



Research article

Hybrid mean values of Dedekind sums via Williams' transformation formula and applications to Kloosterman sums

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Abstract: By combining analytic number theory methods with Williams's classical work, this paper investigated the computational problem of a class of power means involving Dedekind sums, and established a novel explicit conversion identity. Furthermore, we obtained exact formulas for even power means of Kloosterman sums via properties of character sums.

Keywords: Dedekind sums; Kloosterman sums; Williams' transformation formula; asymptotic formula

1. Introduction

For a positive integer q and a natural number h , the classical Dedekind sums $S(h, q)$ is defined as

$$S(h, q) = \sum_{1 \leq a \leq q} \left(\left(\frac{a}{q} \right) \right) \left(\left(\frac{ah}{q} \right) \right),$$

where $((\cdot))$ is given by

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}. \end{cases}$$

This arithmetic function characterizes the transformation behavior of the logarithm of the eta-function under modular maps, and has been a central object in analytic number theory since its introduction (see the foundational works of Rademacher and Grosswald [1,2]). Among its most fundamental properties is the inversion formula, (see [3]): for coprime positive integers h, q , one has the identity

$$S(h, q) + S(q, h) = \frac{h^2 + q^2 + 1}{12hq} - \frac{1}{4}.$$

In addition, investigating its $2m$ -th moment (i.e., $\sum S^{2m}$) provides profound insights into the statistical distribution of Dedekind sums over the complete residue system modulo p , including the frequency of occurrence of their maxima and minima, as well as the overall fluctuation behavior. Motivated by this, many scholars have studied the power mean of Dedekind sums. For example, Conrey et al. [4, Theorem 2] established the asymptotic expansion for even moments:

$$\sum_{\substack{1 \leq h \leq q \\ (h,q)=1}} S^{2m}(h, q) = f_m(q) \left(\frac{q}{12} \right)^{2m} + O\left(\left(q^{\frac{9}{5}} + q^{2m-1+\frac{1}{m+1}} \right) \log^3 q \right), \quad (1.1)$$

where the series of the arithmetic function f_m satisfies

$$\sum_{n=1}^{\infty} \frac{f_m(n)}{n^s} = 2 \frac{\zeta^2(2m)}{\zeta(4m)} \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)} \zeta(s).$$

Jia [5] later sharpened the error term in (1.1) to $O(q^{2m-1})$ for $m \geq 2$. That is, for any integer $m \geq 2$, one has the asymptotic formula

$$\sum_{\substack{1 \leq h \leq q \\ (h,q)=1}} S^{2m}(h, q) = f_m(q) \left(\frac{q}{12} \right)^{2m} + O(q^{2m-1}). \quad (1.2)$$

For $m = 1$, Zhang [6] proved the asymptotic formula

$$\begin{aligned} \sum_{\substack{1 \leq h \leq q \\ (h,q)=1}} S^2(h, q) &= \frac{5}{144} q \phi(q) \prod_{p^\alpha \parallel q} \left(1 + \frac{1}{p} + \frac{1}{p^2} \right)^{-1} \left[\left(1 + \frac{1}{p} \right)^2 - \frac{1}{p^{3\alpha+1}} \right] \\ &\quad + O\left(q \exp\left(\frac{4 \log q}{\log \log q} \right) \right), \end{aligned} \quad (1.3)$$

where $p^\alpha \parallel q$ denotes that p^α divides q , but $p^{\alpha+1}$ does not divide q .

Some related works can be found in [7–10]. This paper studies a previously unexamined variant of such power means: for an odd prime p , let $\bar{x}$ denote the multiplicative inverse of x modulo p (i.e., $\bar{x} \cdot x \equiv 1 \pmod{p}$). Investigating these hybrid moments facilitates connections between Dedekind sums and deeper number-theoretic objects, such as Cochrane sums, Kloosterman sums, and the moments of character sums. We consider

$$\sum_{1 \leq a \leq p-1} \left(\sum_{1 \leq x \leq p-1} S^{2m}(ax + \bar{x}, p) \right)^2. \quad (1.4)$$

We perform the transformation with the help of Williams' important work [11], and then provide an evaluation formula for the hybrid mean value (1.4). As some interesting corollaries, we present two asymptotic formulas for (1.4) with $m = 1$ and 2.

Theorem 1.1. *Let p be an odd prime and m a positive integer. Then,*

$$\sum_{1 \leq a \leq p-1} \left(\sum_{1 \leq x \leq p-1} S^{2m}(ax + \bar{x}, p) \right)^2$$

$$= 2p \sum_{1 \leq x \leq p-1} S^{4m}(x, p) + (p-5) \left(\sum_{1 \leq x \leq p-1} S^{2m}(x, p) \right)^2.$$

By combining relations (1.2) and (1.3) with $m = 1$ and $m = 2$ in Theorem 1.1, we can observe the following asymptotic behaviors for small moments. For instance, when $m = 1$, the left-hand side satisfies:

$$\sum_{1 \leq a \leq p-1} \left(\sum_{1 \leq x \leq p-1} S^2(ax + \bar{x}, p) \right)^2 = \frac{89}{62208} p^5 + O\left(p^4 \exp\left(\frac{4 \log p}{\log \log p}\right)\right).$$

Similarly, when $m = 2$, we obtain:

$$\sum_{1 \leq a \leq p-1} \left(\sum_{1 \leq x \leq p-1} S^4(ax + \bar{x}, p) \right)^2 = \frac{308507}{32553 \times 12^8} p^9 + O(p^8).$$

These examples demonstrate the application of Theorem 1.1 in deriving specific polynomial coefficients, though the focus of this manuscript remains on the general identity established in Theorem 1.1.

To establish the theoretical foundation for our subsequent analysis, we recall several pivotal results concerning the moments of Kloosterman sums over finite fields. Let p be a prime number, and define the exponential sum $e_p(x) = \exp\left(\frac{2\pi i x}{p}\right)$.

Salié [12] established the exact evaluation for the fourth moment:

$$\sum_{0 \leq a \leq p-1} \left(\sum_{1 \leq x \leq p-1} e_p(ax + \bar{x}) \right)^4 = 2p^3 - 3p^2 - 3p. \quad (1.5)$$

Although the sixth-moment identity can be theoretically deduced from Hulek et al. [13, Theorem 3.2], the specific orthogonality calculation is omitted there. For the sake of completeness and to facilitate the reader, we provide the detailed derivation here. First, we establish the following intermediate identity:

$$\sum_{0 \leq a \leq p-1} \left(\sum_{1 \leq x \leq p-1} e_p(ax + \bar{x}) \right)^6 = p^2 \#U(\mathbb{F}_p) - (p-1)^5 - 1,$$

where $U(\mathbb{F}_p)$ denotes the set of points $(x_0, x_1, x_2, x_3, x_4, x_5) \in (\mathbb{F}_p^*)^6$ satisfying $\sum_{i=0}^5 x_i = 0$ and $\sum_{i=0}^5 \bar{x}_i = 0$. Substituting the expression $b_p = p^3 - 19 - \#U(\mathbb{F}_p)$ into the above equation yields the sixth moment formula for $p \geq 5$:

$$\sum_{0 \leq a \leq p-1} \left(\sum_{1 \leq x \leq p-1} e_p(ax + \bar{x}) \right)^6 = 5p^4 - 10p^3 - (b_p + 9)p^2 - 5p, \quad (1.6)$$

where b_p is the integer satisfying $|b_p| < 2p^{\frac{3}{2}}$.

Regarding asymptotic behaviors, Katz's work on Kloosterman sheaves [14] implies the proof of the general asymptotic formula for even moments. Specifically, for any fixed integer $k > 0$, we have:

$$\sum_{0 \leq a \leq p-1} \left(\sum_{1 \leq x \leq p-1} e_p(ax + \bar{x}) \right)^{2k} \sim \frac{1}{k+1} \binom{2k}{k} p^{k+1} \quad \text{as } p \rightarrow \infty.$$

Furthermore, Xi and Yi [15, Theorem 1] refined this by deriving an asymptotic formula with explicit error terms, utilizing Katz's theory and properties of Chebyshev polynomials:

$$\sum_{0 \leq a \leq p-1} \left(\sum_{1 \leq x \leq p-1} e_p(ax + \bar{x}) \right)^{2k} = \frac{1}{k+1} \binom{2k}{k} p^{k+1} + O(p^{k+\frac{1}{2}}). \quad (1.7)$$

Based on the aforementioned results, we derive the following identities:

Corollary 1.2. *Let p be an odd prime. Then,*

$$\sum_{0 \leq a \leq p-1} \left(\sum_{0 \leq x \leq p-1} \left(\frac{x^2 - a}{p} \right) e_p(2x) \right)^4 = p(2p^2 - 3p - 3),$$

where $e_p(y) = \exp\left(\frac{2\pi i y}{p}\right)$ and $i^2 = -1$.

Corollary 1.3. *Let p be an odd prime. There exists an integer b_p with $|b_p| < 2p^{\frac{3}{2}}$ such that*

$$\sum_{0 \leq a \leq p-1} \left(\sum_{0 \leq x \leq p-1} \left(\frac{x^2 - a}{p} \right) e_p(2x) \right)^6 = 5p^4 - 10p^3 - (b_p + 9)p^2 - 5p.$$

In particular, when $p = 3$, the left-hand side equals 66, and $b_3 = -3$ satisfies the required condition.

The inner sum coincides with the classical Kloosterman sums, leading to the following asymptotic formula:

Corollary 1.4. *For any given natural number k and each large prime p , we have*

$$\sum_{0 \leq a \leq p-1} \left(\sum_{0 \leq x \leq p-1} \left(\frac{x^2 - a}{p} \right) e_p(2x) \right)^{2k} = \frac{1}{k+1} \binom{2k}{k} p^{k+1} + O(p^{k+\frac{1}{2}}).$$

2. Some lemmas

All background on elementary and analytic number theory can be found in [12, 13, 16–18]. We only state the key tools needed below.

Lemma 2.1. *Let $G(x)$ be a complex-valued function defined on the integers with period p . For integers a, b, c, A, B, C , define*

$$D = B^2 - 4AC, \quad \Delta = 4aC - 2bB + 4cA, \quad d = b^2 - 4ac.$$

Assume that $p \nmid \Delta^2 - 4Dd$, which is equivalent to

$$p \nmid 16\{(aC - cA)^2 - (aB - bA)(bC - cB)\}.$$

Then,

$$\begin{aligned} & \sum_{\substack{0 \leq x \leq p-1 \\ p \nmid Ax^2 + Bx + C}} G\left(\frac{ax^2 + bx + c}{Ax^2 + Bx + C}\right) \\ &= \sum_{0 \leq x \leq p-1} G(x) + \sum_{0 \leq x \leq p-1} \left(\frac{Dx^2 + \Delta x + d}{p}\right) G(x) - \begin{cases} G(a/A), & \text{if } p \nmid A, \\ 0, & \text{if } p \mid A, \end{cases} \end{aligned}$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre symbol modulo p .

Proof. This identity is a direct specialization of the main theorem established by Williams [11]. We present the proof using the framework of finite transformation formulae for rational functions over $\text{GF}(p)$.

Define the rational function $\theta(x) = \frac{ax^2 + bx + c}{Ax^2 + Bx + C}$ and its associated quadratic polynomial $\theta^*(x) = Dx^2 + \Delta x + d$ as in [11, Section 1]. The condition $p \nmid \Delta^2 - 4Dd$ ensures that $\theta^*(x)$ are non-degenerate, i.e., the discriminant of θ^* is nonzero modulo p .

According to [11, Theorem], for any $F : \text{GF}(p) \rightarrow \mathbb{C}$, the following transformation formula holds:

$$\sum_{0 \leq x \leq p-1} F(x) + \sum_{0 \leq x \leq p-1} \left(\frac{\theta^*(x)}{p}\right) F(x) = \sum_{\substack{0 \leq x \leq p-1 \\ p \nmid Ax^2 + Bx + C}} F(\theta(x)) + \delta(F, \theta),$$

where $\delta(F, \theta)$ is the boundary correction term defined by

$$\delta(F, \theta) = \begin{cases} F(a/A), & \text{if } A \not\equiv 0 \pmod{p}, \\ 0, & \text{if } A \equiv 0 \pmod{p}. \end{cases}$$

Substituting $F(x) = G(x)$ and $\theta^*(x) = Dx^2 + \Delta x + d$ into the above identity yields exactly the desired equation. The term $\sum_{x=0}^{p-1} \left(\frac{Dx^2 + \Delta x + d}{p}\right) G(x)$ arises from the quadratic character count of the solutions to $\theta(x) = y$, while the subtraction of $\delta(G, \theta)$ adjusts for the pole at infinity (when $A \neq 0$) or the absence thereof (when $A = 0$).

For a detailed exposition on the invariant properties of θ and θ^* and the derivation of the boundary term, we refer the reader to [11, 19]. $\square$

Lemma 2.2. Let p be an odd prime and $k \geq 1$ an integer. Then,

$$\begin{aligned} & \sum_{1 \leq a \leq p-1} \left(\sum_{1 \leq x \leq p-1} \left(\frac{x^2 - 4a}{p}\right) S^{2k}(x, p) \right)^2 \\ &= 2p \sum_{1 \leq x \leq p-1} S^{4k}(x, p) - 2 \left(\sum_{1 \leq x \leq p-1} S^{2k}(x, p) \right)^2. \end{aligned}$$

Proof. Using the orthogonality of Legendre symbol:

$$\sum_{0 \leq a \leq p-1} \left(\frac{a^2 + n}{p} \right) = \begin{cases} p-1, & \text{if } p \mid n, \\ -1, & \text{if } p \nmid n, \end{cases} \quad (2.1)$$

and

$$\sum_{1 \leq x \leq p-1} S^{2k}(x, p) = \sum_{1 \leq x \leq p-1} S^{2k}(-x, p), \quad (2.2)$$

from (2.1) and (2.2) we have

$$\begin{aligned} & \sum_{1 \leq a \leq p-1} \left(\sum_{1 \leq x \leq p-1} \left(\frac{x^2 - 4a}{p} \right) S^{2k}(x, p) \right)^2 + \left(\sum_{1 \leq x \leq p-1} S^{2k}(x, p) \right)^2 \\ &= \sum_{0 \leq a \leq p-1} \left(\sum_{1 \leq x \leq p-1} \left(\frac{x^2 + a}{p} \right) S^{2k}(x, p) \right)^2 \\ &= \sum_{1 \leq x \leq p-1} \sum_{1 \leq y \leq p-1} \sum_{0 \leq a \leq p-1} \left(\frac{x^2 + a}{p} \right) \left(\frac{y^2 + a}{p} \right) S^{2k}(x, p) S^{2k}(y, p) \\ &= \sum_{1 \leq x \leq p-1} \sum_{1 \leq y \leq p-1} \sum_{0 \leq a \leq p-1} \left(\frac{a}{p} \right) \left(\frac{y^2 - x^2 + a}{p} \right) S^{2k}(x, p) S^{2k}(y, p) \\ &= \sum_{1 \leq x \leq p-1} \sum_{1 \leq y \leq p-1} \sum_{0 \leq a \leq p-1} \left(\frac{(2a + y^2 - x^2)^2 - (y^2 - x^2)^2}{p} \right) S^{2k}(x, p) S^{2k}(y, p) \\ &= \sum_{1 \leq x \leq p-1} \sum_{1 \leq y \leq p-1} \sum_{0 \leq a \leq p-1} \left(\frac{a^2 - (y^2 - x^2)^2}{p} \right) S^{2k}(x, p) S^{2k}(y, p) \\ &= (p-1) \sum_{\substack{1 \leq x \leq p-1 \\ x^2 \equiv y^2 \pmod{p}}} \sum_{1 \leq y \leq p-1} S^{2k}(x, p) S^{2k}(y, p) - \sum_{\substack{1 \leq x \leq p-1 \\ x^2 \not\equiv y^2 \pmod{p}}} \sum_{1 \leq y \leq p-1} S^{2k}(x, p) S^{2k}(y, p) \\ &= 2p \sum_{1 \leq x \leq p-1} S^{4k}(x, p) - \left(\sum_{1 \leq x \leq p-1} S^{2k}(x, p) \right)^2. \end{aligned}$$

This proves Lemma 2.2. □

Lemma 2.3. Let p be an odd prime and $(a, p) = 1$. Then,

$$\sum_{1 \leq x \leq p-1} e_p(ax + \bar{x}) = \sum_{0 \leq x \leq p-1} \left(\frac{x^2 - 4a}{p} \right) e_p(x) = \sum_{0 \leq x \leq p-1} \left(\frac{x^2 - a}{p} \right) e_p(2x).$$

Proof. Substituting $G(x) = e_p(x)$, $b = 0$, $c = 1$, $A = 0$, $B = 1$, and $C = 0$ into Lemma 2.1, we obtain $D = 1$, $\Delta = 0$, and $d = -4a$, since $p \nmid (\Delta^2 - 4Dd) = 16a$. Moreover,

$$\sum_{0 \leq x \leq p-1} e_p(x) = 0$$

and

$$\sum_{1 \leq x \leq p-1} e_p((ax^2 + 1)\bar{x}) = \sum_{1 \leq x \leq p-1} e_p(ax + \bar{x}).$$

Consequently, we obtain the identity

$$\sum_{1 \leq x \leq p-1} e_p(ax + \bar{x}) = \sum_{0 \leq x \leq p-1} \left(\frac{x^2 - 4a}{p} \right) e_p(x) = \sum_{0 \leq x \leq p-1} \left(\frac{x^2 - a}{p} \right) e_p(2x).$$

This proves Lemma 2.3. $\square$

3. Proofs of the theorems

In this section, we utilize the lemmas established in Section 2 to complete the proofs of all our theorems. We begin with Theorem 1.1. Substituting $G(x) = S^{2k}(x, p)$ into Lemma 2.1, with parameters $1 \leq a \leq p-1$, $b = 0$, $c = 1$, $A = 0$, $B = 1$, and $C = 0$, yields $D = 1$, $\Delta = 0$, and $d = -4a$. As $p \nmid (\Delta^2 - 4Dd) = 16a$, we derive

$$\begin{aligned} & \sum_{1 \leq x \leq p-1} S^{2k}(ax + \bar{x}, p) \\ &= \sum_{0 \leq x \leq p-1} S^{2k}(x, p) + \sum_{0 \leq x \leq p-1} \left(\frac{x^2 - 4a}{p} \right) S^{2k}(x, p) \\ &= \sum_{1 \leq x \leq p-1} S^{2k}(x, p) + \sum_{1 \leq x \leq p-1} \left(\frac{x^2 - 4a}{p} \right) S^{2k}(x, p). \end{aligned} \quad (3.1)$$

To verify the validity of Identity (3.1), we compute both sides for small odd primes p and $k = 1, 2, 3$ under the $a = 1$ (see Tables 1 and 2).

By property of Dirichlet character sums, we have

$$\begin{aligned} & \sum_{1 \leq a \leq p-1} \sum_{1 \leq x \leq p-1} \left(\frac{x^2 - 4a}{p} \right) S^{2k}(x, p) \\ &= \sum_{1 \leq x \leq p-1} \sum_{0 \leq a \leq p-1} \left(\frac{x^2 - 4a}{p} \right) S^{2k}(x, p) - \sum_{1 \leq x \leq p-1} S^{2k}(x, p) \\ &= - \sum_{1 \leq x \leq p-1} S^{2k}(x, p). \end{aligned} \quad (3.2)$$

Combining Eqs (3.1) and (3.2) with Lemma 2.2, we arrive at

$$\begin{aligned} & \sum_{1 \leq a \leq p-1} \left(\sum_{1 \leq x \leq p-1} S^{2k}(ax + \bar{x}, p) \right)^2 = (p-1) \left(\sum_{1 \leq x \leq p-1} S^{2k}(x, p) \right)^2 \\ & + 2 \left(\sum_{1 \leq x \leq p-1} S^{2k}(x, p) \right) \sum_{1 \leq a \leq p-1} \sum_{1 \leq y \leq p-1} \left(\frac{y^2 - 4a}{p} \right) S^{2k}(y, p) \end{aligned}$$

$$\begin{aligned}
& + \sum_{1 \leq a \leq p-1} \left(\sum_{1 \leq x \leq p-1} \left(\frac{x^2 - 4a}{p} \right) S^{2k}(x, p) \right)^2 \\
& = (p-3) \left(\sum_{1 \leq x \leq p-1} S^{2k}(x, p) \right)^2 + 2p \sum_{1 \leq x \leq p-1} S^{4k}(x, p) - 2 \left(\sum_{1 \leq x \leq p-1} S^{2k}(x, p) \right)^2 \\
& = 2p \sum_{1 \leq x \leq p-1} S^{4k}(x, p) + (p-5) \left(\sum_{1 \leq x \leq p-1} S^{2k}(x, p) \right)^2.
\end{aligned}$$

This completes the proof of Theorem 1.1.

Table 1. Mathematica code for identity (3.1).

Mathematica code

(* 1. Basic definitions *)

```
dedekindFrac[x_, p_] := Module[{y = x/p}, If[FractionalPart[y] == 0, 0, y - Floor[y] - 1/2]];
dedekindSum[h_, p_] := Sum[dedekindFrac[a, p] * dedekindFrac[Mod[a * h, p], p], {a, 1, p - 1}];
legendreSymbol[a_, p_] := JacobiSymbol[a, p];
```

(* 2. Three summation formulas *)

```
S1[p_, k_] := Sum[dedekindSum[x + PowerMod[x, -1, p], p]^(2k), {x, 1, p - 1}];
S2[p_, k_] := Sum[dedekindSum[x, p]^(2k), {x, 1, p - 1}];
S3[p_, k_] := Sum[legendreSymbol[x^2 - 4, p] * dedekindSum[x, p]^(2k), {x, 1, p - 1}];
```

(* 3. Parameter settings *)

```
ps = {3, 5, 7, 11};
ks = {1, 2, 3};
```

(* 4. Generate table data *)

```
tableData = Flatten[Table[{p, k, S1[p, k], S2[p, k], S3[p, k]}, {p, ps}, {k, ks}], 1];
```

(* 5. Print results *)

```
Prepend[tableData, {"p", "k", "Σ S^{2k}(x + x̄, p)", "Σ S^{2k}(x, p)", "Σ (x^2-4/p) S^{2k}(x, p)"}].
```

Furthermore, it is worth noting that Lemma 2.3 remains valid even when $a = 0$. By Lemma 2.3 together with existing results (1.5)–(1.7), we obtain the following three identities:

$$\sum_{0 \leq a \leq p-1} \left(\sum_{0 \leq x \leq p-1} \left(\frac{x^2 - a}{p} \right) e_p(2x) \right)^4 = 2p^3 - 3p^2 - 3p,$$

$$\sum_{0 \leq a \leq p-1} \left(\sum_{0 \leq x \leq p-1} \left(\frac{x^2 - a}{p} \right) e_p(2x) \right)^6 = 5p^4 - 10p^3 - (b_p + 9)p^2 - 5p,$$

$$\sum_{0 \leq a \leq p-1} \left(\sum_{0 \leq x \leq p-1} \left(\frac{x^2 - a}{p} \right) e_p(2x) \right)^{2k} = \frac{1}{k+1} \binom{2k}{k} p^{k+1} + O(p^{k+\frac{1}{2}}).$$

These identities yield Corollaries 1.2–1.4.

Table 2. Numerical Verification of Identity (3.1).

p	k	$\sum_{1 \leq x \leq p-1} S^{2k}(x + \bar{x}, p)$	$\sum_{1 \leq x \leq p-1} S^{2k}(x, p)$	$\sum_{1 \leq x \leq p-1} \left(\frac{x^2 - 4}{p} \right) S^{2k}(x, p)$
3	1	$\frac{1}{162}$	$\frac{1}{162}$	0
3	2	$\frac{1}{52488}$	$\frac{1}{52488}$	0
3	3	$\frac{1}{17006112}$	$\frac{1}{17006112}$	0
5	1	0	$\frac{2}{25}$	$-\frac{2}{25}$
5	2	0	$\frac{2}{625}$	$-\frac{2}{625}$
5	3	0	$\frac{2}{15625}$	$-\frac{2}{15625}$
7	1	$\frac{51}{98}$	$\frac{27}{98}$	$\frac{12}{49}$
7	2	$\frac{1251}{19208}$	$\frac{627}{19208}$	$\frac{78}{2401}$
7	3	$\frac{31251}{3764768}$	$\frac{15627}{3764768}$	$\frac{279}{67228}$
11	1	$\frac{61}{242}$	$\frac{293}{242}$	$-\frac{116}{121}$
11	2	$\frac{949}{117128}$	$\frac{52037}{117128}$	$-\frac{6386}{14641}$
11	3	$\frac{18541}{56689952}$	$\frac{11423333}{56689952}$	$-\frac{1425599}{7086244}$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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