



Research article

Spatial properties of transient heat equations on two types of special cylinders

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Abstract: This paper investigates spatial properties of transient heat equations on two special cylinders under non-homogeneous boundary conditions. By introducing a key cross-sectional parameter, the author reveals how the cylinder's geometry affects the asymptotic behavior of solutions. Furthermore, spatial growth and decay rates are derived for different parameter ranges.

Keywords: transient heat equations; cross-sectional parameter; continuous dependence; asymptotic behavior

1. Introduction

The heat conduction equation has always garnered widespread attentions due to its pivotal role in engineering technology and physics. From the thermal design of microelectronic devices and temperature field analysis of large building structures to the thermal protection of aerospace materials, accurately predicting and controlling the heat transfer process is crucial for ensuring system safety and performance optimization. From a mathematical perspective, such problems are typically described by transient heat equations, which depict the evolution of internal temperature within an object as a function of time and spatial variables.

Currently, a significant amount of work is devoted to studying the existence(see [1, 2]), uniqueness(see [3]), blow-up phenomena(see [1, 4–6]), and spatial decay properties of solutions to heat conduction equations on cylinders or exterior domains(see [7, 8]).

However, due to the complexity of the actual physical environment (such as material heterogeneity, irregular geometric shapes, variations in boundary conditions, etc.), solving the transient heat conduction equation often poses significant challenges. This is particularly true under non-homogeneous boundary conditions, where obtaining analytical solutions and conducting qualitative analysis become even more difficult.

Among various engineering structures, cylinders are the most basic and widely used components for load-bearing and heat transfer. In fields such as aerospace, energy and chemical engineering, machinery manufacturing, and civil engineering, cylindrical structures often face drastically changing temperature environments, such as rapid heating of engine components and fire exposure of building structures. During these transient thermal processes, accurately predicting the temperature distribution inside the cylinder is crucial for evaluating its thermal stress, structural integrity, and service life (see [9]).

To simplify calculations, cylinders are often idealized as semi-infinite cylinders, i.e.,

$$R = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 \geq 0\},$$

where D is a bounded convex region on the coordinate plane x_1Ox_2 . When the heat equation satisfies appropriate boundary conditions at the ends of the cylinder R , one often seeks the variation pattern of the solution.

Horgan et al. [10] studied the spatial decay properties of the solution to the initial-boundary value problem of the heat equation in a three-dimensional cylinder, considering only the case where non-zero boundary conditions are imposed at the ends. The influence of the boundary conditions at the ends of the cylinder on the solution is referred to as the “end effect”. Li et al. [11] considered the spatial variation rate of the heat equation in a binary mixture when non-homogeneous boundary conditions are imposed on the side of the cylinder. More attentions are paid to the growth/decay rates of the solution [12]. Additional literature can be found in [13–19].

We have noticed that further research is still needed on the dependence of solutions on geometric parameters, such as the cross-section of a cylinder. We define

$$\Omega = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D(x_3), x_3 \geq a > e\},$$

where $D(x_3)$ is a plane that is parallel to D and related to x_3 .

Compared with the papers in literature (see e.g., [10–13]), we have the following advantages. The cross-section of the cylinder considered in the literature is independent of axial variable, and the cross-sectional area and boundary measurement are constants. The spatial decay (or growth) rate of the solution is usually a single exponential type. At this point, the cross-sectional parameters become a function of z , rendering the classical constant coefficient differential inequality framework ineffective. We establish corresponding variable coefficient nonlinear differential inequalities and classify the growth/decay rates of the solution with respect to nonlinear exponents and cross-sectional parameters, including algebraic, exponential, and logarithmic types, where the latter cannot occur in constant cross-section theory. In addition, this paper proves that the Phragmén-Lindelöf-type alternative structure still holds under the dual complexity of variable cross-section geometry and nonlinear boundary conditions.

This paper aims to systematically study the spatial properties of two special types of cylinders in transient heat conduction equations under non-homogeneous boundary conditions. We focus on the influence of the cylinder cross-section on the asymptotic behavior of the solution by introducing and analyzing a key cross-sectional parameter, revealing the spatial growth/decay patterns of the solution under different geometric configurations. We will derive corresponding spatial growth/decay rates for different ranges of cross-sectional parameters and compare the spatial variation patterns of the solution under different conditions.

2. Preliminary

This paper considers the transient heat equation

$$u_t = \Delta u, \quad \text{in } \Omega, \quad (2.1)$$

where u represents temperature, $u_t = \frac{\partial u}{\partial t}$, and Δ is the three-dimensional Laplace operator. Equation (2.1) satisfies the following initial and boundary conditions:

$$u = 0, \quad \text{in } \Omega \times \{t = 0\}, \quad (2.2)$$

$$\frac{\partial u}{\partial n} + f(u) = 0, \quad \text{on } \partial D(x_3) \times [a, \infty) \times [0, \infty), \quad (2.3)$$

$$u = h(x_1, x_2, t), \quad \text{on } D(a) \times [0, \infty), \quad (2.4)$$

where h is a positive function, $\frac{\partial}{\partial n}$ represents the derivative of the external normal vector, and $f(u)$ satisfies

$$uf(u) \geq \theta |u|^{2p}, \quad \theta > 0, p \geq 1. \quad (2.5)$$

The constant a represents the position of the end of the cylinder. For the convenience of derivation, we will provide its scope in the following text. Let $D(z)$ represent the cross-section of cylinder Ω at $x_3 = z$, i.e.,

$$D(z) = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in D, x_3 = z \geq a\}.$$

Therefore, $D(a)$ represents the finite end of cylinder Ω .

First, we establish an auxiliary function

$$P(z, t) = \int_0^t \int_{D(z)} uu_{,z} dA d\eta, \quad (2.6)$$

where $u_{,z} = \frac{\partial u}{\partial z}$. Letting z_0 be a fixed point on x_3 ,

$$\begin{aligned} P(z, t) - P(z_0, t) &= \int_0^t \int_{z_0}^z \int_{D(\xi)} \frac{\partial}{\partial \xi} (uu_{,\xi}) dA d\xi d\eta \\ &= \int_0^t \int_{z_0}^z \int_{D(\xi)} u_{,\xi}^2 dA d\xi d\eta + \int_0^t \int_{z_0}^z \int_{D(\xi)} uu_{,\xi\xi} dA d\xi d\eta. \end{aligned} \quad (2.7)$$

Using (2.1)–(2.3), from (2.7) we have

$$\begin{aligned} P(z, t) - P(z_0, t) &= \int_0^t \int_{z_0}^z \int_{D(\xi)} |\nabla u|^2 dA d\xi d\eta + \int_0^t \int_{z_0}^z \int_{\partial D(\xi)} uf(u) dl d\xi d\eta \\ &\quad + \frac{1}{2} \int_{z_0}^z \int_{D(\xi)} |\nabla u|^2 dA d\xi \Big|_{\eta=t}. \end{aligned} \quad (2.8)$$

Therefore, we have

$$P_{,z}(z, t) = \int_0^t \int_{D(z)} |\nabla u|^2 dA d\eta + \int_0^t \int_{\partial D(z)} uf(u) dl d\eta + \frac{1}{2} \int_{D(z)} u^2 dA \Big|_{\eta=t}. \quad (2.9)$$

Since $uf(u) \geq 0$, then $\int_0^t \int_{\partial D(z)} uf(u) dld\eta \geq 0$. Using Hölder's inequality, from (2.6) we obtain

$$|P(z, t)| \leq \left(\int_0^t \int_{D(z)} u^2 dAd\eta \int_0^t \int_{D(z)} u_{,z}^2 dAd\eta \right)^{\frac{1}{2}}. \quad (2.10)$$

To deal with the term $\int_0^t \int_{D(z)} u^2 dAd\eta$, we introduce the following inequality (see [20]):

$$\int_{D(z)} \rho^2 dA \leq \frac{1}{2} \left(\int_{\partial D(z)} |\rho| dl \right)^2 + \frac{1}{2} g_1(z) \int_{D(z)} |\nabla_2 \rho|^2 dA,$$

where ρ is a smooth function, $\nabla_2 = (\partial_{x_1}, \partial_{x_2})$, and $g_1(z) = |D(z)|$ represents the area of $D(z)$. Therefore, from (2.10) we have

$$\begin{aligned} |P(z, t)| &\leq \frac{1}{\sqrt{2}} \left[\int_0^t \left(\int_{\partial D(z)} |u| dl \right)^2 d\eta + g_1(z) \int_0^t \int_{D(z)} |\nabla_2 u|^2 dAd\eta \right] \int_0^t \int_{D(z)} u_{,z}^2 dAd\eta \Bigg]^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{2}} |t \partial D(z)|^{1-\frac{1}{2p}} \left[\left(\int_0^t \int_{\partial D(z)} |u|^{2p} dld\eta \right)^{\frac{1}{2p}} \left[\int_0^t \int_{D(z)} u_{,z}^2 dAd\eta \right]^{\frac{1}{2}} \right. \\ &\quad \left. + \frac{1}{\sqrt{2}} \sqrt{g_1(z)} \left[\int_0^t \int_{D(z)} |\nabla_2 u|^2 dAd\eta \int_0^t \int_{D(z)} u_{,z}^2 dAd\eta \right]^{\frac{1}{2}} \right] \\ &\leq \frac{1}{\sqrt{2}} |t \partial D(z)|^{1-\frac{1}{2p}} \left\{ \frac{1}{p+1} \left(\int_0^t \int_{\partial D(z)} |u|^{2p} dld\eta \right)^{\frac{1+p}{2p}} + \frac{p}{p+1} \left[\int_0^t \int_{D(z)} u_{,z}^2 dAd\eta \right]^{\frac{1+p}{2p}} \right\} \\ &\quad + \frac{1}{2\sqrt{2}} \sqrt{g_1(z)} \int_0^t \int_{D(z)} |\nabla u|^2 dAd\eta. \end{aligned} \quad (2.11)$$

Combining (2.5), (2.9), and (2.11), we obtain

$$|P(z, t)| \leq \frac{1}{2\sqrt{2}} \sqrt{g_1(z)} P_{,z}(z, t) + m_1 g_2(z) [P_{,z}(z, t)]^{\frac{p+1}{2p}}, \quad (2.12)$$

where

$$m_1 = \frac{\max\{p, 1\}}{\theta(p+1)\sqrt{2}} |t|^{1-\frac{1}{2p}}, \quad g_2(z) = |\partial D(z)|^{1-\frac{1}{2p}}. \quad (2.13)$$

3. Spatial properties

For the convenience of calculation, we consider a type of cylinder. Assuming the cross-section of the cylinder is defined as

$$D(z) = \{(x_1, x_2) \mid x_1^2 + x_2^2 = b_1 z^2 (\ln z)^{2b_2}, \quad b_1, b_2 > 0\}, \quad (3.1)$$

from (2.13) we have

$$g_1(z) = \pi b_1 z^2 (\ln z)^{2b_2}, \quad g_2(z) = [2\pi b_1 z (\ln z)^{b_2}]^{1-\frac{1}{2p}}. \quad (3.2)$$

Now, Eq (2.12) can be rewritten as

$$\left| P(z, t) \right| \leq m_2 z (\ln z)^{b_2} P_{,z}(z, t) + m_3 [z (\ln z)^{b_2}]^{1-\frac{1}{2p}} \left[P_{,z}(z, t) \right]^{\frac{p+1}{2p}}, \quad (3.3)$$

where $m_2 = \frac{m_1}{2\sqrt{2}} \sqrt{\pi b_1}$, $m_3 = m_1 [2\pi b_1]^{1-\frac{1}{2p}}$.

I. If $\frac{1}{2} < p < 1$, then $1 < \frac{p+1}{2p} < \frac{3}{2}$.

(1). If there exists $z_0 > a$ such that $P(z_0, t) > 0$, since $P_{,z}(z, t) \geq 0$, $P(z, t) \geq P(z_0, t) > 0$, $z \geq z_0 > a$, then (3.3) can be rewritten as

$$P(z, t) \leq m_2 z (\ln z)^{b_2} P_{,z}(z, t) + m_3 [z (\ln z)^{b_2}]^{1-\frac{1}{2p}} \left[P_{,z}(z, t) \right]^{\frac{p+1}{2p}}, \quad z \geq z_0. \quad (3.4)$$

Using the Hölder and the Young inequalities, we obtain

$$\left[P_{,z} \right]^{\frac{p+1}{2p}} = \left[P_{,z} \right]^{1 \cdot \frac{1-p}{2p}} \left[P_{,z} \right]^{2 \cdot \frac{1}{2p}} \leq \frac{1-p}{2p} \left[P_{,z} \right] + \frac{3p-1}{2p} \left[P_{,z} \right]^2. \quad (3.5)$$

Inserting (3.5) into (3.4), we obtain

$$\begin{aligned} P(z, t) &\leq n_1 \left[z (\ln z)^{b_2} + (z (\ln z)^{b_2})^{1-\frac{1}{2p}} \right] \left\{ \left[P_{,z} \right] + \left[P_{,z} \right]^2 \right\} \\ &\leq n_1 \left[z (\ln z)^{b_2} + (z (\ln z)^{b_2})^{1-\frac{1}{2p}} \right] \left\{ \left[P_{,z} + \frac{1}{2} \right]^2 - \frac{1}{4} \right\}, \quad z \geq z_0, \end{aligned} \quad (3.6)$$

where $n_1 = \max\{m_2 + \frac{1-p}{2p} m_3, \frac{3p-1}{2p} m_3\}$. Therefore, we have

$$P_{,z} \geq \left[\frac{P(z, t)}{n_1 \left[z (\ln z)^{b_2} + (z (\ln z)^{b_2})^{1-\frac{1}{2p}} \right]} + \frac{1}{4} \right]^{\frac{1}{2}} - \frac{1}{2}, \quad z \geq z_0. \quad (3.7)$$

Next, we study the asymptotic properties of solutions when the axial variable x_3 approaches infinity. Therefore, we can choose the base parameter a that satisfies

$$n_1 \left[a (\ln a)^{b_2} + a ((\ln a)^{b_2})^{1-\frac{1}{2p}} \right] > 1. \quad (3.8)$$

Since $b_2 > 0$ and $p > \frac{1}{2}$, then

$$\left[z (\ln z)^{b_2} + (z (\ln z)^{b_2})^{1-\frac{1}{2p}} \right]^{\frac{1}{2}} \geq \sqrt{n_1} \left[a (\ln a)^{b_2} + a ((\ln a)^{b_2})^{1-\frac{1}{2p}} \right]^{\frac{1}{2}} > 1. \quad (3.9)$$

Combining (3.7)–(3.9), we obtain

$$P_{,z} \geq \frac{1}{\sqrt{n_1} \left[z (\ln z)^{b_2} + (z (\ln z)^{b_2})^{1-\frac{1}{2p}} \right]^{\frac{1}{2}}} \left\{ \left[P(z, t) + \frac{1}{4} \right]^{\frac{1}{2}} - \frac{1}{2} \right\}, \quad z \geq z_0,$$

or

$$\frac{P_{,z}}{\left[P(z, t) + \frac{1}{4} \right]^{\frac{1}{2}} - \frac{1}{2}} \geq \frac{1}{\sqrt{n_1} \left[z (\ln z)^{b_2} + (z (\ln z)^{b_2})^{1-\frac{1}{2p}} \right]^{\frac{1}{2}}}, \quad z \geq z_0. \quad (3.10)$$

Letting

$$\tau = \tau(z, t) = \left[P(z, t) + \frac{1}{4} \right]^{\frac{1}{2}} - \frac{1}{2},$$

(3.10) can be written as

$$\left[2 + \frac{1}{2\tau} \right] d\tau \geq \frac{1}{\sqrt{n_1} \left[z(\ln z)^{b_2} + (z(\ln z)^{b_2})^{1-\frac{1}{2p}} \right]^{\frac{1}{2}}}, z \geq z_0. \quad (3.11)$$

Integrating (3.11) from z_0 to z , we obtain

$$\begin{aligned} 2 \left[\tau(z, t) - \tau(z_0, t) \right] + \frac{1}{2} \left[\ln \tau(z, t) - \ln \tau(z_0, t) \right] \\ \geq \frac{1}{\sqrt{n_1}} \int_{z_0}^z \frac{1}{\left[\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}} \right]^{\frac{1}{2}}} d\xi, \quad z \geq z_0. \end{aligned} \quad (3.12)$$

Using

$$\tau > \ln(1 + \tau) > \ln \tau$$

and dropping the non-positive term $-2\tau(z_0, t)$, from (3.12) we obtain

$$\tau(z, t) \geq \frac{1}{4} \ln \tau(z_0, t) + \frac{1}{2\sqrt{n_1}} \int_{z_0}^z \frac{1}{\left[\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}} \right]^{\frac{1}{2}}} d\xi. \quad (3.13)$$

Combining the definition of τ and (3.13), we obtain

$$P(z, t) \geq \left\{ \frac{1}{4} \ln \tau(z_0, t) + \frac{1}{2} + \frac{1}{2\sqrt{n_1}} \int_{z_0}^z \frac{1}{\left[\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}} \right]^{\frac{1}{2}}} d\xi \right\}^2 - \frac{1}{4}. \quad (3.14)$$

(2). If for all $z > a > 1$, $P(z, t) < 0$, then (3.3) can be rewritten as

$$-P(z, t) \leq m_2 z (\ln z)^{b_2} P_{,z}(z, t) + m_3 [z (\ln z)^{b_2}]^{1-\frac{1}{2p}} \left[P_{,z}(z, t) \right]^{\frac{p+1}{2p}}, \quad z \geq a. \quad (3.15)$$

Letting

$$\tilde{\tau} = \tilde{\tau}(z, t) = \left[-P(z, t) + \frac{1}{4} \right]^{\frac{1}{2}} - \frac{1}{2}$$

and using a calculation similar to (3.4)–(3.11), it can be obtained from (3.15) that

$$\left[2 + \frac{1}{2\tilde{\tau}} \right] d\tilde{\tau} \leq -\frac{1}{\sqrt{n_1} \left[z(\ln z)^{b_2} + (z(\ln z)^{b_2})^{1-\frac{1}{2p}} \right]^{\frac{1}{2}}}, \quad z \geq a. \quad (3.16)$$

Integrating (3.16) from a to z , we obtain

$$\begin{aligned} 2 \left[\tilde{\tau}(z, t) - \tilde{\tau}(a, t) \right] + \frac{1}{2} \left[\ln \tilde{\tau}(z, t) - \ln \tilde{\tau}(a, t) \right] \\ \leq -\frac{1}{\sqrt{n_1}} \int_a^z \frac{1}{\left[\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}} \right]^{\frac{1}{2}}} d\xi, \quad z \geq a. \end{aligned} \quad (3.17)$$

Dropping the non-negative term $2\tilde{\tau}(z, t)$, from (3.17) we obtain

$$\tilde{\tau}(z, t) \leq \tilde{\tau}(a, t) \exp\{4\tilde{\tau}(a, t)\} \exp\left\{-\frac{2}{\sqrt{n_1}} \int_a^z \frac{1}{\left[\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}}\right]^{\frac{1}{2}}} d\xi\right\}, \quad z \geq a. \quad (3.18)$$

Using

$$\sqrt{c+d} \leq \sqrt{c} + \sqrt{d}, \quad c, d > 0,$$

we have

$$\tilde{\tau}(z, t) \leq \left[-P(z, t)\right]^{\frac{1}{2}}.$$

Therefore, from (3.18) we obtain

$$-P(z, t) \leq \tilde{\tau}^2(a, t) \exp\{8\tilde{\tau}(a, t)\} \exp\left\{-\frac{4}{\sqrt{n_1}} \int_a^z \frac{1}{\left[\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}}\right]^{\frac{1}{2}}} d\xi\right\}. \quad (3.19)$$

From (3.14) and (3.19), we can obtain the following theorem.

Theorem 3.1 If $\frac{1}{2} < p < 1$, then the solutions of Eqs (2.1)–(2.5) either grow algebraically or decay algebraically with the axial variable of the cylinder, that is, either (3.14) holds or (3.19) holds.

From (3.14) and (3.19), the decay/growth rates of the solutions to Eqs (2.1)–(2.5) are related to the parameter b_2 . Next, we will discuss the impact of b_2 on the growth/decay rates of the solutions, and give the following remarks.

Remark 3.1 If $b_2 = 0$, then

$$\int_{z_0}^z \frac{1}{\left[\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}}\right]^{\frac{1}{2}}} d\xi \geq \int_{z_0}^z \frac{1}{\xi + \xi^{1-\frac{1}{2p}}} d\xi = 2p \ln \left| \frac{1 + z^{\frac{1}{2p}}}{1 + z_0^{\frac{1}{2p}}} \right|, \quad z \geq z_0, \quad (3.20)$$

and

$$\int_a^z \frac{1}{\left[\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}}\right]^{\frac{1}{2}}} d\xi = 2p \ln \left| \frac{1 + z^{\frac{1}{2p}}}{1 + a^{\frac{1}{2p}}} \right|, \quad z \geq a. \quad (3.21)$$

According to Theorem 3.1, the solutions of Eqs (2.1)–(2.5) either grow logarithmically or decay algebraically with the axial variable $x_3 \rightarrow \infty$, where the growth and decay rates are at least as fast as $\left[\ln(1 + z^{\frac{1}{2p}})^{\frac{1}{2\sqrt{n_1}}}\right]^2$ and $(1 + z^{\frac{1}{2p}})^{-\frac{8p}{\sqrt{n_1}}}$, respectively.

Remark 3.2 If $0 < b_2 < 1$, since $a > e$ and $\frac{1}{2} \leq 1 - \frac{1}{2p} < 1$, then $(\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}} \leq (\xi(\ln \xi)^{b_2})$. Therefore, using (3.9), we have

$$\begin{aligned} \int_{z_0}^z \frac{1}{\left[\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}}\right]^{\frac{1}{2}}} d\xi &\geq \int_{z_0}^z \frac{\left[a(\ln a)^{b_2} + (a(\ln a)^{b_2})^{1-\frac{1}{2p}}\right]^{\frac{1}{2}}}{\left[\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}}\right]^{\frac{1}{2}}} d\xi \\ &\geq \frac{1}{2} \int_{z_0}^z \frac{1}{\xi(\ln \xi)^{b_2}} d\xi \end{aligned}$$

$$= \frac{1}{2(1-b_2)} [(\ln z)^{1-b_2} - (\ln z_0)^{1-b_2}], \quad z \geq z_0, \quad (3.22)$$

and

$$\int_a^z \frac{1}{\left[\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}} \right]^{\frac{1}{2}}} d\xi \geq \frac{1}{2(1-b_2)} [(\ln z)^{1-b_2} - (\ln a)^{1-b_2}], \quad z \geq a. \quad (3.23)$$

The solutions of Eqs (2.1)–(2.5) either grow logarithmically or decay logarithmically with the axial variable $x_3 \rightarrow \infty$, where the growth and decay rates are at least as fast as $(\ln z)^{2(1-b_2)}$ and $\exp \left\{ -\frac{4}{\sqrt{m_1}} (\ln z)^{1-b_2} \right\}$, respectively.

Remark 3.3 If $b_2 = 1$, then

$$\int_{z_0}^z \frac{1}{\left[\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}} \right]^{\frac{1}{2}}} d\xi \geq \frac{1}{2} \int_{z_0}^z \frac{1}{\xi \ln \xi} d\xi = \frac{1}{2} [\ln |\ln z| - \ln |\ln z_0|], \quad z \geq z_0, \quad (3.24)$$

and

$$\int_a^z \frac{1}{\left[\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}} \right]^{\frac{1}{2}}} d\xi \geq \frac{1}{2} [\ln |\ln z| - \ln |\ln a|], \quad z \geq a. \quad (3.25)$$

The solutions of Eqs (2.1)–(2.5) either grow logarithmically or decay logarithmically with the axial variable $x_3 \rightarrow \infty$, where the growth and decay rates are at least as fast as $(\ln |\ln z|)^2$ and $(|\ln z|)^{-\frac{4}{\sqrt{m_1}}}$, respectively.

II. If $p = 1$, then $\frac{p+1}{2p} = 1$.

(1). If there exists $z_0 > a$ such that $P(z_0, t) > 0$, then (3.3) can be rewritten as

$$P(z, t) \leq \frac{1}{m_4} z (\ln z)^{b_2} P_{,z}(z, t), \quad z \geq z_0, \quad (3.26)$$

where $m_4 = \frac{1}{m_2 + m_3}$. Integrating (3.26) from z_0 to z , we have

$$P(z, t) \geq P(z_0, t) \exp \left\{ m_4 \int_{z_0}^z \frac{1}{\xi (\ln \xi)^{b_2}} d\xi \right\}, \quad z \geq z_0. \quad (3.27)$$

(2). If for all $z > a > e$, $P(z, t) < 0$, then (3.3) can be rewritten as

$$-P(z, t) \leq \frac{1}{m_4} z (\ln z)^{b_2} P_{,z}(z, t), \quad z \geq a. \quad (3.28)$$

Integrating (3.28) from a to z , we have

$$-P(z, t) \leq [-P(a, t)] \exp \left\{ -m_4 \int_a^z \frac{1}{\xi (\ln \xi)^{b_2}} d\xi \right\}, \quad z \geq a. \quad (3.29)$$

From (3.27) and (3.29), we can obtain the following theorem.

Theorem 3.2 If $p = 1$, then the solutions of Eqs (2.1)–(2.5) either grow or decay with the axial variable of the cylinder, that is, either (3.27) holds or (3.29) holds.

Remark 3.4 If $b_2 = 0$, then

$$\int_{z_0}^z \frac{1}{\xi(\ln \xi)^{b_2}} d\xi = \ln \frac{z}{z_0}, \text{ and } \int_a^z \frac{1}{\xi(\ln \xi)^{b_2}} d\xi = \ln \frac{z}{a}.$$

From (3.27) and (3.29), we can conclude that the solutions of Eqs (2.1)–(2.5) either grow algebraically or decay algebraically with the axial variable $x_3 \rightarrow \infty$, where the growth and decay rates are at least as fast as z^{m_4} and z^{-m_4} , respectively.

Remark 3.5 If $0 < b_2 < 1$, then

$$\int_{z_0}^z \frac{1}{\xi(\ln \xi)^{b_2}} d\xi = \frac{1}{1-b_2} [(\ln z)^{1-b_2} - (\ln z_0)^{1-b_2}],$$

and

$$\int_a^z \frac{1}{\xi(\ln \xi)^{b_2}} d\xi = \frac{1}{1-b_2} [(\ln z)^{1-b_2} - (\ln a)^{1-b_2}].$$

From (3.27) and (3.29), the growth and decay rates of (2.1)–(2.5) are at least as fast as $z^{m_4(1-b_2)}$ and $z^{-m_4(1-b_2)}$, respectively.

Remark 3.6 If $b_2 = 1$, then

$$\int_{z_0}^z \frac{1}{\xi(\ln \xi)^{b_2}} d\xi = [\ln \ln z - \ln \ln z_0], \text{ and } \int_a^z \frac{1}{\xi(\ln \xi)^{b_2}} d\xi = [\ln \ln z - \ln \ln a].$$

From (3.27) and (3.29), the solutions of Eqs (2.1)–(2.5) either grow logarithmically or decay logarithmically with the axial variable $x_3 \rightarrow \infty$, where the growth and decay rates are at least as fast as $(\ln z)^{m_4}$ and $(\ln z)^{-m_4}$, respectively.

III. If $p > 1$, then $\frac{1}{2} < \frac{p+1}{2p} < 1$.

(1). If there exists $z_0 > a$ such that $P(z_0, t) > 0$, then we use the Hölder and the Young inequalities to obtain

$$[P_{,z}]^{\frac{p+1}{2p}} = [P_{,z}]^{\frac{1}{2} \cdot \frac{p-1}{p}} [P_{,z}]^{1 \cdot \frac{1}{p}} \leq \frac{p-1}{p} [P_{,z}]^{\frac{1}{2}} + \frac{1}{p} [P_{,z}]. \quad (3.30)$$

Inserting (3.30) into (3.4), we obtain

$$P(z, t) \leq m_5 \left[z(\ln z)^{b_2} + (z(\ln z)^{b_2})^{1-\frac{1}{2p}} \right] \left\{ [P_{,z}]^{\frac{1}{2}} + [P_{,z}] \right\}, z \geq z_0, \quad (3.31)$$

where $m_5 = \max \left\{ \frac{m_1 \sqrt{\pi b_1}}{2\sqrt{2}} + \frac{1}{p} m_1 [2\pi b_1]^{1-\frac{1}{2p}}, \frac{p-1}{p} m_1 [2\pi b_1]^{1-\frac{1}{2p}} \right\}$.

By using calculations similar to (3.8)–(3.11), it can be concluded that

$$P_{,z}(z, t) \geq P(z, t) \frac{1}{m_5 \left[z(\ln z)^{b_2} + (z(\ln z)^{b_2})^{1-\frac{1}{2p}} \right]}, z \geq z_0. \quad (3.32)$$

Integrating (3.32) from z_0 to z , we have

$$P(z, t) \geq P(z_0, t) \exp \left\{ \frac{1}{m_5} \int_{z_0}^z \frac{1}{\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}}} d\xi \right\}, z \geq z_0. \quad (3.33)$$

(2). If for all $z > a > e$, $P(z, t) < 0$, then we can calculate the following result:

$$P_{,z}(z, t) \geq \left[-P(z, t) \right] \frac{1}{m_5 \left[z(\ln z)^{b_2} + (z(\ln z)^{b_2})^{1-\frac{1}{2p}} \right]}, \quad z \geq a.$$

Therefore, we have

$$-P(z, t) \leq \left[-P(a, t) \right] \exp \left\{ -\frac{1}{m_5} \int_{z_0}^z \frac{1}{\xi(\ln \xi)^{b_2} + (\xi(\ln \xi)^{b_2})^{1-\frac{1}{2p}}} d\xi \right\}, \quad z \geq a. \quad (3.34)$$

From (3.25) and (3.26), we can obtain the following theorem.

Theorem 3.3 If $p > 1$, then the solutions of Eqs (2.1)–(2.5) either grow exponentially or decay exponentially with the axial variable of the cylinder, that is, either (3.33) holds or (3.34) holds.

Remark 3.7 If $b_2 = 0$, from the calculations of (3.20) and (3.21), it can be seen that the solutions of Eqs (2.1)–(2.5) either grow algebraically or decay algebraically with the axial variable, where the growth and decay rates are at least as fast as $z^{\frac{1}{m_5}}$ and $z^{-\frac{1}{m_5}}$, respectively.

Remark 3.8 If $0 < b_2 < 1$, then from the calculations of (3.22) and (3.23), the growth and decay rates of (2.1)–(2.5) are at least as fast as $\exp\{\frac{1}{2m_5(1-b_2)}(\ln z)^{1-b_2}\}$ and $\exp\{-\frac{1}{2m_5(1-b_2)}(\ln z)^{1-b_2}\}$, respectively.

Remark 3.9 If $b_2 = 1$, then from the calculations of (3.24) and (3.25), it can be seen that the solutions of Eqs (2.1)–(2.5) either grow logarithmically or decay logarithmically with the axial variable, where the growth and decay rates are at least as fast as $(\ln z)^{\frac{1}{2m_5}}$ and $(\ln z)^{-\frac{1}{2m_5}}$, respectively.

4. The decay properties on a special type of cylinder

In this section, we consider a special type of cylinder under the condition $\frac{1}{2} < p < 1$. The cross-section of the cylinder is defined as

$$D(x_3) = \{(x_1, x_2) | x_1^2 + x_2^2 = r^2(x_3)\}, \quad (4.1)$$

where r satisfies

$$r'(x_3) \leq 0, \quad r(x_3) \geq r_0 \geq 1. \quad (4.2)$$

In view of the definitions of g_1 and g_2 , we have

$$g_1(z) = \pi r^2(z), \quad g_2(z) = (2\pi r(z))^{1-\frac{1}{2p}}. \quad (4.3)$$

Therefore, we can have from (2.12)

$$\left| P(z, t) \right| \leq n_2 r(z) P_{,z}(z, t) + n_2 r(z)^{1-\frac{1}{2p}} \left[P_{,z}(z, t) \right]^{\frac{p+1}{2p}}, \quad (4.4)$$

where $n_2 = \max\{\frac{1}{2\sqrt{2}}\sqrt{\pi}, m_1(2\pi)^{1-\frac{1}{2p}}\}$.

Using the Schwarz inequality, we have

$$\left[P_{,z}(z, t) \right]^{\frac{p+1}{2p}} \leq \frac{1-p}{2p} \left[P_{,z}(z, t) \right] + \frac{3p-1}{2p} \left[P_{,z}(z, t) \right]^2. \quad (4.5)$$

Inserting (4.5) into (4.4), we obtain

$$|P(z, t)| \leq n_3 \left[r(z) + r(z)^{1-\frac{1}{2p}} \right] \left\{ [P_{,z}(z, t)] + [P_{,z}(z, t)] 2 \right\}, \quad (4.6)$$

where $n_3 = n_2 \max\{\frac{3}{2} - \frac{1}{2p}, \frac{1}{2} + \frac{1}{2p}\}$.

By adopting similar calculations as part I of Section 3, we can obtain the following result.

(1). If there exists $z_0 > a$ such that $P(z_0, t) > 0$, then

$$P(z, t) \geq \left\{ \frac{1}{4} \ln \tau(z_0, t) + \frac{1}{2} + \frac{1}{2\sqrt{n_3}} \int_{z_0}^z \frac{1}{[r(\xi) + r^{1-\frac{1}{2p}}(\xi)]^{\frac{1}{2}}} d\xi \right\}^2 - \frac{1}{4}, z \geq z_0. \quad (4.7)$$

(2). If for all $z > a > e$, $P(z, t) < 0$, then

$$-P(z, t) \leq \bar{\tau}^2(a, t) \exp\{8\bar{\tau}(a, t)\} \exp\left\{ -\frac{4}{\sqrt{n_3}} \int_a^z \frac{1}{[r(\xi) + r^{1-\frac{1}{2p}}(\xi)]^{\frac{1}{2}}} d\xi \right\} \quad (4.8)$$

Theorem 4.1 Supposing the system of Eqs (2.1)–(2.5) is defined on the cylinder Ω , where the cross-section of the cylinder satisfies (4.1) and (4.2), then the solutions of Eqs (2.1)–(2.5) either satisfy (4.7) or satisfy (4.8).

5. Conclusions

The results of this paper have direct implications for thermal protection, geometry optimization, and service-life assessment of cylindrical structures. The quantitative relations established here between the cross-sectional parameter b_2 (or the decay rate r') and the spatial growth/decay rates of the solution can be used to predict the axial penetration depth of the temperature field and the extent of the “end effect”, thereby providing a theoretical basis for determining the appropriate length and thickness distribution of thermal protection layers (e.g., ablative coatings or insulating liners). For instance, configurations with faster spatial decay (larger b_2) allow a shorter coverage of the protective layer and thus weight reduction, whereas slower decay calls for enhanced protection. Second, regarding geometry optimization, the paper shows that shrinking cylinders ($r' \leq 0$) and power-law ($z \ln z$) expanding cylinders exhibit qualitatively different asymptotic behaviors. So, designers may therefore actively regulate the axial heat transfer rate by adjusting the growth/decay profile, achieving a lightweight “temperature control by shape” design and avoiding local thermal concentration.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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