



Research article

Duality of uncomplemented null-space and non-closed range, and its impact on the regularization of Mazur-type operators

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Abstract: Up to now, linear operators with uncomplemented null-space have been playing only a minor role in regularization theory for ill-posed inverse problems in Banach spaces. We show that ill-posedness due to an uncomplemented null-space and ill-posedness due to a non-closed range can be regarded as dual concepts, and we may switch between both views at will by restricting or extending the ill-posed operator under consideration in a suitable well-defined way. We apply the duality result to Mazur-type operators, which represent a class of bounded linear operators with uncomplemented null-space. Mazur-type operators in their natural form are not accessible to Tikhonov-type regularization, but after transferring ill-posedness to the range, Tikhonov regularization can be applied. On the other hand, the null-space view of ill-posedness may open up an alternative path for developing new regularization methods for operators ill-posed in the classical sense, that is, for operators ill-posed due to a non-closed range.

Keywords: linear ill-posed problems; non-closed range; uncomplemented null-space; regularization; Mazur-type operators

1. Motivation and preliminaries

We are concerned with linear operator equations

$$Ax = y, \tag{1.1}$$

where $A : X \rightarrow Y$ is a bounded linear operator mapping between Banach spaces X and Y . Here, $y \in Y$ is given and $x \in X$ is being sought. Of particular interest are operators A lacking a continuous (generalized) inverse. To solve such *ill-posed* operator equations, some kind of regularization

technique has to be applied, cf., e.g., [1, 2]. The following definition plays an important role in discussing invertibility of A .

Definition 1. A closed subspace $U \subseteq X$ of a Banach space X is *complemented* if there exists another closed subspace $V \subseteq X$ such that

$$U \cap V = \{0\} \quad \text{and} \quad U + V = X. \quad (1.2)$$

Otherwise, U is *uncomplemented*.

Note that, for each closed subspace $U \subseteq X$, there is always another (possibly non-closed) subspace $V \subseteq X$ such that (1.2) holds true. The difficulty lies in finding a closed subspace V . The Lindenstrauss-Tzaferi Theorem [3] states that a Banach space contains uncomplemented subspaces if and only if it is not isomorphic to a Hilbert space.

If A is injective, then the inverse $A^{-1} : \mathcal{R}(A) \rightarrow X$ is well-defined on the range $\mathcal{R}(A)$ of A and the inverse is bounded if and only if $\mathcal{R}(A)$ is a closed subset of Y . In case of closed range, the inverse may be extended to a bounded linear *generalized inverse* $A^\dagger : Y \rightarrow X$ if the range $\mathcal{R}(A)$ is complemented. But we do not consider such generalizations here; instead, we refer the reader to the original work [4]. Note that the closure of $\mathcal{R}(A)$ is a Banach space on its own. So without loss of generality, we may assume that either $\mathcal{R}(A)$ is dense in Y or $\mathcal{R}(A) = Y$, if Y is chosen suitably.

If A is not injective, that is, A has a non-trivial null-space $\mathcal{N}(A) \neq \{0\}$, we may take some (possibly non-closed) subspace U such that (1.2) holds with $V := \mathcal{N}(A)$. Then the restriction $A|_U : U \rightarrow Y$ is injective and has range $\mathcal{R}(A|_U) = \mathcal{R}(A)$. Thus, the discussion above applies to $A|_U$. But it turns out that even for a closed range $\mathcal{R}(A)$, the corresponding inverse $(A|_U)^{-1}$ may be unbounded. In [5, Prop. 1.11], it has been shown that $(A|_U)^{-1}$ is bounded if and only if U is a complement of $\mathcal{N}(A)$, that is, if and only if U is closed. Consequently, non-closed range $\mathcal{R}(A)$ and uncomplemented null-space $\mathcal{N}(A)$ are two sources of ill-posedness that appear independent of one another at first glance.

Operators with uncomplemented null-space have been playing almost no role in the theory of ill-posed inverse problems during the past decades. Results on ill-posedness phenomena and suitable regularization techniques concentrated on the situation of non-closed range, although uncomplemented null-spaces (and also uncomplemented ranges) have already been mentioned in [4, 6]. There, the possible presence of uncomplemented subspaces has been incorporated into the definition of ill-posedness. That definition is discussed and slightly modified in the recent articles [7, 8] in the light of uncomplemented null-spaces and strictly singular operators.

In the present article's main section, Section 2, we show that ill-posedness in the sense of lacking boundedness of (generalized) inverses resulting from a non-closed range is essentially the same as ill-posedness resulting from an uncomplemented null-space. Hence, both sources of ill-posedness are anything but independent, however they can be regarded as concepts dual to each other. Operators with uncomplemented null-space may be transformed into operators with non-closed range and vice versa.

On the one hand, our results provide a path for applying well-known regularization methods to operators with uncomplemented null-space (by transferring ill-posedness to the range). We illustrate this approach in Section 3 with Mazur-type operators. On the other hand, our results may motivate new regularization methods by transferring ill-posedness to the null-space and developing regularization techniques for such situations.

2. Duality of uncomplemented null-space and non-closed range

The following definition was introduced in [9].

Definition 2. A closed subspace $U \subseteq X$ of a Banach space X is *quasi-complemented* if there exists a closed subspace $V \subseteq X$ such that

$$U \cap V = \{0\} \quad \text{and} \quad U + V \text{ is dense in } X. \quad (2.1)$$

In contrast to complementedness, quasi-complementedness is a rather weak property. In [10], it was shown that in a separable Banach space, each closed subspace is quasi-complemented.

The following theorem holds, in particular, for operators B with uncomplemented null-space. Note that assumed quasi-complementedness of the null-space $\mathcal{N}(B)$ is guaranteed if X is separable.

Theorem 3. Let $B : X \rightarrow Y$ be a bounded operator between Banach spaces X and Y and let $U \subseteq X$ be a quasi-complement of the null-space $\mathcal{N}(B)$. Then the restriction $A := B|_U$ is injective and the range $\mathcal{R}(A)$ of A is dense in the range $\mathcal{R}(B)$ of B .

Proof. We have that $\mathcal{N}(B) \cap U = \{0\}$ and $\mathcal{N}(B) + U$ is dense in X . The former implies that A is injective. The latter, for each $x \in X$, yields sequences $(z^{(k)})_{k \in \mathbb{N}}$ in $\mathcal{N}(B)$ and $(u^{(k)})_{k \in \mathbb{N}}$ in U with $z^{(k)} + u^{(k)} \rightarrow x$ for $k \rightarrow \infty$. From $B(z^{(k)} + u^{(k)}) = B u^{(k)} = A u^{(k)}$, we see that $A u^{(k)} \rightarrow Bx$ if $k \rightarrow \infty$, that is, the range of A is dense in the range of B . $\square$

Corollary 4. Let X be separable. Then for each surjective bounded linear operator $B : X \rightarrow Y$, there is a closed subspace $U \subseteq X$ such that the restriction $A := B|_U$ is injective and the range $\mathcal{R}(A)$ is dense in Y .

Proof. The separability of X yields a quasi-complement U of $\mathcal{N}(B)$. Thus, Theorem 3 applies. $\square$

If B in Corollary 4 is ill-posed due to an uncomplemented null-space, then, up to some extent, the corollary yields an equivalent operator A with possibly non-closed range. Thus, ill-posedness due to an uncomplemented null-space is transferred to ill-posedness due to a non-closed range. That A contains all essential information from B can be seen from the fact that $\mathcal{N}(B) + U$ is dense in X , that is, we only “lose” the null-space of B .

Proposition 5. Let B , U , A be as in Corollary 4. The range $\mathcal{R}(A)$ is closed if and only if U is a complement of the null-space $\mathcal{N}(B)$.

Proof. If $\mathcal{R}(A)$ is closed, that is, $\mathcal{R}(A) = Y$, take $u \in U$ such that $A u = Bx$ for some arbitrary $x \in X$. Then $x = (x - u) + u$ shows that U is a complement of $\mathcal{N}(B)$.

If, on the other hand, U is a complement of $\mathcal{N}(B)$, we find $z \in \mathcal{N}(B)$ and $u \in U$ with $x = z + u$ for some arbitrary $x \in X$. Then $A u = B(z + u) = Bx$ shows that $\mathcal{R}(A) = \mathcal{R}(B) = Y$. $\square$

In other words, the proposition states that, in Corollary 4, ill-posedness due to an uncomplemented null-space occurs for B if and only if ill-posedness due to a non-closed range occurs for A .

Remark 6. From Proposition 5, we cannot deduce that a complemented $\mathcal{N}(B)$ implies a closed $\mathcal{R}(A)$ for all quasi-complements U of $\mathcal{N}(B)$. We only get the existence of at least one complement U of $\mathcal{N}(B)$ such that the corresponding operator A has closed range. For other quasi-complements, $\mathcal{R}(A)$

may be non-closed although $\mathcal{N}(B)$ is complemented. On the other hand, if we know that $\mathcal{R}(A)$ is non-closed for a specific choice of U , we cannot deduce that $\mathcal{N}(B)$ is uncomplemented. Perhaps there is another subspace U that is a complement of $\mathcal{N}(B)$ (and corresponding A has closed range). Note that, even in Hilbert spaces, closed subspaces may have quasi-complements that are not complements (see [11, Section I.15] for an example).

The next theorem is the converse result to Corollary 4. The assumptions could be given in simpler form but we keep them quite general to maintain a high degree of symmetry to Corollary 4. For instance, the assumptions are satisfied if $W := Y \times \{0\}$ with $\|(y, 0)\|_W := \|y\|_Y$, if $C : W \rightarrow Y$ is given by $C(y, 0) := y$, and if $\tilde{X} := Y \times X$ with norm $\|(w, u)\|_{\tilde{X}} := \|w\|_W + \|u\|_X$. The proof of the theorem relies on the Bartle-Graves theorem [12, Cor. 5J.4] stated in the following lemma.

Lemma 7. *Let $B : \tilde{X} \rightarrow Y$ be a surjective bounded linear operator between arbitrary Banach spaces $\tilde{X}$ and Y . Then there exists a continuous (and possibly non-linear) mapping $B^+ : Y \rightarrow \tilde{X}$ such that $B B^+(y) = y$ for all $y \in Y$.*

Proof. See [12, Cor. 5J.4]. □

Theorem 8. *Let $A : X \rightarrow Y$ be an injective bounded linear operator between Banach spaces X and Y with dense range $\mathcal{R}(A)$. Further let the Banach space $\tilde{X} = W + U$ be the direct sum of a closed subspace W , for which there exists a surjective bounded linear operator $C : W \rightarrow Y$, and a closed subspace U isomorphic to X with isomorphism $T : X \rightarrow U$. Then there exists a surjective bounded linear operator $B : \tilde{X} \rightarrow Y$ such that $A = B T$ and U is a quasi-complement of $\mathcal{N}(B)$ in $\tilde{X}$.*

Proof. For $w + u \in \tilde{X}$ with $w \in W$ and $u \in U$, set

$$B(w + u) := C w + A T^{-1} u.$$

Then $B : \tilde{X} \rightarrow Y$ is a well-defined surjective linear operator and $A = B T$. For proving boundedness of B , observe that W and U are a pair of complemented subspaces. Thus, the projection mappings $w + u \mapsto w$ and $w + u \mapsto u$ are continuous (cf. [13, Theorem 3.2.14]) and we obtain

$$\begin{aligned} \|B(w + u)\|_Y &\leq \|C\| \|w\|_{\tilde{X}} + \|A T^{-1}\| \|u\|_{\tilde{X}} \\ &\leq \max\{c_1 \|C\|, c_2 \|A T^{-1}\|\} \|w + u\|_{\tilde{X}} \end{aligned}$$

with constants $c_1 > 0$, $c_2 > 0$.

Injectivity of A ensures $\mathcal{N}(B) \cap U = \{0\}$. To show denseness of $\mathcal{N}(B) + U$, take $w + u \in \tilde{X}$ with $w \in W$ and $u \in U$. By the denseness of $\mathcal{R}(A)$ in Y , there exists a sequence $(u^{(m)})_{m \in \mathbb{N}}$ in U with $B u^{(m)} \rightarrow B(w + u)$ if $m \rightarrow \infty$. Set

$$\tilde{x}^{(m)} := w + u - B^+(B(w + u)) - (u^{(m)} - B^+(B u^{(m)})),$$

where $B^+ : Y \rightarrow \tilde{X}$ is the continuous mapping from Lemma 7. Obviously, $\tilde{x}^{(m)} \in \mathcal{N}(B)$. Further,

$$\tilde{x}^{(m)} + u^{(m)} - (w + u) = B^+(B u^{(m)}) - B^+(B(w + u)) \rightarrow 0$$

if $m \rightarrow \infty$ by continuity of B^+ , that is, $\tilde{x}^{(m)} + u^{(m)} \rightarrow w + u$. □

If A in the theorem is ill-posed due to a non-closed range $\mathcal{R}(A) \neq Y$, the theorem yields an extension B with closed range and possibly uncomplemented null-space. Thus, ill-posedness of A due to a non-closed range is transferred to ill-posedness of B due to an uncomplemented null-space. That B does not “add” more than necessary can be seen from the fact that X is isomorphic to a quasi-complement U of $\mathcal{N}(B)$. This also makes it clear that the choice of C in the theorem does not matter. C is an arbitrary surjective operator without any connection to A and only serves as a tool for “filling the gap” between $\mathcal{R}(A)$ and Y by enlarging the null-space.

Proposition 9. *Let A , B , C , and U be as in Theorem 8. The range $\mathcal{R}(A)$ is closed if and only if U is a complement of the null-space $\mathcal{N}(B)$. Further, $\mathcal{N}(B)$ is complemented in $\tilde{X}$ if $\mathcal{N}(C)$ is complemented in W .*

Proof. We may apply Corollary 4, to B and U to obtain an operator $\tilde{A} = B|_U : U \rightarrow Y$. By Proposition 5 its range $\mathcal{R}(\tilde{A})$ is closed if and only if U is a complement of $\mathcal{N}(B)$. From $A = \tilde{A}T$ with the isomorphism T from Theorem 8, we see that $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(\tilde{A})$ is closed.

If $\mathcal{N}(C)$ is complemented in W , we find a closed subspace V in W such that $\mathcal{N}(C) \cap V = \{0\}$ and $\mathcal{N}(C) + V = W$. We show that V also is a complement of $\mathcal{N}(B)$. Let $\tilde{x} \in \mathcal{N}(B) \cap V$. Then $\tilde{x} \in W$ and $C\tilde{x} = B\tilde{x} = 0$, that is, $\tilde{x} \in \mathcal{N}(C) \cap V$, showing $\tilde{x} = 0$. Now take arbitrary $\tilde{x} \in \tilde{X}$. Then there are $w \in W$ and $u \in U$ such that $\tilde{x} = w + u$. Further, there are $z \in \mathcal{N}(C)$ and $v \in V$ such that $w = z + v$. Surjectivity of C guarantees existence of some $\tilde{w} \in W$ with $C\tilde{w} = -AT^{-1}u$, that is, with $\tilde{w} + u \in \mathcal{N}(B)$. For $\tilde{w}$, we find $\tilde{z} \in \mathcal{N}(C)$ and $\tilde{v} \in V$ such that $\tilde{w} = \tilde{z} + \tilde{v}$. Thus, we have

$$\tilde{x} = z + v + u = \tilde{w} + u + z - \tilde{z} + v - \tilde{v}$$

with $\tilde{w} + u + z - \tilde{z} \in \mathcal{N}(B)$ (note that $\mathcal{N}(C) \subseteq \mathcal{N}(B)$) and $v - \tilde{v} \in V$. This shows $\tilde{X} = \mathcal{N}(B) + V$. $\square$

Analogously to Proposition 5, Proposition 9 states that in Theorem 8, ill-posedness due to an uncomplemented null-space occurs for B if and only if ill-posedness due to a non-closed range occurs for A .

Note that Proposition 9 does not state that $\mathcal{N}(C)$ is complemented if $\mathcal{N}(B)$ is complemented. The authors believe that uncomplemented $\mathcal{N}(C)$ is possible even if $\mathcal{N}(B)$ is complemented. But a rigorous proof is missing.

3. Regularization of Mazur-type operators

In [14, p. 111, item (e)], Banach and Mazur introduced a class of bounded linear operators mapping the sequence space ℓ^1 onto a separable Banach space Y .

Definition 10. A bounded linear operator $B : \ell^1 \rightarrow Y$ mapping into a separable Banach space Y is of *Mazur-type* if there is a sequence $(\zeta^{(k)})_{k \in \mathbb{N}}$ in Y that is dense in the unit sphere and satisfies

$$Bx = \sum_{k=1}^{\infty} x_k \zeta^{(k)} \quad \text{for all } x = (x_1, x_2, \dots) \in \ell^1. \quad (3.1)$$

Mazur-type operators obviously are well-defined and bounded linear operators. Note that each separable Banach space Y and each choice of the sequence $(\zeta^{(k)})_{k \in \mathbb{N}}$, for fixed Y , yield different Mazur-type operators.

Mazur-type operators are surjective and their null-spaces are almost always uncomplemented.

Proposition 11. Let $B : \ell^1 \rightarrow Y$ be a Mazur-type operator with defining sequence $(\zeta^{(k)})_{k \in \mathbb{N}}$. Then B is surjective and the null-space $\mathcal{N}(B)$ is complemented if and only if Y is isomorphic to ℓ^1 .

Proof. Surjectivity is shown in [15, proof of Thm. 2.3.1]. That $\mathcal{N}(B)$ is uncomplemented if Y is not isomorphic to ℓ^1 is a consequence of [15, Cor. 2.3.3]. It remains to show that $\mathcal{N}(B)$ is complemented if Y is isomorphic to ℓ^1 .

Let Y be isomorphic to ℓ^1 and denote by $\tilde{S}$ an isomorphism from ℓ^1 onto Y . Moreover, set $f^{(l)} := \tilde{S} e^{(l)}$ for $l \in \mathbb{N}$, where $e^{(l)}$ has a one at position l and zeros elsewhere. Then $(f^{(l)})_{l \in \mathbb{N}}$ is a basis in Y equivalent to $(e^{(l)})_{l \in \mathbb{N}}$ in ℓ^1 (cf. [13, Prop. 4.3.2]). In particular, we have

$$Y = \left\{ \sum_{l=1}^{\infty} a_l f^{(l)} : a \in \ell^1 \right\}$$

(cf. [13, Def. 4.3.1]). Without loss of generality, we may assume that $(f^{(l)})_{l \in \mathbb{N}}$ is normalized because there are constants $c > 0$ and $C > 0$ such that

$$c = c \|e^{(l)}\|_{\ell^1} \leq \|f^{(l)}\|_Y \leq C \|e^{(l)}\|_{\ell^1} = C \quad \text{for } l \in \mathbb{N}.$$

Normalization of the basis yields

$$\sum_{l=1}^{\infty} a_l f^{(l)} = \sum_{l=1}^{\infty} a_l \|f^{(l)}\|_Y \frac{f^{(l)}}{\|f^{(l)}\|_Y}$$

for coefficients $a_1, a_2, \dots$. Thus, from [13, Thm. 4.3.6], we see that the normalized basis is equivalent to $(e^{(l)})_{l \in \mathbb{N}}$, too.

By [13, Prop. 4.3.4, Prop. 4.3.2], we find a sequence $(k_l)_{l \in \mathbb{N}}$ in $\mathbb{N}$ and an isomorphism $S : Y \rightarrow Y$ such that $(\zeta^{(k_l)})_{l \in \mathbb{N}}$ is a Schauder basis in Y and $\zeta^{(k_l)} = S f^{(l)}$ for $l \in \mathbb{N}$. Thus, $(\zeta^{(k_l)})_{l \in \mathbb{N}}$ is equivalent to $(e^{(l)})_{l \in \mathbb{N}}$, too, resulting in

$$Y = \left\{ \sum_{l=1}^{\infty} a_l \zeta^{(k_l)} : a \in \ell^1 \right\}.$$

Set

$$U := \overline{\text{span}\{e^{(k_l)} : l \in \mathbb{N}\}}^{\ell^1}.$$

Obviously, U is complemented and isomorphic to ℓ^1 and the restriction $A := B|_U$ is surjective. For $x \in \ell^1$, we find $u \in U$ with $A u = B x$, that is, $x - u \in \mathcal{N}(B)$. The resulting decomposition $x = x - u + u$ shows $\ell^1 = \mathcal{N}(B) + U$. Injectivity of A (cf. proof of Theorem 3) implies $\mathcal{N}(B) \cap U = \{0\}$. This indicates that the subspace U is a complement of $\mathcal{N}(B)$ in ℓ^1 , which completes the proof of the proposition. $\square$

Due to their uncomplemented null-space, Mazur-type operators are ill-posed and regularization techniques have to be employed for obtaining stable (approximate) solutions to operator equations $Bx = y$. In [7], it was shown that Mazur-type operators are not accessible to Tikhonov regularization

$$\|Bx - y\|_Y^p + \alpha \|x\|_{\ell^1} \rightarrow \min_{x \in \ell^1} \quad (3.2)$$

with $p \geq 1$ and regularization parameter $\alpha > 0$ because Mazur-type operators are not weak*-to-weak continuous. In particular, it was shown there (cf. [7, Thm. 3.3]) that Tikhonov-regularized solutions exist only for a dense subset of right-hand sides $y \in Y$.

By applying Theorem 3 to a Mazur-type operator B , we obtain an injective bounded linear operator A with non-closed range, which in principle is accessible to Tikhonov regularization. The next proposition shows that indeed such A is weak*-to-weak continuous. Together with the weak* lower-semicontinuity of the norm in ℓ^1 , results on existence, stability, and convergence of Tikhonov-regularized solutions with ℓ^1 penalty apply (cf. [1, 5]). Moreover, the quasi-complement U of the null-space $\mathcal{N}(B)$ required in Theorem 3 can be made explicit in terms of the defining sequence $(\zeta^{(k)})_{k \in \mathbb{N}}$ of the Mazur-type operator under consideration. Weak*-to-weak continuity will be based on the existence of a *shrinking* basis in Y , see, e.g., [15, Def. 3.2.5]. Each reflexive Banach space has a shrinking basis, see [15, Thm. 3.2.13]. For separable Hilbert spaces Y , the proposition was already proven in [16, Prop. 4].

Proposition 12. *Let Y be a Banach space with a Schauder basis. Then for each Mazur-type operator $B : \ell^1 \rightarrow Y$ mapping onto Y , there is a closed and complemented subspace $U \subseteq \ell^1$ isomorphic to ℓ^1 with isomorphism $T : \ell^1 \rightarrow U$ such that $A := BT$ is injective, the range of A is dense in Y , and U is a quasi-complement of $\mathcal{N}(B)$. If Y has a shrinking basis, then A is weak*-to-weak continuous with respect to the predual c_0 of ℓ^1 .*

Proof. Existence of a Schauder basis implies that Y is separable, see [13, Prop. 4.1.6], and that there exists a normalized Schauder basis, see [13, Cor. 4.1.6]. Let $(\zeta^{(k)})_{k \in \mathbb{N}}$ be the sequence defining B and take a normalized Schauder basis $(f^{(k)})_{k \in \mathbb{N}}$ in Y . Then by [13, Prop. 4.3.4, Prop. 4.3.2], we find a sequence $(k_l)_{l \in \mathbb{N}}$ in $\mathbb{N}$ and an isomorphism $S : Y \rightarrow Y$ such that $(\zeta^{(k_l)})_{l \in \mathbb{N}}$ is a Schauder basis in Y and $\zeta^{(k_l)} = S f^{(l)}$ for $l \in \mathbb{N}$.

Set

$$U := \overline{\text{span} \{e^{(k_l)} : l \in \mathbb{N}\}}^{\ell^1}.$$

Then

$$T^{-1} u := (u_{k_l})_{l \in \mathbb{N}} \quad \text{for } u \in U$$

defines the isomorphism $T : \ell^1 \rightarrow U$. Obviously, U is complemented and isomorphic to ℓ^1 . Further, $A T^{-1} = B|_U$ is injective and, thus, A is injective: for $u = \sum_{l=1}^{\infty} u_l e^{(k_l)} \in U$ with $B|_U u = 0$, we have

$$0 = B u = \sum_{l=1}^{\infty} u_l \zeta^{(k_l)} = \sum_{l=1}^{\infty} u_l S f^{(l)} = S \left(\sum_{l=1}^{\infty} u_l f^{(l)} \right)$$

and, thus, $\sum_{l=1}^{\infty} u_l f^{(l)} = 0$, which implies $u = 0$. To show that $\mathcal{R}(A) = \mathcal{R}(A T^{-1}) = \mathcal{R}(B|_U)$ is dense in Y , we take $y \in Y$ and decompose it w. r. t. the basis $(\zeta^{(k_l)})_{l \in \mathbb{N}}$:

$$y = \sum_{l=1}^{\infty} a_l \zeta^{(k_l)}$$

with some sequence of scalars $a_1, a_2, \dots$. Set

$$u^{(m)} = \sum_{l=1}^m a_l e^{(k_l)} \in U.$$

Then $B|_U u^{(m)} = \sum_{l=1}^m a_l \zeta^{(k_l)} \rightarrow y$ if $m \rightarrow \infty$.

$\mathcal{N}(B) \cap U = \{0\}$ and density of $\mathcal{N}(B) + U$ in ℓ^1 can be proven in the same way as in the proof of Theorem 8.

To obtain weak*-to-weak continuity, due to [5, Lemma 9.3], it suffices to show that $A e^{(l)}$ converges weakly to zero for $l \rightarrow \infty$. We have $A e^{(l)} = B e^{(k_l)} = \zeta^{(k_l)} = S f^{(l)}$ and by assumption, $(f^{(l)})_{l \in \mathbb{N}}$ is a shrinking basis. Thus, by [15, Prop. 3.2.7], $(f^{(l)})_{l \in \mathbb{N}}$ converges weakly to zero in Y . Because bounded linear operators are always weak-to-weak continuous, this proves the assertion.

Note that the construction of U is based on the fact that each Banach space has a normalized basis $(f^{(l)})_{l \in \mathbb{N}}$. But in the proposition we require existence of a shrinking basis without being normalized. In fact, normalizing a shrinking basis yields a shrinking basis again. This can be seen, for instance, from [13, Prop. 4.4.7, Prop. 4.1.5]. $\square$

Depending on the choice of the Schauder basis for Y in Proposition 12, we obtain different subspaces U and, thus, different operators A . Consequently, Tikhonov regularization and other regularization methods for A will yield different solutions and will show different stability and convergence behavior depending on the choice of the Schauder basis.

Note that Mazur-type operators are not only an example to which Corollary 4 can be applied. They may also serve as a canonical surjection C in Theorem 8 if $W = \ell^1$. Choosing $W \neq Y$ provides some control over the properties of the extended domain $\tilde{X}$ of the operator B . If Y is not isomorphic to ℓ^1 , this choice of W always yields an uncomplemented null-space $\mathcal{N}(B)$ by Proposition 9.

4. Discussion and future challenges

Corollary 4 and Theorem 8 show that there is a tight one-to-one correspondence between injective operators A with dense range and surjective operators B . In both cases, A is a restriction of B and B is an extension of A (up to the isomorphism T). The operator A can be regarded as the *essential part* of B in the sense that the domain of A is a quasi-complement of the null-space of B .

With the results presented in this article, one may transfer ill-posedness due to a non-closed range to ill-posedness expressed in terms of a relationship between the null-space and one of its quasi-complements, and vice versa. See Remark 6 for a more-detailed interpretation of this duality.

Whether characterizing ill-posedness in terms of properties of the null-space offers added value depends on whether future ideas emerge on how to algorithmically overcome ill-posedness caused by an uncomplemented null-space. The Bartle-Graves theorem formulated in Lemma 7 always yields a (possibly nonlinear) continuous inverse, even if there is no bounded linear inverse. Attempting to algorithmically approximate such a Bartle-Graves inverse could prove to be a promising path and a rich source of new regularization methods. But at the moment for the authors, it is not clear how to tackle this task.

Transferring ill-posedness to the null-space may even yield new regularization methods in Hilbert spaces. The resulting surjective operator has a continuous Bartle-Graves inverse, which might become accessible to algorithmic evaluation or approximation in the future. To transfer ill-posedness of an operator $A : X \rightarrow Y$ between Hilbert spaces to the null-space in Theorem 8, one may choose $W := Y \times \{0\}$ with norm $\|(y, 0)\|_W := \|y\|_Y$, $C : W \rightarrow Y$ with $C(y, 0) := y$, and $\tilde{X} := Y \times X$ with norm

$$\|(w, x)\|_{\tilde{X}} := \sqrt{\|w\|_W^2 + \|x\|_X^2}.$$

Then the domain $\tilde{X}$ of the extension B is a Hilbert space again with inner product

$$\langle (w_1, x_1), (w_2, x_2) \rangle_{\tilde{X}} = \langle w_1, w_2 \rangle_W + \langle x_1, x_2 \rangle_X,$$

where $\langle (y_1, 0), (y_2, 0) \rangle_W = \langle y_1, y_2 \rangle_Y$.

Remember Remark 6, which states that although each closed subspace of a Hilbert space is complemented, there might be quasi-complements that are not complements. Thus, even in Hilbert spaces, the null-space $\mathcal{N}(B)$ may have a quasi-complement U that is not a complement. In this sense, $\mathcal{N}(B)$ can be regarded as “uncomplemented with respect to U ”.

Mazur-type operators are a rather large class of operators because every separable Banach space not isomorphic to ℓ^1 allows one to construct Mazur-type operators along the lines of [15, proof of Thm. 2.3.1], see also [15, Cor. 2.3.3]. Next to Mazur-type operators, the authors are not currently aware of any other class of operators with uncomplemented null-space. Of particular interest are practically relevant operators with this property.

Presenting concrete applications of Mazur-type operators and verifying them through numerical experiments lay outside the scope of this article, which was dominated by functional analysis. However, this may represent an interesting challenge for future research, where for example the inverse problem of reconstructing blurred and noisy images could be modeled by operators mapping from ℓ^1 . This field is very important and has several different directions, as there are the ℓ^1 -regularization with sparsity constraints [17], regularization approaches for ℓ^p to ℓ^q mappings [18], advanced Tikhonov regularization procedures [19], and models incorporating boundary conditions [20].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

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