_0 u_i^2$ such that

$$u_i^2 = \frac{\partial^2 u_i}{\partial t^2}(x, 0) = \frac{1}{\rho(x)} \left\{ \frac{\partial}{\partial x_j} \left(\alpha_{ijmn} \varepsilon_{mn}(\mathbf{u}^0) \right) + \frac{\partial}{\partial x_j} (\gamma_{ij} \theta^0) + \rho(x) F_i(x, 0) \right\}. \quad (3.20)$$

It is clear from assumption (iv), we have $\widetilde{u}_0^1 \in L^2(\Omega)$. Then we set

$$\widetilde{u}_i^{N1} := \mathcal{P}(\widetilde{u}_i^1) \rightarrow \widetilde{u}_i^1 \text{ strongly in } L^2(\Omega). \quad (3.21)$$

Therefore, the solution $(\widetilde{u}_i, \widetilde{\theta})$ of system (3.14)–(3.19) is approximated as:

$$\begin{aligned} \widetilde{u}_i^N(\cdot, t) &= \sum_{k=1}^N \widetilde{d}_k^N(t) w_i^k \text{ for all } w_i^k \in \mathcal{V}_0^N, \\ \widetilde{\theta}^N(\cdot, t) &= \sum_{k=1}^N \widetilde{e}_k^N(t) \phi^k \text{ for all } \phi^k \in \mathcal{V}^N, \end{aligned}$$

for all $k = 1, 2, \dots, N$, where $\widetilde{d}_k^N(t) = (\widetilde{u}_i^N, w_i^k)$ and $\widetilde{e}_k^N(t) = (\widetilde{\theta}^N, \phi^k)$ such that

$$\left(\rho(x) \partial_t^2 \widetilde{u}_i^N, w_i^k \right) + \left(\alpha_{ijmn} \varepsilon_{nm}(\widetilde{\mathbf{u}}^N), \varepsilon_{ij}(\mathbf{w}^k) \right) - \left(\gamma_{ij} \widetilde{\theta}^N, \varepsilon_{ij}(\mathbf{w}^k) \right) = (\rho(x) \widetilde{F}_i, w_i^k), \quad (3.22)$$

for $k = 1, \dots, N$, together with

$$\left(\rho(x) c_E \partial_t \widetilde{\theta}^N, \phi^k \right) + \left(T_0 \gamma_{ij} \varepsilon_{ij}(\partial_t \widetilde{\mathbf{u}}^N), \phi^k \right) + \left(\kappa_{ij} \partial_{x_j} \theta^N, \partial_{x_i} \phi^k \right) = 0, \quad (3.23)$$

for $k = 1, \dots, N$. Multiplying Eq (3.22) by $\widetilde{d}_k^N$ and summing from $k = 1$ to N , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left\{ \rho \left(\partial_t \widetilde{u}_i^N \right)^2 + \alpha_{ijmn} \varepsilon_{ij}(\widetilde{\mathbf{u}}^N) \varepsilon_{mn}(\widetilde{\mathbf{u}}^N) \right\} dx - \int_{\Omega} \widetilde{\theta}^N \gamma_{ij} \varepsilon_{ij} \left(\partial_t \widetilde{\mathbf{u}}^N \right) dx \\ = \int_{\Omega} \rho \widetilde{F}_i \partial_t \widetilde{u}_i^N dx. \end{aligned} \quad (3.24)$$

From Eq (3.15), we can write

$$\gamma_{ij} \varepsilon_{ij} \left(\partial_t \widetilde{\mathbf{u}}^N \right) = \frac{1}{T_0} \left\{ \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial \theta^N}{\partial x_j} \right) - \rho c_E \frac{\partial \widetilde{\theta}^N}{\partial t} \right\}. \quad (3.25)$$

Substituting (3.25) into (3.24), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left\{ \rho \left(\partial_t \widetilde{u}_i^N \right)^2 + \alpha_{ijmn} \varepsilon_{ij}(\widetilde{\mathbf{u}}^N) \varepsilon_{mn}(\widetilde{\mathbf{u}}^N) + \frac{\rho c_E}{T_0} (\widetilde{\theta}^N)^2 + \frac{\tau_0}{T_0} k_{ij} \frac{\partial \theta^N}{\partial x_i} \frac{\partial \theta^N}{\partial x_j} \right\} dx \\ + \frac{1}{T_0} \int_{\Omega} k_{ij} \frac{\partial \theta^N}{\partial x_i} \frac{\partial \theta^N}{\partial x_j} dx = \int_{\Omega} \rho \widetilde{F}_i \partial_t \widetilde{u}_i^N dx. \end{aligned}$$

Integration over $(0, t)$ gives

$$E^N(t) + \frac{1}{T_0} \int_0^t \int_{\Omega} k_{ij} \frac{\partial \theta^N}{\partial x_i} \frac{\partial \theta^N}{\partial x_j} dx ds = E^N(0) + \int_0^t \int_{\Omega} \rho \widetilde{F}_i \partial_t \widetilde{u}_i^N dx ds. \quad (3.26)$$

From assumptions (i)–(iv), we have

$$\begin{aligned} E^N(t) &\geq \int_{\Omega} \left\{ c_{\rho} \left(\partial_t \widetilde{u}_i^N \right)^2 + c_{\alpha} c_p \left(\partial_{x_j} \widetilde{\mathbf{u}}^N \right)^2 + \frac{\rho c_E}{T_0} (\widetilde{\theta}^N)^2 + \frac{\tau_0 c_{\kappa}}{T_0} \left(\frac{\partial \theta^N}{\partial x_i} \right)^2 \right\} dx \\ &\geq c_1 \left\{ \left\| \partial_t \widetilde{u}_i^N(t) \right\|_{L^2(\Omega)}^2 + \left\| \partial_{x_j} \widetilde{u}_i^N(t) \right\|_{L^2(\Omega)}^2 + \left\| \widetilde{\theta}^N(t) \right\|_{L^2(\Omega)}^2 + \left\| \partial_{x_j} \theta^N(t) \right\|_{L^2(\Omega)}^2 \right\}, \end{aligned} \quad (3.27)$$

where $c_1 = \min \left\{ c_{\rho}, c_{\alpha} c_p, \frac{\rho c_E}{T_0}, \frac{\tau_0 c_{\kappa}}{T_0} \right\}$ and c_p is the Korn's inequality constant.

$$\begin{aligned} E^N(0) &\leq \int_{\Omega} \left\{ C_{\rho} \left(\partial_t \widetilde{u}_i^N(0) \right)^2 + C_{\alpha} C_p \left(\partial_{x_j} \widetilde{\mathbf{u}}^N(0) \right)^2 + \frac{\rho(x) c_E}{T_0} (\widetilde{\theta}^N(0))^2 + \frac{\tau_0 C_{\kappa}}{T_0} \left(\frac{\partial \theta^N}{\partial x_i}(0) \right)^2 \right\} dx \\ &\leq c_2 \left\{ \left\| \widetilde{u}_i^{N1} \right\|_{L^2(\Omega)}^2 + \left\| \partial_{x_j} \widetilde{u}_i^{N0} \right\|_{L^2(\Omega)}^2 + \left\| \widetilde{\theta}^{N0} \right\|_{L^2(\Omega)}^2 + \left\| \partial_{x_j} \theta^{N0} \right\|_{L^2(\Omega)}^2 \right\}, \end{aligned} \quad (3.28)$$

where $c_2 = \max \left\{ C_{\rho}, C_{\alpha} C_p, \frac{\rho c_E}{T_0}, \frac{\tau_0 C_{\kappa}}{T_0} \right\}$ and c_p is the Korn's inequality constant.

$$\begin{aligned} \int_0^t \int_{\Omega} \rho \widetilde{F}_i \partial_t \widetilde{u}_i^N dx ds &\leq C_{\rho} \int_0^T \int_{\Omega} |\widetilde{F}_i \partial_t \widetilde{u}_i^N| dx ds \\ &\leq C_{\rho} \int_0^T \left\{ \|F_i\|_{L^2(\Omega)}^2 + \tau_0^2 \|\partial_t F_i\|_{L^2(\Omega)}^2 \right\} dt + C_{\rho} \int_0^T \|\partial_t \widetilde{u}_i^N\|_{L^2(\Omega)}^2 dt. \end{aligned} \quad (3.29)$$

From inequalities (3.27)–(3.29), equality (3.26), and Gronwall's inequality, we have

$$\begin{aligned} & \|\partial_t \widetilde{u}_i^N(t)\|_{L^2(\Omega)}^2 + \|\partial_{x_j} \widetilde{u}_i^N(t)\|_{L^2(\Omega)}^2 + \|\widetilde{\theta}^N(t)\|_{L^2(\Omega)}^2 + \|\partial_{x_j} \theta^N(t)\|_{L^2(\Omega)}^2 \\ & + \int_0^T \|\partial_{x_j} \theta^N(s)\|_{L^2(\Omega)}^2 ds \leq C. \end{aligned} \quad (3.30)$$

These estimates imply the following boundedness;

$$\begin{aligned} & \partial_t \widetilde{u}_i^N \text{ is bounded in } L^\infty(0, T; L^2(\Omega)), \\ & \widetilde{u}_i^N \text{ is bounded in } L^\infty(0, T; \mathcal{V}_0(\Omega)), \\ & \widetilde{\theta}^N \text{ is bounded in } L^\infty(0, T; L^2(\Omega)), \\ & \theta^N \text{ is bounded in } L^2(0, T; \mathcal{V}(\Omega)). \end{aligned} \quad (3.31)$$

To obtain boundedness for $\partial_t \widetilde{\theta}^N$ and $\partial_{t^2} \widetilde{u}_i^N$, we consider

$$\begin{aligned} \widetilde{u}_i^2 &= \frac{\partial^2 \widetilde{u}_i}{\partial t^2}(x, 0) = \frac{1}{\rho} \left\{ \frac{\partial}{\partial x_j} \left(\alpha_{ijmn} \varepsilon_{mn}(\widetilde{\mathbf{u}}^0) \right) + \frac{\partial}{\partial x_j} (\gamma_{ij} \widetilde{\theta}^0) + \rho \widetilde{F}_i(x, 0) \right\}, \\ \widetilde{\theta}^1 &= \frac{\partial \widetilde{\theta}}{\partial t}(x, 0) = \frac{1}{\rho(x)c_E} \left\{ -T_0 \gamma_{ij} \varepsilon_{ij}(\widetilde{\mathbf{u}}^1) + \frac{\partial}{\partial x_i} \left(\kappa_{ij} \frac{\partial \theta^0}{\partial x_j} \right) \right\}. \end{aligned} \quad (3.32)$$

By assumptions (i)–(iv), it is easy to see that $\widetilde{u}_0^2 \in L^2(\Omega)$ and $\widetilde{\theta}^1 \in L^2(\Omega)$. Thus, as before, we define

$$\begin{aligned} \widetilde{u}_i^{N2} &= \mathcal{P}(\widetilde{u}_i^2) \rightarrow \widetilde{u}_i^2 \text{ strongly in } L^2(\Omega), \\ \widetilde{\theta}^{N1} &= \mathcal{Q}(\widetilde{\theta}^1) \rightarrow \widetilde{\theta}^1 \text{ strongly in } L^2(\Omega). \end{aligned} \quad (3.33)$$

Let us now differentiate (3.22) and (3.23) with respect to time. We have

$$\left(\rho \partial_{t^3}^3 \widetilde{u}_i^N, w_i^k \right) + \left(\alpha_{ijmn} \varepsilon_{nm} \left(\partial_t \widetilde{\mathbf{u}}^N \right), \varepsilon_{ij}(\mathbf{w}^k) \right) - \left(\gamma_{ij} \partial_t \widetilde{\theta}^N, \varepsilon_{ij}(\mathbf{w}^k) \right) = \left(\rho \partial_t \widetilde{F}_i, w_i^k \right), \quad (3.34)$$

for $k = 1, \dots, N$, together with

$$\left(\rho c_E \partial_{t^2}^2 \widetilde{\theta}^N, \phi^k \right) + \left(T_0 \gamma_{ij} \varepsilon_{ij} \left(\partial_{t^2}^2 \widetilde{\mathbf{u}}^N \right), \phi^k \right) + \left(\kappa_{ij} \partial_t \partial_{x_j} \theta^N, \partial_{x_i} \phi^k \right) = 0, \quad (3.35)$$

for $k = 1, \dots, N$. Multiplying Eq (3.34) by $\widetilde{d}_k^N$ and summing from $k = 1$ to N , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left\{ \rho \left(\partial_{t^2}^2 \widetilde{u}_i^N \right)^2 + \alpha_{ijmn} \varepsilon_{ij} \left(\partial_t \left(\widetilde{\mathbf{u}}^N \right) \right) \varepsilon_{mn} \left(\partial_t \left(\widetilde{\mathbf{u}}^N \right) \right) \right\} dx \\ & - \int_{\Omega} \widetilde{\theta}^N \gamma_{ij} \partial_{t^2}^2 \left(\varepsilon_{ij}(\widetilde{\mathbf{u}}^N) \right) dx = \int_{\Omega} \rho \partial_t \widetilde{F}_i \partial_{t^2}^2 \widetilde{u}_i^N dx. \end{aligned} \quad (3.36)$$

From Eq (3.35), we can write

$$\gamma_{ij} \varepsilon_{ij} \left(\partial_{t^2}^2 \widetilde{\mathbf{u}}^N \right) = \frac{1}{T_0} \left\{ \frac{\partial}{\partial x_i} \left(\kappa_{ij} \frac{\partial^2 \theta^N}{\partial t \partial x_j} \right) - \rho c_E \frac{\partial^2 \theta^N}{\partial t^2} \right\}. \quad (3.37)$$

Substituting (3.37) into (3.36), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left\{ \rho \left(\partial_{t^2}^2 \widetilde{u}_i^N \right)^2 + \alpha_{ijmn} \varepsilon_{ij} \left(\partial_t \left(\widetilde{\mathbf{u}}^N \right) \right) \varepsilon_{mn} \left(\partial_t \left(\widetilde{\mathbf{u}}^N \right) \right) + \frac{\rho c_E}{T_0} \left(\partial_t \widetilde{\theta}^N \right)^2 \right. \\ & \quad \left. + \frac{\tau_0}{T_0} k_{ij} \frac{\partial^2 \theta^N}{\partial t \partial x_i} \frac{\partial^2 \theta^N}{\partial t \partial x_j} \right\} dx + \frac{1}{T_0} \int_{\Omega} k_{ij} \frac{\partial^2 \theta^N}{\partial t \partial x_i} \frac{\partial^2 \theta^N}{\partial t \partial x_j} dx \\ & = \int_{\Omega} \rho \partial_t \widetilde{F}_i \partial_{t^2}^2 \widetilde{u}_i^N dx. \end{aligned} \quad (3.38)$$

Integrating both sides of (3.38) over $(0, t)$, we have

$$\begin{aligned} & \int_{\Omega} \left\{ \rho \left(\partial_{t^2}^2 \widetilde{u}_i^N \right)^2 + \alpha_{ijmn} \varepsilon_{ij} \left(\partial_t \left(\widetilde{\mathbf{u}}^N \right) \right) \varepsilon_{mn} \left(\partial_t \left(\widetilde{\mathbf{u}}^N \right) \right) + \frac{\rho c_E}{T_0} \left(\partial_t \widetilde{\theta}^N \right)^2 \right. \\ & \quad \left. + \frac{\tau_0}{T_0} k_{ij} \frac{\partial^2 \theta^N}{\partial t \partial x_i} \frac{\partial^2 \theta^N}{\partial t \partial x_j} \right\} dx + \frac{1}{T_0} \int_0^t \int_{\Omega} k_{ij} \frac{\partial^2 \theta^N}{\partial t \partial x_i} \frac{\partial^2 \theta^N}{\partial t \partial x_j} dx ds \\ & = \int_{\Omega} \left\{ \rho \left(\widetilde{u}_i^{N2} \right)^2 + \alpha_{ijmn} \varepsilon_{ij} \left(\widetilde{\mathbf{u}}^{N1} \right) \varepsilon_{mn} \left(\widetilde{\mathbf{u}}^{N1} \right) + \frac{\rho c_E}{T_0} \left(\widetilde{\theta}^{N1} \right)^2 \right. \\ & \quad \left. + \frac{\tau_0}{T_0} k_{ij} \frac{\partial \theta^{N1}}{\partial x_i} \frac{\partial \theta^{N1}}{\partial x_j} \right\} dx + \int_0^t \int_{\Omega} \rho \partial_t \widetilde{F}_i \partial_{t^2}^2 \widetilde{u}_i^N dx ds. \end{aligned} \quad (3.39)$$

From this and assumptions (i) and (ii), we have

$$\begin{aligned} & c_1 \left\{ \left\| \partial_{t^2}^2 \widetilde{u}_i^N(t) \right\|_{L^2(\Omega)}^2 + \left\| \partial_t \partial_{x_j} \widetilde{u}_i^N(t) \right\|_{L^2(\Omega)}^2 + \left\| \partial_t \widetilde{\theta}^N(t) \right\|_{L^2(\Omega)}^2 + \left\| \partial_t \partial_{x_j} \theta^N(t) \right\|_{L^2(\Omega)}^2 \right\} \\ & + \frac{c_k}{T_0} \int_0^T \left\| \partial_t \partial_{x_j} \theta^N(t) \right\|_{L^2(\Omega)}^2 dt \leq C_{\rho} \int_0^T \left\{ \left\| F_i \right\|_{L^2(\Omega)}^2 + \tau_0^2 \left\| \partial_{t^2}^2 F_i \right\|_{L^2(\Omega)}^2 \right\} dt \\ & + C_{\rho} \int_0^T \left\| \partial_{t^2}^2 \widetilde{u}_i^N \right\|_{L^2(\Omega)}^2 dt + c_2 \left\{ \left\| \widetilde{u}_i^{N2} \right\|_{L^2(\Omega)}^2 + \left\| \partial_{x_j} \widetilde{u}_i^{N1} \right\|_{L^2(\Omega)}^2 + \left\| \widetilde{\theta}^{N1} \right\|_{L^2(\Omega)}^2 \right. \\ & \quad \left. + \left\| \partial_{x_j} \theta^{N1} \right\|_{L^2(\Omega)}^2 \right\}, \end{aligned} \quad (3.40)$$

where $c_1 = \min \left\{ c_{\rho}, c_{\alpha} c_p, \frac{\rho c_E}{T_0}, \frac{\tau_0 c_k}{T_0} \right\}$ and $c_2 = \max \left\{ C_{\rho}, C_{\alpha} C_p, \frac{\rho c_E}{T_0}, \frac{\tau_0 c_k}{T_0} \right\}$. From Gronwall's inequality, we have

$$\begin{aligned} & \left\| \partial_{t^2}^2 \widetilde{u}_i^N(t) \right\|_{L^2(\Omega)}^2 + \left\| \partial_t \partial_{x_j} \widetilde{u}_i^N(t) \right\|_{L^2(\Omega)}^2 + \left\| \partial_t \widetilde{\theta}^N(t) \right\|_{L^2(\Omega)}^2 + \left\| \partial_t \partial_{x_j} \theta^N(t) \right\|_{L^2(\Omega)}^2 \\ & + \int_0^T \left\| \partial_t \partial_{x_j} \theta^N(s) \right\|_{L^2(\Omega)}^2 dt \leq C. \end{aligned} \quad (3.41)$$

This gives:

$$\begin{aligned} & \partial_{t^2}^2 \widetilde{u}_i^N \text{ is bounded in } L^{\infty}(0, T; L^2(\Omega)), \\ & \partial_t \widetilde{u}_i^N \text{ is bounded in } L^{\infty}(0, T; \mathcal{V}_0(\Omega)), \\ & \partial_t \widetilde{\theta}^N \text{ is bounded in } L^{\infty}(0, T; L^2(\Omega)), \\ & \partial_t \theta^N \text{ is bounded in } L^2(0, T; \mathcal{V}(\Omega)). \end{aligned} \quad (3.42)$$

Passing to the limit

From the boundedness obtained in the preceding section and the weak-weak* compactness, we are able to construct subsequences $u_i^{N_k}$ of u_i^N , θ^{N_k} of θ^N such that

$$\begin{aligned}\widetilde{u}_i^{N_k} &\rightarrow \widetilde{u}_i \text{ weakly}^* \text{ in } L^\infty(0, T; \mathcal{V}_0(\Omega)), \\ \partial_t \widetilde{u}_i^{N_k} &\rightarrow \partial_t \widetilde{u}_i \text{ weakly}^* \text{ in } L^\infty(0, T; \mathcal{V}_0(\Omega)) \cap L^2(0, T; L^2(\Omega)), \\ \partial_{t^2}^2 \widetilde{u}_i^{N_k} &\rightarrow \partial_{t^2}^2 \widetilde{u}_i \text{ weakly}^* \text{ in } L^\infty(0, T; L^2(\Omega)), \\ \widetilde{\theta}^{N_k} &\rightarrow \widetilde{\theta} \text{ weakly in } L^2(0, T; \mathcal{V}(\Omega)), \\ \partial_t \widetilde{\theta}^{N_k} &\rightarrow \partial_t \widetilde{\theta} \text{ weakly in } L^2(0, T; \mathcal{V}(\Omega)), \\ \theta^{N_k} &\rightarrow \theta \text{ weakly in } L^2(0, T; \mathcal{V}(\Omega)), \\ \partial_t \theta^{N_k} &\rightarrow \partial_t \theta \text{ weakly in } L^2(0, T; \mathcal{V}(\Omega)).\end{aligned}\tag{3.43}$$

Integrating equations (3.34) and (3.35) over $(0, t)$, we obtain

$$\begin{aligned}(\rho \partial_{t^2}^2 \widetilde{u}_i^N, w_i^N) + (\alpha_{ijmn} \varepsilon_{nm}(\widetilde{\mathbf{u}}^N), \varepsilon_{ij}(\mathbf{w}^N)) - (\gamma_{ij} \widetilde{\theta}^N, \varepsilon_{ij}(\mathbf{w}^N)) - (\rho \widetilde{F}_i, w_i^N) \\ = (\rho \widetilde{u}_i^{N^2}, w_i^N) + (\alpha_{ijmn} \varepsilon_{nm}(\widetilde{\mathbf{u}}^{N^0}), \varepsilon_{ij}(\mathbf{w}^N)) \\ - (\gamma_{ij} \widetilde{\theta}^{N^0}, \varepsilon_{ij}(\mathbf{w}^N)) - (\rho \widetilde{F}_i(0), w_i^N)\end{aligned}\tag{3.44}$$

and

$$\begin{aligned}(\rho c_E \partial_t \widetilde{\theta}^N, \phi^N) + (T_0 \gamma_{ij} \varepsilon_{ij}(\partial_t \widetilde{\mathbf{u}}^N), \phi^N) + (\kappa_{ij} \partial_{x_j} \theta^N, \partial_{x_i} \phi^N) \\ = (\rho c_E \widetilde{\theta}^{N^1}, \phi^N) + (T_0 \gamma_{ij} \varepsilon_{ij}(\widetilde{\mathbf{u}}^{N^1}), \phi^N) + (\kappa_{ij} \partial_{x_j} \theta^{N^0}, \partial_{x_i} \phi^N).\end{aligned}\tag{3.45}$$

From the definitions of $\mathcal{V}_0^N(\Omega)$ and $\mathcal{V}^N(\Omega)$, it is easy to see that $w_i^N \rightarrow w_i$ in $\mathcal{V}_0(\Omega)$ and $\phi^N \rightarrow \phi$ in $\mathcal{V}(\Omega)$ as N goes to ∞ . Therefore, we multiply (3.44) and (3.45) by $\psi \in \mathcal{D}(0, T)$, replace N by N_k , integrate over $(0, T)$, and take the limit when k goes to ∞ , and we have

$$\begin{aligned}\int_0^T \left((\rho \partial_{t^2}^2 \widetilde{u}_i, w_i) + (\alpha_{ijmn} \varepsilon_{nm}(\widetilde{\mathbf{u}}), \varepsilon_{ij}(\mathbf{w})) - (\gamma_{ij} \widetilde{\theta}, \varepsilon_{ij}(\mathbf{w})) - (\rho \widetilde{F}_i, w_i) \right) \psi(t) dt \\ = \int_0^T \left((\rho \widetilde{u}_i^2 - \partial_{x_j} \alpha_{ijmn} \varepsilon_{nm}(\widetilde{\mathbf{u}}^0) - \partial_{x_j} \gamma_{ij} \widetilde{\theta}^0 - \rho \widetilde{F}_i(0), w_i) \right) \psi(t) dt\end{aligned}\tag{3.46}$$

and

$$\begin{aligned}\int_0^T \left((\rho c_E \partial_t \widetilde{\theta}, \phi) + (T_0 \gamma_{ij} \varepsilon_{ij}(\partial_t \widetilde{\mathbf{u}}), \phi) + (\kappa_{ij} \partial_{x_j} \theta, \partial_{x_i} \phi) \right) \psi(t) dt \\ = \int_0^T \left((\rho c_E \widetilde{\theta}^{N^1} + T_0 \gamma_{ij} \varepsilon_{ij}(\widetilde{\mathbf{u}}^1) - \partial_{x_i} \kappa_{ij} \partial_{x_j} \theta^0, \phi) \right) \psi(t) dt.\end{aligned}\tag{3.47}$$

From (3.46), (3.47), and (3.32), we have

$$\int_0^T \left((\rho \partial_{t^2}^2 \widetilde{u}_i, w_i \psi) + (\alpha_{ijmn} \varepsilon_{nm}(\widetilde{\mathbf{u}}), \varepsilon_{ij}(\mathbf{w}) \psi) - (\gamma_{ij} \widetilde{\theta}, \varepsilon_{ij}(\mathbf{w}) \psi) - (\rho \widetilde{F}_i, w_i \psi) \right) dt = 0\tag{3.48}$$

and

$$\int_0^T \left((\rho c_E \partial_t \bar{\theta}, \phi \psi) + (T_0 \gamma_{ij} \varepsilon_{ij}(\partial_t \bar{\mathbf{u}}), \phi \psi) + (\kappa_{ij} \partial_{x_j} \bar{\theta}, \partial_{x_i} \phi \psi) \right) dt = 0. \quad (3.49)$$

The integral equations (3.48) and (3.49) are nothing but (3.14) and (3.15) in the sense of distribution. We now show that the initial conditions are satisfied. From Lemma 3.4, we have

$$\int_0^T (\partial_t \bar{u}_i^{N_k}(t), w_i^{N_k} \psi(t)) dt = -(\bar{u}_i^{N_k}(0), w_i^{N_k} \psi(0)) - \int_0^T (\bar{u}_i^{N_k}(t), w_i^{N_k} \dot{\psi}(t)) dt.$$

We pass to the limit using the convergences (3.6) and (3.43), and the fact that $w_i^{N_k} \rightarrow w_i$ in $\mathcal{V}_0(\Omega)$, to get

$$\int_0^T (\partial_t \bar{u}_i(t), w_i \psi(t)) dt = -(\bar{u}_i^0, w_i \psi(0)) - \int_0^T (\bar{u}_i(t), w_i \dot{\psi}(t)) dt. \quad (3.50)$$

Integration by parts gives

$$\int_0^T (\partial_t \bar{u}_i(t), w_i \psi(t)) dt = -(\bar{u}_i(0), w_i \psi(0)) - \int_0^T (\bar{u}_i(t), w_i \dot{\psi}(t)) dt. \quad (3.51)$$

From (3.50), (3.51), and density arguments, we have $\bar{u}_i(0) = \bar{u}_i^0$. To show that $\partial_t \bar{u}_i(0) = \bar{u}_i^1$, $\bar{\theta}(0) = \bar{\theta}^0$, we have

$$\int_0^T (\partial_t^2 \bar{u}_i^{N_k}(t), w_i^{N_k} \psi(t)) dt = -(\partial_t \bar{u}_i^{N_k}(0), w_i^{N_k} \psi(0)) - \int_0^T (\partial_t \bar{u}_i^{N_k}(t), w_i^{N_k} \dot{\psi}(t)) dt.$$

We pass to the limit using the convergences (3.21), (3.43) and the fact that $w_i^{N_k} \rightarrow w_i$ in $\mathcal{V}_0(\Omega)$ to obtain

$$\int_0^T (\partial_t^2 \bar{u}_i(t), w_i \psi(t)) dt = -(\bar{u}_i^1, w_i \psi(0)) - \int_0^T (\partial_t \bar{u}_i(t), w_i \dot{\psi}(t)) dt. \quad (3.52)$$

Integration by parts gives

$$\int_0^T (\partial_t^2 \bar{u}_i(t), w_i \psi(t)) dt = -(\partial_t \bar{u}_i(0), w_i \psi(0)) - \int_0^T (\partial_t \bar{u}_i(t), w_i \dot{\psi}(t)) dt. \quad (3.53)$$

Density arguments give $\partial_t \bar{u}_i(0) = \bar{u}_i^1$. Note that the operators $\alpha : \mathcal{V}_0(\Omega) \rightarrow \mathcal{V}'_0(\Omega)$ such that $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \alpha_{ijmn} \varepsilon_{ij}(\mathbf{u}) \varepsilon_{mn}(\mathbf{v}) dx$, $\kappa : \mathcal{V}(\Omega) \rightarrow \mathcal{V}'(\Omega)$ such that $\langle \kappa \phi, \psi \rangle = \int_{\Omega} \kappa_{ij} \partial_{x_j} \phi \partial_{x_i} \psi dx$, and $\gamma : L^2(\Omega) \rightarrow \mathcal{V}'_0(\Omega)$ such that $\langle \gamma \phi, \mathbf{v} \rangle = \int_{\Omega} \phi \gamma_{ij} \varepsilon_{ij}(\mathbf{v}) dx$ are continuous linear operators. Also, note that $w_i \psi$ is in the space $\mathcal{D}(0, T; \mathcal{V}_0(\Omega))$, which is dense in $L^2(0, T; \mathcal{V}_0(\Omega))$, and $\phi \psi \in \mathcal{D}(0, T; \mathcal{V}_0(\Omega))$ is in the space $\mathcal{D}(0, T; \mathcal{V}(\Omega))$, which is dense in $L^2(0, T; \mathcal{V}(\Omega))$. Therefore, Eqs (3.48) and (3.49) can be written as:

$$\rho \partial_t^2 \bar{u}_i - \partial_{x_j} (\alpha_{ijmn} \varepsilon_{nm}(\bar{\mathbf{u}})) + \partial_{x_j} \gamma_{ij} \bar{\theta} = \rho \bar{F}_i \text{ in } L^2(0, T; \mathcal{V}'_0(\Omega)), \quad (3.54)$$

$$\rho c_E \partial_t \bar{\theta} + T_0 \gamma_{ij} \varepsilon_{ij}(\partial_t \bar{\mathbf{u}}) - \partial_{x_i} (\kappa_{ij} \partial_{x_j} \bar{\theta}) = 0 \text{ in } L^2(0, T; \mathcal{V}'(\Omega)). \quad (3.55)$$

Now we multiply Eq (3.54) by $\partial_t \bar{\mathbf{u}}$ and integrate over $(0, t)$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left\{ \rho (\partial_t \bar{u}_i)^2 + \alpha_{ijmn} \varepsilon_{ij}(\bar{\mathbf{u}}) \varepsilon_{mn}(\bar{\mathbf{u}}) \right\} dx - \int_{\Omega} \bar{\theta} \gamma_{ij} \varepsilon_{ij}(\partial_t \bar{\mathbf{u}}) dx \\ = \int_{\Omega} \rho \bar{F}_i \partial_t \bar{u}_i dx. \end{aligned} \quad (3.56)$$

From Eq (3.55), we can write

$$\gamma_{ij}\varepsilon_{ij}(\partial_t\widetilde{\mathbf{u}}) = \frac{1}{T_0} \left\{ \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial \theta}{\partial x_j} \right) - \rho c_E \frac{\partial \widetilde{\theta}}{\partial t} \right\}. \quad (3.57)$$

Substituting (3.57) into (3.56), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left\{ \rho (\partial_t \widetilde{u}_i)^2 + \alpha_{ijmn} \varepsilon_{ij}(\widetilde{\mathbf{u}}) \varepsilon_{mn}(\widetilde{\mathbf{u}}) + \frac{\rho c_E}{T_0} (\widetilde{\theta})^2 + \frac{\tau_0}{T_0} \int_{\Omega} k_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} dx \right\} dx \\ + \frac{1}{T_0} \int_0^t \int_{\Omega} k_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} dx ds = \int_0^t \int_{\Omega} \rho \widetilde{F}_i \partial_t \widetilde{u}_i dx ds. \end{aligned}$$

Integration over $(0, t)$ gives

$$E(t) + \frac{1}{T_0} \int_0^t \int_{\Omega} k_{ij} \frac{\partial \theta}{\partial x_i} \frac{\partial \theta}{\partial x_j} dx ds = E(0) + \int_0^t \int_{\Omega} \rho \widetilde{F}_i \partial_t \widetilde{u}_i dx ds.$$

As in (3.30) and (3.41), we have the following estimates to the solution of our model:

$$\begin{aligned} \|\partial_t \widetilde{u}_i(t)\|_{L^2(\Omega)}^2 + \|\partial_{x_j} \widetilde{u}_i(t)\|_{L^2(\Omega)}^2 + \|\widetilde{\theta}(t)\|_{L^2(\Omega)}^2 + \|\partial_{x_j} \theta(t)\|_{L^2(\Omega)}^2 \\ + \int_0^T \|\partial_{x_j} \theta(s)\|_{L^2(\Omega)}^2 ds \leq C \end{aligned} \quad (3.58)$$

and

$$\begin{aligned} \|\partial_t^2 \widetilde{u}_i(t)\|_{L^2(\Omega)}^2 + \|\partial_t \partial_{x_j} \widetilde{u}_i(t)\|_{L^2(\Omega)}^2 + \|\partial_t \widetilde{\theta}(t)\|_{L^2(\Omega)}^2 + \|\partial_t \partial_{x_j} \theta(t)\|_{L^2(\Omega)}^2 \\ + \int_0^T \|\partial_t \partial_{x_j} \theta(s)\|_{L^2(\Omega)}^2 ds \leq C. \end{aligned} \quad (3.59)$$

We now obtain the uniqueness of the solution. This is done in a similar way as in the proof of the a priori estimates. We assume that (u_i, θ) and (v_i, ϑ) are two solutions of the system (2.5)–(2.8) and let $V_i = u_i - v_i$ and $\varphi = \theta - \vartheta$. Then system (3.14)–(3.19) gives the following.

$$\rho(x) \frac{\partial^2 \widetilde{V}_i}{\partial t^2} - \frac{\partial}{\partial x_j} \left(\alpha_{ijmn} \varepsilon_{mn}(\widetilde{\mathbf{V}}) \right) - \frac{\partial}{\partial x_j} (\gamma_{ij} \widetilde{\varphi}) = 0, \quad \text{in } \Omega \times (0, T), \quad (3.60)$$

$$\rho(x) c_E \frac{\partial \widetilde{\varphi}}{\partial t} + T_0 \gamma_{ij} \varepsilon_{ij}(\partial_t \widetilde{\mathbf{V}}) - \frac{\partial}{\partial x_i} \left(\kappa_{ij} \frac{\partial \varphi}{\partial x_j} \right) = 0, \quad \text{in } \Omega \times (0, T). \quad (3.61)$$

The boundary conditions are

$$\widetilde{V}_i = 0 \text{ on } S_1 \times (0, T), \quad \alpha_{ijmn} \varepsilon_{mn}(\widetilde{\mathbf{V}}) v_j = 0 \text{ on } S_2 \times (0, T), \quad (3.62)$$

and

$$\bar{\varphi} = 0 \text{ on } \partial\Omega \times (0, T). \quad (3.63)$$

The initial conditions are

$$\bar{V}_i(x, 0) = 0, \quad \partial_t \bar{V}_i(x, 0) = 0, \quad x \in \Omega, \quad (3.64)$$

and

$$\varphi(x, 0) = 0, \quad \bar{\varphi}(x, 0) = 0, \quad x \in \Omega. \quad (3.65)$$

Again, we multiply Eq (3.60) by $\partial_t \bar{V}$ and integrate over $(0, t)$;

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left\{ \rho (\partial_t \bar{V}_i)^2 + \alpha_{ijmn} \varepsilon_{ij}(\bar{\mathbf{V}}) \varepsilon_{mn}(\bar{\mathbf{V}}) \right\} dx - \int_{\Omega} \bar{\varphi} \gamma_{ij} \varepsilon_{ij}(\partial_t \bar{\mathbf{V}}) dx = 0. \quad (3.66)$$

From Eq (3.61), we can write

$$\gamma_{ij} \varepsilon_{ij}(\partial_t \bar{\mathbf{V}}) = \frac{1}{T_0} \left\{ \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial \varphi}{\partial x_j} \right) - \rho c_E \frac{\partial \bar{\varphi}}{\partial t} \right\}. \quad (3.67)$$

Substituting (3.67) into (3.66), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left\{ \rho (\partial_t \bar{V}_i)^2 + \alpha_{ijmn} \varepsilon_{ij}(\bar{\mathbf{V}}) \varepsilon_{mn}(\bar{\mathbf{V}}) + \frac{\rho c_E}{T_0} (\bar{\varphi})^2 + \frac{\tau_0}{T_0} \int_{\Omega} k_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx \right\} dx \\ & + \frac{1}{T_0} \int_0^t \int_{\Omega} k_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx ds = 0. \end{aligned}$$

Integration over $(0, t)$ gives

$$\begin{aligned} & \int_{\Omega} \left\{ \rho (\partial_t \bar{V}_i)^2 + \alpha_{ijmn} \varepsilon_{ij}(\bar{\mathbf{V}}) \varepsilon_{mn}(\bar{\mathbf{V}}) + \frac{\rho c_E}{T_0} (\bar{\varphi})^2 + \frac{\tau_0}{T_0} \int_{\Omega} k_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx \right\} dx \\ & + \frac{1}{T_0} \int_0^t \int_{\Omega} k_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx = 0. \end{aligned} \quad (3.68)$$

Therefore, from assumptions (i) and (ii), we have that

$$\begin{aligned} & \int_{\Omega} \left\{ \rho (\partial_t \bar{V}_i)^2 + \alpha_{ijmn} \varepsilon_{ij}(\bar{\mathbf{V}}) \varepsilon_{mn}(\bar{\mathbf{V}}) + \frac{\rho c_E}{T_0} (\bar{\varphi})^2 + \frac{\tau_0}{T_0} \int_{\Omega} k_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx \right\} dx \\ & + \frac{1}{T_0} \int_0^t \int_{\Omega} k_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx \\ & \leq \int_{\Omega} \left\{ c_{\rho} (\partial_t \bar{V}_i)^2 + c_{\alpha} c_p (\partial_{x_j} \bar{\mathbf{V}})^2 + \frac{\rho c_E}{T_0} (\bar{\varphi})^2 + \frac{\tau_0 c_{\kappa}}{T_0} \left(\frac{\partial \varphi}{\partial x_i} \right)^2 \right\} dx \\ & + \frac{c_{\kappa}}{T_0} \int_0^t \int_{\Omega} \left(\frac{\partial \varphi}{\partial x_i} \right)^2 dx ds = 0. \end{aligned} \quad (3.69)$$

This gives $\theta = \vartheta$ and $\bar{V}_i = 0$. But $\bar{V}_i = 0$ implies $V_i + \tau_0 \partial_t V_i = 0$, which has the solution $V(x, t) = V(x, 0)e^{-\frac{t}{\tau_0}}$. But $V(x, 0) = 0$. Thus $v_i = u_i$ and the model possesses a unique solution.

Remark 3.5. The existence and uniqueness result requires $\tau_0 > 0$, together with the positivity and boundedness of the density, elasticity tensor, and thermal conductivity tensor, which ensure the coercivity of the energy functional. In particular, the coercivity constant depends on $\tau_0 c_\kappa / T_0$, while the inverse transformation introduces an additional quantitative dependence on $1/\tau_0$. The tensor $\gamma_{ij}(x) \in W^{1,\infty}(\Omega)$ affects the constants in the higher-order estimates through its spatial variation. However, the thermoelastic coupling terms cancel in the energy identity, so no smallness assumption on γ_{ij} is required. Thus, for every fixed $\tau_0 > 0$ and admissible $\gamma_{ij}(x)$, spatial heterogeneity affects the quantitative bounds but does not compromise the existence and uniqueness of the solution.

Remark 3.6. Note that our analysis is performed on the system (3.14)–(3.19), which was obtained by applying the operator $\tilde{f} := (1 + \tau_0 \partial_t)f$ to Eq (2.6). So, to get back to our original system (2.5)–(2.8), we apply the inverse operator $f = (1 + \tau_0 \partial_t)\tilde{f} := \frac{1}{\tau_0} \int_0^t e^{\frac{-(t-s)}{\tau_0}} \tilde{f}(s) ds + e^{\frac{-t}{\tau_0}} \tilde{f}(0)$ to (3.14), assuming that the initial and boundary conditions of the original system are satisfied.

Remark 3.7. The estimates derived for the transformed variables $\tilde{\cdot}$ can be readily translated into estimates for the original solution by invoking Lemma 3.3. Thus, from estimates (3.58) and (3.59), and assumption (iv), we have the following boundedness;

$$u_i \in L^\infty(0, T; H_0^1(\Omega)), \quad \partial_t u_i \in L^\infty(0, T; L^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)), \quad \partial_{tt} u_i \in L^\infty(0, T; L^2(\Omega)),$$

$$\theta \in L^\infty(0, T; L^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)), \quad \partial_t \theta \in L^\infty(0, T; L^2(\Omega)), \quad \partial_{tt} \theta \in L^\infty(0, T; L^2(\Omega)).$$

Remark 3.8. The regularity stated in Theorem 2.1 represents the basic continuity regularity of the weak solution. The a priori estimates established later in the proof provide additional regularity under the assumptions imposed on the initial data and the body force. In particular, estimates (3.58) and (3.59), together with Lemma 3.3 and the inverse transformation, yield the stronger bounds stated in Remark 3.7. Thus, Remark 3.7 should be understood as a refinement of the regularity asserted in Theorem 2.1, rather than as a different solution concept.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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