



Research article

Analytical and dynamical investigation of novel solutions and conserved vectors for a generalized 2D B-type Kadomtsev-Petviashvili equation of fluid dynamics

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Abstract: Nonlinear partial differential equations are of particular interest to mathematicians and physicists because of their ability to model a wide range of complex nonlinear phenomena. Consequently, obtaining their exact analytical solutions is of considerable importance. In this study, we invoked the Lie symmetry method to construct new exact explicit solutions to a generalized 2D B-type Kadomtsev-Petviashvili (2D-gnBKP) equation. The Lie symmetry method provides a systematic framework for identifying continuous symmetries admitted by a differential equation and, subsequently, reducing its order and complexity. First, the infinitesimal generators were derived and using these, a one-dimensional optimal system of subalgebras was constructed. The optimal system enables the classification of inequivalent symmetry reductions and avoids redundant solution representations. Thereafter, symmetry reductions were performed that reduce the 2D-gnBKP equation to multiple ordinary differential equations. These reduced equations were then analyzed using a variety of analytical techniques. In particular, exact solutions were obtained through direct integration, the simplest equation, (G'/G) -expansion, and Kudryashov's techniques. Finally, for the first time in the literature, we constructed conserved vectors for the equation using the multiplier approach and Ibragimov's method. The resulting conserved quantities include physically meaningful conservation laws related to momentum and energy, thereby highlighting important structural and physical properties of the model.

Keywords: generalized 2D B-type Kadomtsev-Petviashvili; Lie point symmetries; optimal system; exact solutions; conserved vectors

1. Introduction

Nonlinear partial differential equations (NPDEs) are an important component in applied science and engineering, with applications in mechanics, ecological and economic systems, control processes, epidemiology, etc. [1–3]. Currently, the study of exact solutions, particularly solitary wave solutions, for NPDEs plays a significant role due to their impact in the natural world, for example, in numerous areas of science and engineering [4–6]. Many problems in nuclear physics, applied science, fluid mechanics, ecology, plasma, and nonlinear optics can be represented and expressed using NPDEs [7–9]. Thus, numerous physicists and mathematicians developed a variety of reliable and efficient methods for finding exact explicit solutions to NPDEs. Some of these methods are the inverse scattering method [10], Painlevé expansion method [11], (G'/G) -expansion method [12], Hirota bilinear technique [13,14], Cole-Hopf transformation [15], tanh function technique [16], Riccati equation technique [17], Kudryashov's techniques [18,19], homogeneous balance technique [20], Jacobian elliptic function expansion technique [21], sine-cosine method [22], bifurcation method [23], extended trial equation technique and modified auxiliary equation method [24], simplest equation method [25,26], F-expansion technique [27], first integral technique [28], power series solution technique [29], Darboux transformation [30], a novel computational algorithm for efficiently solving partial differential equations [31], exp-function method [32], and Lie symmetry method [33–35].

The Lie symmetry technique is an effective mathematical tool for analyzing and solving differential equations (DEs). It was uncovered by Sophus Lie (1842–1899), a mathematician from Norway. Drawing inspiration from Évariste Galois (1811–1832), a French mathematician, Lie developed symmetry-based methods, which uncovered that the many integration techniques, once considered ad hoc, could in fact be derived systematically. For partial differential equations (PDEs), the technique focuses on finding the symmetries of PDEs, which can then be used to reduce the independent variables in PDEs, making the solving process easier [36–38].

Since a Lie algebra of a PDE usually contains infinitely many subalgebras, many of which lead to equivalent (repetitive) group-invariant solutions, to avoid this redundancy, we construct an optimal system, which is a minimal set of inequivalent subalgebras that ensure we get only the distinct invariant solutions. These analyses can be used to derive transformations that keep the PDEs invariant. From these transformations, invariant solutions can be derived [33,35].

Conservation laws are extremely significant in the study of DEs and their influence extends to many areas of science and engineering. Conservation laws are mathematical representations of physical rules like conservation of mass, momentum, and energy [38]. Several strategies have been introduced to construct conservation laws. These include the multiplier method [35], Noether's theorem [39], and Ibragimov's theorem [40,41]. It is important to recognize that the usefulness of Noether's theorem depends on the existence of a Lagrangian of DEs. However, many DEs do not have a Lagrangian representation, making Noether's theorem ineffective for establishing conservation laws. In such cases, the multiplier method and Ibragimov's theorem are extremely useful in constructing conservation laws for DEs.

The 2D integrable Kadomtsev-Petviashvili (KP) equation [42] is

$$u_{tx} - u_{yy} + 6(uu_x)_x + u_{xxx} = 0, \quad (1.1)$$

which describes the evolution of long waves with small amplitude and slow dependence on the transverse

coordinate [43]. Its Hirota bilinear form is

$$(D_x^4 + D_t D_x - D_y^2) f \cdot f = 0, \quad (1.2)$$

where $f = f(t, x, y)$ and D_x, D_t, D_y represent the Hirota bilinear operators [14]. After substituting u by u_x and integrating with regard to x [44], the potential KP equation

$$u_{tx} - u_{yy} + 6u_x u_{xx} + u_{xxxx} = 0, \quad (1.3)$$

is produced from (1.1) [43]. Equation (1.3) represents the transmission of nonlinear waves in a 2D space [45], and it may model various physical applications, like nonlinear optics, plasma physics, and fluid dynamics. Also, the 2D integrable B-type Kadomtsev-Petviashvili (BKP) equation [46] is

$$u_{tx} - u_{yy} + (15u_x^3 + 15u_x u_{xxx} + u_{xxxx})_x + 5(u_{xxxy} + 3(u_x u_y)_x) = 0, \quad (1.4)$$

which has been used to model the shallow water wave in fluids and the electrostatic wave potential in plasmas, and it depicts the interactions between exponentially confined structures [47, 48]. Its Hirota bilinear form is

$$(D_x^6 + 5D_x^3 D_y + D_t D_x - D_y^2) f \cdot f = 0, \quad (1.5)$$

which is obtained through the substitution $u = 2(\ln f)_x$ into equation (1.4). Equations (1.1) and (1.4) possess infinite Lie algebra structures, as well as N -soliton solutions via their Hirota bilinear forms [49]. According to the Hirota bilinear form (1.5), the soliton solutions for (1.4) were examined using the Pfaffian approach [50] and the τ -function approach [51].

In this study, we extend the classical BKP equation given in (1.4) to a more general form, which we refer to as the generalized two-dimensional B-type Kadomtsev-Petviashvili (2D-gnBKP) equation:

$$u_{tx} - u_{yy} + (abu_x^3 + abu_x u_{xxx} + u_{xxxx})_x + b(u_{xxxy} + a(u_x u_y)_x) = 0, \quad (1.6)$$

where a and b are constants. This generalization is introduced to provide a broader framework for investigating the underlying nonlinear dynamics and to encompass a wider class of solutions and physical or mathematical phenomena than those described by the standard BKP equation. In particular, the proposed 2D-gnBKP equation preserves the essential structural features of the original BKP equation while incorporating parameters that allow for greater flexibility in the modeling and analysis of nonlinear wave interactions.

We first perform Lie symmetry analysis of 2D-gnBKP equation (1.6) and discover its exact solutions. We also compute conserved vectors for (1.6) using the multiplier approach and Ibragimov's method. As a result, the structure of the paper is as follows: Section 2 presents point symmetries of (1.6). In Section 3, exact solutions to (1.6) are derived through the use of the simplest equation approach, (G'/G) -expansion technique, and Kudryashov's method. The derivation of conserved vectors for (1.6) utilizing Ibragimov's method and the multiplier approach is covered in Section 4. Section 5 presents the discussions and conclusions.

2. Point symmetries and optimal system of (1.6)

We start by computing the point symmetries of the 2D-gnBKP equation (1.6). Thereafter, we construct the optimal system of one-dimensional sub-algebras using these symmetries. The optimal system is then utilized to perform symmetry reductions and to construct exact solutions of Eq (1.6).

2.1. Point symmetries of (1.6)

Consider the one-parameter group of transformations

$$\bar{t} = t + \nu \xi^1, \quad \bar{x} = x + \nu \xi^2, \quad \bar{y} = y + \nu \xi^3, \quad \bar{u} = u + \nu \eta,$$

with $\xi^i, i = 1, 2, 3, \eta$ depending on (t, x, y, u) , and ν being a small parameter. The associated vector field is then given by

$$R = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u}$$

and it is a point symmetry of (1.6) if the invariance condition

$$R^{[6]} \left\{ u_{tx} - u_{yy} + \left(abu_x^3 + abu_x u_{xxx} + u_{xxxx} \right)_x + b \left(u_{xxy} + a \left(u_x u_y \right)_x \right) \right\} \Big|_{(1.6)} = 0 \quad (2.1)$$

holds. Here, $R^{[6]}$ represents the sixth extension of R , which is defined as

$$\begin{aligned} R^{[6]} = R &+ \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_y \frac{\partial}{\partial u_y} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \zeta_{xy} \frac{\partial}{\partial u_{xy}} + \zeta_{yy} \frac{\partial}{\partial u_{yy}} + \zeta_{xxx} \frac{\partial}{\partial u_{xxx}} \\ &+ \zeta_{xxxx} \frac{\partial}{\partial u_{xxxx}} + \zeta_{xxy} \frac{\partial}{\partial u_{xxy}} + \zeta_{xxx} \frac{\partial}{\partial u_{xxx}} + \zeta_{xxxxx} \frac{\partial}{\partial u_{xxxxx}}, \end{aligned} \quad (2.2)$$

where ζ' s are given by [34]

$$\zeta_{j_1, j_2, \dots, j_k} = D_{j_k} \left(\zeta_{j_1, j_2, \dots, j_{k-1}} \right) - u_{j_1, j_2, \dots, j_{k-1}} D_{j_k} (\xi^n) \quad (\text{sum on } n), \quad (2.3)$$

and the total differential operators D_t , D_x , and D_y are defined as

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + u_{ijk} \frac{\partial}{\partial u_{jk}} + \dots, \quad (2.4)$$

where $(x^1, x^2, x^3) = (t, x, y)$. Expanding (2.1) and separating on derivatives of u , we obtain the system of over-determined seventeen linear and homogeneous PDEs, given as

$$\begin{aligned} \xi_x^1 &= 0, \quad \xi_u^1 = 0, \quad \xi_y^1 = 0, \quad \xi_t^1 + 5\eta_u = 0, \quad \xi_y^2 = 0, \quad \xi_u^2 = 0, \quad \xi_t^2 - ab\eta_y = 0, \quad \xi_x^2 + \eta_u = 0, \\ \xi_t^3 &= 0, \quad \xi_x^3 = 0, \quad \xi_u^3 = 0, \quad \xi_y^3 + 3\eta_u = 0, \quad \eta_x = 0, \quad \eta_{tu} = 0, \quad \eta_{yy} = 0, \quad \eta_{yu} = 0, \quad \eta_{uu} = 0. \end{aligned}$$

Solving the above system of PDEs, we obtain the Lie algebra of the 2D-gnBKP equation (1.6) that is spanned by

$$\begin{aligned} R_1 &= \frac{\partial}{\partial t}, \quad R_2 = \frac{\partial}{\partial x}, \quad R_3 = \frac{\partial}{\partial y}, \quad R_4 = \frac{\partial}{\partial u}, \\ R_5 &= abt \frac{\partial}{\partial x} + y \frac{\partial}{\partial u}, \quad R_6 = 5t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y} - u \frac{\partial}{\partial u}. \end{aligned} \quad (2.5)$$

We see that point symmetries (2.5) lead to the corresponding one-parameter group of transformations given by

$$T_1 : \bar{t} = t + \nu, \quad \bar{x} = x, \quad \bar{y} = y, \quad \bar{u} = u,$$

$$\begin{aligned}
T_2 : \bar{t} &= t, \bar{x} = x + v, \bar{y} = y, \bar{u} = u, \\
T_3 : \bar{t} &= t, \bar{x} = x, \bar{y} = y + v, \bar{u} = u, \\
T_4 : \bar{t} &= t, \bar{x} = x, \bar{y} = y, \bar{u} = u + v, \\
T_5 : \bar{t} &= t, \bar{x} = x + abvt, \bar{y} = y, \bar{u} = u + vy, \\
T_6 : \bar{t} &= te^{5v}, \bar{x} = xe^v, \bar{y} = ye^{3v}, \bar{u} = ue^{-v}.
\end{aligned}$$

The above transformations indicate that if $u = H(t, x, y)$ is a solution of (1.6), then so are the functions

$$\begin{aligned}
u_1(t, x, y) &= H(t - v, x, y, u), \\
u_2(t, x, y) &= H(t, x - v, y, u), \\
u_3(t, x, y) &= H(t, x, y - v, u), \\
u_4(t, x, y) &= H(t, x, y) + v, \\
u_5(t, x, y) &= H(t, x - abvt, y) + vy, \\
u_6(t, x, y) &= e^{-v}H(te^{-5v}, xe^{-v}, ye^{-3v}).
\end{aligned}$$

2.2. Optimal system (OS) of one-dimensional subalgebras

An OS of one-dimensional subalgebras provides a structured way to find a set of non-equivalent, one-parameter subalgebras within a Lie algebra. This set is “optimal” in the sense that any other one-parameter subalgebra is equivalent to one of these under a transformation within the adjoint representation. This classification is crucial for finding symmetry-reduced solutions and invariant solutions of differential equations.

One of the methods for computing an OS of one-dimensional subalgebras is described in [35]. It gives a systematic method for classifying inequivalent one-dimensional subalgebras of a Lie algebra, which is pivotal for carrying out symmetry reductions of differential equations. First, we need to compute the commutator table of all symmetries and the adjoint representation of subalgebras. Recall that the commutator of R_i and R_j is defined as $[R_i, R_j] = R_i R_j - R_j R_i$. Thus, all the commutation relations for (2.5) are presented in Table 1.

Table 1. Commutator table of (2.5).

$[R_i, R_j]$	R_1	R_2	R_3	R_4	R_5	R_6
R_1	0	0	0	0	abR_2	$5R_1$
R_2	0	0	0	0	0	R_2
R_3	0	0	0	0	R_4	$3R_3$
R_4	0	0	0	0	0	$-R_4$
R_5	$-abR_2$	0	$-R_4$	0	0	$-4R_5$
R_6	$-5R_1$	$-R_2$	$-3R_3$	R_4	$4R_5$	0

The entries of the adjoint representation are computed using

$$\text{Ad}(\exp(vR_i))R_j = R_j - v[R_i, R_j] + \frac{v^2}{2}[R_i, [R_i, R_j]] - \dots.$$

The results are tabulated in Table 2.

Table 2. Adjoint representation of subalgebras.

$[R_i, R_j]$	R_1	R_2	R_3	R_4	R_5	R_6
R_1	R_1	R_2	R_3	R_4	$-abvR_2 + R_5$	$-5vR_1 + R_6$
R_2	R_1	R_2	R_3	R_4	R_5	$-vR_2 + R_6$
R_3	R_1	R_2	R_3	R_4	$-vR_4 + R_5$	$-3vR_3 + R_6$
R_4	R_1	R_2	R_3	R_4	R_5	$vR_4 + R_6$
R_5	$R_1 + abvR_2$	R_2	$R_3 + vR_4$	R_3	R_5	$4vR_5 + R_6$
R_6	$e^{5v}R_1$	e^vR_2	$e^{3v}R_3$	$e^{-v}R_4$	$e^{-4v}R_5$	R_6

Consequently, using the Tables 1 and 2, and following the procedure given in [35], we infer that the OS of one-dimensional subalgebras is

$$\{R_6, R_2 + a_1R_4, R_1 + a_1R_2 + a_4R_4, R_2 + a_1R_3 + a_2R_4, R_1 + a_2R_2 + a_3R_3 + a_4R_4 + a_5R_5\}. \quad (2.6)$$

3. Symmetry reductions and solutions of (1.6) using an OS

We use an OS of one-dimensional subalgebras (2.6) to perform symmetry reductions and determine exact explicit solutions of the 2D-gnBKP equation (1.6). The symmetry reductions will lead us to nonlinear ordinary differential equations (NODEs), which are then solved by using different techniques to obtain exact solutions of (1.6).

3.1. Case 1. R_6

We first consider the symmetry $R_6 = 5t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}$. The associated Lagrangian system yields the invariants

$$\tau = t^{-1/5}x, \quad \xi = t^{-3/5}y, \quad F(\tau, \xi) = t^{1/5}u,$$

and using these invariants, Eq (1.6) is transformed into the NPDE

$$5abF_{\tau\tau}F_{\xi} - 5F_{\xi\xi} - \tau F_{\tau\tau} - 2F_{\tau} - 3\xi F_{\tau\xi} + 15abF_{\tau}^2F_{\tau\tau} + 5abF_{\tau}F_{\tau\xi} + 5bF_{\tau\tau\xi} + 5abF_{\tau}F_{\tau\tau\tau} + 5abF_{\tau\tau}F_{\tau\tau\tau} + 5F_{\tau\tau\tau\tau\tau} = 0. \quad (3.1)$$

Equation (3.1) has two point symmetries, viz.,

$$S_1 = \frac{\partial}{\partial F}, \quad S_2 = 5ab\frac{\partial}{\partial \tau} + \xi\frac{\partial}{\partial F}.$$

The symmetry S_1 gives us no solution. However, the symmetry S_2 provides us with invariants $\epsilon = \xi$, $G(\epsilon) = F - \tau\xi/(5ab)$, which transform NPDE (3.1) into

$$G_{\epsilon\epsilon} + \frac{4\epsilon}{25ab} = 0,$$

whose solution is $G(\epsilon) = -2\epsilon^3/(75ab) + C_1\epsilon + C_2$, where C_1, C_2 are constants. Thus, the exact explicit solution of 2D-gnBKP equation (1.6) is

$$u(t, x, y) = \frac{1}{5ab}xyt^{-1} - \frac{2}{75ab}y^3t^{-2} + C_1yt^{-4/5} + C_2t^{-1/5}. \quad (3.2)$$

The graph of solution (3.2) is portrayed in Figure 1.

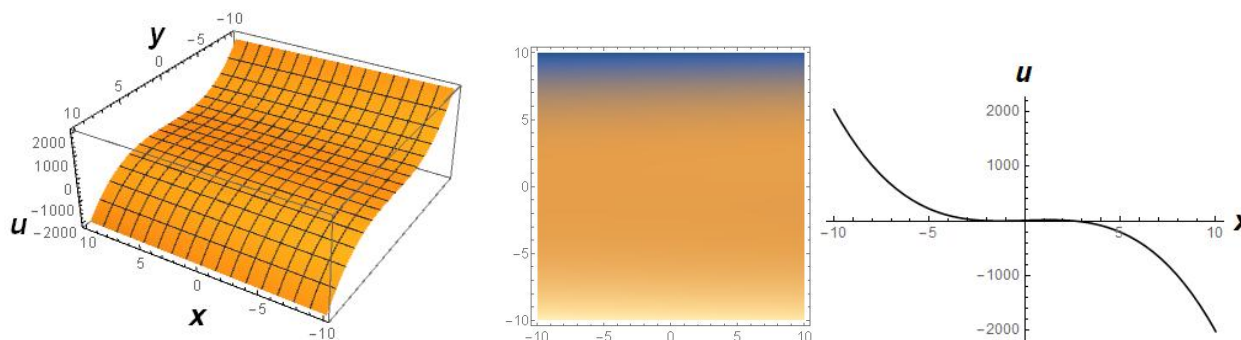


Figure 1. The 3D and 2D portrait of solution (3.2), where $a = 5$, $b = 1$, $C_1 = 1$, $C_2 = 5$, $t = 0.05$, $-10 \leq x, y \leq 10$.

3.2. Case 2. $R = R_2 + a_1 R_4$

The symmetry $R = \frac{\partial}{\partial x} + a_1 \frac{\partial}{\partial u}$, where a_1 is a constant, gives the invariants $\psi = t$, $\theta = y$, $A(\psi, \theta) = u - a_1 x$, which transform the 2D-gnBKP equation (1.6) into $A_{\theta\theta} = 0$, whose solution is $A(\psi, \theta) = \theta B(\psi) + C(\psi)$, where $B(\psi)$ and $C(\psi)$ are arbitrary functions of ψ . Hence, the exact solution of 2D-gnBKP equation (1.6) is

$$u(t, x, y) = a_1 x + yB(t) + C(t). \quad (3.3)$$

The graphic portrait of solution (3.3) is illustrated in Figure 2.

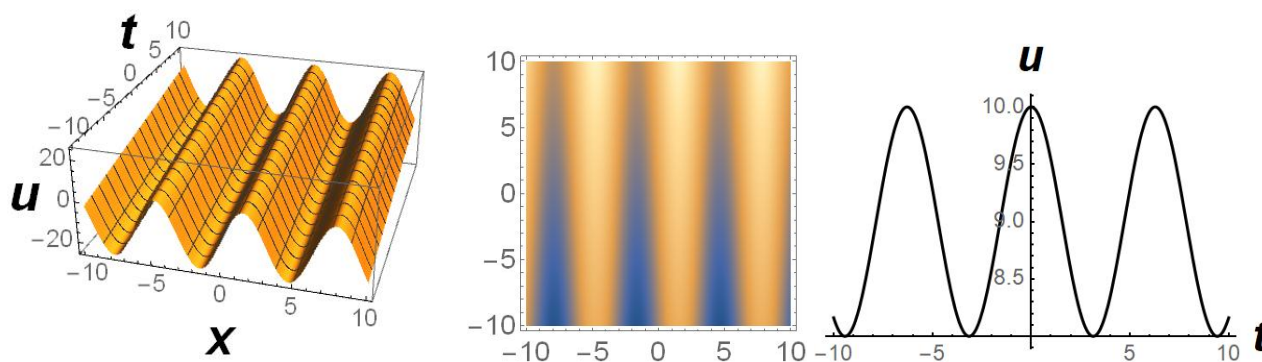


Figure 2. The 3D and 2D portrait of solution (3.3), where $B(t) = \sin(t)$, $C(t) = \cos(t)$ with $a_1 = 1$, $y = 15$, $-10 \leq t, x \leq 10$.

3.3. Case 3. $R = R_1 + a_1 R_2 + a_2 R_4$

The symmetry $R = \frac{\partial}{\partial t} + a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial u}$, with a_1, a_2 being constants, gives invariants $\Theta = x - a_1 t$, $\Phi = y$, $Q(\Theta, \Phi) = u - a_2 t$, which transform the 2D-gnBKP equation (1.6) into the NPDE

$$\begin{aligned} & abQ_{\Theta\Theta}Q_{\Phi} + abQ_{\Theta}Q_{\Theta\Phi} - a_1Q_{\Theta\Theta} - Q_{\Phi\Phi} + 3abQ_{\Theta}^2Q_{\Theta\Theta} + abQ_{\Theta\Theta}Q_{\Theta\Theta\Theta} \\ & + abQ_{\Theta}Q_{\Theta\Theta\Theta\Theta} + bQ_{\Theta\Theta\Theta\Phi} + Q_{\Theta\Theta\Theta\Theta\Theta\Theta} = 0. \end{aligned} \quad (3.4)$$

The NPDE (3.4) has four symmetries, viz.,

$$X_1 = \frac{\partial}{\partial \Theta}, \quad X_2 = \frac{\partial}{\partial \Phi}, \quad X_3 = \frac{\partial}{\partial Q}, \quad X_4 = ab\Theta \frac{\partial}{\partial \Theta} + 3ab\Phi \frac{\partial}{\partial \Phi} + (4a_1\Phi - abQ) \frac{\partial}{\partial Q}.$$

3.3.1. Symmetry reduction using $X_1 = \frac{\partial}{\partial \Theta}$

From the symmetry X_1 , we achieve the invariants $\sigma = \Phi$, $M(\sigma) = Q$, and using these invariants, NPDE (3.4) is transformed into $M_{\sigma\sigma} = 0$, whose solution is $M(\sigma) = C_1\sigma + C_2$. Hence, the exact solution of 2D-gnBKP equation (1.6) is

$$u(t, x, y) = a_2t + C_1y + C_2$$

with C_1, C_2 as constants.

3.3.2. Symmetry reduction using $X_2 = \frac{\partial}{\partial \Phi}$

The symmetry X_2 produces the invariants $\rho = \Theta$, $Y(\rho) = Q$, which transform NPDE (3.4) into the sixth-order NLODE

$$3abY'^2Y'' - a_1Y'' + abY''Y''' + abY'Y'''' + Y'''''' = 0. \quad (3.5)$$

Solutions of (3.5) via the simplest equation method

We employ the simplest equation method [25, 26] to find closed-form solutions of Eq (3.5). The equations used here are the Bernoulli equation:

$$\mathcal{R}'(\rho) = m\mathcal{R}(\rho) + l\mathcal{R}^2(\rho) \quad (3.6)$$

and the Riccati equation:

$$\mathcal{R}'(\rho) = q\mathcal{R}^2(\rho) + r\mathcal{R}(\rho) + s, \quad (3.7)$$

with m, l, q, r, s as constants. We search for solutions of the type

$$Y(\rho) = \sum_{i=0}^M C_i(\mathcal{R}(\rho))^i, \quad (3.8)$$

where $\mathcal{R}(\rho)$ solves the Bernoulli or Riccati equation, M is an integer that is positive, and $C_0, \dots, C_M$ are unknown constants.

The exact solutions of (3.6) are given by [26]

$$\mathcal{R}(\rho) = \frac{-mC_1}{l(C_1 + \cosh(m\{\rho + \rho_0\}) - \sinh(m\{\rho + \rho_0\}))}$$

and

$$\mathcal{R}(\rho) = \frac{-m(\cosh(m\{\rho + \rho_0\}) + \sinh(m\{\rho + \rho_0\}))}{l(C_2 + \cosh(m\{\rho + \rho_0\}) + \sinh(m\{\rho + \rho_0\}))},$$

where C_1 , C_2 , and ρ_0 are constants.

Riccati equation (3.7) has solutions [26]

$$\mathcal{R}(\rho) = -\frac{r}{2q} - \frac{z}{2q} \tanh\left(\frac{z}{2}\{\rho + \rho_0\}\right) \quad (3.9)$$

and

$$\mathcal{R}(\rho) = -\frac{r}{2q} - \frac{z}{2q} \tanh\left(\frac{z}{2}\rho\right) + \frac{\operatorname{sech}\left(\frac{z}{2}\rho\right)}{\rho_0 \cosh\left(\frac{z}{2}\rho\right) - \frac{2q}{z} \sinh\left(\frac{z}{2}\rho\right)} \quad (3.10)$$

with $z^2 = r^2 - 4qs$.

Solution of (3.5) via the Bernoulli equation

The balancing procedure gives $M = 1$ when applied to (3.5) and hence (3.8) becomes

$$Y(\rho) = C_0 + C_1 \mathcal{R}(\rho). \quad (3.11)$$

We substitute $Y(\rho)$ into (3.5), and by using Bernoulli equation (3.6), hereafter we equate the coefficients of like powers of $\mathcal{R}$ to zero. We get

$$\begin{aligned} m^4 - a_1 &= 0, \\ 6ablC_1 + abC_1^2 + 120l &= 0, \\ 20ablC_1 + 3abC_1^2 + 420l^2 &= 0, \\ 166ablC_1 + 27abC_1^2 + 3360l^2 &= 0, \\ 6ablmC_1 + abmC_1^2 + 120l^2m &= 0, \\ 2abm^4C_1 + 63lm^4 - 3la_1 &= 0, \\ 26ablm^4C_1 + 3abm^4C_1^2 + 602l^2m^4 - 2l^2a_1 &= 0. \end{aligned}$$

The above algebraic equations yield

$$C_0 = C_0, \quad C_1 = -2l, \quad a_1 = m^4, \quad ab = 15.$$

Therefore, the exact solutions of 2D-gnBKP equation (1.6) are

$$u(t, x, y) = C_0 - 2l \left\{ \frac{-mC_1}{l(C_1 + \cosh(m\{\rho + \rho_0\}) - \sinh(m\{\rho + \rho_0\}))} \right\} + a_2t \quad (3.12)$$

and

$$u(t, x, y) = C_0 - 2l \left\{ \frac{-m(\cosh(m\{\rho + \rho_0\}) + \sinh(m\{\rho + \rho_0\}))}{l(C_2 + \cosh(m\{\rho + \rho_0\}) + \sinh(m\{\rho + \rho_0\}))} \right\} + a_2t, \quad (3.13)$$

where $\rho = x - a_1t$, C_1 , C_2 , and ρ_0 are constants. The portrait of the solution (3.12) is shown in Figure 3, whereas the portrait of solution (3.13) is depicted in Figure 4.

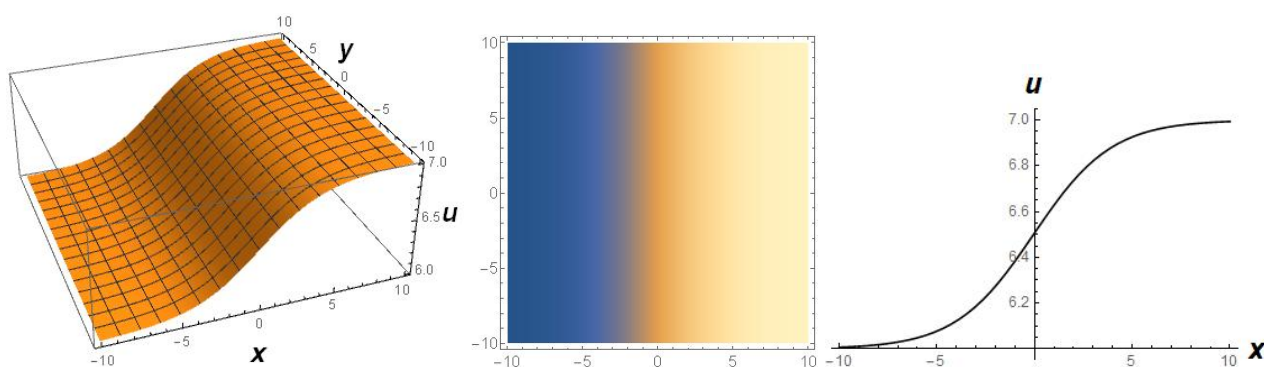


Figure 3. The 3D and 2D portraits of solution (3.12), where $m = 0.5$, $l = 0.5$, $\rho_0 = 1.5$, $C_0 = 5$, $a_1 = a_2 = 1$, $C_1 = 0.5$, $t = 1$, $-10 \leq x, y \leq 10$.

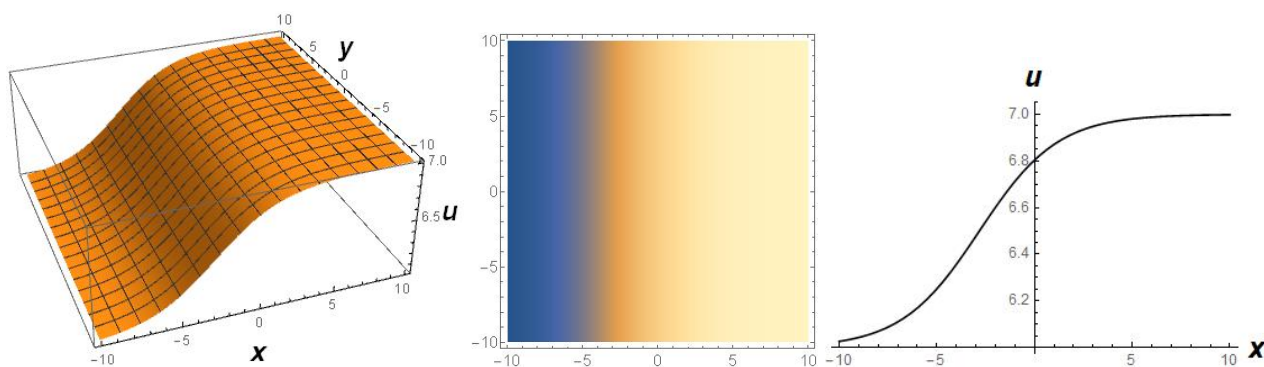


Figure 4. The 3D and 2D portraits of solution (3.13), where $m = 0.5$, $l = 0.5$, $\rho_0 = 1.5$, $C_0 = 5$, $a_1 = a_2 = 1$, $C_2 = 0.5$, $t = 1$, $-10 \leq x, y \leq 10$.

Solutions of (3.5) via the Riccati equation

Following the above procedure, but using the Riccati equation (3.7), we obtain

$$C_0 = C_0, \quad C_1 = -2q, \quad ab = 15, \quad a_1 = 16q^2s^2 - 8qr^2s + r^4.$$

Consequently, the exact solutions of 2D-gnBKP equation (1.6) are

$$u(t, x, y) = C_0 - 2q \left[-\frac{r}{2q} - \frac{z}{2q} \tanh \left(\frac{z}{2} \{\rho + \rho_0\} \right) \right] + a_2 t \quad (3.14)$$

and

$$u(t, x, y) = C_0 - 2q \left\{ -\frac{r}{2q} - \frac{z}{2q} \tanh \left(\frac{z}{2} \rho \right) + \frac{\operatorname{sech} \left(\frac{z}{2} \rho \right)}{\rho_0 \cosh \left(\frac{z}{2} \rho \right) - \frac{2q}{z} \sinh \left(\frac{z}{2} \rho \right)} \right\} + a_2 t, \quad (3.15)$$

where $\rho = x - a_1 t$ with C_0, ρ_0 being constants. The portrait of solution (3.14) is depicted in Figure 5, while the portrait of solution (3.15) is depicted in Figure 6.

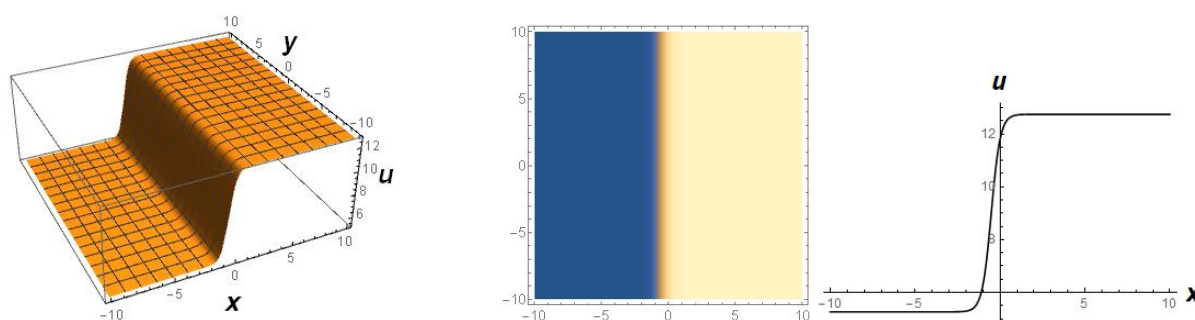


Figure 5. The 3D and 2D portraits of the kink solution (3.14), where $q = 0.5$, $r = 4$, $s = 1$, $\rho_0 = 1.5$, $a_1 = a_2 = 1$, $C_0 = 5$, $t = 0.005$, $-10 \leq x, y \leq 10$.

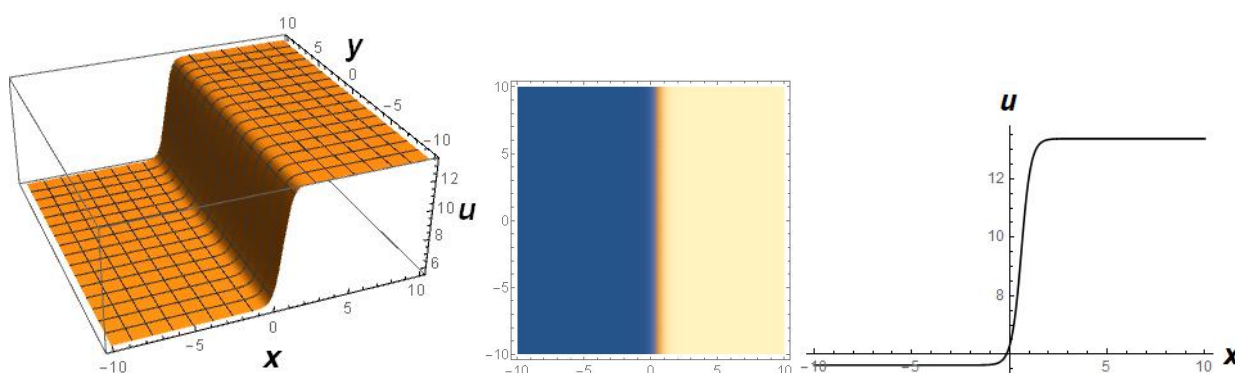


Figure 6. The 3D and 2D portraits of a kink solution (3.15), where $q = 0.5$, $r = 0.5$, $s = 0.5$, $\rho_0 = 1.5$, $a_1 = a_2 = 1$, $C_0 = 5$, $t = 0.5$, $-10 \leq x, y \leq 10$.

3.3.3. Symmetry reduction using $X = X_1 + \kappa X_2$

Now using the symmetry $X = \frac{\partial}{\partial \Theta} + \kappa \frac{\partial}{\partial \Phi}$ with κ a constant, we achieve the invariants $\Omega = \Phi - \kappa \Theta$, $Q(\Omega) = S$, which transform NPDE (3.4) into the NODE

$$2ab\kappa^2 S' S'' - (1 + a_1 \kappa^2) S'' + 3ab\kappa^4 S'^2 S'' - ab\kappa^5 S'' S''' - ab\kappa^5 S' S'''' + \kappa^6 S''''' = 0. \quad (3.16)$$

Solutions of (3.16) via the (G'/G) -expansion technique

We utilize the (G'/G) -expansion technique [12] to obtain exact solutions of (3.16). We assume that a solution of Eq (3.16) is

$$S(\Omega) = \sum_{i=0}^N A_i \left(\frac{G'(\Omega)}{G(\Omega)} \right)^i, \quad (3.17)$$

with A_i being parameters to be found and N is a positive nonzero constant. The function $G(\Omega)$ solves

$$G'' + \lambda G' + \mu G = 0, \quad (3.18)$$

with μ, λ as constants. The balancing procedure when applied to (3.16) gives $N = 1$ and so (3.17) becomes

$$S(\Omega) = A_0 + A_1 \frac{G'(\Omega)}{G(\Omega)}. \quad (3.19)$$

Substituting (3.19) into (3.16), engaging (3.18), and subsequently comparing the coefficients of powers of $G'(\Omega)/G(\Omega)$ gives eight algebraic equations in A_0, A_1 , given by

$$\begin{aligned}
6\kappa + A_1 &= 0, \\
24b\kappa^3 + 4ab\kappa^2 A_1 - 166ab\kappa^5 \lambda^2 A_1 - 92ab\kappa^5 \mu A_1 - 27ab\kappa^4 \lambda^2 A_1^2 - 18ab\kappa^4 \mu A_1^2 &= 0, \\
24b\kappa^3 + 4ab\kappa^2 A_1 - 166ab\kappa^5 \lambda^2 A_1 - 92ab\kappa^5 \mu A_1 - 27ab\kappa^4 \lambda^2 A_1^2 - 18ab\kappa^4 \mu A_1^2 &= 0, \\
b\kappa^3 \lambda^2 + 8b\kappa^3 \mu^2 + \kappa^2 \mu a_1 + 2ab\kappa \mu A_1 - 2ab\kappa^5 \lambda^2 \mu A_1 - 10ab\kappa^5 \mu^2 A_1 - 3ab\kappa^4 \mu^2 A_1^2 + 1 &= 0, \\
\lambda^2 + b\kappa^3 \lambda^4 + 2\mu + 22b\kappa^3 \lambda^2 \mu + 16b\kappa^3 \mu^2 + \kappa^2 \lambda^2 a_1 + 2\kappa^2 \mu a_1 + 4ab\kappa^2 \lambda^2 \mu A_1 - 4ab\kappa^5 \lambda^4 \mu A_1 \\
+ 4ab\kappa^2 \mu^2 A_1 - 42ab\kappa^5 \lambda^2 \mu^2 A_1 - 20ab\kappa^5 \mu^3 A_1 - 9ab\kappa^4 \lambda^2 \mu^2 A_1^2 - 6ab\kappa^4 \mu^3 A_1^2 &= 0, \\
3\lambda + 15b\kappa^3 \lambda^3 + 60b\kappa^3 \lambda \mu + 3\kappa^2 \lambda a_1 + 2ab\kappa^2 \lambda^3 A_1 - 2ab\kappa^5 \lambda^5 A_1 + 12ab\kappa^2 \lambda \mu A_1 - 58ab\kappa^5 \lambda^3 \mu A_1 \\
- 114ab\kappa^5 \lambda \mu^2 A_1 - 9ab\kappa^4 \lambda^3 \mu A_1^2 - 27ab\kappa^4 \lambda \mu^2 A_1^2 &= 0, \\
50b\kappa^3 \lambda^2 + 40b\kappa^3 \mu + 2\kappa^2 a_1 + 8ab\kappa^2 \lambda^2 A_1 - 26ab\kappa^5 \lambda^4 A_1 + 8ab\kappa^2 \mu A_1 - 192ab\kappa^5 \lambda^2 \mu A_1 \\
- 76ab\kappa^5 \mu^2 A_1 - 3ab\kappa^4 \lambda^4 A_1^2 - 36ab\kappa^4 \lambda^2 \mu A_1^2 - 18ab\kappa^4 \mu^2 A_1^2 + 2 &= 0, \\
60b\kappa^3 \lambda A_1 + 10ab\kappa^2 \lambda A_1^2 - 100ab\kappa^5 \lambda^3 A_1^2 - 230ab\kappa^5 \lambda \mu A_1^2 - 15ab\kappa^4 \lambda^3 A_1^3 - 45ab\kappa^4 \lambda \mu A_1^3 &= 0.
\end{aligned}$$

The above equations give

$$A_0 = A_0, \quad A_1 = -6\kappa, \quad a = -\frac{1}{1 - \kappa^3 \lambda^2 + 4\mu \kappa^3}, \quad b = \frac{(\kappa + 1)(\kappa - 1) + \kappa^3(\lambda^2 - 4\mu)}{\lambda^2 - 4\mu - \kappa^3(72\mu - 24\lambda^4 - 96\mu^2)}.$$

Hence, the three exact solutions of 2D-gnBKP equation (1.6) are given below.

Case 1. For $\lambda^2 - 4\mu > 0$, we secure a hyperbolic function solution

$$u(t, x, y) = A_0 - 6\kappa \left\{ -\frac{\lambda}{2} + \delta \left(\frac{C_1 \sinh(\delta \Omega) + C_2 \cosh(\delta \Omega)}{C_1 \cosh(\delta \Omega) + C_2 \sinh(\delta \Omega)} \right) \right\}, \quad (3.20)$$

with C_1, C_2 being constants, $\Omega = y - \kappa(x - a_1 t)$, and $\delta = \frac{1}{2} \sqrt{\lambda^2 - 4\mu}$.

Case 2. For $\lambda^2 - 4\mu < 0$, we acquire a trigonometric function solution

$$u(t, x, y) = A_0 - 6\kappa \left\{ -\frac{\lambda}{2} + \delta \left(\frac{-C_1 \sin(\delta \Omega) + C_2 \cos(\delta \Omega)}{C_1 \cos(\delta \Omega) + C_2 \sin(\delta \Omega)} \right) \right\}, \quad (3.21)$$

with C_1, C_2 being constants, $\Omega = y - \kappa(x - a_1 t)$, and $\delta = \frac{1}{2} \sqrt{4\mu - \lambda^2}$.

Case 3. For $\lambda^2 - 4\mu = 0$, we get a rational function solution

$$u(t, x, y) = A_0 - 6\kappa \left\{ -\frac{\lambda}{2} + \frac{C_2}{C_1 + \Omega C_2} \right\}, \quad (3.22)$$

with C_1, C_2 being constants, $\Omega = y - \kappa(x - a_1 t)$, and $\delta = \frac{1}{2} \sqrt{\lambda^2 - 4\mu}$.

The portrait of solution (3.20) is depicted in Figure 7, the portrait of solution (3.21) is depicted in Figure 8, whereas the portrait of solution (3.22) is given in Figure 9.

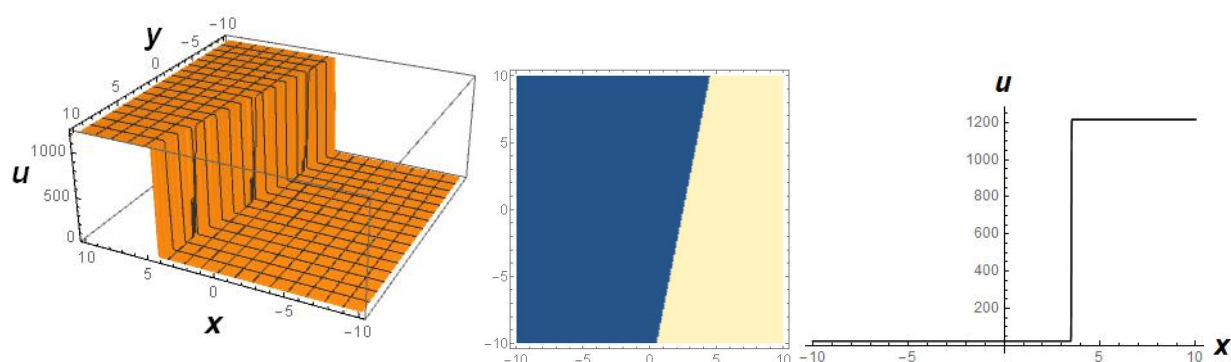


Figure 7. The portraits of the hyperbolic function solution (3.20), where $C_1 = 10$, $C_2 = 15$, $\lambda = 40$, $\mu = 5$, $a_1 = 5$, $\kappa = 5$, $A_0 = 20$, $t = 0.5$, $-10 \leq x, y \leq 10$.

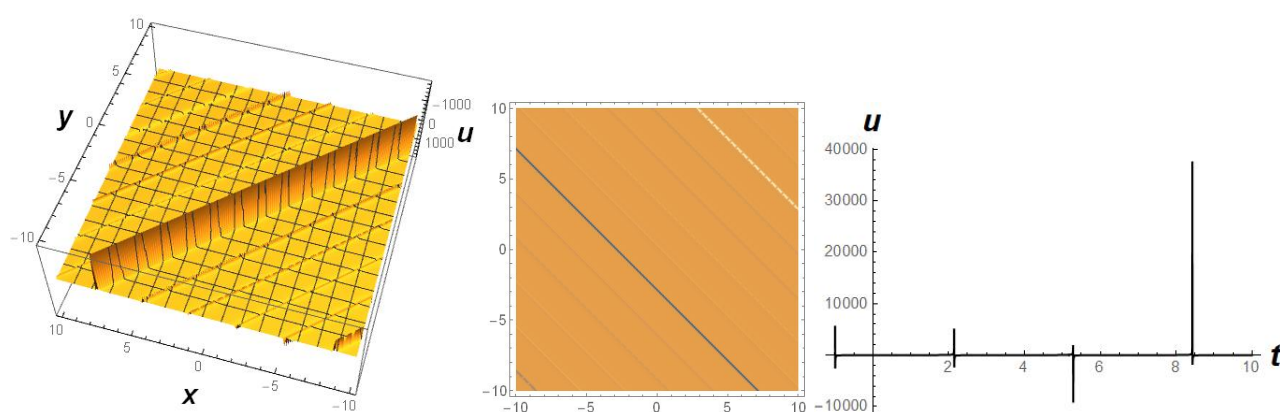


Figure 8. The portraits of the trigonometric function solution (3.21), where $C_1 = 1$, $C_2 = 0$, $\lambda = 0$, $\mu = 1$, $a_1 = 1$, $\kappa = -1$, $A_0 = 0$, $t = 5$, $-10 \leq x, y \leq 10$.

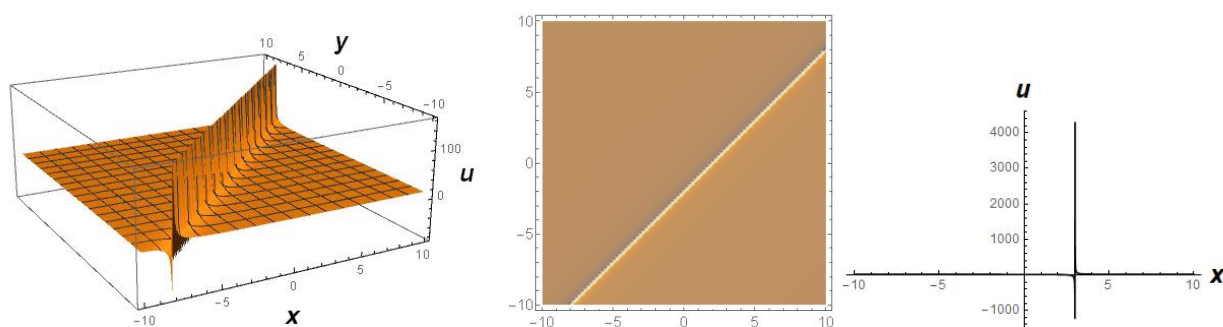


Figure 9. The portraits of the rational function solution (3.22), where $C_1 = 1$, $C_2 = 1$, $\lambda = 4$, $\mu = 4$, $a_1 = 1$, $\kappa = 1$, $A_0 = 2$, $t = 1$, $-10 \leq x, y \leq 10$.

Note that symmetries X_3 and X_4 do not produce any results of importance.

3.4. Case 4. $R = R_2 + a_1 R_3 + a_2 R_4$

The associated characteristic equations of symmetry $R = \frac{\partial}{\partial x} + a_1 \frac{\partial}{\partial y} + a_2 \frac{\partial}{\partial u}$, with a_1, a_2 being constants, yield the invariants

$$\delta = t, \quad \alpha = x - \frac{1}{a_1}y, \quad H(\delta, \alpha) = u - \frac{a_2}{a_1}y,$$

and using these invariants, the 2D-gnBKP equation (1.6) transforms into

$$\begin{aligned} & a_1^2 H_{\delta\alpha} - 2aba_1 H_\alpha H_{\alpha\alpha} + (aba_1 a_2 - 1) H_{\alpha\alpha} + 3aba_1^2 H_\alpha^2 H_{\alpha\alpha} - ba_1 H_{\alpha\alpha\alpha} \\ & + aba_1^2 H_{\alpha\alpha} H_{\alpha\alpha\alpha} + aba_1^2 H_\alpha H_{\alpha\alpha\alpha} + a_1^2 H_{\alpha\alpha\alpha\alpha} = 0. \end{aligned} \quad (3.23)$$

The NPDE (3.23) has four symmetries given by

$$T_1 = \frac{\partial}{\partial \delta}, \quad T_2 = \frac{\partial}{\partial \alpha}, \quad T_3 = G(\delta) \frac{\partial}{\partial H}, \quad T_4 = F(\delta) \frac{\partial}{\partial H}.$$

3.4.1. Symmetry reduction using $T = T_1 + \omega T_2$

The symmetry $T = \frac{\partial}{\partial \delta} + \omega \frac{\partial}{\partial \alpha}$, with ω a constant, gives the invariants $\beta = \alpha - \omega\delta$, $M(\beta) = H$, that transform NPDE (3.23) into the NODE

$$\begin{aligned} & (aba_1 a_2 - \omega a_1^2 - 1)M'' - ba_1 M'''' - 2aba_1 M' M'' + aba_1^2 M'' M''' \\ & + 3aba_1^2 M'^2 M'' + aba_1^2 M' M'''' + a_1^2 M'''''' = 0. \end{aligned} \quad (3.24)$$

Solution of (3.24) by means of Kudryashov's technique

We invoke Kudryashov's technique [18] to compute the exact solution of Eq (3.24). We assume that the solution of (3.24) is

$$M(\beta) = \sum_{i=0}^P \mathcal{A}_i \mathcal{B}^i(\beta) \quad (3.25)$$

with $\mathcal{A}_i, \dots, \mathcal{A}_P$ being constants to be found, $P > 0$, and $\mathcal{B}(\beta)$ solves the Riccati equation

$$\mathcal{B}'(\beta) = \mathcal{B}^2(\beta) - \mathcal{B}(\beta), \quad (3.26)$$

whose solution is

$$\mathcal{B}(\beta) = \frac{1}{1 + \cosh(\beta) + \sinh(\beta)}. \quad (3.27)$$

Applying the balancing method on Eq (3.24) allows us to find P . Thus, we get $P = 1$ so the solution (3.25) is expressed as

$$M(\beta) = \mathcal{A}_0 + \mathcal{A}_1 \mathcal{B}(\beta). \quad (3.28)$$

Inserting (3.28) into (3.24), invoking (3.26), and then equating the coefficients of the like powers of $\mathcal{B}$ to zero gives

$$ab\mathcal{A}_1^2 + 6ab\mathcal{A}_1 + 120 = 0,$$

$$\begin{aligned}
ab\mathcal{A}_1^2 + 6ab\mathcal{A}_1 + 120 &= 0, \\
aba_1a_2 - a_1^2\omega - a_1b + a_1^2 - 1 &= 0, \\
3aba_1\mathcal{A}_1^2 + 20aba_1\mathcal{A}_1 - 2ab\mathcal{A}_1 - 12b + 420a_1 &= 0, \\
27aba_1\mathcal{A}_1^2 + 166aba_1\mathcal{A}_1 - 4abB_1 - 24b + 3360a_1 &= 0, \\
2aba_1\mathcal{A}_1 - 2aba_1^2\mathcal{A}_1 - 3aba_1a_2 + 3a_1^2\omega + 15a_1b - 63a_1^2 + 3 &= 0, \\
3aba_1^2\mathcal{A}_1^2 + 26aba_1^2\mathcal{A}_1 - 8aba_1\mathcal{A}_1 + 2aba_1a_2 - 2a_1^2\omega - 50a_1b + 602a_1^2 - 2 &= 0.
\end{aligned}$$

Using Maple to solve these algebraic equations, one potential set of values for $\mathcal{A}_0, \mathcal{A}_1$ is

$$\mathcal{A}_0 = \mathcal{A}_0, \quad \mathcal{A}_1 = -\frac{2(3b - 5a_1 - 10)}{b - 5a_1}, \quad a = -\frac{3(3b - 5a_1)^2}{b(a_1 - 1)(3b - 5a_1 - 10)},$$

$$\begin{aligned}
\omega = -(3b^2a_1^2 + 3b^2a_1a_2 - 8ba_1^3 - 30ba_1^2a_2 + 5a_1^4 + 75a_1^3a_2 - 3b^2a_1 - 2ba_1^2 + 5a_1^3 + 13ba_1 \\
- 15a_1^2 - 3b - 5a_1 + 10)/(a_1 - 1)(3b - 5a_1 - 10)a_1^2
\end{aligned}$$

with $b \neq 5a_1$, $a_1 \neq 0, -1$, $b \neq 0$, and $3b - 5a_1 \neq 10$. Thus, the exact solution of 2D-gnBKP equation (1.6) is

$$u(t, x, y) = \mathcal{A}_0 - \frac{2(3b - 5a_1 - 10)}{b - 5a_1} \left\{ \frac{1}{1 + \cosh(x - (y/a_1) - \omega t) + \sinh(x - (y/a_1) - \omega t)} \right\} + \frac{a_2}{a_1}y \quad (3.29)$$

with $\mathcal{A}_0$ being a constant. The portrait of solution (3.29) is depicted in Figure 10.

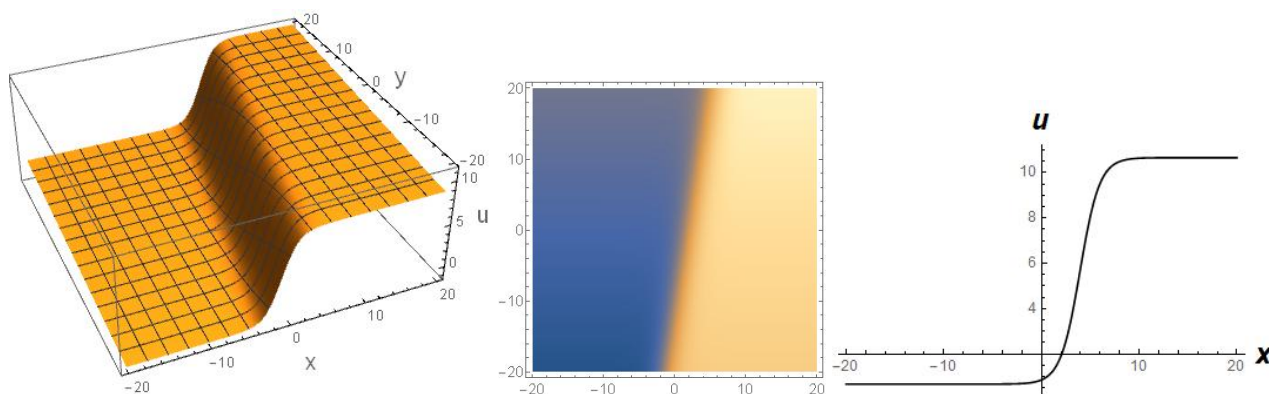


Figure 10. The 3D and 2D portraits of solution (3.29), where $b = 5$, $a_1 = 6$, $a_2 = 0.4$, $\mathcal{A}_0 = 10$, $t = 2$, $-20 \leq x, y \leq 20$.

3.5. Case 5. $R = R_1 + a_1R_2 + a_2R_3 + a_3R_4 + a_4R_5$

The symmetry R can be written as $R = \frac{\partial}{\partial t} + (a_1 + aba_4t) \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + (a_3 + a_4y) \frac{\partial}{\partial u}$. On solving its corresponding characteristic equations, we get invariants

$$\varphi = y - a_2t, \quad \varrho = x - a_1t - \frac{1}{2}aba_4t^2, \quad W(\varphi, \varrho) = u + \frac{1}{2}a_2a_4t^2 - (a_4y + a_3)t,$$

which transform 2D-gnBKP equation (1.6) into the NPDE

$$3abW_{\varrho}^2W_{\varrho\varrho} + abW_{\varrho}W_{\varrho\varrho\varrho} + abW_{\varrho}W_{\varphi\varrho} + bW_{r\varrho\varrho\varrho} + abW_{\varphi}W_{\varrho\varrho} + abW_{\varrho\varrho}W_{\varrho\varrho\varrho} - a_1W_{\varrho\varrho} - a_2W_{\varphi\varrho} - W_{\varphi\varphi} + W_{\varrho\varrho\varrho\varrho\varrho} = 0. \quad (3.30)$$

NPDE (3.30) has three translation symmetries, viz.,

$$P_1 = \frac{\partial}{\partial\varphi}, \quad P_2 = \frac{\partial}{\partial\varrho}, \quad P_3 = \frac{\partial}{\partial W}.$$

3.5.1. Symmetry reduction using $P_1 = \frac{\partial}{\partial\varphi}$

The symmetry P_1 gives the invariants $\Gamma = \varrho$, $Z(\Gamma) = W$, which transform NPDE (3.30) into the NODE

$$3abZ'^2Z'' - a_1Z'' + abZ''Z''' + abZ'Z'''' + Z'''''' = 0. \quad (3.31)$$

NODE (3.31) is similar to NODE (3.5), and using the above procedure, its solution is given by

$$u(t, x, y) = \mathcal{D}_0 - 2f \left\{ -\frac{g}{2f} - \frac{m}{2f} \tanh \left(\frac{1}{2}m(\Gamma + e) \right) \right\} - \frac{1}{2}a_2a_4t^2 + (a_4y + a_3)t, \quad (3.32)$$

where $ab = 15$, $a_1 = 16f^2h^2 - 8fhg^2 + g^4$, $\Gamma = x - a_1t - \frac{1}{2}aba_4t^2$, $m = \sqrt{g^2 - 4fh}$, and $\mathcal{D}_0, e, f, g, h$ are constants. The portrait of solution (3.32) is depicted in Figure 11.

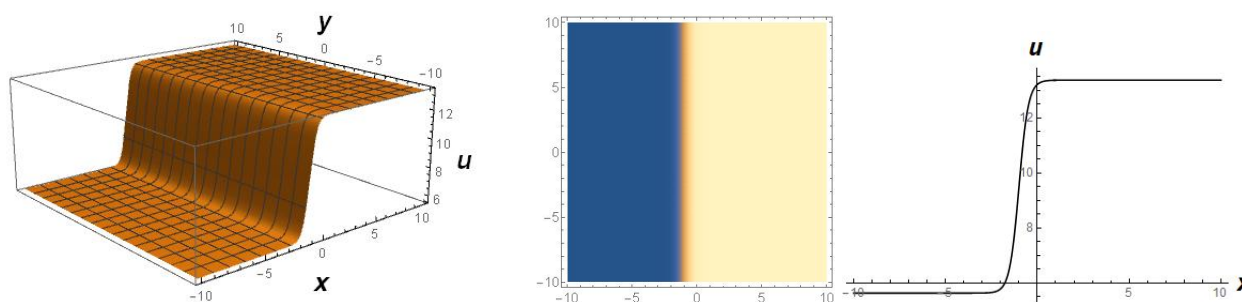


Figure 11. The 3D and 2D portraits of solution (3.32), where $q = 0.5$, $r = 4$, $s = 0.5$, $k = 1.5$, $a_2 = 15$, $a_3 = 1$, $a_4 = 0$, $\mathcal{D}_0 = 5$, $t = 0.5$, $-10 \leq x, y \leq 10$.

3.5.2. Symmetry reduction using $P_2 = \frac{\partial}{\partial\varrho}$

From the symmetry P_2 , we get the invariants $\varsigma = \varphi$, $\mathcal{M}(\sigma) = W$, which transform NPDE (3.30) into $M_{\varsigma\varsigma} = 0$, whose solution is $M(\varsigma) = K_1\varsigma + K_2$, where K_1 and K_2 are constants. Hence the exact solution of 2D-gnBKP equation (1.6) is

$$u(t, x, y) = -\frac{1}{2}a_2a_4t^2 + (a_4y + a_3)t + K_1(y - a_2t) + K_2.$$

3.5.3. Symmetry reduction using $P = P_1 + \vartheta P_2$

The symmetry $P = \frac{\partial}{\partial \varphi} + \vartheta \frac{\partial}{\partial \varrho}$, with ϑ a constant, yields the invariants $\delta = \varrho - \vartheta \varphi$, $\mathcal{Z}(\delta) = W$, which transform NPDE (3.30) into the sixth-order NODE

$$3ab\mathcal{Z}'^2\mathcal{Z}'' - (a_1 + \vartheta^2 - a_2\vartheta)\mathcal{Z}'' - 2ab\vartheta\mathcal{Z}'\mathcal{Z}'' - b\vartheta\mathcal{Z}'''' + ab\mathcal{Z}''\mathcal{Z}''' + ab\mathcal{Z}'\mathcal{Z}'''' + \mathcal{Z}'''''' = 0. \quad (3.33)$$

NODE (3.33) is similar to NODE (3.24), and using the above procedure, its solution can be written as

$$\mathcal{Z}(\delta) = \mathcal{K}_0 - \frac{2(3b\vartheta - 10\vartheta - 5)}{b\vartheta - 5} \left\{ \frac{1}{1 + \exp(\delta)} \right\}, \quad (3.34)$$

where $\mathcal{K}_0$ is a constant, $a_1 = 1 - b\vartheta - \vartheta^2 + a_2\vartheta$, and $a = \frac{3(b\vartheta - 5)^2}{b(\vartheta - 1)(3b\vartheta - 10\vartheta - 5)}$. Hence, the exact solution of 2D-gnBKP equation (1.6) is

$$u(t, x, y) = \mathcal{K}_0 - \frac{2(3b\vartheta - 10\vartheta - 5)}{b\vartheta - 5} \left\{ \frac{1}{1 + \exp(\delta)} \right\} - \frac{1}{2}a_4t^2a_2 + (a_4y + a_3)t, \quad (3.35)$$

where $\delta = x - \frac{1}{2}aba_4t^2 - a_1t - \vartheta(y - a_2t)$. The portrait of solution (3.35) is depicted in Figure 12.

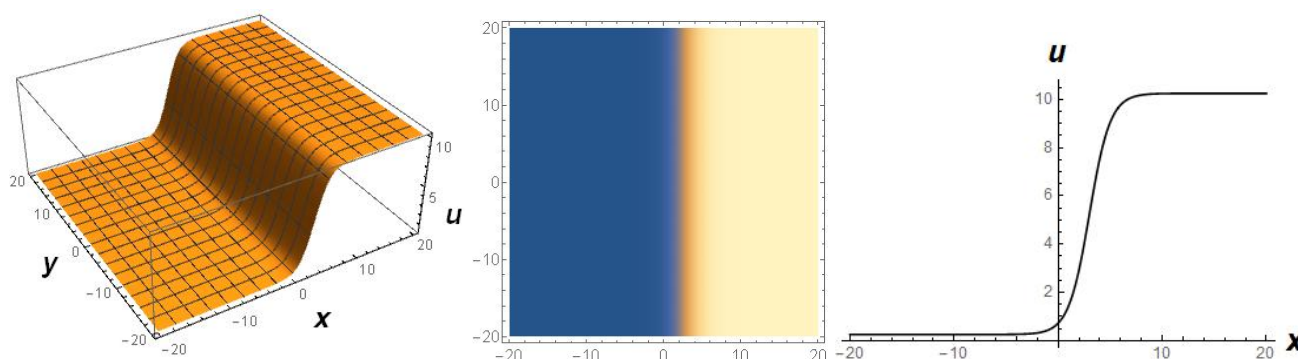


Figure 12. The 3D and 2D portrait of solution (3.35), where $b = 5$, $a_1 = 6$, $a_2 = 0.4$, $a_3 = 0.5$, $a_4 = 0$, $\mathcal{K}_0 = 10$, $\vartheta = 0.5$, $t = 0.5$, $-20 \leq x, y \leq 20$.

4. Conserved vectors of (1.6)

In this section, we construct conserved quantities of 2D-gnBKP equation (1.6) through the application of the multiplier approach and Ibragimov's method. It should be noted that both of these approaches are applicable to any PDE, regardless of whether it has a Lagrangian formulation or not. This is to say that the availability of variational principles is not necessary for obtaining the conserved vectors of a specific PDE.

4.1. Conserved vectors of (1.6) via the multiplier approach

We derive conserved quantities of 2D-gnBKP equation (1.6) using the multiplier approach [35]. We look for first-order multipliers $\Lambda = \Lambda(t, x, y, u, u_t, u_x, u_y)$. The determining equation for Λ is given as

$$\frac{\delta}{\delta u} \left[\Lambda \left\{ u_{tx} - u_{yy} + \left(abu_x^3 + abu_x u_{xxx} + u_{xxxx} \right)_x + b \left(u_{xxx} + a \left(u_x u_y \right)_x \right) \right\} \right] = 0, \quad (4.1)$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_t D_x \frac{\partial}{\partial u_{tx}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_y \frac{\partial}{\partial u_{xy}} + D_y^2 \frac{\partial}{\partial u_{yy}} - D_x^3 \frac{\partial}{\partial u_{xxx}} + \dots$$

is the Euler-Lagrange operator. Thus, expanding (4.1) and separating based on derivatives of u , we get a system of six linear homogeneous PDEs:

$$\Lambda_x = 0, \Lambda_u = 0, \Lambda_{u_t} = 0, \Lambda_{u_x} = 0, \Lambda_{u_y} = 0, \Lambda_{u_{yy}} = 0.$$

Solving the above PDEs for Λ , we get $\Lambda = V(t)y + W(t)$, which is a combination of two multipliers $\Lambda_1 = V(t)y$, $\Lambda_2 = W(t)$ with $V(t)$, $W(t)$ being functions of t . Now to determine conserved vectors, we use

$$\Lambda \left\{ u_{tx} - u_{yy} + \left(abu_x^3 + abu_x u_{xxx} + u_{xxxxx} \right)_x + b \left(u_{xxy} + a \left(u_x u_y \right)_x \right) \right\} = D_t Y^t + D_x Y^x + D_y Y^y, \quad (4.2)$$

where Y^t is the conserved density, and Y^x and Y^y are spatial fluxes. We have two cases:

Case 1. The conserved vector (Y_1^t, Y_1^x, Y_1^y) that relates to $\Lambda_1 = V(t)y$ is given by

$$\begin{aligned} Y_1^t &= \frac{1}{2} V(t) y u_x, \\ Y_1^x &= \frac{3}{4} b V(t) y u_{xy} + ab V(t) y u_x^3 - \frac{1}{4} ab V(t) u u_x + ab V(t) y u_x u_{xxx} - \frac{1}{4} b V(t) u_{xx} \\ &\quad - \frac{1}{4} ab V(t) y u u_{xy} + \frac{1}{2} V(t) y u_t + \frac{3}{4} ab V(t) y u_x u_y - \frac{1}{2} V'(t) y u + V(t) y u_{xxxxx}, \\ Y_1^y &= V(t) u - V(t) y u_y + \frac{1}{4} ab V(t) y u_x^2 + \frac{1}{4} ab V(t) y u u_{xx} + \frac{1}{4} b V(t) y u_{xxx}. \end{aligned}$$

Case 2. The corresponding conserved vector (Y_2^t, Y_2^x, Y_2^y) associated with $\Lambda_2 = W(t)$ is

$$\begin{aligned} Y_2^t &= \frac{1}{2} W(t) u_x, \\ Y_2^x &= \frac{1}{2} W(t) u_t - \frac{1}{2} W'(t) u + ab W(t) u_x u_{xxx} + ab W(t) u_x^3 - \frac{1}{4} ab W(t) u u_{xy} \\ &\quad + \frac{3}{4} ab W(t) u_x u_y + \frac{3}{4} b W(t) u_{xxy} + W(t) u_{xxxxx}, \\ Y_2^y &= \frac{1}{4} ab W(t) u_x^2 - W(t) u_y + \frac{1}{4} ab W(t) u u_{xx} + \frac{1}{4} b W(t) u_{xxx}. \end{aligned}$$

4.2. Conserved vectors of (1.6) via Ibragimov's method

Applying Ibragimov's method [40, 41], we compute the conserved quantities of 2D-gnBKP equation (1.6). We begin by writing the adjoint equation of (1.6) that is given by

$$E^* = \frac{\delta}{\delta u} \left[v \left\{ u_{tx} - u_{yy} + \left(abu_x^3 + abu_x u_{xxx} + u_{xxxxx} \right)_x + b \left(u_{xxy} + a \left(u_x u_y \right)_x \right) \right\} \right] = 0, \quad (4.3)$$

where we introduce the new variable $v = v(t, x, y)$. In this case, the Euler operator is given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_t D_x \frac{\partial}{\partial u_{tx}} + D_x^2 \frac{\partial}{\partial u_{xx}} + D_x D_y \frac{\partial}{\partial u_{xy}}$$

$$+ D_y^2 \frac{\partial}{\partial u_{yy}} + D_y D_x^3 \frac{\partial}{\partial u_{xxx}} + D_x^6 \frac{\partial}{\partial u_{xxxxx}} + \cdots,$$

where D_t , D_x , and D_y are specified in (2.4) [41]. Expanding (4.3) gives the adjoint equation as

$$E^* = v_{tx} - v_{yy} + abu_x v_{xy} + abu_y v_{xx} + 3abu_x^2 v_{xx} + 4abv_{xx} u_{xxx} + 3abu_{xx} v_{xxx} + bv_{xxx} u_{xy} + 2abv_x (u_{xy} + 3u_x u_{xx} + u_{xxx}) + abu_x v_{xxxx} + v_{xxxxx}. \quad (4.4)$$

Thereafter, by [41], we see that Eq (1.6) and its adjoint (4.4) have the Lagrangian given as

$$\mathcal{L} = v \left\{ u_{tx} - u_{yy} + (abu_x^3 + abu_x u_{xxx} + u_{xxxx})_x + b(u_{xxy} + a(u_x u_y)_x) \right\}. \quad (4.5)$$

We recall that Eq (1.6) admits six Lie point symmetries given by (2.5). Therefore, using the results given in [41], we construct the conserved vectors (J_i^t, J_i^x, J_i^y) of (1.6) for the given Lagrangian (4.5). We have the following six cases:

Case 1. The conserved vector (J_1^t, J_1^x, J_1^y) associated to symmetry R_1 is given by

$$\begin{aligned} J_1^t &= abvu_x u_{xy} - vu_{yy} + abvu_y u_{xx} + 3abvu_x^2 u_{xx} + abvu_{xx} u_{xxx} + bv u_{xxy} + \frac{1}{2} v_x u_t \\ &\quad + abvu_x u_{xxx} + vu_{xxxx} + \frac{1}{2} vu_{tx}, \\ J_1^x &= \frac{1}{2} abv_y u_x u_t + abu_y v_x u_t + 3abu_x^2 v_x u_t + \frac{1}{2} abvu_{xy} u_t + 2abu_{xx} v_{xx} u_t + \frac{3}{4} bv_{xxy} u_t \\ &\quad + 2abv_x u_{xxx} u_t + abu_x v_{xxx} u_t + v_{xxxx} u_t + \frac{1}{2} u_t v_t - \frac{1}{2} abvu_x u_{ty} - \frac{1}{4} bv_{xx} u_{ty} - abvu_y u_{tx} \\ &\quad - 3abvu_x^2 u_{tx} - \frac{1}{2} bv_{xy} u_{tx} - abv_x u_{xx} u_{tx} - abu_x v_{xx} u_{tx} - abvu_{xxx} u_{tx} - v_{xxxx} u_{tx} \\ &\quad + \frac{1}{2} bv_x u_{txy} + \frac{1}{4} bv_y u_{txx} + abu_x v_x u_{txx} + v_{xxx} u_{txx} - \frac{3}{4} bv u_{txxy} - abvu_x u_{txxx} - v_{xx} u_{txxx} \\ &\quad + v_x u_{txxxx} - vu_{txxxx} - \frac{1}{2} vu_{tt}, \\ J_1^y &= \frac{1}{2} abu_x v_x u_t - v_y u_t - \frac{1}{2} abvu_{xx} u_t + \frac{1}{4} bv_{xxx} u_t + vu_{ty} - \frac{1}{2} abvu_x u_{tx} - \frac{1}{4} bv_{xx} u_{tx} \\ &\quad + \frac{1}{4} bv_x u_{txx} - \frac{1}{4} bv u_{txxx}. \end{aligned}$$

Case 2. The conserved vector (J_2^t, J_2^x, J_2^y) linked to symmetry R_2 is

$$\begin{aligned} J_2^t &= \frac{1}{2} u_x v_x - \frac{1}{2} vu_{xx}, \\ J_2^x &= \frac{1}{2} abv_y u_x^2 - vu_{yy} + abu_y u_x v_x + 3abu_x^3 v_x + abvu_x u_{xy} - \frac{1}{2} bv_{xy} u_{xx} - abv_x u_{xx}^2 \\ &\quad - \frac{1}{4} bu_{xy} v_{xx} + abu_x u_{xx} v_{xx} + \frac{1}{2} bv_x u_{xxy} + \frac{3}{4} bu_x v_{xxy} + \frac{1}{4} bv_y u_{xxx} + 3abu_x v_x u_{xxx} \\ &\quad + abu_x^2 v_{xxx} + u_{xxx} v_{xxx} + \frac{1}{4} bv u_{xxy} - v_{xx} u_{xxx} - u_{xx} v_{xxx} + v_x u_{xxxx} + u_x v_{xxxx} \\ &\quad + \frac{1}{2} u_x v_t + \frac{1}{2} vu_{tx}, \end{aligned}$$

$$J_2^y = \frac{1}{2}abu_x^2v_x - v_yu_x + vu_{xy} - abvu_xu_{xx} - \frac{1}{4}bu_{xx}v_{xx} + \frac{1}{4}bv_xu_{xxx} + \frac{1}{4}bu_xv_{xxx} - \frac{1}{4}bvu_{xxxx}.$$

Case 3. The conserved vector (J_3^t, J_3^x, J_3^y) corresponding to symmetry R_3 takes the form

$$\begin{aligned} J_3^t &= \frac{1}{2}u_yv_x - \frac{1}{2}vu_{xy}, \\ J_3^x &= \frac{1}{2}abu_yv_yu_x - \frac{1}{2}abvu_{yy}u_x + abu_y^2v_x + 3abu_yu_x^2v_x - \frac{1}{2}abvu_yu_{xy} - 3abvu_x^2u_{xy} \\ &\quad - \frac{1}{2}bu_{xy}v_{xy} + \frac{1}{2}bv_xu_{xyy} - abv_xu_{xy}u_{xx} - \frac{1}{4}bu_{yy}v_{xx} - abu_xu_{xy}v_{xx} + 2abu_yu_{xx}v_{xx} \\ &\quad + \frac{1}{4}bv_yu_{xxy} + abu_xv_xu_{xxy} + \frac{3}{4}bu_yv_{xxy} - \frac{3}{4}bvu_{xxyy} + 2abu_yv_xu_{xxx} - abvu_{xy}u_{xxx} \\ &\quad + abu_yu_xv_{xxx} + u_{xxy}v_{xxx} - abvu_xu_{xxy} - v_{xx}u_{xxy} - u_{xy}v_{xxx} + v_xu_{xxxxy} + u_yv_{xxxxy} \\ &\quad - vu_{xxxxxy} + \frac{1}{2}u_yv_t - \frac{1}{2}vu_{ty}, \\ J_3^y &= \frac{1}{2}abu_yu_xv_x - u_yv_y + \frac{1}{2}abvu_xu_{xy} + \frac{1}{2}abvu_yu_{xx} + 3abvu_x^2u_{xx} - \frac{1}{4}bu_{xy}v_{xx} + \frac{1}{4}bv_xu_{xxy} \\ &\quad + abvu_{xx}u_{xxx} + \frac{1}{4}bu_yv_{xxx} + \frac{3}{4}bvu_{xxy} + abvu_xu_{xxx} + vu_{xxxxx} + vu_{tx}. \end{aligned}$$

Case 4. The conserved vector (J_4^t, J_4^x, J_4^y) associated to symmetry R_4 is

$$\begin{aligned} J_4^t &= -\frac{1}{2}v_x, \\ J_4^x &= -\frac{1}{2}abv_yu_x - abu_yv_x - 3abu_x^2v_x - \frac{1}{2}abvu_{xy} - 2abu_{xx}v_{xx} - \frac{3}{4}bv_{xxy} - 2abv_xu_{xxx} \\ &\quad - abu_xv_{xxx} - v_{xxxxx} - \frac{1}{2}v_t, \\ J_4^y &= v_y - \frac{1}{2}abu_xv_x + \frac{1}{2}abvu_{xx} - \frac{1}{4}bv_{xxx}. \end{aligned}$$

Case 5. The conserved vector (J_5^t, J_5^x, J_5^y) corresponding to symmetry R_5 is expressed as

$$\begin{aligned} J_5^t &= \frac{1}{2}abtu_xv_x - \frac{1}{2}yv_x - \frac{1}{2}abtvu_{xx}, \\ J_5^x &= \frac{1}{2}abtvu_{tx} - abtvu_{yy} - \frac{1}{2}abyv_yu_x + \frac{1}{2}a^2b^2tv_yu_x^2 - abyu_yv_x + a^2b^2tu_yu_xv_x - 3abyu_x^2v_x \\ &\quad + 3a^2b^2tu_x^3v_x - \frac{1}{2}abyvu_{xy} + a^2b^2tvu_xu_{xy} - \frac{1}{2}ab^2tv_{xy}u_{xx} - a^2b^2tv_xu_{xx}^2 + \frac{1}{4}bv_{xx} \\ &\quad - \frac{1}{4}ab^2tu_{xy}v_{xx} - 2abyu_{xx}v_{xx} + a^2b^2tu_xu_{xx}v_{xx} + \frac{1}{2}ab^2tv_xu_{xxy} - \frac{3}{4}byv_{xxy} + \frac{3}{4}ab^2tu_xv_{xxy} \\ &\quad + \frac{1}{4}ab^2tv_yu_{xxx} - 2abyv_xu_{xxx} + 3a^2b^2tu_xv_xu_{xxx} - abyu_xv_{xxx} + a^2b^2tu_x^2v_{xxx} + abtu_{xxx}v_{xxx} \\ &\quad + \frac{1}{4}ab^2tvu_{xxy} - abtv_{xx}u_{xxx} - abtu_{xx}v_{xxx} + abtv_xu_{xxxx} - yv_{xxxxx} + abtu_xv_{xxxxx} \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}yv_t + \frac{1}{2}abtu_xv_t, \\
J_5^y = & \frac{1}{2}a^2b^2tu_x^2v_x - v + yv_y - abtv_yu_x - \frac{1}{2}abyu_xv_x + abtvu_{xy} + \frac{1}{2}abyvu_{xx} - \frac{1}{4}byv_{xxx} \\
& - a^2b^2tvu_xu_{xx} - \frac{1}{4}ab^2tu_{xx}v_{xx} + \frac{1}{4}ab^2tv_xu_{xxx} + \frac{1}{4}ab^2tu_xv_{xxx} - \frac{1}{4}ab^2tvu_{xxxx}.
\end{aligned}$$

Case 6. Finally, the conserved vector (J_6^t, J_6^x, J_6^y) associated to symmetry R_6 is

$$\begin{aligned}
J_6^t = & \frac{1}{2}uv_x - 5tvu_{yy} - vu_x + \frac{3}{2}yu_yv_x + \frac{1}{2}xu_xv_x - \frac{3}{2}yvu_{xy} + 5abtvu_xu_{xy} - \frac{1}{2}xvu_{xx} \\
& + 5abtvu_yu_{xx} + 15abtvu_x^2u_{xx} + 5abtvu_{xx}u_{xxx} + 5btvu_{xxxy} + 5abtvu_xu_{xxxx} \\
& + 5tvu_{xxxxx} + \frac{5}{2}tv_xu_t + \frac{5}{2}tvu_{tx}, \\
J_6^x = & \frac{1}{2}abuv_yu_x - xvu_{yy} - 4abvu_yu_x + \frac{3}{2}abyu_yv_yu_x - \frac{3}{2}abyvu_{yy}u_x + \frac{1}{2}abxv_yu_x^2 \\
& - 6abvu_x^3 + abuu_yv_x + 3abyu_y^2v_x + abxu_yu_xv_x + 3abuu_x^2v_x + 9abyu_yu_x^2v_x \\
& + 3abxu_x^3v_x + \frac{1}{2}abuvu_{xy} - \frac{3}{2}abyvu_yu_{xy} + abxvu_xu_{xy} - 9abyvu_x^2u_{xy} + \frac{5}{2}bv_xu_{xy} \\
& - bu_xv_{xy} - \frac{3}{2}byu_{xy}v_{xy} + \frac{3}{2}byv_xu_{xyy} + \frac{3}{4}bv_yu_{xx} + abu_xv_xu_{xx} - 3abyv_xu_{xy}u_{xx} \\
& - \frac{1}{2}bxv_{xy}u_{xx} - abxv_xu_{xx}^2 - bu_yv_{xx} - \frac{3}{4}byu_{yy}v_{xx} - 2abu_x^2v_{xx} - \frac{1}{4}bxu_{xy}v_{xx} \\
& - 3abyu_xu_{xy}v_{xx} + 2abuu_{xx}v_{xx} + 6abyu_yu_{xx}v_{xx} + abxu_xu_{xx}v_{xx} - \frac{9}{2}bvu_{xxy} \\
& + \frac{3}{4}byv_yu_{xxy} + \frac{1}{2}bxv_xu_{xxy} + 3abyu_xv_xu_{xxy} + \frac{3}{4}buv_{xxy} + \frac{9}{4}byu_yv_{xxy} + \frac{3}{4}bxu_xv_{xxy} \\
& - \frac{9}{4}byvu_{xxyy} + \frac{1}{4}bxv_yu_{xxx} - 6abvu_xu_{xxx} + 2abuv_xu_{xxx} + 6abyu_yv_xu_{xxx} + 3abxu_xv_xu_{xxx} \\
& - 3abyvu_{xy}u_{xxx} - 4v_{xx}u_{xxx} + abuu_xv_{xxx} + 3abyu_yu_xv_{xxx} + abxu_x^2v_{xxx} + 3u_{xx}v_{xxx} \\
& + 3yu_{xxy}v_{xxx} + xu_{xxx}v_{xxx} + \frac{1}{4}bxvu_{xxxy} - 3abyvu_xu_{xxxy} - 3yv_{xx}u_{xxxy} + 5v_xu_{xxxx} \\
& - xv_{xx}u_{xxxx} - 2u_xv_{xxxx} - 3yu_{xy}v_{xxxx} - xu_{xx}v_{xxxx} + 3yv_xu_{xxxxy} - 6vu_{xxxxx} \\
& + xv_xu_{xxxxx} + uv_{xxxxx} + 3yu_yv_{xxxxx} + xu_xv_{xxxxx} - 3yvu_{xxxxxy} - 3vu_t + \frac{5}{2}abtv_yu_xu_t \\
& + 5abtv_yv_xu_t + 15abtv_x^2v_xu_t + \frac{5}{2}abtvu_{xy}u_t + 10abtv_{xx}v_{xx}u_t + 10abtv_xu_{xxx}u_t \\
& + 5abtv_xv_{xxx}u_t + 5tv_{xxxx}u_t + \frac{1}{2}uv_t + \frac{15}{4}btv_{xxy}u_t + \frac{3}{2}yu_yv_t + \frac{1}{2}xu_xv_t + \frac{5}{2}tu_tv_t \\
& - \frac{3}{2}yvu_{ty} - \frac{5}{2}abtvu_xu_{ty} - \frac{5}{4}btv_{xx}u_{ty} + \frac{1}{2}xvu_{tx} - 5abtvu_yu_{tx} - 15abtvu_x^2u_{tx} \\
& - \frac{5}{2}btv_{xy}u_{tx} - 5abtv_xu_{xx}u_{tx} - 5abtv_xv_{xx}u_{tx} - 5abtvu_{xxx}u_{tx} - 5tv_{xxxx}u_{tx} \\
& + \frac{5}{2}btv_xu_{txy} + \frac{5}{4}btv_yu_{txx} + 5abtv_xv_xu_{txx} + 5tv_{xxx}u_{txx} - \frac{15}{4}btvu_{txxy} - 5abtvu_xu_{txxx}
\end{aligned}$$

$$\begin{aligned}
& -5tv_{xx}u_{txxx} + 5tv_xu_{txxxx} - 5tvu_{txxxx} - \frac{5}{2}tvu_{tt}, \\
J_6^y = & 4vu_y - uv_y - 3yu_yv_y - xv_yu_x - abvu_x^2 + \frac{1}{2}abuu_xv_x + \frac{3}{2}abyu_yu_xv_x + \frac{1}{2}abxu_x^2v_x \\
& + xv u_{xy} + \frac{3}{2}abyvu_xu_{xy} - \frac{1}{2}abuvu_{xx} + \frac{3}{2}abyvu_yu_{xx} - abxvu_xu_{xx} + 9abyvu_x^2u_{xx} \\
& + \frac{3}{4}bv_xu_{xx} - \frac{1}{2}bu_xv_{xx} - \frac{3}{4}byu_{xy}v_{xx} - \frac{1}{4}bxu_{xx}v_{xx} + \frac{3}{4}byv_xu_{xxy} - bv u_{xxx} + \frac{1}{4}bxv_xu_{xxx} \\
& + 3abyvu_{xx}u_{xxx} + \frac{1}{4}buv_{xxx} + \frac{3}{4}byu_yv_{xxx} + \frac{1}{4}bxu_xv_{xxx} + \frac{9}{4}byvu_{xxy} - \frac{1}{4}bxvu_{xxx} \\
& + 3abyvu_xu_{xxx} + 3yv u_{xxxx} - 5tv_yu_t + \frac{5}{2}abtu_xv_xu_t - \frac{5}{2}abtvu_{xx}u_t + \frac{5}{4}btv_{xxx}u_t \\
& + 5tvu_{ty} + 3yv u_{tx} - \frac{5}{2}abtvu_xu_{tx} - \frac{5}{4}btv_{xx}u_{tx} + \frac{5}{4}btv_xu_{txx} - \frac{5}{4}btvu_{txxx}.
\end{aligned}$$

5. Conclusions

In this paper, we presented a comprehensive study of the sixth-order 2D generalized nonlinear B-type Kadomtsev-Petviashvili (2D-gnBKP) equation (1.6). First, the Lie point symmetries of Eq (1.6) were systematically computed and thereafter employed to construct an optimal system of one-dimensional subalgebras. The optimal system was then utilized to perform symmetry reductions, through which the 2D-gnBKP equation (1.6) was reduced to several NODEs. By solving the resulting reduced NODEs, we constructed a variety of new exact solutions for (1.6). To accomplish these solutions, various powerful analytical techniques were invoked, including direct integration, the simplest equation approach, the (G'/G) -expansion method, and Kudryashov's method. As a consequence, numerous classes of exact explicit solutions were successfully obtained, such as hyperbolic function solutions, rational function solutions, trigonometric function solutions, exponential-type solutions, and kink-wave solutions. These solutions disclosed the rich dynamical behavior and nonlinear wave structures connected with the considered model. Furthermore, to better illustrate the physical features and propagation behavior of the obtained solutions, graphical representations were depicted in Figures 1–12.

In addition, conserved vectors for 2D-gnBKP equation (1.6) were derived by invoking two distinct and effective approaches, namely, the multiplier method and Ibragimov's conservation theorem. Using the multiplier approach, two independent multipliers were found, which yielded two local conservation laws. On the other hand, Ibragimov's method generated six local conservation laws associated with physical quantities such as momentum and energy. The conserved vectors obtained in this work further demonstrated the mathematical richness and physical significance of the 2D-gnBKP equation and may play an important role in the analysis of integrability, stability, and nonlinear wave propagation phenomena.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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