



Research article

Variable-exponent modular delays in state-dependent delay differential equations: an abstract semigroup formulation

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Abstract: We study state-dependent delay differential equations whose delays are generated by variable-exponent modular functionals. Because translations are not, in general, bounded on $L^{p(\cdot)}(-h_0, 0; \mathbb{R}^n)$, the classical delay shift semigroup cannot be constructed directly on the variable-exponent space. We therefore combine the Ito-Kappel semigroup framework with the modular structure of $L^{p(\cdot)}$. We establish an equivalent abstract semilinear formulation, prove local well-posedness, and obtain well-posedness on finite intervals under stronger Lipschitz and growth assumptions. For modular state-dependent delays, we verify the required local Lipschitz properties and derive sufficient conditions for decay relative to the zero equilibrium and for conditional incremental contraction. Numerical experiments illustrate the role of the variable exponent through the modular delay and the distinction between these two sufficient estimates.

Keywords: state-dependent delay equations; variable-exponent modular delays; variable-exponent Lebesgue spaces; Ito-Kappel framework; semigroup formulation; well-posedness

1. Introduction

The theory of strongly continuous semigroups provides a fundamental framework for the analysis of evolution equations in Banach spaces. In particular, an abstract Cauchy problem of the form

$$\frac{d}{dt}u(t) = A u(t), \quad t \geq 0, \quad u(0) = u_0, \quad (1.1)$$

where A is a possibly unbounded linear operator, can be studied by means of the C_0 -semigroup generated by A . This approach provides a systematic treatment of existence, uniqueness, regularity, and qualitative properties of solutions; see [1, 2].

Variable-exponent Lebesgue spaces $L^{p(\cdot)}(-h_0, 0; \mathbb{R}^n)$ generalize the classical Lebesgue spaces by allowing the integrability exponent to vary with the underlying variable. These spaces arise naturally

in problems with nonstandard growth and heterogeneous constitutive behavior, including models of electrorheological fluids, image processing, and nonlinear elasticity; see [3–6]. A central role is played by the variable-exponent modular and the associated Luxemburg norm, which determine the modular and norm structure of $L^{p(\cdot)}$ and its Sobolev counterpart; see [7–9].

Delay differential equations constitute another class of evolution equations in which the dynamics depend on the present state and on the corresponding state segment. For a fixed maximal delay $h_0 > 0$, classical formulations use, for example,

$$C([-h_0, 0]; \mathbb{R}^n) \quad \text{or} \quad \mathbb{R}^n \times L^p(-h_0, 0; \mathbb{R}^n),$$

with the corresponding state segment at time t represented by

$$x_t(\theta) = x(t + \theta), \quad -h_0 \leq \theta \leq 0.$$

This representation leads to an abstract evolution equation governed by a suitable delay C_0 -semigroup; see [10–14].

State-dependent delay equations arise when the delay depends on the current state segment of the solution. Such equations occur naturally in biological and physiological models involving maturation, development, and feedback mechanisms with state-dependent characteristic times; see [15, 16].

In the present work, the delay is determined by a functional built from the variable-exponent modular

$$\rho_{p(\cdot)}(\phi) = \int_{-h_0}^0 |\phi(\theta)|^{p(\theta)} d\theta.$$

Thus, the contribution of the state segment to the delay is weighted nonuniformly through the exponent $p(\theta)$, so that the nonlinear dependence on $\phi(\theta)$ may vary across the delay interval.

The exponent function $p(\cdot)$ therefore enters directly into the delay functional, rather than only through the underlying function space. Its variation across the delay interval produces a position-dependent nonlinear contribution of the state segment to the delay.

A principal difficulty is that the translation properties used in the classical delay-semigroup construction generally fail in $L^{p(\cdot)}(-h_0, 0; \mathbb{R}^n)$ when $p(\cdot)$ is nonconstant. In particular, translation operators need not be bounded, so the classical delay shift semigroup cannot, in general, be constructed directly on the variable-exponent phase space. Moreover, elements of $L^{p(\cdot)}$ are equivalence classes of measurable functions, and point evaluation therefore requires additional regularity; see [7].

Accordingly, we do not attempt to construct a C_0 -semigroup directly on the variable-exponent phase space. Instead, the semigroup and modular structures are treated separately. The delay evolution is realized on the auxiliary constant-exponent space

$$H_- = \mathbb{R}^n \times L^{p^-}(-h_0, 0; \mathbb{R}^n),$$

where p^- denotes the essential infimum of $p(\cdot)$; see Section 2. Following Ito–Kappel [13], the space H_- carries the delay C_0 -semigroup, whereas the variable-exponent dependence is retained in the modular state-dependent delay.

The variable-exponent analysis therefore reduces to verifying the well-definedness and local Lipschitz properties of the modular delay operator on the admissible regularity space. Once these

properties are established, the corresponding semilinear problem can be treated within the Ito-Kappel framework.

The spaces introduced above play complementary roles. The variable-exponent spaces

$$L^{p(\cdot)}(-h_0, 0; \mathbb{R}^n) \quad \text{and} \quad W^{1,p(\cdot)}(-h_0, 0; \mathbb{R}^n)$$

encode the modular structure and the corresponding Sobolev regularity, whereas X and W_∞ provide, respectively, continuity for point evaluation and the Lipschitz regularity required by the state-dependent delay. The auxiliary space H_- carries the delay C_0 -semigroup. The relevant continuous embeddings are established in Section 2.

Within this framework, we establish an abstract semilinear formulation equivalent to the original delay equation and obtain local well-posedness and continuous dependence on the initial data. Under stronger Lipschitz and linear-growth assumptions, we obtain well-posedness on finite intervals. We further derive a sufficient condition for exponential decay relative to the zero equilibrium and a stronger sufficient condition for conditional incremental contraction between two solutions.

The numerical experiments illustrate the distinction between the two sufficient conditions and the influence of $p(\cdot)$ through the modular delay. The Ito-Kappel semidiscretization is complemented by a refinement study. Throughout, failure of a sufficient condition is not interpreted as its converse.

The paper is organized as follows. Section 2 recalls the variable-exponent Lebesgue and Sobolev spaces and the modular properties used in the analysis. Section 3 introduces the state-dependent delay problem and its Ito-Kappel semigroup formulation. Section 4 establishes local well-posedness and well-posedness on finite intervals under stronger assumptions. Section 5 treats the modular state-dependent delay, derives the decay and incremental contraction estimates, and presents the numerical results. The final section discusses the results and possible extensions.

2. Preliminaries on variable-exponent spaces

This section collects the variable-exponent results used in the sequel. We first recall the Lebesgue-space and modular structure, then introduce the associated Sobolev space, and finally discuss the phase-space difficulty that motivates the auxiliary Ito-Kappel formulation of Section 3.

2.1. Variable-exponent Lebesgue spaces and modular structure

Let

$$I := (-h_0, 0), \quad h_0 > 0,$$

equipped with the Lebesgue measure, and let

$$p : I \rightarrow [1, \infty)$$

be measurable. We set

$$p^- := \operatorname{ess\,inf}_{\theta \in I} p(\theta), \quad p^+ := \operatorname{ess\,sup}_{\theta \in I} p(\theta). \quad (2.1)$$

The variable-exponent Lebesgue space $L^{p(\cdot)}(I; \mathbb{R}^n)$ consists of all measurable functions $u : I \rightarrow \mathbb{R}^n$ such that

$$\rho_{p(\cdot)}(\lambda u) < \infty \quad \text{for some } \lambda > 0,$$

where

$$\rho_{p(\cdot)}(u) := \int_I |u(\theta)|^{p(\theta)} d\theta = \int_{-h_0}^0 |u(\theta)|^{p(\theta)} d\theta. \quad (2.2)$$

We refer to $\rho_{p(\cdot)}$ as the variable-exponent modular. The associated Luxemburg norm is introduced below and provides the Banach-space norm on $L^{p(\cdot)}(I; \mathbb{R}^n)$.

Throughout the paper, we impose the bounded-exponent condition

$$1 < p^- \leq p^+ < \infty. \quad (2.3)$$

For the basic theory of variable-exponent Lebesgue spaces, including their modular and norm structure, we refer to [7–9]. For the general theory of modular function spaces, see [17, 18].

The Luxemburg norm associated with $\rho_{p(\cdot)}$ is defined by

$$\|u\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}. \quad (2.4)$$

Equipped with this norm, $L^{p(\cdot)}(I; \mathbb{R}^n)$ is a Banach space. Under the standing assumption (2.3), it is moreover separable and reflexive; see [7].

We shall repeatedly use the elementary fact that bounded functions have finite $p(\cdot)$ -modular. Indeed, if

$$|u(\theta)|_{\mathbb{R}^n} \leq C \quad \text{for a.e. } \theta \in I,$$

then

$$\begin{aligned} \rho_{p(\cdot)}(u) &= \int_I |u(\theta)|_{\mathbb{R}^n}^{p(\theta)} d\theta \\ &\leq \int_I C^{p(\theta)} d\theta \\ &\leq h_0 \max \{C^{p^-}, C^{p^+}\} < \infty. \end{aligned}$$

In particular, every bounded state segment has finite $p(\cdot)$ -modular.

We also recall the variable-exponent Hölder inequality that will be used below; see [7, Lemma 3.2.20].

Proposition 2.1 (Variable-exponent Hölder inequality). *Let*

$$p, q, s : I \rightarrow [1, \infty)$$

be measurable exponents satisfying

$$\frac{1}{s(\theta)} = \frac{1}{p(\theta)} + \frac{1}{q(\theta)} \quad \text{for a.e. } \theta \in I.$$

Then, for scalar-valued functions

$$u \in L^{p(\cdot)}(I), \quad v \in L^{q(\cdot)}(I),$$

one has

$$\rho_{s(\cdot)}(uv) \leq \rho_{p(\cdot)}(u) + \rho_{q(\cdot)}(v), \quad (2.5)$$

$$\|uv\|_{s(\cdot)} \leq 2 \|u\|_{p(\cdot)} \|v\|_{q(\cdot)}. \quad (2.6)$$

The first estimate follows from the pointwise Young inequality, whereas the second is the variable-exponent Hölder inequality.

The functional $\rho_{p(\cdot)}$ in (2.2) is convex. In particular,

$$\rho_{p(\cdot)}(\alpha u + (1 - \alpha)v) \leq \alpha \rho_{p(\cdot)}(u) + (1 - \alpha) \rho_{p(\cdot)}(v), \quad 0 \leq \alpha \leq 1.$$

Moreover,

$$\rho_{p(\cdot)}(u) = 0 \iff u = 0 \quad \text{a.e. on } I,$$

and

$$\rho_{p(\cdot)}(\alpha u) = \rho_{p(\cdot)}(u), \quad |\alpha| = 1.$$

Thus, $\rho_{p(\cdot)}$ is a convex regular modular in the terminology of modular function spaces; see [17].

Recall that a modular ρ satisfies the Δ_2 -condition if there exists $K > 0$ such that

$$\rho(2u) \leq K\rho(u) \tag{2.7}$$

for all admissible u . Under the bounded-exponent assumption (2.3), $\rho_{p(\cdot)}$ satisfies

$$\rho_{p(\cdot)}(2u) \leq 2^{p^+} \rho_{p(\cdot)}(u). \tag{2.8}$$

Consequently,

$$\rho_{p(\cdot)}(u + v) \leq 2^{p^+-1} (\rho_{p(\cdot)}(u) + \rho_{p(\cdot)}(v)). \tag{2.9}$$

2.2. Variable-exponent Sobolev regularity

We recall the variable-exponent Sobolev space associated with $L^{p(\cdot)}(I; \mathbb{R}^n)$. This regularity is used to relate the modular structure to the continuous phase spaces arising in the delay equation.

Definition 2.2. *The variable-exponent Sobolev space is*

$$W^{1,p(\cdot)}(I; \mathbb{R}^n) := \{u \in L^{p(\cdot)}(I; \mathbb{R}^n) : u' \in L^{p(\cdot)}(I; \mathbb{R}^n)\},$$

where u' denotes the weak derivative.

We equip $W^{1,p(\cdot)}(I; \mathbb{R}^n)$ with the norm

$$\|u\|_{W^{1,p(\cdot)}} := \|u\|_{p(\cdot)} + \|u'\|_{p(\cdot)},$$

and with the modular

$$\rho_{W^{1,p(\cdot)}}(u) := \rho_{p(\cdot)}(u) + \rho_{p(\cdot)}(u').$$

Under (2.3), because I is bounded, one has the continuous embeddings

$$W^{1,p(\cdot)}(I; \mathbb{R}^n) \hookrightarrow W^{1,p^-}(I; \mathbb{R}^n) \hookrightarrow C([-h_0, 0]; \mathbb{R}^n). \tag{2.10}$$

Hence, every element of $W^{1,p(\cdot)}(I; \mathbb{R}^n)$ admits a continuous representative on $[-h_0, 0]$, and point evaluation is well-defined for this representative.

This regularity does not remove the semigroup obstruction. For nonconstant $p(\cdot)$, translation operators need not be bounded on $L^{p(\cdot)}(I; \mathbb{R}^n)$; see [7, Proposition 3.6.1]. Consequently, the classical delay shift semigroup cannot, in general, be constructed directly on the variable-exponent phase space. This motivates the auxiliary semigroup construction introduced below.

2.3. Phase-space difficulties and the auxiliary semigroup structure

The choice of phase space for delay equations is already delicate in the constant-exponent case. An L^p -type phase space is well-adapted to the semigroup formulation but does not, by itself, allow point evaluation, whereas $C([-h_0, 0]; \mathbb{R}^n)$ provides point evaluation but does not, in general, lead directly to the same abstract Cauchy-problem formulation; see [13, 14, 19].

As noted in subsection 2.2, the lack of suitable translation properties prevents, in general, the direct construction of the classical delay shift semigroup on $L^{p(\cdot)}(I; \mathbb{R}^n)$. We therefore introduce the variable-exponent phase space

$$H := \mathbb{R}^n \times L^{p(\cdot)}(I; \mathbb{R}^n), \quad (2.11)$$

together with an auxiliary constant-exponent phase space for the Ito–Kappel semigroup formulation.

For point evaluation and the regularity required by the state-dependent delay operator, we introduce

$$X := \{(\phi(0), \phi) : \phi \in C([-h_0, 0]; \mathbb{R}^n)\}, \quad (2.12)$$

$$W_\infty := \{(\phi(0), \phi) : \phi \in W^{1,\infty}(I; \mathbb{R}^n)\}. \quad (2.13)$$

The auxiliary phase space carrying the delay C_0 -semigroup is

$$H_- := \mathbb{R}^n \times L^{p^-}(I; \mathbb{R}^n). \quad (2.14)$$

Because $p(\theta) \geq p^-$ almost everywhere on the bounded interval I , one has the continuous embedding

$$H \hookrightarrow H_-. \quad (2.15)$$

Thus, H retains the variable-exponent modular structure, H_- carries the auxiliary delay C_0 -semigroup, and X and W_∞ provide, respectively, point evaluation and the Lipschitz regularity required by the state-dependent delay operator. The principal notation is summarized in Table 1.

Table 1. Principal notation used throughout the paper.

Notation	Meaning
$I = (-h_0, 0)$	Delay interval, where $h_0 > 0$ is the maximal delay.
$p(\cdot), p^-, p^+$	Measurable variable exponent and its essential lower and upper bounds; see (2.1).
$L^{p(\cdot)}(I; \mathbb{R}^n)$	Variable-exponent Lebesgue space.
$\rho_{p(\cdot)}$	Variable-exponent modular defined in (2.2).
$\ \cdot\ _{p(\cdot)}$	Luxemburg norm defined in (2.4).
$W^{1,p(\cdot)}(I; \mathbb{R}^n)$	Variable-exponent Sobolev space.
H	Variable-exponent product space defined in (2.11).
H_-	Auxiliary constant-exponent state space defined in (2.14); it carries the C_0 -semigroup used in the delay formulation.
X	Continuous state space defined in (2.12), used for point evaluation.
W	Variable-exponent regularity space defined in (3.4).
W_∞	Lipschitz regularity space defined in (2.13).

Continued on the next page

Notation	Meaning
y_t	State segment defined by $y_t(\theta) = y(t + \theta)$, $-h_0 \leq \theta \leq 0$.
$h(t, \phi)$	General state-dependent delay with values in $[0, h_0]$.
$\eta(\phi)$	Variable-exponent modular delay considered in Section 5.
$G(t, \phi)$	Nonlinear delay operator defined in (3.6).
B	Bounded injection operator defined in (3.7).
$(S(t))_{t \geq 0}$	C_0 -semigroup on H_- associated with the delay formulation.
A_-	Infinitesimal generator of $(S(t))_{t \geq 0}$ on H_- .
$L_{\beta, p}$	Local Lipschitz constant associated with the modular on the bounded set determined by β .

3. Delay equations with time- and state-dependent delays

3.1. Delay equation and phase-space formulation

We consider the state-dependent delay equation

$$\dot{y}(t) = f(t, y(t), y(t - h(t, y_t))), \quad t > 0, \quad (3.1)$$

where

$$f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad h : [0, T] \times C([-h_0, 0]; \mathbb{R}^n) \rightarrow [0, h_0],$$

and y_t denotes the state segment defined in Table 1.

Following [13, Section 11.2], we assume that f is measurable in t and locally Lipschitz continuous in its state variables, and that h is measurable in t and locally Lipschitz continuous with respect to its state-segment argument. These assumptions are denoted by (B1)–(B2). We consider delays of the form

$$h(t, \phi) = \Theta(t, \rho_{p(\cdot)}(\phi)), \quad (3.2)$$

where $\rho_{p(\cdot)}$ is defined in (2.2), and

$$\Theta : [0, T] \times [0, \infty) \rightarrow [0, h_0]$$

is measurable in t and locally Lipschitz continuous in its second variable.

Let $\beta > 0$, and let $\phi, \psi \in X$ satisfy

$$\|\phi\|_X, \|\psi\|_X \leq \beta.$$

Then,

$$\rho_{p(\cdot)}(\phi), \rho_{p(\cdot)}(\psi) \leq R_{\beta, p},$$

where

$$R_{\beta, p} := h_0 \max \{\beta^{p^-}, \beta^{p^+}\}.$$

Because Θ is locally Lipschitz continuous in its second variable, there exists $L_{\Theta, \beta} > 0$ such that

$$|\Theta(t, r_1) - \Theta(t, r_2)| \leq L_{\Theta, \beta} |r_1 - r_2|, \quad r_1, r_2 \in [0, R_{\beta, p}]$$

for almost every $t \in [0, T]$.

Moreover, the local Lipschitz estimate for the modular gives

$$|\rho_{p(\cdot)}(\phi) - \rho_{p(\cdot)}(\psi)| \leq h_0 L_{\beta,p} \|\phi - \psi\|_X,$$

where

$$L_{\beta,p} = p^+ \max \{ \beta^{p^- - 1}, \beta^{p^+ - 1} \}.$$

Consequently,

$$|h(t, \phi) - h(t, \psi)| \leq L_{\Theta, \beta} h_0 L_{\beta,p} \|\phi - \psi\|_X. \quad (3.3)$$

Thus the delay operator (3.2) is locally Lipschitz with respect to its state-segment argument on bounded subsets of X .

We also introduce the variable-exponent regularity space

$$W := \{(\phi(0), \phi) : \phi \in W^{1,p(\cdot)}(I; \mathbb{R}^n)\}. \quad (3.4)$$

By (2.10),

$$W \hookrightarrow X \hookrightarrow H.$$

Hence, elements of W admit continuous representatives on $[-h_0, 0]$, and point evaluation is well-defined.

We set

$$U := \mathbb{R}^n. \quad (3.5)$$

Identifying $(\phi(0), \phi) \in X$ with its continuous representative ϕ , define

$$G(t, \phi) := f(t, \phi(0), \phi(-h(t, \phi))), \quad (t, \phi) \in [0, T] \times X. \quad (3.6)$$

Because $h(t, \phi) \in [0, h_0]$, the point evaluation in (3.6) is well-defined.

We also define

$$Bu := (u, 0), \quad u \in U. \quad (3.7)$$

Then,

$$B \in \mathcal{L}(U, H_-).$$

The operators B and G , together with the delay C_0 -semigroup on H_- , determine the abstract semilinear problem considered below.

3.2. Auxiliary delay semigroup

As discussed in subsection 2.3, the classical delay shift semigroup cannot, in general, be constructed directly on the variable-exponent phase space H . We therefore use the auxiliary constant-exponent phase space H_- defined in (2.14), with

$$H \hookrightarrow H_-.$$

The space H_- carries the delay C_0 -semigroup used in the Ito–Kappel formulation, and the variable-exponent dependence remains in the modular delay operator (3.2).

For $(\eta, \phi) \in H_-$ and $t \geq 0$, define

$$S(t)(\eta, \phi) := (\eta, g_t), \quad (3.8)$$

where

$$g_t(\theta) = g(t + \theta), \quad -h_0 \leq \theta \leq 0,$$

and

$$g(s) = \begin{cases} \phi(s), & -h_0 \leq s < 0, \\ \eta, & s \geq 0. \end{cases} \quad (3.9)$$

This is the delay C_0 -semigroup associated with the homogeneous equation

$$\dot{y}(t) = 0.$$

Recall the Lipschitz regularity space W_∞ defined in (2.13). We equip it with the norm

$$\|(\phi(0), \phi)\|_{W_\infty} := |\phi(0)|_{\mathbb{R}^n} + \|\phi'\|_{L^\infty(I; \mathbb{R}^n)}.$$

Then,

$$W_\infty \hookrightarrow X \hookrightarrow H_-, \quad W_\infty \hookrightarrow W^{1,p(\cdot)}(I; \mathbb{R}^n).$$

The space W_∞ is invariant under the delay semigroup,

$$S(t)W_\infty \subset W_\infty, \quad t \geq 0,$$

but the restriction of $S(t)$ to W_∞ is not, in general, strongly continuous with respect to the W_∞ -norm; see [20, Theorem 10.2.3] and [13, Section 11.1]. Thus, W_∞ serves as a regularity space for the state-dependent delay operator rather than as the phase space of a C_0 -semigroup.

The next subsection verifies the assumptions of [13, Section 11.1] for the spaces and operators introduced above.

3.3. Verification of the Ito–Kappel assumptions

We verify that the spaces and operators introduced above satisfy assumptions (A1)–(A3) of [13, Section 11.1].

Lemma 3.1. *Assume (B1)–(B2). Let H_- , X , W_∞ , and U be defined by (2.14), (2.12), (2.13), and (3.5), respectively, and let B , G , and $(S(t))_{t \geq 0}$ be given by (3.7), (3.6), and (3.8)–(3.9). Then, (A1)–(A3) of [13, Section 11.1] hold.*

Proof. Verification of (A1). By [13, Section 11.2], $(S(t))_{t \geq 0}$ is a C_0 -semigroup on H_- , and

$$S(t)X \subset X, \quad S(t)W_\infty \subset W_\infty, \quad t \geq 0.$$

Moreover, the restrictions $S(t)|_X$ form a C_0 -semigroup on X . As noted in subsection 3.2, W_∞ is invariant under $S(t)$, although the restriction of $S(t)$ to W_∞ is not, in general, strongly continuous in the W_∞ -norm.

For every $T > 0$, there exists $M_1 > 0$ such that

$$\|S(t)\|_{\mathcal{L}(W_\infty)} \leq M_1, \quad 0 \leq t \leq T.$$

Hence, (A1) holds.

Verification of (A2). By (3.7),

$$B \in \mathcal{L}(U, H_-).$$

Let

$$F \in L^{p^-}(0, T; U),$$

and define

$$z(t) := \int_0^t S(t-s)BF(s) ds, \quad 0 \leq t \leq T.$$

A direct computation gives

$$z(t) = \left(\int_0^t F(s) ds, w_t \right),$$

where

$$w_t(\theta) = \begin{cases} \int_0^{t+\theta} F(s) ds, & t + \theta > 0, \\ 0, & t + \theta \leq 0. \end{cases}$$

Because

$$w_t(0) = \int_0^t F(s) ds,$$

and $w_t \in C([-h_0, 0]; \mathbb{R}^n)$, one has

$$z(t) \in X, \quad 0 \leq t \leq T.$$

If $(p^-)'$ denotes the conjugate exponent of p^- , then Hölder's inequality yields

$$\begin{aligned} \|z(t)\|_X &= \|w_t\|_{C([-h_0, 0]; \mathbb{R}^n)} \\ &\leq \int_0^t |F(s)|_{\mathbb{R}^n} ds \\ &\leq t^{1/(p^-)'} \|F\|_{L^{p^-}(0, t; U)} \\ &\leq T^{1/(p^-)'} \|F\|_{L^{p^-}(0, T; U)}. \end{aligned}$$

Thus, the first estimate in (A2) follows.

If $F \in L^\infty(0, T; U)$, then $w_t \in W^{1, \infty}(I; \mathbb{R}^n)$, and for almost every $\theta \in I$,

$$w'_t(\theta) = \begin{cases} F(t + \theta), & \max\{-h_0, -t\} < \theta < 0, \\ 0, & -h_0 < \theta < \min\{-t, 0\}. \end{cases}$$

Hence,

$$z(t) \in W_\infty, \quad 0 \leq t \leq T,$$

and there exists $M_3 > 0$, depending only on T , such that

$$\left\| \int_0^t S(t-s)BF(s) ds \right\|_{W_\infty} \leq M_3 \|F\|_{L^\infty(0, T; U)}, \quad 0 \leq t \leq T.$$

Therefore, (A2) holds.

Verification of (A3). By (3.6), the point evaluations entering G are well-defined on X , and the measurability of $t \mapsto G(t, \psi)$ follows from the assumptions on f , h , and Θ .

Let $\beta > 0$, and let $\phi, \psi \in X$ satisfy

$$\|\phi\|_X, \|\psi\|_X \leq \beta, \quad \phi \in W_\infty.$$

By the local Lipschitz continuity of f ,

$$\begin{aligned} |G(t, \phi) - G(t, \psi)|_{\mathbb{R}^n} &\leq C_\beta \left(|\phi(0) - \psi(0)|_{\mathbb{R}^n} \right. \\ &\quad \left. + |\phi(-h(t, \phi)) - \psi(-h(t, \psi))|_{\mathbb{R}^n} \right). \end{aligned}$$

Furthermore,

$$\begin{aligned} |\phi(-h(t, \phi)) - \psi(-h(t, \psi))|_{\mathbb{R}^n} &\leq |\phi(-h(t, \phi)) - \phi(-h(t, \psi))|_{\mathbb{R}^n} \\ &\quad + |\phi(-h(t, \psi)) - \psi(-h(t, \psi))|_{\mathbb{R}^n}. \end{aligned}$$

Because $\phi \in W_\infty$,

$$|\phi(-h(t, \phi)) - \phi(-h(t, \psi))|_{\mathbb{R}^n} \leq \|\phi'\|_{L^\infty(I; \mathbb{R}^n)} |h(t, \phi) - h(t, \psi)|.$$

Moreover,

$$|\phi(-h(t, \psi)) - \psi(-h(t, \psi))|_{\mathbb{R}^n} \leq \|\phi - \psi\|_X.$$

Using (3.3), we obtain

$$|\phi(-h(t, \phi)) - \psi(-h(t, \psi))|_{\mathbb{R}^n} \leq \left(1 + L_{\Theta, \beta} h_0 L_{\beta, p} \|\phi'\|_{L^\infty(I; \mathbb{R}^n)} \right) \|\phi - \psi\|_X.$$

Consequently,

$$|G(t, \phi) - G(t, \psi)|_{\mathbb{R}^n} \leq k_\beta (1 + \|\phi\|_{W_\infty}) \|\phi - \psi\|_X$$

for some $k_\beta > 0$ and for almost every $t \in [0, T]$.

The corresponding local growth estimate follows from (B1). Therefore, (A3) holds, and the proof is complete. $\square$

Remark 3.2. As noted in subsection 3.2, W_∞ is invariant under $(S(t))_{t \geq 0}$, but the restriction of $S(t)$ to W_∞ is not, in general, strongly continuous with respect to the W_∞ -norm. Accordingly, solutions are sought in

$$C([0, T]; X) \cap B([0, T]; W_\infty),$$

where

$$\|u\|_{B([0, T]; W_\infty)} := \sup_{0 \leq t \leq T} \|u(t)\|_{W_\infty}.$$

The space W defined in (3.4) retains the variable-exponent regularity, whereas the semigroup analysis is carried out on H_- , X , and W_∞ .

3.4. Mild solutions, local well-posedness, and equivalence

Let

$$\Phi = (\phi(0), \phi) \in W_\infty.$$

Following [13, Section 11.1], a function

$$u \in C([0, T]; X) \cap B([0, T]; W_\infty)$$

is called a mild solution if

$$u(t) = S(t)\Phi + \int_0^t S(t-s)B G(s, u(s)) ds, \quad 0 \leq t \leq T. \quad (3.10)$$

The integral in (3.10) is a Bochner integral in H_- , and Lemma 3.1 ensures that the integral term belongs to X .

The following result follows from [13, Theorem 11.2] and Lemma 3.1.

Theorem 3.3 (Local well-posedness). *Assume (B1)–(B2). Then, for every $\Phi \in W_\infty$, there exists $\tau > 0$ such that (3.10) admits a unique mild solution,*

$$u \in C([0, \tau]; X) \cap B([0, \tau]; W_\infty).$$

Moreover, the solution depends continuously on the initial datum Φ in the W_∞ -norm.

Proof. By Lemma 3.1, the assumptions (A1)–(A3) of [13, Section 11.1] hold. Hence, [13, Theorem 11.2] applies. $\square$

Let A_- denote the infinitesimal generator of $(S(t))_{t \geq 0}$ on H_- . For the delay semigroup (3.8)–(3.9),

$$D(A_-) = \{(\eta, \phi) \in H_- : \phi \in W^{1,p^-}(I; \mathbb{R}^n), \eta = \phi(0)\},$$

and

$$A_-(\eta, \phi) = (0, \phi');$$

see [13, 14].

Because

$$W^{1,\infty}(I; \mathbb{R}^n) \hookrightarrow W^{1,p^-}(I; \mathbb{R}^n) \hookrightarrow C([-h_0, 0]; \mathbb{R}^n),$$

it follows that

$$W_\infty \hookrightarrow D(A_-) \hookrightarrow X \hookrightarrow H_-.$$

Thus, each $\Phi \in W_\infty$ belongs to $D(A_-)$ and has the regularity required by the state-dependent delay operator.

Theorem 3.4 (Equivalence with the delay equation). *Let*

$$\Phi = (\phi(0), \phi) \in W_\infty.$$

(i) Suppose that

$$u \in C([0, T]; X) \cap B([0, T]; W_\infty)$$

satisfies (3.10). Writing

$$u(t) = (\xi(t), v_t),$$

define

$$y(t) = \begin{cases} \xi(t), & 0 \leq t \leq T, \\ \phi(t), & -h_0 \leq t \leq 0. \end{cases}$$

Then,

$$y \in W^{1,\infty}(-h_0, T; \mathbb{R}^n), \quad v_t = y_t, \quad 0 \leq t \leq T,$$

and

$$y(t) = \phi(0) + \int_0^t f(s, y(s), y(s - h(s, y_s))) ds, \quad 0 \leq t \leq T. \quad (3.11)$$

Moreover,

$$y_t \in W^{1,p(\cdot)}(I; \mathbb{R}^n) \subset L^{p(\cdot)}(I; \mathbb{R}^n), \quad (y(t), y_t) \in H.$$

(ii) Conversely, suppose that

$$y \in W^{1,\infty}(-h_0, T; \mathbb{R}^n), \quad y(t) = \phi(t), \quad -h_0 \leq t \leq 0,$$

and that y satisfies (3.11). Then,

$$u(t) := (y(t), y_t), \quad 0 \leq t \leq T,$$

belongs to

$$C([0, T]; X) \cap B([0, T]; W_\infty),$$

and satisfies (3.10), and

$$u(t) \in H, \quad 0 \leq t \leq T.$$

Proof. Assume first that u satisfies (3.10). Taking the first component gives

$$\xi(t) = \phi(0) + \int_0^t G(s, u(s)) ds.$$

By (A3),

$$G(\cdot, u(\cdot)) \in L^\infty(0, T; \mathbb{R}^n),$$

and therefore,

$$y \in W^{1,\infty}(-h_0, T; \mathbb{R}^n).$$

For $\theta \in [-h_0, 0]$, the second component of (3.10), together with (3.8)–(3.9), gives

$$v_t(\theta) = \begin{cases} \phi(t + \theta), & t + \theta \leq 0, \\ \phi(0) + \int_0^{t+\theta} G(s, u(s)) ds, & t + \theta > 0. \end{cases}$$

Hence,

$$v_t(\theta) = y(t + \theta),$$

so that $v_t = y_t$. By (3.6),

$$G(s, u(s)) = f(s, y(s), y(s - h(s, y_s))),$$

and (3.11) follows.

Because

$$y_t \in W^{1,\infty}(I; \mathbb{R}^n),$$

Subsection 2.2 yields

$$y_t \in W^{1,p(\cdot)}(I; \mathbb{R}^n) \subset L^{p(\cdot)}(I; \mathbb{R}^n),$$

and therefore,

$$(y(t), y_t) \in H.$$

This proves (i).

Conversely, let y satisfy the hypotheses of (ii), and define

$$u(t) := (y(t), y_t).$$

Because

$$y \in W^{1,\infty}(-h_0, T; \mathbb{R}^n),$$

one has

$$u \in C([0, T]; X) \cap B([0, T]; W_\infty),$$

and, by Subsection 2.2,

$$u(t) \in H, \quad 0 \leq t \leq T.$$

Finally, using (3.11), (3.6), and (3.8)–(3.9), we obtain

$$u(t) = S(t)\Phi + \int_0^t S(t-s)B G(s, u(s)) ds.$$

Thus, u satisfies (3.10), which proves (ii). □

4. Well-posedness of the abstract problem

4.1. Local well-posedness

In the previous section, the state-dependent delay equation (3.1) was written as the abstract semilinear problem (3.10) following [13, Section 11.1]. The delay C_0 -semigroup acts on the auxiliary phase space H_- , whereas the variable-exponent structure is retained through H and the corresponding regularity spaces. The relations among the principal spaces are summarized in Figure 1.

By Lemma 3.1, the assumptions (A1)–(A3) of [13, Section 11.1] hold. Hence, Theorem 3.3, together with Theorem 3.4, yields the following local well-posedness result for the delay equation.

$$\begin{array}{ccccc}
 W_\infty & \xrightarrow{\hookrightarrow} & X & \xrightarrow{\hookrightarrow} & H_- \\
 \downarrow \hookrightarrow & & & & \uparrow \hookrightarrow \\
 W^{1,p(\cdot)}(I; \mathbb{R}^n) & & & & H
 \end{array}$$

Figure 1. Continuous embeddings among the principal phase and regularity spaces. The space H_- carries the delay C_0 -semigroup, H retains the variable-exponent modular structure, and W_∞ provides the regularity required by the state-dependent delay operator.

Corollary 4.1 (Local well-posedness of the delay equation). *Assume (B1)–(B2). Then, for every*

$$\phi \in W^{1,\infty}(I; \mathbb{R}^n),$$

there exist $\tau > 0$ and a unique

$$y \in W^{1,\infty}(-h_0, \tau; \mathbb{R}^n)$$

such that

$$y(t) = \phi(t), \quad -h_0 \leq t \leq 0,$$

and y satisfies (3.11) on $[0, \tau]$. Equivalently,

$$\dot{y}(t) = f(t, y(t), y(t - h(t, y_t)))$$

for almost every $t \in (0, \tau)$.

Moreover,

$$y_t \in W^{1,p(\cdot)}(I; \mathbb{R}^n) \subset L^{p(\cdot)}(I; \mathbb{R}^n), \quad 0 \leq t \leq \tau,$$

and hence,

$$(y(t), y_t) \in H, \quad 0 \leq t \leq \tau.$$

Proof. The result follows from Theorem 3.3 and Theorem 3.4. $\square$

The result is local. Well-posedness on a finite interval $[0, T]$ requires the stronger assumptions introduced in Subsection 4.2.

4.2. Well-posedness on a finite interval

The local result above follows from (B1)–(B2) and does not, in general, ensure existence on an arbitrary finite interval $[0, T]$. For well-posedness on $[0, T]$, we impose the following stronger conditions on G .

Assumption 4.2. *There exist constants $L_G, a_G, b_G \geq 0$ such that, for every $t \in [0, T]$ and $\phi, \psi \in X$,*

$$|G(t, \phi) - G(t, \psi)|_{\mathbb{R}^n} \leq L_G \|\phi - \psi\|_X,$$

and

$$|G(t, \phi)|_{\mathbb{R}^n} \leq a_G + b_G \|\phi\|_X.$$

A contraction argument in a Bielecki norm gives the following result.

Theorem 4.3 (Well-posedness on a finite interval). *Assume (B1)–(B2) and Assumption 4.2. Then, for every $\Phi \in W_\infty$, Problem (3.10) admits a unique mild solution,*

$$u \in C([0, T]; X) \cap B([0, T]; W_\infty).$$

The corresponding solution y of (3.1) satisfies

$$y \in W^{1,\infty}(-h_0, T; \mathbb{R}^n),$$

and, for every $t \in [0, T]$,

$$y_t \in W^{1,p(\cdot)}(I; \mathbb{R}^n) \subset L^{p(\cdot)}(I; \mathbb{R}^n).$$

Proof. For $K > 0$, equip $C([0, T]; X)$ with the Bielecki norm

$$\|u\|_K := \sup_{0 \leq t \leq T} e^{-Kt} \|u(t)\|_X.$$

Define

$$(\mathcal{J}u)(t) := S(t)\Phi + \int_0^t S(t-s)B G(s, u(s)) ds.$$

By Assumption 4.2 and (A1)–(A2),

$$\mathcal{J} : C([0, T]; X) \rightarrow C([0, T]; X).$$

For $u, v \in C([0, T]; X)$, the estimate in (A2) and the Lipschitz condition on G yield

$$\begin{aligned} e^{-Kt} \|(\mathcal{J}u)(t) - (\mathcal{J}v)(t)\|_X &\leq L_G \int_0^t e^{-K(t-s)} e^{-Ks} \|u(s) - v(s)\|_X ds \\ &\leq \frac{L_G}{K} \|u - v\|_K. \end{aligned}$$

Hence,

$$\|\mathcal{J}u - \mathcal{J}v\|_K \leq \frac{L_G}{K} \|u - v\|_K.$$

Choosing $K > L_G$, $\mathcal{J}$ is a contraction. Banach's fixed-point theorem therefore gives a unique

$$u \in C([0, T]; X)$$

satisfying (3.10).

Because $\Phi \in W_\infty$ and, by Assumption 4.2,

$$G(\cdot, u(\cdot)) \in L^\infty(0, T; U),$$

the W_∞ -estimates in (A1)–(A2) imply

$$u \in B([0, T]; W_\infty).$$

Thus,

$$u \in C([0, T]; X) \cap B([0, T]; W_\infty).$$

The assertions for y follow from Theorem 3.4. □

Remark 4.4. *Assumption 4.2 is stronger than the local assumptions (B1)–(B2) and yields well-posedness on every finite interval $[0, T]$. Thus, Corollary 4.1 provides local well-posedness, whereas Theorem 4.3 gives well-posedness on $[0, T]$ under Assumption 4.2; the latter does not assert global existence on $[0, \infty)$.*

5. Application: A variable-exponent state-dependent delay

We apply the preceding results to a state-dependent delay generated by the variable-exponent modular $\rho_{p(\cdot)}$ in (2.2). Let p satisfy (2.3), and define

$$\eta(\psi) := h_0 \frac{\rho_{p(\cdot)}(\psi)}{1 + \rho_{p(\cdot)}(\psi)}, \quad \psi \in L^{p(\cdot)}(I; \mathbb{R}). \quad (5.1)$$

Because $\rho_{p(\cdot)}(\psi) \geq 0$,

$$0 \leq \eta(\psi) \leq h_0,$$

so that

$$\eta : L^{p(\cdot)}(I; \mathbb{R}) \longrightarrow [0, h_0]$$

is well-defined.

We consider

$$\begin{cases} y'(t) = f(t, y(t), y(t - \eta(y_t))), & 0 < t \leq T, \\ y(\theta) = \phi(\theta), & -h_0 \leq \theta \leq 0, \end{cases} \quad (5.2)$$

with

$$\phi \in W^{1,\infty}(I; \mathbb{R}).$$

Because

$$W^{1,\infty}(I; \mathbb{R}) \hookrightarrow L^{p(\cdot)}(I; \mathbb{R}),$$

the quantity $\eta(y_t)$ is well-defined for the state segments arising from the semigroup formulation.

We verify the assumptions of Corollary 4.1. In the scalar case, (3.6) becomes

$$G(t, \psi) = f(t, \psi(0), \psi(-\eta(\psi))), \quad (t, \psi) \in [0, T] \times X.$$

Assume that

$$f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

is measurable in t and locally Lipschitz continuous in its state variables. Thus, for every $\beta > 0$, there exists $L_{f,\beta} > 0$ such that

$$|f(t, x_1, z_1) - f(t, x_2, z_2)| \leq L_{f,\beta}(|x_1 - x_2| + |z_1 - z_2|)$$

for almost every $t \in [0, T]$ whenever

$$|x_i|, |z_i| \leq \beta, \quad i = 1, 2.$$

Let $\beta > 0$, and let $\phi, \psi \in X$ satisfy

$$\|\phi\|_X, \|\psi\|_X \leq \beta.$$

For almost every $\theta \in I$, set

$$g_\theta(s) := |s|^{p(\theta)}.$$

Because $p(\theta) \geq p^- > 1$,

$$g'_\theta(s) = p(\theta)|s|^{p(\theta)-2}s.$$

For $|s| \leq \beta$,

$$|g'_\theta(s)| \leq L_{\beta,p},$$

where

$$L_{\beta,p} := p^+ \max \{ \beta^{p^- - 1}, \beta^{p^+ - 1} \}. \quad (5.3)$$

Hence, by the mean value theorem,

$$| |\phi(\theta)|^{p(\theta)} - |\psi(\theta)|^{p(\theta)} | \leq L_{\beta,p} |\phi(\theta) - \psi(\theta)|$$

for almost every $\theta \in I$. Therefore,

$$|\rho_{p(\cdot)}(\phi) - \rho_{p(\cdot)}(\psi)| \leq h_0 L_{\beta,p} \|\phi - \psi\|_X. \quad (5.4)$$

Define

$$q(s) := h_0 \frac{s}{1+s}, \quad s \geq 0.$$

Because

$$|q'(s)| = \frac{h_0}{(1+s)^2} \leq h_0,$$

q is globally Lipschitz on $[0, \infty)$. Using

$$\eta = q \circ \rho_{p(\cdot)},$$

we obtain

$$|\eta(\phi) - \eta(\psi)| \leq h_0^2 L_{\beta,p} \|\phi - \psi\|_X. \quad (5.5)$$

Thus, η is locally Lipschitz on bounded subsets of X .

Let now

$$\phi, \psi \in W^{1,\infty}(I; \mathbb{R}), \quad \|\phi\|_X, \|\psi\|_X \leq \beta, \quad \|\phi'\|_{L^\infty(I)} \leq \beta.$$

Then,

$$\begin{aligned} |\phi(-\eta(\phi)) - \psi(-\eta(\psi))| &\leq \|\phi'\|_{L^\infty(I)} |\eta(\phi) - \eta(\psi)| + \|\phi - \psi\|_X \\ &\leq (1 + \beta h_0^2 L_{\beta,p}) \|\phi - \psi\|_X. \end{aligned}$$

Hence,

$$\psi \mapsto \psi(-\eta(\psi))$$

is locally Lipschitz on bounded subsets of $W^{1,\infty}(I; \mathbb{R})$ with respect to the X -norm.

Consequently, for almost every $t \in [0, T]$,

$$|G(t, \phi) - G(t, \psi)| \leq C_\beta \|\phi - \psi\|_X,$$

where

$$C_\beta := L_{f,\beta} (2 + \beta h_0^2 L_{\beta,p}). \quad (5.6)$$

Assume further that, for every $\beta > 0$, there exists $M_{f,\beta} > 0$ such that

$$|f(t, x, z)| \leq M_{f,\beta}$$

for almost every $t \in [0, T]$ whenever

$$|x|, |z| \leq \beta.$$

Then, if $\|\psi\|_X \leq \beta$,

$$|G(t, \psi)| \leq M_{f, \beta}$$

for almost every $t \in [0, T]$. Thus, G satisfies the local Lipschitz and local boundedness assumptions required in Corollary 4.1.

Corollary 5.1. *Assume that f satisfies the local Lipschitz and local boundedness conditions above, and let*

$$\phi \in W^{1, \infty}(I; \mathbb{R}).$$

Then, there exists $\tau > 0$ such that (5.2) admits a unique solution,

$$y \in W^{1, \infty}(-h_0, \tau; \mathbb{R}).$$

Moreover, for every $t \in [0, \tau]$,

$$y_t \in W^{1, p(\cdot)}(I; \mathbb{R}) \subset L^{p(\cdot)}(I; \mathbb{R}), \quad \eta(y_t) \in [0, h_0].$$

Thus, the modular $\rho_{p(\cdot)}$ defines a state-dependent delay compatible with the regularity obtained above.

5.1. Incremental contraction for the modular delay equation

We derive a sufficient condition for exponential contraction between two solutions of a vector-valued modular delay equation. The dependence on the variable exponent enters through the local Lipschitz estimate (5.5) for the delay functional.

Let $Y = \mathbb{R}^n$, endowed with the Euclidean norm $\|\cdot\|$, and consider

$$\begin{cases} y'(t) = -Dy(t) + By(t - \eta(y_t)), & t > 0, \\ y(\theta) = \phi(\theta), & -h_0 \leq \theta \leq 0, \end{cases} \quad (5.7)$$

where

$$D, B \in \mathbb{R}^{n \times n}, \quad \phi \in W^{1, \infty}(I; Y),$$

and η is the modular delay functional defined by (5.1), with the Euclidean norm used in the vector-valued modular.

We assume the following:

(H1) The symmetric part of D is positive definite, and

$$\alpha := \lambda_{\min} \left(\frac{D + D^T}{2} \right) > 0.$$

(H2) The norm $\|B\|$ denotes the matrix norm induced by the Euclidean norm on Y .

Let $T > 0$, and let y and z be two solutions of (5.7) on $[-h_0, T]$, corresponding to initial functions

$$\phi, \psi \in W^{1, \infty}(I; Y),$$

respectively, in the solution class established above. By Theorem 3.4,

$$y, z \in W^{1,\infty}(-h_0, T; Y).$$

For these two solutions, define

$$\beta_T := \max \left\{ \sup_{0 \leq t \leq T} \|y_t\|_X, \sup_{0 \leq t \leq T} \|z_t\|_X, \sup_{0 \leq t \leq T} \|\partial_\theta y_t\|_{L^\infty(I; Y)}, \sup_{0 \leq t \leq T} \|\partial_\theta z_t\|_{L^\infty(I; Y)} \right\}. \quad (5.8)$$

The preceding $W^{1,\infty}$ -regularity implies that all four quantities in (5.8) are finite. Hence,

$$\beta_T < \infty.$$

Consequently,

$$y_t, z_t \in \mathcal{B}_{\beta_T}, \quad 0 \leq t \leq T,$$

where

$$\mathcal{B}_{\beta_T} := \left\{ \varphi \in W^{1,\infty}(I; Y) : \|\varphi\|_X \leq \beta_T, \|\partial_\theta \varphi\|_{L^\infty(I; Y)} \leq \beta_T \right\}.$$

Therefore, applying (5.5) with $\beta = \beta_T$, we obtain, for $\varphi, \psi \in \mathcal{B}_{\beta_T}$,

$$|\eta(\varphi) - \eta(\psi)| \leq h_0^2 L_{\beta_T, p} \|\varphi - \psi\|_X, \quad (5.9)$$

where

$$L_{\beta_T, p} := p^+ \max \left\{ \beta_T^{p^- - 1}, \beta_T^{p^+ - 1} \right\}. \quad (5.10)$$

Thus, no separate boundedness or regularity hypothesis on the solution segments is required on the fixed interval $[0, T]$. Once the two solutions exist on $[-h_0, T]$ in the solution class established above, their $W^{1,\infty}$ -regularity yields the finite constant β_T . The variable exponent enters the incremental contraction estimate through the local Lipschitz constant $L_{\beta_T, p}$.

Theorem 5.2 (Conditional incremental contraction). *Assume (H1)–(H2). Let $T > 0$, and let y and z be two solutions of (5.7) on $[-h_0, T]$, corresponding to initial functions*

$$\phi, \psi \in W^{1,\infty}(I; Y),$$

respectively, in the solution class established above. Let β_T be defined by (5.8). If

$$\alpha > \|B\| \left(1 + \beta_T h_0^2 L_{\beta_T, p} \right), \quad (5.11)$$

then there exists a unique $\omega > 0$ satisfying

$$\omega = \alpha - \|B\| \left(1 + \beta_T h_0^2 L_{\beta_T, p} \right) e^{\omega h_0}, \quad (5.12)$$

and

$$|y(t) - z(t)| \leq e^{-\omega t} \|\phi - \psi\|_X, \quad 0 \leq t \leq T. \quad (5.13)$$

Proof. Set

$$w(t) := y(t) - z(t).$$

Subtracting the equations satisfied by y and z , we obtain

$$w'(t) = -Dw(t) + B[y(t - \eta(y_t)) - z(t - \eta(z_t))]. \quad (5.14)$$

We first estimate the delayed term. Adding and subtracting $y(t - \eta(z_t))$, we obtain

$$\begin{aligned} & |y(t - \eta(y_t)) - z(t - \eta(z_t))| \\ & \leq |y(t - \eta(y_t)) - y(t - \eta(z_t))| + |w(t - \eta(z_t))|. \end{aligned}$$

By the definition of β_T in (5.8),

$$\|\partial_\theta y_t\|_{L^\infty(I; Y)} \leq \beta_T, \quad 0 \leq t \leq T.$$

Moreover, by the regularity of the solutions established above,

$$y_t \in W^{1,\infty}(I; Y), \quad 0 \leq t \leq T.$$

Hence, y_t admits a Lipschitz continuous representative and

$$\partial_\theta y_t(\theta) = y'(t + \theta)$$

for almost every $\theta \in I$. Therefore,

$$\begin{aligned} & |y(t - \eta(y_t)) - y(t - \eta(z_t))| \\ & = |y_t(-\eta(y_t)) - y_t(-\eta(z_t))| \\ & \leq \|\partial_\theta y_t\|_{L^\infty(I; Y)} |\eta(y_t) - \eta(z_t)| \\ & \leq \beta_T h_0^2 L_{\beta_T, p} \|y_t - z_t\|_X, \end{aligned}$$

where in the last step, we used (5.9).

Moreover, because

$$\eta(z_t) \in [0, h_0],$$

we have

$$|w(t - \eta(z_t))| \leq \|w_t\|_X.$$

Because

$$y_t - z_t = w_t,$$

we conclude that

$$|y(t - \eta(y_t)) - z(t - \eta(z_t))| \leq (1 + \beta_T h_0^2 L_{\beta_T, p}) \|w_t\|_X. \quad (5.15)$$

Define

$$q_T := \|B\| (1 + \beta_T h_0^2 L_{\beta_T, p}). \quad (5.16)$$

By (H2), $\|B\|$ is the matrix norm induced by the Euclidean norm on Y . Hence,

$$|B\xi| \leq \|B\| |\xi|, \quad \xi \in Y.$$

Applying this estimate to (5.15), we obtain

$$\left| B[y(t - \eta(y_t)) - z(t - \eta(z_t))] \right| \leq q_T \|w_t\|_X. \quad (5.17)$$

We next estimate the contribution of D . Decompose D into its symmetric and skew-symmetric parts:

$$D = \frac{D + D^\top}{2} + \frac{D - D^\top}{2}.$$

Because

$$\left(\frac{D - D^\top}{2} \right)^\top = -\frac{D - D^\top}{2},$$

the matrix $(D - D^\top)/2$ is skew-symmetric. Hence, for every $w \in Y = \mathbb{R}^n$,

$$w^\top \frac{D - D^\top}{2} w = 0.$$

Therefore,

$$\begin{aligned} \langle Dw, w \rangle &= \left\langle \frac{D + D^\top}{2} w, w \right\rangle + \left\langle \frac{D - D^\top}{2} w, w \right\rangle \\ &= \left\langle \frac{D + D^\top}{2} w, w \right\rangle. \end{aligned}$$

By (H1),

$$\alpha = \lambda_{\min} \left(\frac{D + D^\top}{2} \right) > 0.$$

Consequently,

$$\langle Dw, w \rangle \geq \alpha |w|^2, \quad w \in Y. \quad (5.18)$$

Set

$$v(t) := |w(t)|, \quad -h_0 \leq t \leq T.$$

Because w is absolutely continuous, v is also absolutely continuous. For almost every $t \in (0, T)$ such that $w(t) \neq 0$, the chain rule for the Euclidean norm gives

$$v'(t) = \left\langle \frac{w(t)}{|w(t)|}, w'(t) \right\rangle.$$

Using (5.14), (5.18), and (5.17), we obtain

$$\begin{aligned} v'(t) &= -\frac{\langle Dw(t), w(t) \rangle}{|w(t)|} \\ &\quad + \left\langle \frac{w(t)}{|w(t)|}, B[y(t - \eta(y_t)) - z(t - \eta(z_t))] \right\rangle \\ &\leq -\alpha |w(t)| + q_T \|w_t\|_X \\ &= -\alpha v(t) + q_T \|w_t\|_X. \end{aligned}$$

Because w is absolutely continuous, one has

$$w'(t) = 0$$

for almost every t in the level set

$$\{t \in (0, T) : w(t) = 0\}.$$

Consequently,

$$v'(t) = 0$$

for almost every such t , and the same differential inequality holds for almost every $t \in (0, T)$. Thus,

$$v'(t) \leq -\alpha v(t) + q_T \|w_t\|_X. \quad (5.19)$$

Because

$$\|w_t\|_X = \sup_{-h_0 \leq \theta \leq 0} |w(t + \theta)| = \sup_{t-h_0 \leq s \leq t} v(s),$$

(5.19) becomes

$$v'(t) \leq -\alpha v(t) + q_T \sup_{t-h_0 \leq s \leq t} v(s) \quad (5.20)$$

for almost every $t \in (0, T)$.

By (5.11) and (5.16),

$$\alpha > q_T.$$

Hence, Halanay's inequality [21, p. 378] yields a unique $\omega > 0$ satisfying

$$\omega = \alpha - q_T e^{\omega h_0},$$

that is,

$$\omega = \alpha - \|B\| \left(1 + \beta_T h_0^2 L_{\beta_T, p}\right) e^{\omega h_0},$$

which is precisely (5.12). Moreover,

$$v(t) \leq e^{-\omega t} \sup_{-h_0 \leq s \leq 0} v(s), \quad 0 \leq t \leq T.$$

Finally,

$$\sup_{-h_0 \leq s \leq 0} v(s) = \|\phi - \psi\|_X,$$

and therefore,

$$|y(t) - z(t)| \leq e^{-\omega t} \|\phi - \psi\|_X, \quad 0 \leq t \leq T.$$

This proves (5.13). □

Remark 5.3. Theorem 5.2 is a finite-interval conditional incremental contraction result. For solutions in the class established above,

$$y, z \in W^{1, \infty}(-h_0, T; Y),$$

and therefore, the quantity β_T defined by (5.8) is finite.

In general, β_T depends on the two solutions and on the interval $[0, T]$. Hence, the contraction estimate is valid on each fixed interval $[0, T]$ on which the solutions exist, and (5.11) is satisfied. The result does not, by itself, provide a bound uniform in T , and therefore does not imply global exponential contraction on $[0, \infty)$.

Because $\rho_{p(\cdot)}(0) = 0$, one has $\eta(0) = 0$. Hence, $y \equiv 0$ is an equilibrium of (5.7). We next derive a sufficient condition for exponential decay relative to this equilibrium.

Proposition 5.4 (Decay relative to the zero equilibrium). *Assume (H1)–(H2), and let y be a solution of (5.7) on $[-h_0, T]$, corresponding to the initial function $\phi \in W^{1,\infty}(I; Y)$. If*

$$\alpha > \|B\|, \quad (5.21)$$

then

$$|y(t)| \leq e^{-\gamma t} \|\phi\|_X, \quad 0 \leq t \leq T, \quad (5.22)$$

where $\gamma > 0$ is the unique solution of

$$\gamma = \alpha - \|B\|e^{\gamma h_0}. \quad (5.23)$$

Proof. Set

$$v(t) := |y(t)|, \quad -h_0 \leq t \leq T.$$

By the regularity of the solution established above, y is absolutely continuous on $[-h_0, T]$, and hence, v is absolutely continuous.

For almost every $t \in (0, T)$ such that $y(t) \neq 0$, the chain rule for the Euclidean norm gives

$$v'(t) = \left\langle \frac{y(t)}{|y(t)|}, y'(t) \right\rangle.$$

Using (5.7), we obtain

$$\begin{aligned} v'(t) &= -\frac{\langle Dy(t), y(t) \rangle}{|y(t)|} \\ &\quad + \left\langle \frac{y(t)}{|y(t)|}, B y(t - \eta(y_t)) \right\rangle. \end{aligned}$$

By (H1),

$$\langle Dy(t), y(t) \rangle \geq \alpha |y(t)|^2,$$

and by (H2),

$$|B y(t - \eta(y_t))| \leq \|B\| |y(t - \eta(y_t))|.$$

Therefore,

$$v'(t) \leq -\alpha v(t) + \|B\| |y(t - \eta(y_t))|.$$

Because

$$0 \leq \eta(y_t) \leq h_0,$$

we have

$$|y(t - \eta(y_t))| \leq \sup_{t-h_0 \leq s \leq t} |y(s)| = \sup_{t-h_0 \leq s \leq t} v(s).$$

Hence,

$$v'(t) \leq -\alpha v(t) + \|B\| \sup_{t-h_0 \leq s \leq t} v(s) \quad (5.24)$$

for almost every $t \in (0, T)$.

As in the proof of Theorem 5.2, the same inequality holds for almost every t such that $y(t) = 0$. Therefore, (5.24) holds for almost every $t \in (0, T)$.

Because

$$\alpha > \|B\|,$$

Halanay's inequality [21, p. 378] yields a unique $\gamma > 0$ satisfying

$$\gamma = \alpha - \|B\|e^{\gamma h_0},$$

together with

$$v(t) \leq e^{-\gamma t} \sup_{-h_0 \leq s \leq 0} v(s), \quad 0 \leq t \leq T.$$

Finally,

$$\sup_{-h_0 \leq s \leq 0} v(s) = \|\phi\|_X,$$

and therefore,

$$|y(t)| \leq e^{-\gamma t} \|\phi\|_X, \quad 0 \leq t \leq T.$$

This proves (5.22). □

Remark 5.5. The conditions (5.11) and (5.21) concern different estimates.

The condition

$$\alpha > \|B\| \left(1 + \beta_T h_0^2 L_{\beta_T, p}\right)$$

yields exponential contraction between two solutions. It therefore requires control of the variation of the state-dependent delay and depends on the solution-dependent quantity β_T and on the local Lipschitz constant $L_{\beta_T, p}$.

For decay relative to the zero equilibrium, no comparison between two different delays is required. It is sufficient to use

$$0 \leq \eta(y_t) \leq h_0,$$

which leads to the simpler condition

$$\alpha > \|B\|.$$

Thus, the zero-equilibrium decay condition does not involve $L_{\beta_T, p}$ and is independent of the variable exponent $p(\cdot)$. By contrast, the incremental contraction condition depends on $p(\cdot)$ through $L_{\beta_T, p}$.

5.2. Numerical illustration of Halanay decay and incremental contraction

We illustrate numerically the sufficient decay condition (5.21) and the finite-interval incremental contraction condition (5.11), together with the dependence of the modular delay on $p(\cdot)$. The computations are interpreted only with respect to these sufficient conditions; failure of either condition is not regarded as a converse statement.

Throughout, we set

$$h_0 = 1, \quad B = \frac{1}{2}I_2, \quad \|B\| = \frac{1}{2},$$

and consider

$$D_\alpha = \begin{pmatrix} \alpha & -3 \\ 3 & \alpha \end{pmatrix}.$$

Because

$$\frac{D_\alpha + D_\alpha^\top}{2} = \alpha I_2,$$

the constant in (H1) is precisely α .

For the numerical comparison, let y and z denote the two solutions under consideration on the fixed interval $[-h_0, T]$, and let β_T be the corresponding quantity defined in (5.8). Define the two margins

$$H := \alpha - \|B\|, \quad I_T := \alpha - \|B\| \left(1 + \beta_T h_0^2 L_{\beta_T, p}\right). \quad (5.25)$$

Here,

$$L_{\beta_T, p} = p^+ \max \left\{ \beta_T^{p^- - 1}, \beta_T^{p^+ - 1} \right\}.$$

Thus,

$$H > 0$$

is equivalent to the sufficient condition (5.21) for exponential decay relative to the zero equilibrium. This margin depends only on α and $\|B\|$ and is independent of the variable exponent $p(\cdot)$.

On the other hand,

$$I_T > 0$$

is equivalent to the sufficient condition (5.11) for exponential contraction between the particular pair of solutions y and z on the fixed interval $[0, T]$. In contrast to H , the margin I_T depends on the trajectories through β_T and on the variable exponent through $L_{\beta_T, p}$.

Accordingly, I_T is not a parameter-only stability margin: It is a finite-interval, solution-dependent sufficient contraction margin. The numerical values reported below are therefore computed from the corresponding trajectories on the prescribed simulation interval $[0, T]$.

5.2.1. Effect of the variable exponent on the incremental condition

We fix

$$D = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}, \quad \alpha = 2, \quad \|B\| = \frac{1}{2}.$$

Let $p^- = 1.5$, and $p^+ \in \{2, 3, 4\}$. For each value of p^+ , we take

$$p(\theta) = 1.5 + (p^+ - 1.5)(\theta + 1), \quad -1 \leq \theta \leq 0, \quad (5.26)$$

so that

$$p(-1) = 1.5, \quad p(0) = p^+.$$

Thus, the three computations differ only through the exponent entering the modular delay.

The initial segment is

$$\phi(\theta) = \begin{pmatrix} 0.2125 + 0.0200 \cos(\pi(\theta + 1)) \\ 0.0500 + 0.0200 \sin(2\pi(\theta + 1)) \end{pmatrix}, \quad -1 \leq \theta \leq 0. \quad (5.27)$$

To evaluate the finite-interval bound entering Theorem 5.2, we reconstruct the numerical solution segments and compute their X -norms and state-segment derivatives. For the trajectory y_N , the corresponding maxima are reported in Table 2.

Table 2. Finite-interval numerical bounds for the reconstructed trajectory y_N .

p^+	$\max_t \ y_{N,t}\ _\infty$	$\max_t \ \partial_\theta y_{N,t}\ _\infty$
2	0.23972	0.62159
3	0.23972	0.62144
4	0.23972	0.62272

Thus, for this trajectory,

$$\sup_{0 \leq t \leq T} \|y_{N,t}\|_\infty < \infty, \quad \sup_{0 \leq t \leq T} \|\partial_\theta y_{N,t}\|_\infty < \infty,$$

in agreement with the finite-interval regularity used in the definition of β_T .

For the incremental contraction estimate, however, β_T must be computed from both numerical trajectories y_N and z_N . More precisely,

$$\beta_T = \max \left\{ \max_t \|y_{N,t}\|_\infty, \max_t \|z_{N,t}\|_\infty, \max_t \|\partial_\theta y_{N,t}\|_\infty, \max_t \|\partial_\theta z_{N,t}\|_\infty \right\}. \quad (5.28)$$

Once β_T is obtained from the two reconstructed trajectories, we compute

$$L_{\beta_T, p} = p^+ \max \left\{ \beta_T^{p^- - 1}, \beta_T^{p^+ - 1} \right\},$$

and the finite-interval incremental contraction margin

$$\delta_{\text{inc}, T}(p^+) := 2 - \frac{1}{2} \left(1 + \beta_T L_{\beta_T, p} \right). \quad (5.29)$$

Here, we use $h_0 = 1$.

Thus,

$$\delta_{\text{inc}, T}(p^+) > 0$$

is equivalent to the sufficient incremental contraction condition (5.11) for the corresponding pair of numerical trajectories on $[0, T]$. Because β_T is trajectory-dependent, the incremental margin is no longer determined by p^+ alone.

For all three values of p^+ ,

$$H = \alpha - \|B\| = 1.5 > 0.$$

Therefore, the sufficient zero-equilibrium decay condition (5.21) holds in every case. This condition is independent of β_T and of $p(\cdot)$, whereas the stronger incremental contraction condition depends on both the computed trajectory bound β_T and the variable exponent through $L_{\beta_T, p}$.

As before, failure of

$$\delta_{\text{inc}, T}(p^+) > 0$$

does not imply failure of incremental contraction, as (5.11) is only a sufficient condition.

5.2.2. Ito–Kappel semidiscretization and numerical implementation

The numerical approximation is based on the Ito–Kappel semidiscretization of the delay phase space [13]. For $N = 120$, the discrete state belongs to $\mathbb{R}^{2(N+1)}$, and the corresponding continuous state segment is reconstructed by the first-order spline operator E_N . With the grid

$$\theta_j = -\frac{jh_0}{N}, \quad j = 0, \dots, N,$$

the discrete approximation of the shift generator is

$$A_N \xi = \left(0, \frac{N}{h_0}(\xi_0 - \xi_1), \dots, \frac{N}{h_0}(\xi_{N-1} - \xi_N) \right),$$

and the discrete injection operator is

$$B_N z = (z, 0, \dots, 0).$$

Thus, the first component contains the delayed nonlinear term, and the remaining components approximate translation of the state segment.

The semidiscrete system is integrated with MATLAB `ode15s`, using maximum internal step 10^{-2} , relative tolerance 10^{-7} , and absolute tolerance 10^{-9} . The modular $\rho_{p(\cdot)}(y_t)$ is approximated by the composite trapezoidal rule with 160 quadrature points on $[-1, 0]$. The delayed value is evaluated from the same piecewise-linear reconstruction $E_N \Phi_N(t)$ at $-\eta(y_t)$.

For the finite-interval incremental contraction diagnostic, the same reconstructed segments are used to approximate the quantities entering β_T . In particular, for each computed trajectory, we evaluate

$$\|y_{N,t}\|_\infty \quad \text{and} \quad \|\partial_\theta y_{N,t}\|_\infty$$

over the numerical time grid. When two trajectories y_N and z_N are compared, the numerical counterpart of (5.8) is taken as

$$\beta_{T,N} := \max \left\{ \max_t \|y_{N,t}\|_\infty, \max_t \|z_{N,t}\|_\infty, \max_t \|\partial_\theta y_{N,t}\|_\infty, \max_t \|\partial_\theta z_{N,t}\|_\infty \right\}.$$

The corresponding numerical incremental margin is then evaluated from

$$I_{T,N} = \alpha - \|B\| \left(1 + \beta_{T,N} h_0^2 L_{\beta_{T,N},p} \right),$$

where

$$L_{\beta_{T,N},p} = p^+ \max \left\{ \beta_{T,N}^{p^--1}, \beta_{T,N}^{p^+-1} \right\}.$$

Thus, the numerical contraction diagnostic is computed from the reconstructed trajectories rather than from a prescribed bound β .

The variable exponent enters the semidiscrete problem through the modular delay,

$$p(\cdot) \mapsto \rho_{p(\cdot)}(y_t) \mapsto \eta(y_t) = h_0 \frac{\rho_{p(\cdot)}(y_t)}{1 + \rho_{p(\cdot)}(y_t)}.$$

For the exponent functions in (5.26), the initial delay values are approximately

$$\eta(y_0) = 0.06809, \quad 0.04126, \quad 0.02851$$

for $p^+ = 2, 3, 4$, respectively. Thus, varying $p(\cdot)$ changes the modular and hence the state-dependent delay, even when the qualitative decay behavior remains unchanged.

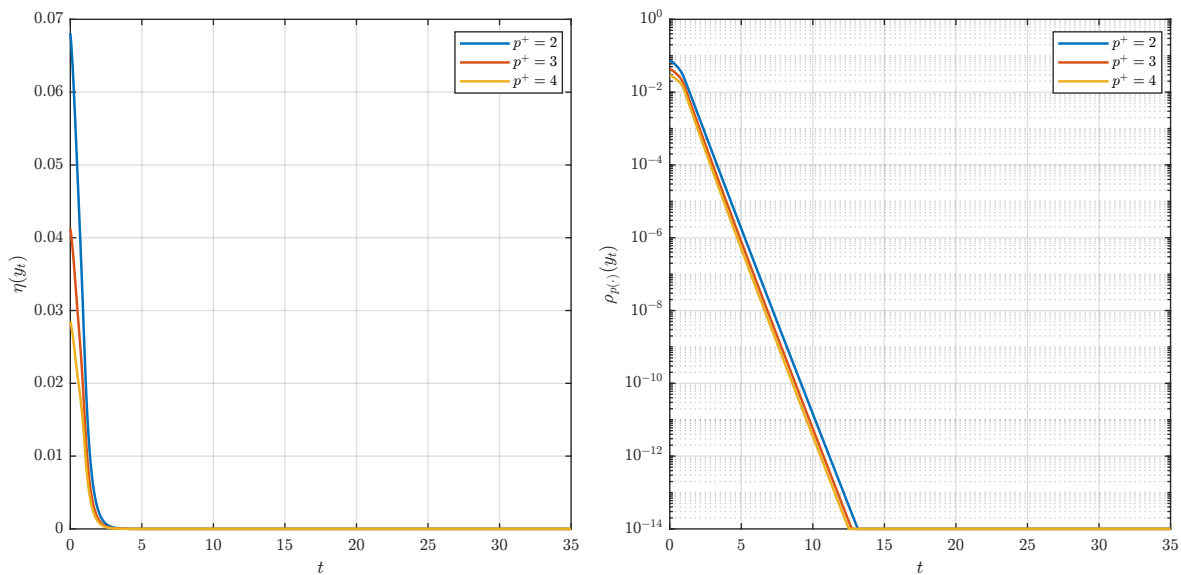


Figure 2. Variable-exponent modular and corresponding state-dependent delay for $p^+ = 2, 3, 4$. The exponent function changes $\rho_{p(\cdot)}(y_t)$ and hence $\eta(y_t)$, whereas the remaining parameters of the semidiscrete system are fixed.

5.2.3. Halanay decay versus incremental contraction

We compare three parameter regimes for the sufficient decay condition (5.21) and the finite-interval incremental contraction condition (5.11). We fix

$$h_0 = 1, \quad B = \frac{1}{2}I_2, \quad p^- = 1.5,$$

and use the same modular delay.

For each case, the zero-equilibrium decay margin is

$$H := \alpha - \|B\|,$$

whereas the numerical finite-interval incremental contraction margin is defined by

$$I_{T,N} := \alpha - \|B\| \left(1 + \beta_{T,N} h_0^2 L_{\beta_{T,N},p} \right),$$

where

$$L_{\beta_{T,N},p} = p^+ \max \{ \beta_{T,N}^{p^- - 1}, \beta_{T,N}^{p^+ - 1} \},$$

and $\beta_{T,N}$ is computed from the two reconstructed numerical trajectories according to the discrete counterpart of (5.8).

For the computations below, $N = 120$, and $T = 35$. In particular,

$$\beta_{T,N} = \max \left\{ \max_t \|y_{N,t}\|_\infty, \max_t \|z_{N,t}\|_\infty, \max_t \|\partial_\theta y_{N,t}\|_\infty, \max_t \|\partial_\theta z_{N,t}\|_\infty \right\}.$$

The corresponding parameter values and numerical margins are listed in Table 3.

Table 3. Zero-equilibrium decay margin $H = \alpha - \|B\|$ and finite-interval incremental-contraction margin $I_{T,N}$, computed with $N = 120$ on $[0, 35]$. Positive values correspond to the respective sufficient conditions.

Case	α	p^+	H	$\beta_{T,N}$	$I_{T,N}$	Sufficient condition(s)
I	2.0	2	1.50	0.62159	1.00993	decay and incremental contraction
II	0.7	4	0.20	0.58379	-0.69210	decay only
III	0.3	4	-0.20	2.10790	-39.68444	neither

In Case I,

$$H = 1.50 > 0, \quad I_{T,N} = 1.00993 > 0.$$

Thus, both sufficient conditions hold. The computed trajectory decays toward the zero equilibrium, and the distance between the two numerical solutions decreases, consistently with Proposition 5.4 and Theorem 5.2.

In Case II,

$$H = 0.20 > 0, \quad I_{T,N} = -0.69210 < 0.$$

Thus, the sufficient zero-equilibrium decay condition holds, whereas the sufficient incremental contraction condition is not satisfied. For the selected initial segments, the numerical distance between the two solutions nevertheless decreases. This does not contradict Theorem 5.2, since the incremental contraction condition is only sufficient.

In Case III,

$$H = -0.20 < 0, \quad I_{T,N} = -39.68444 < 0.$$

Thus, neither sufficient condition applies. In this case,

$$\beta_{T,N} = 2.10790,$$

which is substantially larger than in Cases I and II and produces the large negative value of the incremental margin. Numerically, the state amplitude increases over the simulated interval, and the distance between the two selected trajectories initially increases. These observations are specific to the chosen parameters and are not inferred from the failure of the sufficient inequalities.

Figure 3 shows the three regimes. The upper panels display the state amplitude, and the lower panels display the distance between two solutions generated from distinct initial segments.

The corresponding phase portraits are shown in Figure 4. In Case I, the trajectory approaches the origin rapidly; in Case II, it exhibits an inward spiral; and in Case III, it moves away from the origin over the reported interval.

For Case III, this behavior is also consistent with the linearization at the zero equilibrium. Because

$$\eta(y_t) \longrightarrow 0 \quad \text{as} \quad y_t \longrightarrow 0,$$

the linearized equation at the origin is governed by

$$-D_\alpha + B = \begin{pmatrix} -\alpha + \frac{1}{2} & 3 \\ -3 & -\alpha + \frac{1}{2} \end{pmatrix}.$$

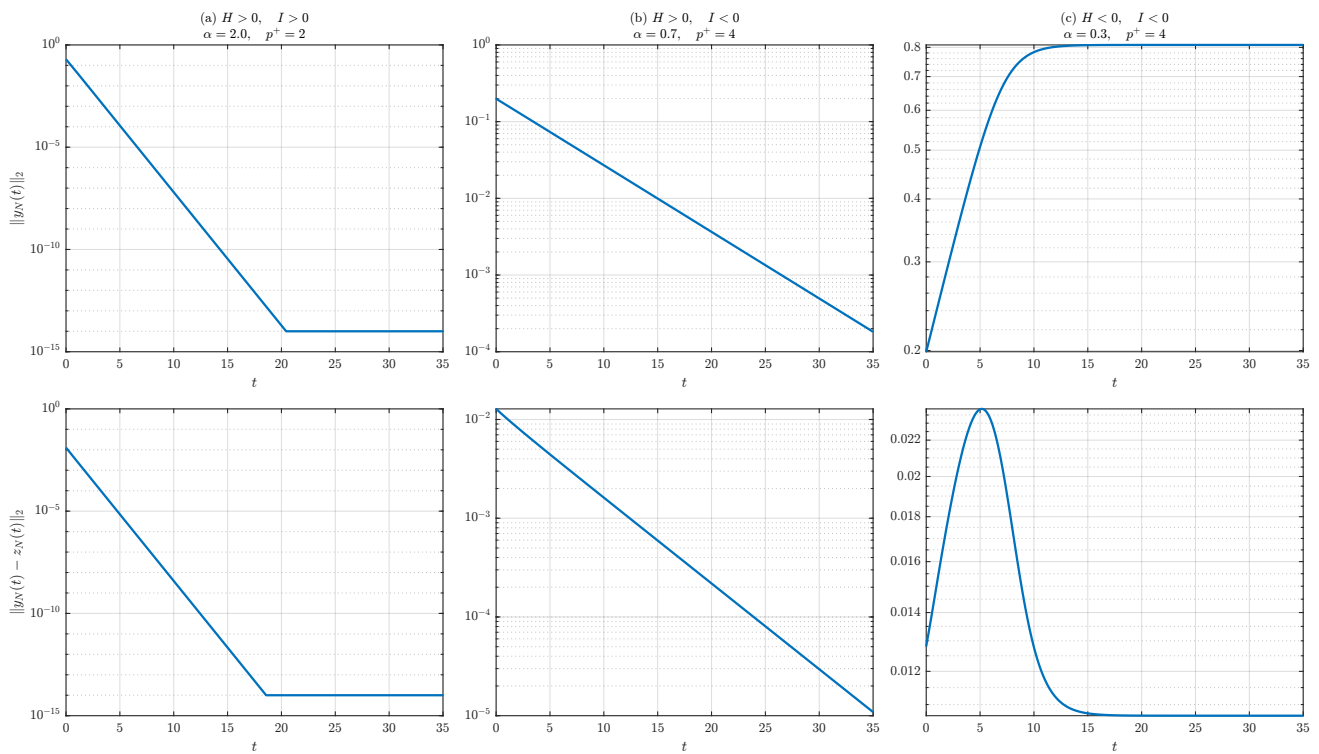


Figure 3. Comparison of the zero-equilibrium decay and finite-interval incremental contraction conditions. Left: Both sufficient conditions hold ($H = 1.50$, $I_{T,N} = 1.00993$). Middle: Only the zero-equilibrium decay condition holds ($H = 0.20$, $I_{T,N} = -0.69210$). Right: Neither sufficient condition holds ($H = -0.20$, $I_{T,N} = -39.68444$). The upper panels show the state amplitude and the lower panels the distance between two numerical solutions.

For $\alpha = 0.3$, the eigenvalues are

$$\lambda_{1,2} = 0.2 \pm 3i,$$

and hence,

$$\operatorname{Re} \lambda_{1,2} = 0.2 > 0.$$

Thus, the zero equilibrium is linearly unstable for this parameter choice, consistent with the observed growth of the solution away from the origin. This conclusion is independent of the failure of the sufficient conditions $H > 0$ and $I_{T,N} > 0$.

The comparison emphasizes the different roles of the two estimates. Proposition 5.4 concerns decay relative to the zero equilibrium and depends only on the parameter margin

$$H = \alpha - \|B\|.$$

By contrast, Theorem 5.2 concerns the distance between two solutions on a fixed finite interval and depends on the trajectory-dependent quantity β_T . Its numerical counterpart therefore uses $\beta_{T,N}$ and the finite-interval margin $I_{T,N}$. Failure of the incremental condition alone yields no conclusion about contraction or stability.

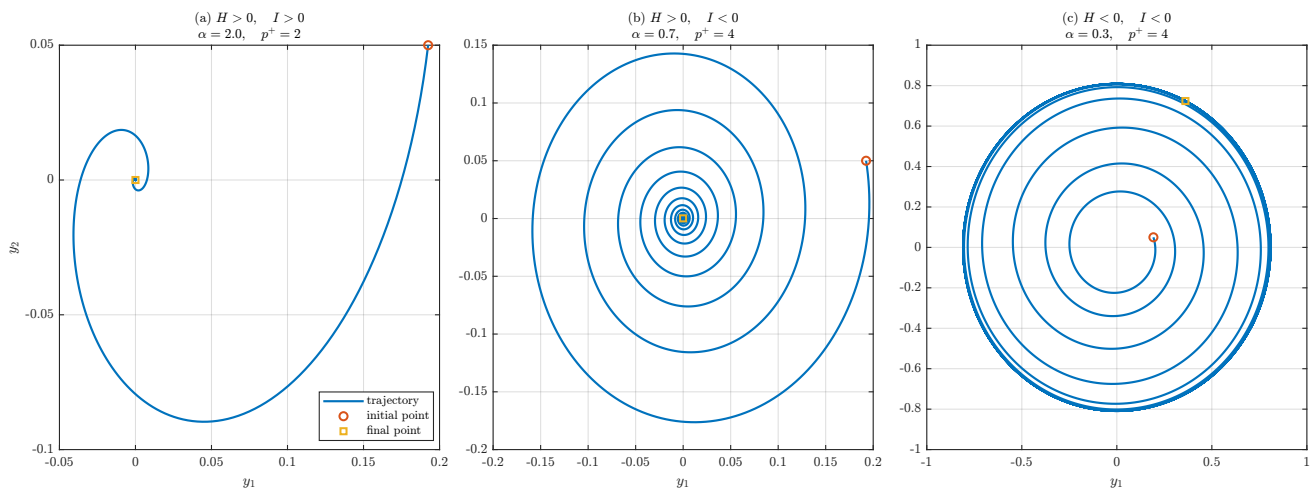


Figure 4. Phase portraits for the three regimes in Table 3. Left: Both sufficient conditions hold. Middle: Only the zero-equilibrium decay condition holds. Right: Neither sufficient condition holds; in this case, the zero equilibrium is also linearly unstable.

5.2.4. Refinement study

We assess the numerical approximation by comparing

$$N = 60, \quad N = 120, \quad N = 240$$

with a reference computation using

$$N_{\text{ref}} = 360.$$

The maximum errors in the state and in the state-dependent delay, computed on the common time grid, are reported in Table 4.

Table 4. Refinement of the Ito–Kappel semidiscretization. Errors are measured relative to the reference computation with $N_{\text{ref}} = 360$.

p^+	N	$\max_t \ y_N(t) - y_{\text{ref}}(t)\ $	$\max_t \eta_N(t) - \eta_{\text{ref}}(t) $
2	60	1.3711×10^{-4}	1.5088×10^{-3}
2	120	5.7459×10^{-5}	6.7351×10^{-4}
2	240	1.5280×10^{-5}	1.8179×10^{-4}
3	60	9.8663×10^{-5}	1.1773×10^{-3}
3	120	3.7523×10^{-5}	5.4086×10^{-4}
3	240	9.3812×10^{-6}	1.5033×10^{-4}
4	60	7.1774×10^{-5}	9.6986×10^{-4}
4	120	2.7715×10^{-5}	4.5635×10^{-4}
4	240	7.0124×10^{-6}	1.2857×10^{-4}

For each value of p^+ , both error measures decrease under refinement. This provides a numerical consistency check for the Ito–Kappel reconstruction and for the evaluation of the variable-exponent modular delay.

The refinement study supports the numerical resolution used in the preceding experiments. In particular, it shows that both the computed state trajectory and the associated state-dependent delay stabilize under refinement of the semidiscrete state-segment representation. Together with the preceding computations, these results illustrate the role of the variable exponent through the modular delay.

The two analytical estimates retain their distinct interpretations. Proposition 5.4 provides a sufficient condition for decay relative to the zero equilibrium, whereas Theorem 5.2 provides a finite-interval sufficient condition for incremental contraction between two solutions through the trajectory-dependent quantity β_T . In the numerical implementation, the corresponding finite-dimensional quantity $\beta_{T,N}$ is computed from the reconstructed solution segments and is used only in the incremental contraction diagnostic.

6. Discussion

We considered state-dependent delay equations with the modular delay (5.1). Because translations need not be bounded on $L^{p(\cdot)}(I; \mathbb{R}^n)$, the delay C_0 -semigroup is realized on the auxiliary phase space H_- , following the Ito–Kappel construction, and the variable-exponent dependence is retained through the modular delay.

The variable-exponent contribution consists in establishing the well-definedness and local Lipschitz continuity of the modular delay on the admissible phase space. These properties permit the use of the Ito–Kappel theory and yield local well-posedness. Under Assumption 4.2, well-posedness holds on finite intervals.

The qualitative results give two distinct sufficient conditions. Proposition 5.4 concerns decay relative to the zero equilibrium under (5.21), whereas Theorem 5.2 gives the finite-interval conditional incremental contraction under (5.11). The latter involves the trajectory-dependent quantity β_T and the local Lipschitz constant $L_{\beta_T,p}$, because variations of the state-dependent delay must be estimated along the two solution trajectories. For every fixed interval $[0, T]$ on which the two solutions exist in the solution class established above, their $W^{1,\infty}$ -regularity implies that $\beta_T < \infty$. Thus, no separate boundedness assumption or positive-invariance hypothesis for a prescribed set $\mathcal{B}_\beta$ is required.

The resulting incremental contraction condition is nevertheless conditional in a different sense: The quantity β_T generally depends on the particular pair of solutions and on the interval $[0, T]$. Therefore, Theorem 5.2 does not by itself provide a bound that is uniform for all $T > 0$ and hence does not establish global exponential contraction on $[0, \infty)$. A global result of that type would require an additional uniform a priori bound or another argument providing control of β_T independently of T .

By contrast, the sufficient condition for decay relative to the zero equilibrium is

$$\alpha > \|B\|$$

and does not require an estimate of the variation of the delay functional. It therefore does not involve $L_{\beta_T,p}$ and is independent of the variable exponent $p(\cdot)$. This distinction explains why the zero-equilibrium decay condition may hold even when the stronger incremental contraction condition is not satisfied.

The numerical experiments reflect this distinction. The zero-equilibrium margin

$$H = \alpha - \|B\|$$

is determined entirely by the system matrices, whereas the numerical incremental margin

$$I_{T,N} = \alpha - \|B\| \left(1 + \beta_{T,N} h_0^2 L_{\beta_{T,N},p} \right)$$

is evaluated on a fixed computational interval using the reconstructed pair of trajectories. Consequently, the numerical incremental diagnostic is solution dependent, in agreement with the analytical formulation of Theorem 5.2.

For the three numerical regimes considered above, the computed values

$$I_{T,N} = 1.00993, \quad I_{T,N} = -0.69210, \quad I_{T,N} = -39.68444$$

correspond, respectively, to a case in which both sufficient conditions hold, a case in which only the zero-equilibrium decay condition holds, and a case in which neither sufficient condition holds. These numerical values also illustrate the effect of the trajectory-dependent quantity $\beta_{T,N}$, particularly in the third regime, where the larger finite-interval bound produces a substantially more negative incremental margin. Failure of the sufficient incremental condition, however, gives no converse conclusion concerning contraction.

The analysis is carried out under the bounded-exponent assumption (2.3). If $p^+ = \infty$, the Δ_2 -condition (2.7) may fail, and the usual relation between modular and norm convergence is no longer available in the same form. The unbounded-exponent case therefore requires a separate modular analysis.

Possible extensions include distributed and fractional delays, integrated and regularized semigroup formulations, variable-exponent Sobolev delay operators, and approximation of modular delay equations. Further directions include linearization, spectral analysis, positivity, bifurcation, asymptotic behavior, and the derivation of global a priori bounds that could lead to time-uniform incremental contraction criteria.

Use of AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the preparation of this manuscript.

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Conflict of interest

The author declares that he has no competing interests.

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