



Research article

Global dynamics and optimal control in a brucellosis transmission model with distributed incubation delays

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Abstract: In this paper, the global dynamics and optimal control strategies in a brucellosis transmission model with distribution incubation delays were investigated. The basic reproduction number of the model was first calculated, and then the threshold criteria for the global asymptotic stability of disease-free and endemic equilibria were established. Furthermore, the several effective measures for controlling the spread of diseases are introduced, and then an optimal control model that incorporates these measures was proposed. By providing a reasonable objective function, some optimal control strategies were established. Through numerical simulation of these optimal control strategies, reasonable prevention, control, and treatment measures for brucellosis were proposed.

Keywords: brucellosis model; global dynamics; optimal control strategy; distribution delay; latency

1. Introduction

Brucellosis is a widespread zoonotic disease worldwide [1]. This bacterial infection not only leads to severe and debilitating health conditions but also imposes significant economic burdens, resulting from patients' inability to engage in daily activities [2] and reduced productivity in livestock farming [3]. Infected animals, particularly sheep and cattle, act as the main reservoir for brucellosis transmission. The bacteria can be transmitted via multiple pathways, including broken skin or mucous membranes, the digestive tract, and the respiratory tract [4]. Although brucellosis has a low mortality rate, its impact on patients' labor ability and quality of life cannot be ignored. Therefore, studying the transmission mechanism and prevention and control strategies of this disease has important practical significance.

Dynamic modeling has long been instrumental in characterizing and understanding the epidemiology of brucellosis. Brucellosis involves a transmission chain that encompasses animals, the environment, and humans, with complex underlying mechanisms, making it highly suitable for

quantitative study using dynamic models [5–11]. In [12], Zhang et al. combined an ordinary differential equation model and a deep neural network model to develop a hybrid model. In [13], Wang et al. studied a stochastic brucellosis model that includes vaccination and the spread of environmental pollutants to investigate the impact of environmental factors on disease transmission. In [14], Kang et al. developed a mathematical model to simulate the transmission dynamics of brucellosis in Indian cattle and evaluated the effectiveness of various control measures. Based on brucellosis infection data from Jilin Province, China. Nie et al. established a susceptibility-exposure-infection-pathogen dynamics model that incorporates external transfer rates to characterize the spread of brucellosis in dairy cattle [15]. In [16], Zhang et al. constructed a mathematical framework integrating economic variables, through which they estimated the costs associated with vaccination, test-and-cull, and trace-and-cull strategies in sheep populations.

We know that when sheep are infected with *Brucella*, in the initial stage, phagocytes in the immune tissue will engulf the bacteria, so there will be an initial incubation period of *Brucella* during which the infected sheep do not have infectivity. However, these phagocytes often cannot effectively eliminate the pathogen, leading to the continuous existence and reproduction of the pathogen within the cells [17]. As the pathogen accumulates in the body, the infected sheep will enter a latent period with infectivity, and during this period, latent individuals usually do not show clinical symptoms, and thus are also called asymptomatic infected. The duration from initial infection to becoming an asymptomatic infected may range from a few weeks to a year [18]. Once the bacterial load within the cells reaches a critical level, brucella will cause the phagocytes to rupture, thereby rapidly spreading to the surrounding tissues [19]. As symptoms appear or are detected through artificial examination, sheep in the latent period will become diseased sheep. Furthermore, the secretions and excretions of infected animals can also become a source of transmission because they can cause environmental pollution [20]. Healthy animals that come into contact with contaminated feed, water, or other substances are prone to brucellosis. We already know that in the modeling of infectious disease dynamics, the latent period of a disease can be represented by a time delay. Moreover, there have been some research works on the transmission dynamics models of brucellosis with delay effects (see [5, 6, 21–26] and the references cited therein). Particularly, in [5], the authors established a periodic time dynamic model with latent delay coupled with the livestock breeding and temperature characteristics of Xinjiang Region. The global dynamics of periodic solutions are demonstrated by the theory of monotone dynamical system. The monthly human brucellosis cases and average temperature data of Xinjiang from 2015 to 2020 are fitted, and the actual transmission dynamics are discussed. Several important and novel findings are proposed. In [21], the authors proposed a class of brucellosis transmission model with seasonal alternation, density-dependent growth, stage-structure, maturation delay, and time-varying incubation. The threshold criteria for disease extinction and persistence are established. The theoretical results are illustrated by numerical simulation, from which we find that the brucellosis transmission would be overestimated (or underestimated) if we ignore the influence of time-varying incubation or maturation delay. In [23], the authors proposed an animal-human infection model of brucellosis with three distributed delays, representing the latent period of brucellosis in infected animal and human population and the survival time of brucella in the environment. The threshold criteria for the global stability of disease-free and endemic equilibria are established. By introducing four control variables, the corresponding optimal control model is proposed, and the existence of an optimal control solution is derived. In [26], the author constructed a

dynamic model with discrete-time delays to reflect the incubation period of the disease and the delayed effects of intervention measures, as well as the environmental transmission mechanism. Threshold criteria for controlling the prevalence of the disease were obtained. A key contribution of this work is the formulation and analysis of a mixed state-control delay optimal control problem, and the optimal strategies, including vaccination, culling, environmental disinfection, and public awareness campaigns, are derived.

Inspired by the above articles, in this paper, we propose a model of brucella transmission in sheep that considers the distributed incubation delays and the impact of environmental pathogen contamination.

$$\begin{cases} dS = [A - \beta S(E + I) - \phi S B - (\mu + \nu)S + \delta V]dt, \\ dV = [\nu S - (\mu + \delta)V - \varepsilon \beta V(E + I) - \varepsilon \phi V B]dt, \\ dE = \left[\int_0^\infty \beta [S(t - \tau) + \varepsilon V(t - \tau)] [E(t - \tau)f_1(\tau) + I(t - \tau)f_2(\tau)] e^{-\mu\tau} d\tau \right. \\ \quad \left. + \int_0^\infty \phi [S(t - \tau) + \varepsilon V(t - \tau)] B(t - \tau)f_3(\tau) e^{-\mu\tau} d\tau - (\mu + \sigma)E \right]dt, \\ dI = [\sigma E - (\mu + c)I]dt, \\ dB = [k(E + I) - (d + n)B]dt, \end{cases} \quad (1.1)$$

where $S(t)$, $V(t)$, $E(t)$, and $I(t)$ represent the numbers of susceptible, vaccinated, latent infected (or referred to as asymptomatic infected), and infected sheep at time t , respectively, and $B(t)$ represents the amount of brucella in the environment at time t . Moreover, model (1) is a system of ordinary differential equations with infinite distributed delays.

In model (1), the infinite distributed delay is used to represent that due to the varying resistance levels of the sheep, the time it takes for them to transform into infectious latent individuals after being infected can also be different, thus presenting a distribution state on interval $(-\infty, 0]$, where $e^{-\mu\tau}$ indicates the probability that sheep will be alive after τ time units due to natural culling. The kernel functions $f_i(\tau)$ ($i = 1, 2, 3$) represent the transformation probabilities of susceptible and vaccinated individuals, respectively, who are initially infected at time $t - \tau$ by infectious latent individuals, infected individuals, and environmental pathogens, and then become infectious latent individuals at time t . Therefore, $f_i(\tau)$ ($i = 1, 2, 3$) are defined for $\tau \in [0, \infty)$ and are nonnegative and integrable, and satisfy $\int_0^\infty f_i(\tau) d\tau = 1$.

In addition, it should be noted that the use of infinitely distributed time delays in model (1) is to facilitate unified discussion and analysis using the theory of infinite time delay functional differential equations. In practice, we can set a maximum age of death for sheep and environmental pathogens based on the actual situation, such as an enough large constant $\tau_0 > 0$, so that when the age τ is greater than τ_0 , the value of the kernel functions $f_i(\tau)$ ($i = 1, 2, 3$) are set to zero. In particular, the functions $f_i(\tau)$ ($i = 1, 2, 3$) can be taken as:

$$f_i(\tau) = \begin{cases} \frac{\gamma_i}{1 - e^{-\gamma_i\tau_0}} e^{-\gamma_i\tau}, & 0 \leq \tau \leq \tau_0, \\ 0, & \tau > \tau_0, \end{cases}$$

where γ_i is a positive constant. In this way, model (1) can be degraded into the following model with

bounded distributed delays:

$$\begin{cases} dS = [A - \beta S(E + I) - \phi S B - (\mu + \nu)S + \delta V]dt, \\ dV = [\nu S - (\mu + \delta)V - \epsilon \beta V(E + I) - \epsilon \phi V B]dt, \\ dE = \left[\int_0^{\tau_0} \beta [S(t - \tau) + \epsilon V(t - \tau)] [E(t - \tau)f_1(\tau) + I(t - \tau)f_2(\tau)] e^{-\mu\tau} d\tau \right. \\ \quad \left. + \int_0^{\tau_0} \phi [S(t - \tau) + \epsilon V(t - \tau)] B(t - \tau)f_3(\tau) e^{-\mu\tau} d\tau - (\mu + \sigma)E \right]dt, \\ dI = [\sigma E - (\mu + c)I]dt, \\ dB = [k(E + I) - (d + n)B]dt. \end{cases}$$

Furthermore, we can also select the kernel function as $f_1(s) = \delta(s - \tau_1)$, $f_2(s) = \delta(s - \tau_2)$, $f_3(s) = \delta(s - \tau_3)$, where τ_i ($i = 1, 2, 3$) are positive constants and $\delta(\cdot)$ is the dirac delta function, then model (1) will degenerate into the form of discrete time delay (see model (5.1) in Section 5 below).

The biological description of parameters in model (1) can be found in Table 1 below. All these parameters are nonnegative constants, and $A > 0$, $\beta > 0$, $\mu > 0$, and $d > 0$.

Table 1. Description of parameters in model (1).

Parameter	Biological description
A	The recruitment rate of sheep population
μ	The natural elimination rate of sheep
c	The clearance rate of infected sheep
δ	The failure rate of sheep vaccines
ν	The vaccination rate of sheep
β	Sheep to sheep infection rate
ϵ	The failure rate of the vaccinated sheep, $0 \leq \epsilon \leq 1$
ϕ	Brucella in the environment to susceptible sheep infection rate
σ	The rate of clinical outcome of latent infected sheep
k	Brucella shedding rate by latent infected and infected sheep
d	The natural clearance rate of brucella in the environment
n	The artificial clearance rate of environmental pathogens

For model (1), we first investigate the global dynamical behavior. We will calculate the basic reproduction number of the model, and then further establish the threshold criteria for the global asymptotic stability of disease-free and endemic equilibria. Next, we will introduce several effective strategies for controlling the spread of diseases, and further establish an optimal control model that incorporates these strategies. By providing a reasonable objective function, an effective optimal control strategy can be established. Last, through numerical simulation of these optimal control strategies, reasonable prevention, control, and treatment strategies for brucellosis are proposed.

This paper is structured as follows: In Section 2, we prove the existence of positivity and boundedness for model (1). In Section 3, a basic reproduction number and equilibrium of model (1) is obtained. In addition, based on basic reproduction number, we prove that disease-free equilibrium P_0 and endemic equilibrium P^* is globally asymptotically stable. In Section 4, the control problem of the

delay model is given. Furthermore, in Sections 5 and 6, we will present sensitivity analysis and the numerical examples and simulations to validate the main results obtained in this paper. Finally, a brief conclusion is given in Section 7.

2. Basic properties

We denote by $C((-\infty, 0], \mathbb{R}^5)$ the space of all continuous functions $\phi : (-\infty, 0] \rightarrow \mathbb{R}^5$. Since model (1) is the functional differential equations with infinite distributed delays, a certain suitable phase space is required to ensure the existence, uniqueness and continuous dependence of solutions with respect to any initial values. Choose a constant $\Delta > 0$ with $\Delta < \mu$ such that $\int_0^\infty f_i(\tau)e^{-(\mu-\Delta)\tau}d\tau < \infty$ ($i = 1, 2, 3$). The phase space of model (1) is defined by the Banach space as follows:

$$C_\Delta((-\infty, 0], \mathbb{R}_+^5) = \left\{ \phi \in C((-\infty, 0], \mathbb{R}_+^5) : \phi(\theta)e^{\Delta\theta} \text{ is uniformly continuous for } \theta \in (-\infty, 0], \right. \\ \left. \text{and } \sup_{\theta \leq 0} |\phi(\theta)|e^{\Delta\theta} < \infty \right\},$$

with the norm $\|\phi\| = \sup_{\theta \leq 0} |\phi(\theta)|e^{\Delta\theta}$ (see, e.g., [27]). Denote by $C_\Delta((-\infty, 0], \mathbb{R}_+^5)$ the positive cone of $C_\Delta((-\infty, 0], \mathbb{R}^5)$, where $\mathbb{R}_+^5 = \{(x_1, x_2, x_3, x_4, x_5) : x_i \geq 0, i = 1, 2, 3, 4, 5\}$. In addition, for the convenience of presentation, we denote $\eta_i = \int_0^\infty f_i(\tau)e^{-\mu\tau}d\tau$ for $i = 1, 2, 3$.

In model (1), based on biological significance, we assume that any solution satisfies the following initial condition

$$S(\theta) = \phi_1(\theta), V(\theta) = \phi_2(\theta), E(\theta) = \phi_3(\theta), I(\theta) = \phi_4(\theta), B(\theta) = \phi_5(\theta), \theta \in (-\infty, 0], \quad (2.1)$$

where $\phi_i(\theta)$ ($i = 1, 2, 3, 4, 5$) are bounded continuous functions defined on $(-\infty, 0]$.

For model (1), as the basic properties, we first discuss the nonnegativity and boundedness of solutions. We have the following conclusion:

Lemma 2.1. *For any initial condition (2.1), model (1) has a unique nonnegative solution defined for all $t \in [0, +\infty)$ and is ultimately bounded.*

Proof. Since $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \in C_\Delta((-\infty, 0], \mathbb{R}_+^5)$, based on the fundamental theory of functional differential equations with infinite delays [28], it follows that model (1) has a unique solution $(S(t), V(t), E(t), I(t), B(t))$ with initial condition (2.1) defined in an interval $[0, T_0)$. When $\phi_i(0) > 0$ for $i = 1, 2, 3, 4, 5$, by means of the auxiliary function $m(t) = \max\{S(t), V(t), E(t), I(t), B(t)\}$, we can prove that $(S(t), V(t), E(t), I(t), B(t))$ is positive in the existence interval $[0, T_0)$. When $\phi_i(0) \geq 0$ for $i = 1, 2, 3, 4, 5$, by utilizing the continuity of solutions with respect to the initial values we can derive that $(S(t), V(t), E(t), I(t), B(t))$ is nonnegative in the existence interval $[0, T_0)$.

Next, we prove the boundedness of solution $(S(t), V(t), E(t), I(t), B(t))$. From the first two equations of model (1), we can derive that

$$d(S(t) + V(t)) \leq [A - \mu(S(t) + V(t))]dt. \quad (2.2)$$

This implies that $S(t) + V(t) \leq \max\{S(0) + V(0), \frac{A}{\mu}\}$ for all $t \in [0, T_0)$. Hence, $S(t)$ and $V(t)$ are bounded on $[0, T_0)$.

Define the Lyapunov function as follows:

$$F(t) = \int_0^\infty e^{-\mu\tau} [f_1(\tau) + f_2(\tau) + f_3(\tau)] [S(t-\tau) + V(t-\tau)] d\tau + E(t) + I(t).$$

We have

$$\begin{aligned} F(t) &= \int_{-\infty}^t e^{-\mu(t-u)} [f_1(t-u) + f_2(t-u) + f_3(t-u)] [S(u) + V(u)] du + E(t) + I(t) \\ &= \int_{-\infty}^0 e^{-\mu(t-u)} [f_1(t-u) + f_2(t-u) + f_3(t-u)] [\phi_1(u) + \phi_2(u)] du \\ &\quad + \int_0^t e^{-\mu(t-u)} [f_1(t-u) + f_2(t-u) + f_3(t-u)] [S(u) + V(u)] du + E(t) + I(t). \end{aligned}$$

From $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \in C_\Delta((-\infty, 0], \mathbb{R}_+^5)$ and $0 < \Delta < \mu$, we can prove that $F(t)$ is defined for any $t \in [0, T_0)$ and is continuously differentiable.

Calculating the derivative of $F(t)$ along any nonnegative solutions of model (1), we can derive that

$$\begin{aligned} \frac{d}{dt} F(t) &= \int_0^\infty e^{-\mu\tau} [f_1(\tau) + f_2(\tau) + f_3(\tau)] [A - \mu[S(t-\tau) + V(t-\tau)] \\ &\quad - \beta[S(t-\tau) + \epsilon V(t-\tau)] (E(t-\tau) + I(t-\tau)) - \phi[S(t-\tau) \\ &\quad + \epsilon V(t-\tau)] B(t-\tau)] d\tau + \int_0^\infty \beta[S(t-\tau) + \epsilon V(t-\tau)] [E(t-\tau) f_1(\tau) \\ &\quad + I(t-\tau) f_2(\tau)] e^{-\mu\tau} d\tau + \int_0^\infty \phi[S(t-\tau) + \epsilon V(t-\tau)] \\ &\quad \times B(t-\tau) f_3(\tau) e^{-\mu\tau} d\tau - \mu E - (\mu + c) I \\ &\leq A \sum_{i=1}^3 \eta_i - \mu F(t). \end{aligned} \tag{2.3}$$

From (2.3), we can easily obtain that $F(t)$ is bounded on $[0, T_0)$. This implies that $E(t)$ and $I(t)$ are also bounded on $[0, T_0)$.

Last, from the five equation of model (1), we can derive that $B(t)$ is bounded on $[0, T_0)$. Thus, we finally derive that solution $(S(t), V(t), E(t), I(t), B(t))$ is bounded on $[0, T_0)$. Therefore, solution $(S(t), V(t), E(t), I(t), B(t))$ is defined for all $u \in [0, \infty)$.

From (2.3), we further derive $\limsup_{t \rightarrow \infty} F(t) \leq \frac{A}{\mu} \sum_{i=1}^3 \eta_i$, which implies that $\limsup_{t \rightarrow \infty} (E(t) + I(t)) \leq \frac{A}{\mu} \sum_{i=1}^3 \eta_i$. From (2.2), it is clear that $\limsup_{t \rightarrow \infty} (S(t) + V(t)) \leq \frac{A}{\mu}$. Finally, from the equation of B in model (1), we directly derive $\limsup_{t \rightarrow \infty} (B(t)) \leq \frac{kA}{\mu(d+n)} \sum_{i=1}^3 \eta_i$. This implies that solution $(S(t), V(t), E(t), I(t), B(t))$ is bounded. $\square$

It is easy to see that model (1) has a disease-free equilibrium $P_0 = (S_0, V_0, 0, 0, 0)$ with $S_0 = \frac{A(\mu+\delta)}{\mu(\mu+\delta+\nu)}$.

and $V_0 = \frac{A\nu}{\mu(\mu+\delta+\nu)}$. Using Linearized model (1) at equilibrium P_0 , we obtain

$$\begin{cases} dE = \left[\int_0^\infty \beta[S_0 + \epsilon V_0][E(t-\tau)f_1(\tau) + I(t-\tau)f_2(\tau)]e^{-\mu\tau}d\tau \right. \\ \quad \left. + \int_0^\infty \phi[S_0 + \epsilon V_0]B(t-\tau)f_3(\tau)e^{-\mu\tau}d\tau - (\mu + \sigma)E \right]dt, \\ dI = [\sigma E - (\mu + c)I]dt, \\ dB = [k(E + I) - (d + n)B]dt. \end{cases} \quad (2.4)$$

Define the matrices as follows

$$A_1 = \begin{pmatrix} \mu + \sigma & 0 & 0 \\ -\sigma & \mu + c & 0 \\ -k & -k & d + n \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \beta(S_0 + \epsilon V_0)\eta_1 & \beta(S_0 + \epsilon V_0)\eta_2 & \phi(S_0 + \epsilon V_0)\eta_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Based on the article [29], we can define the basic reproduction number of model (1) as $R_0 = \rho(A_2A_1^{-1})$, the spectral radius of $A_2A_1^{-1}$. By calculating, we further derive

$$R_0 = \left[\frac{(d+n)\beta\eta_1 + k\phi\eta_3}{(d+n)(\mu+\sigma)} + \frac{(d+n)\beta\sigma\eta_2 + k\sigma\phi\eta_3}{(d+n)(\mu+c)(\mu+\sigma)} \right] (S_0 + \epsilon V_0).$$

Furthermore, on the existence of endemic equilibrium for model (1), we have the following conclusion:

Lemma 2.2. Assume $R_0 > 1$, then, except for P_0 , model (1) has a unique endemic equilibrium $P^* = (S^*, V^*, E^*, I^*, R^*)$, where

$$S^* = \alpha_3 - \epsilon V^*, \quad E^* = \frac{A - (\mu + \nu)S^* + \delta V^*}{S^*(\beta(1 + \alpha_1) + \phi\alpha_3)}, \quad I^* = \alpha_1 E^*, \quad B^* = \alpha_2 E^*,$$

and V^* is the positive root of Eq (2.8), where Eq (2.8) and α_i ($i = 1, 2, 3$) are given in the following proof. Particularly, when $\epsilon = 0$, we also have $S^* = \alpha_3$ and $V^* = \frac{\nu\alpha_3}{\mu+\delta}$.

Proof. We know that endemic equilibrium $P^* = (S^*, V^*, E^*, I^*, R^*)$ satisfies the equation as follows:

$$\begin{cases} A - \beta S(E + I) - \phi S B - (\mu + \nu)S + \delta V = 0, \\ \nu S - (\mu + \delta)V - \epsilon\beta V(E + I) - \epsilon\phi V B = 0, \\ \beta(S + \epsilon V)(E\eta_1 + I\eta_2) + \phi(S + \epsilon V)B\eta_3 - (\mu + \sigma)E = 0, \\ \sigma E - (\mu + c)I = 0, \\ k(E + I) - (d + n)B = 0. \end{cases} \quad (2.5)$$

From the last two equations of (2.5) we directly have $I = \alpha_1 E$ and $B = \alpha_2 E$, where $\alpha_1 = \frac{\sigma}{\mu+c}$ and $\alpha_2 = \frac{k(1+\alpha_1)}{d+n}$. From the third equation of (2.5), we further get

$$\beta(S + \epsilon V)(\eta_1 E + \alpha_1 \eta_2 E) + \phi(S + \epsilon V)\alpha_2 \eta_3 E - (\mu + \sigma)E = 0. \quad (2.6)$$

When $E \neq 0$, from (2.6), we obtain $S + \epsilon V = \frac{\mu + \sigma}{\beta(\eta_1 + \alpha_1 \eta_2) + \phi \alpha_2 \eta_3} := \alpha_3$. Hence, we have $S = \alpha_3 - \epsilon V$. From the first equation of (2.5), we can obtain $\beta S(E + \alpha_1 E) + \phi \alpha_2 S E = A - (\mu + \nu)S + \delta V$, and then from the second equation of (2.5), we further obtain

$$\nu S^2 - (\mu + \delta)S V - \epsilon V(A - (\mu + \nu)S + \delta V) = 0. \quad (2.7)$$

Substituting $S = \alpha_3 - \epsilon V$ into (2.7) we derive

$$H(V) := \epsilon \mu (1 - \epsilon) V^2 + (\epsilon \mu \alpha_3 - \epsilon \nu \alpha_3 - (\mu + \delta) \alpha_3 - \epsilon A) V + \nu \alpha_3 = 0. \quad (2.8)$$

When $\epsilon = 0$, from (2.8) we have $H(V) = \nu \alpha_3^2 - (\mu + \delta) \alpha_3 V$. Hence, $H(V) = 0$ has a unique solution $V^* = \frac{\nu \alpha_3}{\mu + \delta}$. This implies that model (1) has a unique possible endemic equilibrium $P^* = (S^*, V^*, E^*, I^*, R^*)$ with

$$S^* = \alpha_3, \quad V^* = \frac{\nu \alpha_3}{\mu + \delta}, \quad E^* = \frac{A - (\mu + \nu)S^* + \delta V^*}{S^*(\beta(1 + \alpha_1) + \phi \alpha_3)}, \quad I^* = \alpha_1 E^*, \quad B^* = \alpha_2 E^*. \quad (2.9)$$

Now, we need to prove that this P^* is an endemic equilibrium when $R_0 > 1$. It is clear that we only need to show $E^* > 0$ when $R_0 > 1$. In fact, by the expressions of α_1 , α_2 , and α_3 , through detailed calculating, we can derive

$$\begin{aligned} A - (\mu + \nu)S^* + \delta V^* &= A - (\mu + \nu)\alpha_3 + \delta \frac{\nu \alpha_3}{\mu + \delta} \\ &= A - \frac{\mu(\mu + \nu + \delta)(\mu + \sigma)(\mu + c)(d + n)}{(\mu + \delta)[(d + n)\beta((\mu + c)\eta_1 + \sigma \eta_2) + \phi k \eta_3(\mu + c + \sigma)]}. \end{aligned}$$

Since $S_0 = \frac{A(\mu + \delta)}{\mu(\mu + \delta + \nu)}$, we further can derive

$$\begin{aligned} &A - (\mu + \nu)S^* + \delta V^* \\ &= \frac{A}{S_0} \frac{(d + n)(\mu + \sigma)(\mu + c)}{(d + n)\beta((\mu + c)\eta_1 + \sigma \eta_2) + k(\mu + c + \sigma)\phi \eta_3} (R_0 - 1). \end{aligned}$$

Therefore, $A - (\mu + \nu)S^* + \delta V^* > 0$, which implies $E^* > 0$ when $R_0 > 1$.

When $\epsilon > 0$, we have $H(0) = \nu \alpha_3^2 > 0$, $H(\frac{\alpha_3}{\epsilon}) = -\alpha_3(A + \frac{\delta}{\epsilon} \alpha_3) < 0$ and $H(+\infty) = +\infty$, which implies that $H(V) = 0$ has two positive roots $V^* \in (0, \frac{\alpha_3}{\epsilon})$ and $V^{**} > \frac{\alpha_3}{\epsilon}$. Thus, $S = \alpha_3 - \epsilon V^{**} < 0$. Hence, V^{**} is excluded. Thus, model (1) has a unique possible endemic equilibrium $P^* = (S^*, V^*, E^*, I^*, R^*)$ with $V^* \in (0, \frac{\alpha_3}{\epsilon})$ and

$$S^* = \alpha_3 - \epsilon V^*, \quad E^* = \frac{A - (\mu + \nu)S^* + \delta V^*}{S^*(\beta(1 + \alpha_1) + \phi \alpha_3)}, \quad I^* = \alpha_1 E^*, \quad B^* = \alpha_2 E^*.$$

Moreover, we also need to prove $A - (\mu + \nu)S^* + \delta V^* > 0$ when $R_0 > 1$, which implies that P^* is an endemic equilibrium. Suppose that $A - (\mu + \nu)S^* + \delta V^* \leq 0$, then from (2.7), we derive $\nu \alpha_3 - (\nu \epsilon + \mu + \delta)V^* \leq 0$. Hence, we have $V^* \geq \frac{\nu \alpha_3}{\nu \epsilon + \mu + \delta}$. However, from the above argument, it follows that

$$\begin{aligned} A - (\mu + \nu)S^* + \delta V^* &> A - (\mu + \nu)\alpha_3 + \delta \frac{\nu \alpha_3}{\mu + \delta} \\ &= \frac{A}{S_0} \frac{(d + n)(\mu + \sigma)(\mu + c)}{(d + n)\beta((\mu + c)\eta_1 + \sigma \eta_2) + k(\mu + c + \sigma)\phi \eta_3} (R_0 - 1) > 0. \end{aligned}$$

This leads to a contradiction. Therefore, $A - (\mu + \nu)S^* + \delta V^* > 0$ when $R_0 > 1$. This completes the proof. $\square$

3. Global stability

In this section, we investigate the global stability of disease-free and endemic equilibria of model (1). To proceed, we first introduce the function $g(x) = x - 1 - \ln x$ for $x \in (0, \infty)$, which will be used in the construction of Lyapunov functionals of this section. It is easy to see that $g(x) > g(1) = 0$ for any $x > 0$ with $x \neq 1$.

Theorem 3.1. Assume that $\eta_1 = \eta_2 = \eta_3 := \eta$ and $R_0 \leq 1$, then disease-free equilibrium $E_0 = (S_0, V_0, 0, 0, 0)$ is globally asymptotically stable.

Proof. We first prove the local asymptotic stability of equilibrium $E_0 = (S_0, V_0, 0, 0, 0)$ when $R_0 < 1$. Using Linearized model (1) at equilibrium P_0 , we obtain

$$\begin{cases} dS = [-\beta S_0(E + I) - \phi S_0 B - (\mu + \nu)S + \delta V]dt, \\ dV = [\nu S - (\mu + \delta)V - \varepsilon \beta V_0(E + I) - \varepsilon \phi V_0 B]dt, \\ dE = \left[\int_0^\infty \beta[S_0 + \varepsilon V_0][E(t - \tau)f_1(\tau) + I(t - \tau)f_2(\tau)]e^{-\mu\tau}d\tau \right. \\ \quad \left. + \int_0^\infty \phi[S_0 + \varepsilon V_0]B(t - \tau)f_3(\tau)e^{-\mu\tau}d\tau - (\mu + \sigma)E \right]dt, \\ dI = [\sigma E - (\mu + c)I]dt, \\ dB = [k(E + I) - (d + n)B]dt. \end{cases} \quad (3.1)$$

Substituting $(S, V, E, I, B) = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)e^{\lambda t}$ into (3.1), and then eliminating $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$, and $e^{\lambda t}$, we can derive the characteristic equation

$$H_1(\lambda)H_2(\lambda) = 0,$$

where

$$H_1(\lambda) = \lambda^2 + (2\mu + \nu + \delta)\lambda + \mu(\mu + \nu + \delta)$$

and

$$\begin{aligned} H_2(\lambda) = & (\mu + \sigma + \lambda)(\mu + c + \lambda)(d + n + \lambda) \\ & - \beta(S_0 + \varepsilon V_0) \left[(\mu + c + \lambda)(d + n + \lambda)\hat{\eta}_1(\mu + \lambda) \right. \\ & \left. + \sigma(d + n + \lambda)\hat{\eta}_2(\mu + \lambda) + \frac{k\phi}{\beta}(\mu + \sigma + c + \lambda)\hat{\eta}_3(\mu + \lambda) \right], \end{aligned} \quad (3.2)$$

where $\hat{\eta}_i(\mu + \lambda) = \int_0^\infty f_i(\tau)e^{-\tau(\mu + \lambda)}d\tau$ for $i = 1, 2, 3$. Moreover, two roots of $H_1(\lambda) = 0$ have negative real parts. Consider the equation $H_2(\lambda) = 0$. When $R_0 > 1$, we have $H_2(0) = (\mu + \sigma)(\mu + c)(d + n)(1 - R_0) < 0$. Since $\lim_{\lambda \rightarrow +\infty} \hat{\eta}_i(\mu + \lambda) = 0$ for $i = 1, 2, 3$, we further have $\lim_{\lambda \rightarrow +\infty} H_2(\lambda) = +\infty$. Hence, $H_2(\lambda) = 0$ has a positive root. This implies that equilibrium P_0 is unstable.

Let $R_0 < 1$. Suppose that $H_2(\lambda) = 0$ has a complex root $\lambda_0 = a + bi$ with the real part $a \geq 0$. Thus, $(\mu + \sigma + \lambda_0)(\mu + c + \lambda_0)(d + n + \lambda_0) \neq 0$. Hence, we derive that

$$\begin{aligned} 1 = & \left[\frac{(d + n + \lambda_0)\beta\hat{\eta}_1(\mu + \lambda_0) + k\phi\hat{\eta}_3(\mu + \lambda_0)}{(d + n + \lambda_0)(\mu + \sigma + \lambda_0)} \right. \\ & \left. + \frac{(d + n + \lambda_0)\beta\sigma\hat{\eta}_2(\mu + \lambda_0) + k\sigma\phi\hat{\eta}_3(\mu + \lambda_0)}{(d + n + \lambda_0)(\mu + c + \lambda_0)(\mu + \sigma + \lambda_0)} \right] (S_0 + \varepsilon V_0). \end{aligned}$$

Since $\hat{\eta}_i(\mu + \lambda_0) \leq \eta_i$ for $i = 1, 2, 3$, $|\mu + \sigma + \lambda_0| \geq \mu + \sigma$, $|\mu + c + \lambda_0| \geq \mu + c$ and $|d + n + \lambda_0| \geq d + n$, we further have the norm

$$\begin{aligned} & \left\| \frac{(d + n + \lambda_0)\beta\hat{\eta}_1(\mu + \lambda_0) + k\phi\hat{\eta}_3(\mu + \lambda_0)}{(d + n + \lambda_0)(\mu + \sigma + \lambda_0)} \right. \\ & \quad \left. + \frac{(d + n + \lambda_0)\beta\sigma\hat{\eta}_2(\mu + \lambda_0) + k\sigma\phi\hat{\eta}_3(\mu + \lambda_0)}{(d + n + \lambda_0)(\mu + c + \lambda_0)(\mu + \sigma + \lambda_0)} \right\| (S_0 + \epsilon V_0) \\ & \leq \left[\frac{|d + n + \lambda_0|\beta\eta_1 + k\phi\eta_3}{|d + n + \lambda_0|(\mu + \sigma)} + \frac{|d + n + \lambda_0|\beta\sigma\eta_2 + k\sigma\phi\eta_3}{|d + n + \lambda_0|(\mu + c)(\mu + \sigma)} \right] (S_0 + \epsilon V_0) \\ & \leq \left[\frac{(d + n)\beta\eta_1 + k\phi\eta_3}{(d + n)(\mu + \sigma)} + \frac{(d + n)\beta\sigma\eta_2 + k\sigma\phi\eta_3}{(d + n)(\mu + c)(\mu + \sigma)} \right] (S_0 + \epsilon V_0) \\ & = R_0 < 1, \end{aligned}$$

which leads to a contradiction. This shows that all roots of $H_2(\lambda) = 0$ have negative real parts. Therefore, P_0 is locally asymptotically stable.

Define a Lyapunov functional as follows

$$\begin{aligned} V_1 = & S - S_0 - S_0 \ln \frac{S}{S_0} + V - V_0 - V_0 \ln \frac{V}{V_0} + \frac{1}{\eta} E + \frac{\phi(S_0 + \epsilon V_0)}{d + n} B \\ & + \frac{(d + n)(\mu + \sigma) - [(d + n)\beta\eta + k\phi\eta](S_0 + \epsilon V_0)}{\eta(d + n)\sigma} I \\ & + \frac{1}{\eta} \int_0^\infty e^{-\mu s} f_1(s) \int_{t-s}^t \beta[S(\tau) + \epsilon V(\tau)] E(\tau) d\tau ds \\ & + \frac{1}{\eta} \int_0^\infty e^{-\mu s} f_2(s) \int_{t-s}^t \beta[S(\tau) + \epsilon V(\tau)] I(\tau) d\tau ds \\ & + \frac{1}{\eta} \int_0^\infty e^{-\mu s} f_3(s) \int_{t-s}^t \phi[S(\tau) + \epsilon V(\tau)] B(\tau) d\tau ds. \end{aligned}$$

From the boundedness of solution $(S(t), V(t), E(t), I(t), B(t))$, we know that V_1 is defined for any $t \geq 0$ and is continuously differentiable. From $R_0 \leq 1$, we can derive $(d + n)(\mu + \sigma) - [(d + n)\beta\eta + k\phi\eta](S_0 + \epsilon V_0) > 0$. Therefore, V_1 is positive definite in $\mathbb{R}_+^5$.

Calculating the derivative of V_1 along solutions of model (1), it follows that

$$\begin{aligned}
 \frac{dV_1}{dt} = & A(S - S_0)\left(\frac{1}{S} - \frac{1}{S_0}\right) + \delta(S - S_0)\left(\frac{V}{S} - \frac{V_0}{S_0}\right) - \beta S(E + I) - \phi S B \\
 & + \beta S_0(E + I) + \phi S_0 B + \nu(V - V_0)\left(\frac{S}{V} - \frac{S_0}{V_0}\right) - \epsilon \beta V(E + I) - \epsilon \phi V B \\
 & + \epsilon \beta V_0(E + I) + \epsilon \phi V_0 B - \frac{(\mu + \sigma)E}{\eta} + \beta(S + \epsilon V)(E + I) + \phi(S + \epsilon V)B \\
 & + \frac{k\phi(S_0 + \epsilon V_0)(E + I)}{d + n} - \phi(S_0 + \epsilon V_0)B \\
 & + \frac{(d + n)(\mu + \sigma) - [(d + n)\beta\eta + k\phi\eta](S_0 + \epsilon V_0)}{\eta(d + n)} E \\
 & - \frac{(d + n)(\mu + \sigma) - [(d + n)\beta\eta + k\phi\eta](S_0 + \epsilon V_0)}{\eta(d + n)\sigma} (\mu + c)I \\
 = & A(S - S_0)\left(\frac{1}{S} - \frac{1}{S_0}\right) + \delta(S - S_0)\left(\frac{V}{S} - \frac{V_0}{S_0}\right) + \nu(V - V_0)\left(\frac{S}{V} - \frac{S_0}{V_0}\right) \\
 & + \left(\beta + \frac{k\phi}{d + n} + \frac{[(d + n)\beta\eta + k\phi\eta](\mu + c)}{\eta(d + n)\sigma}\right)(S_0 + \epsilon V_0)I - \frac{\mu + \delta)(\mu + c)}{\eta\sigma} I \\
 = & \frac{(\mu + c)(\mu + \sigma)}{\eta\sigma} [R_0 - 1]I + \mu V_0\left[3 - \frac{S_0}{S} - \frac{S V_0}{V S_0} - \frac{V}{V_0}\right] \\
 & + \delta V_0\left[2 - \frac{V S_0}{S V_0} - \frac{S V_0}{V S_0}\right] + \mu S_0\left[2 - \frac{S}{S_0} - \frac{S_0}{S}\right].
 \end{aligned} \tag{3.3}$$

When $R_0 \leq 1$, we have $\frac{dV_1}{dt} \leq 0$ for all $S \geq 0$, $V \geq 0$, $E \geq 0$, $I \geq 0$, and $B \geq 0$, and $\frac{dV_1}{dt} = 0$ implies that $S = S_0$ and $V = V_0$. For any solution $(S(t), V(t), E(t), I(t), B(t))$, when $S(t) \equiv S_0$ and $V(t) \equiv V_0$, we further have that $A - \beta S_0(E(t) + I(t)) - \phi S_0 B(t) - (\mu + \nu)S_0 + \delta V_0 \equiv 0$. Since $A - (\mu + \nu)S_0 + \delta V_0 = 0$, we have $\beta(E(t) + I(t)) + \phi B(t) \equiv 0$. This implies that $E(t) \equiv 0$, $I(t) \equiv 0$ and $B(t) \equiv 0$ for all $t \geq 0$. Therefore, by the LaSalle's invariance principle [30], equilibrium P_0 is globally asymptotically stable. $\square$

Theorem 3.2. Assume that $\eta_1 = \eta_2 = \eta_3 := \eta$ and $R_0 > 1$, then endemic equilibrium $P^* = (S^*, V^*, E^*, I^*, B^*)$ is globally asymptotically stable.

Proof. Let $g(x) = x - 1 - \ln x$, it follows that $g(x) > 0$ for any $x > 0$ and $x \neq 1$. Define the Lyapunov functional as follows

$$V_2 = V_{21} + V_{22} + V_{23} + V_{24} + V_{25} + V_{26},$$

where

$$\begin{aligned}
 V_{21} = & S - S^* - S^* \ln \frac{S}{S^*}, \quad V_{22} = V - V^* - V^* \ln \frac{V}{V^*}, \\
 V_{23} = & \frac{(S^* + \epsilon V^*)I^*(\beta E^* + \beta I^* + \phi B^*)}{\sigma E^*(E^* + I^*)} (I - I^* - I^* \ln \frac{I}{I^*}), \\
 V_{24} = & \frac{\phi(S^* + \epsilon V^*)B^*}{k(E^* + I^*)} (B - B^* - B^* \ln \frac{B}{B^*}), \quad V_{25} = \frac{1}{\eta} \{E - E^* - E^* \ln \frac{E}{E^*}\}, \\
 V_{26} = & \frac{\beta S^* I^*}{\eta} \int_0^\infty e^{-\mu\tau} f_2(\tau) \int_{t-\tau}^t \left(\frac{SI}{S^* I^*} - 1 - \ln \frac{SI}{S^* I^*}\right) d\tau ds
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\beta S^* E^*}{\eta} \int_0^\infty e^{-\mu\tau} f_1(\tau) \int_{t-\tau}^t \left(\frac{SE}{S^* E^*} - 1 - \ln \frac{SE}{S^* E^*} \right) d\tau ds \\
& + \frac{\beta \epsilon V^* E^*}{\eta} \int_0^\infty e^{-\mu\tau} f_1(\tau) \int_{t-\tau}^t \left(\frac{VE}{V^* E^*} - 1 - \ln \frac{VE}{V^* E^*} \right) d\tau ds \\
& + \frac{\beta \epsilon V^* I^*}{\eta} \int_0^\infty e^{-\mu\tau} f_2(\tau) \int_{t-\tau}^t \left(\frac{VI}{V^* I^*} - 1 - \ln \frac{VI}{V^* I^*} \right) d\tau ds \\
& + \frac{\phi S^* B^*}{\eta} \int_0^\infty e^{-\mu\tau} f_3(\tau) \int_{t-\tau}^t \left(\frac{SB}{S^* B^*} - 1 - \ln \frac{SB}{S^* B^*} \right) d\tau ds \\
& + \frac{\phi \epsilon V^* B^*}{\eta} \int_0^\infty e^{-\mu\tau} f_3(\tau) \int_{t-\tau}^t \left(\frac{VB}{V^* B^*} - 1 - \ln \frac{VB}{V^* B^*} \right) d\tau ds.
\end{aligned}$$

Calculating the derivatives of V_{2i} ($i = 1, 2, \dots, 6$) along solutions of model (1), combining the following equations of endemic equilibrium P^* ,

$$\begin{cases}
A - \beta S^*(E^* + I^*) - \phi S^* B^* - (\mu + \nu)S^* + \delta V^* = 0, \\
\nu S^* - (\mu + \delta)V^* - \epsilon \beta V^*(E^* + I^*) - \epsilon \phi V^* B^* = 0, \\
\beta[S^* + \epsilon V^*][E^* \eta_1 + I^* \eta_2] + \phi[S^* + \epsilon V^*]B^* \eta_3 - (\mu + \sigma)E^* = 0, \\
\sigma E^* - (\mu + c)I^* = 0, \\
k(E^* + I^*) - (d + n)B^* = 0,
\end{cases}$$

we derive that

$$\begin{aligned}
\frac{dV_{21}}{dt} &= A \left(2 - \frac{S^*}{S} - \frac{S}{S^*} \right) + \delta V + \delta V^* - \beta S^* \frac{V}{S} - \frac{\beta V^*}{S^*} S - \beta S(E + I) - \phi S B \\
&\quad + \beta S(E^* + I^*) + \phi S B^* + \beta S^*(E + I) - \beta S^*(E^* + I^*) + \phi S^* B - \phi S^* B^*, \\
\frac{dV_{22}}{dt} &= \nu S - \frac{S^*}{V^*} \nu V - \epsilon \beta V(E + I) + \epsilon \beta V(E^* + I^*) - \epsilon \phi V B + \epsilon \phi B^* V \\
&\quad - \frac{S}{V} \nu V^* + \nu S^* + \epsilon \beta V^*(E + I) - \epsilon \beta V^*(E^* + I^*) + \epsilon \phi V^* B - \epsilon \phi V^* B^*, \\
\frac{dV_{23}}{dt} &= \frac{(S^* + \epsilon V^*)(\beta E^* + \beta I^* + \phi B^*)I^*}{E^*(E^* + I^*)} E - \frac{(S^* + \epsilon V^*)(\beta E^* + \beta I^* + \phi B^*)}{E^* + I^*} I \\
&\quad - \frac{(S^* + \epsilon V^*)(\beta E^* + \beta I^* + \phi B^*)I^{*2}}{E^*(E^* + I^*)} \times \frac{E}{I} + \frac{(S^* + \epsilon V^*)(\beta E^* + \beta I^* + \phi B^*)I^*}{E^* + I^*}, \\
\frac{dV_{24}}{dt} &= \frac{\phi(S^* + \epsilon V^*)B^*}{E^* + I^*} (E + I) - \frac{\phi(S^* + \epsilon V^*)B^{*2}}{B(E^* + I^*)} (E + I) - \phi(S^* + \epsilon V^*)B \\
&\quad + \phi(S^* + \epsilon V^*)B^*, \\
\frac{dV_{23}}{dt} &= \frac{1}{\eta} \int_0^\infty e^{-\mu\tau} [S(t - \tau) + \epsilon V(t - \tau)] [\beta E(t - \tau) f_1(\tau) + \beta I(t - \tau) f_2(\tau) \\
&\quad + \phi B(t - \tau) f_3(\tau)] d\tau - \frac{E^*}{\eta E} \int_0^\infty e^{-\mu\tau} [S(t - \tau) + \epsilon V(t - \tau)] \\
&\quad \times [\beta E(t - \tau) f_1(\tau) + \beta I(t - \tau) f_2(\tau) + \phi B(t - \tau) f_3(\tau)] d\tau
\end{aligned}$$

$$\begin{aligned}
& - \frac{(S^* + \epsilon V^*)(\beta I^* + \beta E^* + \phi B^*)}{E^*} E + (S^* + \epsilon V^*)(\beta I^* + \beta E^* + \phi B^*), \\
\frac{dV_{26}}{dt} = & \frac{\beta S^* I^*}{\eta} \int_0^\infty e^{-\mu\tau} f_2(\tau) \left[\frac{SI}{S^* I^*} - \frac{S(t-\tau)I(t-\tau)}{S^* I^*} + \ln \frac{S(t-\tau)I(t-\tau)}{SI} \right] d\tau \\
& + \frac{\beta S^* E^*}{\eta} \int_0^\infty e^{-\mu\tau} f_1(\tau) \left[\frac{SE}{S^* E^*} - \frac{S(t-\tau)E(t-\tau)}{S^* E^*} + \ln \frac{S(t-\tau)E(t-\tau)}{SE} \right] d\tau \\
& + \frac{\epsilon \beta V^* E^*}{\eta} \int_0^\infty e^{-\mu\tau} f_1(\tau) \left[\frac{VE}{V^* E^*} - \frac{V(t-\tau)E(t-\tau)}{V^* E^*} + \ln \frac{V(t-\tau)E(t-\tau)}{VE} \right] d\tau \\
& + \frac{\epsilon \beta V^* I^*}{\eta} \int_0^\infty e^{-\mu\tau} f_2(\tau) \left[\frac{VI}{V^* I^*} - \frac{V(t-\tau)I(t-\tau)}{V^* I^*} + \ln \frac{V(t-\tau)I(t-\tau)}{VI} \right] d\tau \\
& + \frac{\phi S^* B^*}{\eta} \int_0^\infty e^{-\mu\tau} f_3(\tau) \left[\frac{SB}{S^* B^*} - \frac{S(t-\tau)B(t-\tau)}{S^* B^*} + \ln \frac{S(t-\tau)B(t-\tau)}{SB} \right] d\tau \\
& + \frac{\phi \epsilon V^* B^*}{\eta} \int_0^\infty e^{-\mu\tau} f_3(\tau) \left[\frac{VB}{V^* B^*} - \frac{V(t-\tau)B(t-\tau)}{V^* B^*} + \ln \frac{V(t-\tau)B(t-\tau)}{VB} \right] d\tau.
\end{aligned}$$

Thus, we further have

$$\begin{aligned}
\frac{dV_2}{dt} = & \frac{dV_{21}}{dt} + \frac{dV_{22}}{dt} + \frac{dV_{23}}{dt} + \frac{dV_{24}}{dt} + \frac{dV_{25}}{dt} + \frac{dV_{26}}{dt} \\
= & \mu S^* \left[2 - \frac{S}{S^*} - \frac{S^*}{S} \right] + \delta V^* \left[2 - \frac{V}{S} \frac{S^*}{V^*} - \frac{S}{S^*} \frac{V^*}{V} \right] + \mu V^* \left[3 - \frac{S^*}{S} - \frac{S}{S^*} \frac{V^*}{V} - \frac{V}{V^*} \right] \\
& + \frac{\beta S^* I^*}{\eta} \int_0^\infty e^{-\mu\tau} f_2(\tau) \left[3 - \frac{S^*}{S} - \frac{I^* E}{IE^*} - \frac{E^* S(t-\tau)I(t-\tau)}{ES^* I^*} \right. \\
& \left. + \ln \frac{S(t-\tau)I(t-\tau)}{SI} \right] d\tau + \frac{\beta S^* E^*}{\eta} \int_0^\infty e^{-\mu\tau} f_1(\tau) \left[2 - \frac{S^*}{S} \right. \\
& \left. - \frac{S(t-\tau)E(t-\tau)}{S^* E} + \ln \frac{S(t-\tau)E(t-\tau)}{SE} \right] d\tau + \frac{\phi S^* B^* I^*}{\eta(E^* + I^*)} \int_0^\infty e^{-\mu\tau} f_3(\tau) \\
& \times \left[4 - \frac{S^*}{S} - \frac{I^* E}{IE^*} - \frac{B^* I}{BI^*} - \frac{E^* S(t-\tau)B(t-\tau)}{ES^* B^*} + \ln \frac{S(t-\tau)B(t-\tau)}{SB} \right] d\tau \\
& + \frac{\phi S^* B^* E^*}{\eta(E^* + I^*)} \int_0^\infty e^{-\mu\tau} f_3(\tau) \left[3 - \frac{S^*}{S} - \frac{B^* E}{BE^*} - \frac{E^* S(t-\tau)B(t-\tau)}{ES^* B^*} \right. \\
& \left. + \ln \frac{S(t-\tau)B(t-\tau)}{SB} \right] d\tau + \frac{\epsilon \beta V^* E^*}{\eta} \int_0^\infty e^{-\mu\tau} f_1(\tau) \left[3 - \frac{S^*}{S} - \frac{S}{S^*} \frac{V^*}{V} \right. \\
& \left. - \frac{E^* V(t-\tau)E(t-\tau)}{V^* E} + \ln \frac{V(t-\tau)E(t-\tau)}{VE} \right] d\tau \\
& + \frac{\epsilon \beta V^* I^*}{\eta} \int_0^\infty e^{-\mu\tau} f_2(\tau) \left[4 - \frac{S^*}{S} - \frac{S}{S^*} \frac{V^*}{V} - \frac{I^* E}{IE^*} \right. \\
& \left. - \frac{E^* V(t-\tau)I(t-\tau)}{V^* I^* E} + \ln \frac{V(t-\tau)I(t-\tau)}{VI} \right] d\tau \\
& + \frac{\epsilon \phi V^* B^* I^*}{\eta(E^* + I^*)} \int_0^\infty e^{-\mu\tau} f_3(\tau) \left[5 - \frac{S^*}{S} - \frac{S}{S^*} \frac{V^*}{V} - \frac{B^* I}{BI^*} - \frac{I^* E}{IE^*} \right. \\
& \left. - \frac{E^* V(t-\tau)B(t-\tau)}{EV^* B^*} + \ln \frac{V(t-\tau)B(t-\tau)}{VB} \right] d\tau
\end{aligned}$$

$$\begin{aligned}
& + \frac{\epsilon \phi V^* B^* E^*}{\eta(E^* + I^*)} \int_0^\infty e^{-\mu\tau} f_3(\tau) \left[4 - \frac{S^*}{S} - \frac{S V^*}{S^* V} - \frac{B^* E}{B E^*} \right. \\
& \quad \left. - \frac{E^* V(t-\tau) B(t-\tau)}{E V^* B^*} + \ln \frac{V(t-\tau) B(t-\tau)}{V B} \right] d\tau \\
& = \mu S^* \left[2 - \frac{S}{S^*} - \frac{S^*}{S} \right] + \delta V^* \left[2 - \frac{V S^*}{S V^*} - \frac{S V^*}{S^* V} \right] + \mu V^* \left[3 - \frac{S^*}{S} - \frac{S V^*}{S^* V} - \frac{V}{V^*} \right] \\
& \quad - \frac{\beta S^* I^*}{\eta} \int_0^\infty e^{-\mu\tau} f_2(\tau) \left[g\left(\frac{S^*}{S}\right) + g\left(\frac{I^* E}{I E^*}\right) + g\left(\frac{E^* S(t-\tau) I(t-\tau)}{E S^* I^*}\right) \right] d\tau \\
& \quad - \frac{\beta S^* E^*}{\eta} \int_0^\infty e^{-\mu\tau} f_1(\tau) \left[g\left(\frac{S^*}{S}\right) + g\left(\frac{S(t-\tau) E(t-\tau)}{S^* E}\right) \right] d\tau \\
& \quad - \frac{\phi S^* B^* I^*}{\eta(E^* + I^*)} \int_0^\infty e^{-\mu\tau} f_3(\tau) \left[g\left(\frac{S^*}{S}\right) + g\left(\frac{I^* E}{I E^*}\right) + g\left(\frac{B^* I}{B I^*}\right) \right. \\
& \quad \left. + g\left(\frac{E^* S(t-\tau) B(t-\tau)}{E S^* B^*}\right) \right] d\tau - \frac{\phi S^* B^* E^*}{\eta(E^* + I^*)} \int_0^\infty e^{-\mu\tau} f_3(\tau) \left[g\left(\frac{S^*}{S}\right) \right. \\
& \quad \left. + g\left(\frac{B^* E}{B E^*}\right) + g\left(\frac{E^* S(t-\tau) B(t-\tau)}{E S^* B^*}\right) \right] d\tau - \frac{\epsilon \beta V^* E^*}{\eta} \int_0^\infty e^{-\mu\tau} f_1(\tau) \left[g\left(\frac{S^*}{S}\right) \right. \\
& \quad \left. + g\left(\frac{S V^*}{S^* V}\right) + g\left(\frac{E^* V(t-\tau) E(t-\tau)}{V^* E}\right) \right] d\tau - \frac{\epsilon \beta V^* I^*}{\eta} \int_0^\infty e^{-\mu\tau} f_2(\tau) \left[g\left(\frac{S^*}{S}\right) \right. \\
& \quad \left. + g\left(\frac{S V^*}{S^* V}\right) + g\left(\frac{I^* E}{I E^*}\right) + g\left(\frac{E^* V(t-\tau) I(t-\tau)}{V^* I^* E}\right) \right] d\tau \\
& \quad - \frac{\epsilon \phi V^* B^* I^*}{\eta(E^* + I^*)} \int_0^\infty e^{-\mu\tau} f_3(\tau) \left[g\left(\frac{S^*}{S}\right) - g\left(\frac{S V^*}{S^* V}\right) + g\left(\frac{B^* I}{B I^*}\right) + g\left(\frac{I^* E}{I E^*}\right) \right. \\
& \quad \left. + g\left(\frac{E^* V(t-\tau) B(t-\tau)}{E V^* B^*}\right) \right] d\tau - \frac{\epsilon \phi V^* B^* E^*}{\eta(E^* + I^*)} \int_0^\infty e^{-\mu\tau} f_3(\tau) \left[g\left(\frac{S^*}{S}\right) \right. \\
& \quad \left. + g\left(\frac{S V^*}{S^* V}\right) + g\left(\frac{B^* E}{B E^*}\right) + g\left(\frac{E^* V(t-\tau) B(t-\tau)}{E V^* B^*}\right) \right] d\tau \leq 0,
\end{aligned}$$

for all $S > 0$, $V > 0$, $E > 0$, $I > 0$, and $B > 0$. When $\frac{dV_2}{dt} = 0$, we derive that $S = S^*$, $V = V^*$, $E = E^*$, $I = I^*$ and $B = B^*$. Therefore, by the LaSalle's invariance principle, equilibrium P^* is globally asymptotically stable. $\square$

Remark 3.1. It is difficult to determine the local asymptotic stability of endemic equilibrium P^* directly using linearization, because the characteristic equation is a five times beyond equation. It is very difficult to prove that all roots of it have negative real parts.

Remark 3.2. In the above theorems, an additional assumption is introduced, namely: $\eta_1 = \eta_2 = \eta_3 := \eta$. However, the numerical examples given in Section 6 indicate that when $\eta_1 \neq \eta_2 \neq \eta_3$ the conclusions in Theorems 3.1 and 3.2 may be true. Therefore, we can boldly assume that as long as $R_0 \leq 1$, the disease-free equilibrium is globally stable, while when $R_0 > 1$, the endemic equilibrium is globally stable.

4. Optimal control analysis

Theorems 3.1 and 3.2 indicate that as long as we adopt certain effective control measures in terms of disease prevention and treatment, reducing the basic reproduction number R_0 to less than one, the

asymptomatic infected individuals, infected individuals, and pathogens in the environment can be brought to zero, thereby controlling the spread of the disease. However, from the expression of the basic reproduction number R_0 , we can see that merely implementing control measures, which causes a change in one of the key parameters of the model, such as the infection rate β or clearance rate c of infected sheep, etc., is not sufficient to reduce the basic reproduction number R_0 to less than one. This is because there are multiple ways for diseases to spread, such as transmission by asymptomatic carriers, transmission by infected individuals, and transmission through environmental pathogens. Therefore, a comprehensive control approach must be adopted, where multiple control measures simultaneously exert influence on several key parameters of the model. Only in this way can the goal of effectively controlling the spread of diseases be achieved. However, a key problem with the comprehensive control approach that employs multiple control measures simultaneously is that when medical resources are limited or funds are insufficient, it is impossible for all the control measures to fully exert their functions; for instance: Hospital beds are limited, so not all patients can receive complete treatment; or due to limited funds, not all susceptible individuals can be vaccinated, and so on. Therefore, an optimal control strategy must be proposed, that is, an effective plan within a reasonable time interval, such as $[0, T]$, which minimizes the cost of all control measures while reducing the number of asymptomatic infected individuals, infected individuals, and environmental pathogens to a reasonable and controllable range.

To determine which parameters of the model are the key parameters, we will employ the method of sensitivity analysis. By calculating the sensitivity values of model parameters with respect to the basic reproduction number (see Table 3 and Figure 1 in Section 5), we can observe that factors such as infection rate, treatment rate, vaccination rate, the rate of clinical outcome of asymptomatic infected, and environmental pathogen clearance rate have relatively high sensitivity. Therefore, we introduce control variables for these parameters.

Based on the above discussions, we propose the following model with control variables:

$$\begin{cases} dS = [A - (1 - u_1(t))\beta S(E + I) - \phi S B - (\mu + \nu u_3(t))S + \delta V]dt, \\ dV = [\nu u_3(t)S - (\mu + \delta)V - (1 - u_1(t))\epsilon\beta V(E + I) - \epsilon\phi V B]dt, \\ dE = \left[\int_0^\infty (1 - u_1(t))\beta[S(t - \tau) + \epsilon V(t - \tau)][E(t - \tau)f_1(\tau) + I(t - \tau)f_2(\tau)]e^{-\mu\tau}d\tau \right. \\ \quad \left. + \int_0^\infty \phi[S(t - \tau) + \epsilon V(t - \tau)]B(t - \tau)f_3(\tau)e^{-\mu\tau}d\tau - (\mu + \sigma u_2(t))E \right]dt, \\ dI = [\sigma u_2(t)E - (\mu + cu_5(t))I]dt, \\ dB = [k(E + I) - (d + nu_4(t))B]dt, \end{cases} \quad (4.1)$$

where $u_i(t)$ ($i = 1, 2, 3, 4, 5$) are control variables, which are described as follows:

- $u_1(t)$ is used to reduce the infection rate of susceptible and vaccinated individuals by asymptomatic and symptomatic infected individuals.
- $u_2(t)$ is used to strengthen medical screening rate for asymptomatic infected sheep.
- $u_3(t)$ is used to strengthen the vaccination rate of susceptible sheep.
- $u_4(t)$ is used to strengthen the clearance rate of environmental pathogens.
- $u_5(t)$ is used to strengthen the clearance rate of infected sheep.

In addition, some practical implementation of each control variable u_i can be simply described as follows: Control measure u_1 can be achieved by timely screening and isolating infected sheep, adding

anti-brucellosis drugs to feed and drinking water in a timely manner, and dividing large flocks into smaller ones for decentralized breeding to reduce the infection rate. Control measure u_2 can be implemented through regular random screening of low-risk sheep flocks and comprehensive medical screening of high-risk sheep flocks, etc. For control measure u_3 , it should be implemented on the basis of medical screening. According to the quantity and supply method of vaccines, susceptible flocks should be vaccinated from high-risk to low-risk flocks in accordance with the medical coverage ratio for controlling disease transmission. Control measure u_4 can be implemented by strengthening the spraying of disinfectants in the living environment of the flock, disinfecting feed and drinking water, and promptly cleaning up manure, etc. Last, control measure u_5 can be implemented by strengthening the monitoring of sheep flocks, promptly identifying infected sheep, and immediately isolating, disinfecting, slaughtering, burning, or deeply burying them.

The control variable u_1 is required to satisfy $0 \leq u_1(t) \leq 1$, where the lower bound $u_1 = 0$ represents the actual situation before the implementation of the control measure, i.e., the infection rate is not reduced, and the upper bound $u_1 = 1$ represents that the infection rate is reduced to zero. The control variables u_i ($i = 2, 3, 4, 5$) are required to satisfy the inequalities $1 \leq u_i(t) \leq \bar{u}_i$, where the lower bound $u_i = 1$ represents the actual situation before the implementation of the control measures, and the upper bound $u_i = \bar{u}_i$ represents the maximum strength that the control measures can be implemented.

Our control objective is to minimize the total cost of the above five control measures on a time interval $[0, T]$ while minimizing the number of the total number of infectious latent, infected, and environmental pathogens. Therefore, we define the objective functional as

$$J(u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)) = \int_0^T [A_1 E + A_2 I + A_3 B + \frac{1}{2}(B_1 u_1^2(t) + B_2 u_2^2(t) + B_3 u_3^2(t) + B_4 u_4^2(t) + B_5 u_5^2(t))] dt, \quad (4.2)$$

where A_i ($i = 1, 2, 3$) are positive weight constants for (E, I, B) . $B_i u_i(t)$ ($i = 1, 2, 3, 4, 5$) represents the cost required for implementing control measure $u_i(t)$ at time t . Our goal is to find the optimal control variables $(u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t), u_5^*(t)) \in U$ such that

$$J(u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t), u_5^*(t)) = \min\{J(u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)) : (u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)) \in U\}, \quad (4.3)$$

where the control set U is given as

$$U = \{(u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)) : 0 \leq u_1(t) \leq 1, 1 \leq u_2(t) \leq \bar{u}_2, 1 \leq u_3(t) \leq \bar{u}_3, 1 \leq u_4(t) \leq \bar{u}_4, 1 \leq u_5(t) \leq \bar{u}_5, t \in [0, T]\},$$

and T is the final time of implemented control policies.

In the objective functional $J(u_1, u_2, u_3, u_4, u_5)$, the linear term $A_1 E + A_2 I + A_3 B$ represents a weighted sum of infectious latent, infected, and environmental pathogens, where A_1 , A_2 and, A_3 are weight coefficients. Our control goal is to minimize the amount of accumulation of this weighted sum on time interval $[0, T]$. The quadratic term $\frac{1}{2}(B_1 u_1^2(t) + B_2 u_2^2(t) + B_3 u_3^2(t) + B_4 u_4^2(t) + B_5 u_5^2(t))$ represents the total cost of implementing all five control measures required, which is consistent with the quadratic cost formulation adopted in related epidemic optimal control models [31]. Specifically, in practical disease management, the marginal expenditure of an intervention grows as its intensity increases, so

a convex quadratic cost is a standard and reasonable assumption. The positive weights B_i reflect the relative economic burden of the different control measures. Our control goal is to minimize the amount of accumulation of this total cost on interval $[0, T]$.

Based on the results given in [32, 33] for the existence of optimal control of differential equation control models, we can directly establish the following conclusion.

Theorem 4.1. *There exist the optimal control variables $u^*(t) = (u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) \in U$ and the corresponding optimal solution $(S^*, V^*, E^*, I^*, B^*)$ such that $f(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) = \min_{(u_1, u_2, u_3, u_4, u_5) \in U} f(u_1, u_2, u_3, u_4, u_5)$, subject to the control system (4.1).*

Based on this existence theorem and the Pontryagin's extremum theory, we can establish the following calculation method for the optimal control pair $u^*(t) = (u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) \in U$.

Theorem 4.2. *Let $(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*)$ be optimal control variables, and $(S^*, V^*, E^*, I^*, B^*)$ be corresponding optimal solution of control system (4.1). Then there exists an adjoint variable $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t))$ that satisfies the following adjoint equations:*

$$\begin{aligned}
 \frac{d\lambda_1}{dt} &= \lambda_1(t)[(1 - u_1^*)\beta(E^* + I^*) + \phi B^* + \mu + \nu u_3^*] - \lambda_2(t)\nu u_3^* \\
 &\quad - \chi_{[0, T-\tau]} \lambda_3(t + \tau) \left[\int_0^\infty (1 - u_1^*)\beta[E^*(t - \tau)f_1(\tau) + I^*(t - \tau)f_2(\tau)]e^{-\mu\tau} d\tau \right. \\
 &\quad \left. + \int_0^\infty \phi B^*(t - \tau)f_3(\tau)e^{-\mu\tau} d\tau \right], \\
 \frac{d\lambda_2}{dt} &= -\lambda_1(t)\delta + \lambda_2(t)[\delta + \mu + \epsilon\phi B^* + (1 - u_1^*)\epsilon\beta(E^* + I^*)] \\
 &\quad - \chi_{[0, T-\tau]} \lambda_3(t + \tau) \left[\int_0^\infty (1 - u_1^*)\epsilon\beta[E^*(t - \tau)f_1(\tau) + I^*(t - \tau)f_2(\tau)]e^{-\mu\tau} d\tau \right. \\
 &\quad \left. + \int_0^\infty \epsilon\phi B^*(t - \tau)f_3(\tau)e^{-\mu\tau} d\tau \right], \\
 \frac{d\lambda_3}{dt} &= \lambda_1(t)(1 - u_1^*)\beta S^* + \lambda_2(t)(1 - u_1^*)\epsilon\beta V^* + \lambda_3(t)(\mu + \sigma * u_2^*) - \lambda_4(t)\sigma * u_2^* - \lambda_5(t)k \\
 &\quad - \chi_{[0, T-\tau]} \lambda_3(t + \tau) \int_0^\infty (1 - u_1^*)\beta[S^*(t - \tau) + \epsilon V^*(t - \tau)]e^{-\mu\tau} f_1(\tau) d\tau - A_1, \\
 \frac{d\lambda_4}{dt} &= \lambda_1(t)(1 - u_1^*)\beta S^* + \lambda_2(t)(1 - u_1^*)\epsilon\beta V^* - \chi_{[0, T-\tau]} \lambda_3(t + \tau) \\
 &\quad \times \int_0^\infty (1 - u_1^*)\beta[S^*(t - \tau) + \epsilon V^*(t - \tau)]e^{-\mu\tau} f_2(\tau) d\tau + \lambda_4(\mu + u_5^*c) - \lambda_5k - A_2, \\
 \frac{d\lambda_5}{dt} &= \lambda_1(t)\phi B^* + \lambda_2(t)\epsilon\phi V^* - \chi_{[0, T-\tau]} \lambda_3(t + \tau) \\
 &\quad \times \int_0^\infty \phi[S^*(t - \tau) + \epsilon V^*(t - \tau)]e^{-\mu\tau} f_3(\tau) d\tau + \lambda_5(d + nu_4^*) - A_3,
 \end{aligned} \tag{4.4}$$

with transversality conditions $\lambda_i(T) = 0$ ($i = 1, 2, 3, 4, 5$), where $\chi_{[0, T-\tau]}$ is the characteristic function defined in the proof below. Furthermore, the optimal control variables $u^*(t) = (u_1^*, u_2^*, u_3^*, u_4^*, u_5^*) \in U$

can be given in the following forms:

$$\begin{aligned} u_1^* &= \min\{\max\{k_1, 0\}, 1\}, \quad u_2^* = \min\{\max\{k_2, 1\}, \bar{u}_2\}, \quad u_3^* = \min\{\max\{k_3, 1\}, \bar{u}_3\}, \\ u_4^* &= \min\{\max\{k_4, 1\}, \bar{u}_4\}, \quad u_5^* = \min\{\max\{k_5, 1\}, \bar{u}_5\}, \end{aligned}$$

where

$$\begin{aligned} k_1 &= \frac{\lambda_3(t) \int_0^\infty \beta[S^*(t-\tau) + \epsilon V^*(t-\tau)][E^*(t-\tau)f_1(\tau) + I^*(t-\tau)f_2(\tau)]e^{-\mu\tau} d\tau}{B_1} \\ &\quad - \frac{(\lambda_1(t)\beta S^* + \lambda_2(t)\epsilon\beta V^*)(E^* + I^*)}{B_1}, \\ k_2 &= \frac{(\lambda_3(t) - \lambda_4(t))\sigma E^*}{B_2}, \quad k_3 = \frac{(\lambda_1(t) - \lambda_2(t))\nu S^*}{B_3}, \quad k_4 = \frac{\lambda_5(t)nB^*}{B_4}, \quad k_5 = \frac{\lambda_4(t)cI^*}{B_5}. \end{aligned} \quad (4.5)$$

Proof. First, for any interval $[a, b]$, let $\chi_{[a,b]}(t)$ be the characteristic function of $[a, b]$ defined by

$$\chi_{[a,b]}(t) = \begin{cases} 1, & \text{if } t \in [a, b], \\ 0, & \text{otherwise.} \end{cases}$$

Define the corresponding Hamiltonian as

$$\begin{aligned} H &= A_1 E + A_2 I + A_3 B + \frac{1}{2}(B_1 u_1^2(t) + B_2 u_2^2(t) + B_3 u_3^2(t) + B_4 u_4^2(t) + B_5 u_5^2(t)) \\ &\quad + \lambda_1 \frac{dS}{dt} + \lambda_2 \frac{dV}{dt} + \lambda_3 \frac{dE}{dt} + \lambda_4 \frac{dI}{dt} + \lambda_5 \frac{dB}{dt}. \end{aligned} \quad (4.6)$$

Let $(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*)$ be the given optimal control functions and $(S^*, V^*, E^*, I^*, B^*)$ be the corresponding optimal solution of systems (4.1)–(4.3). By Pontryagin's Maximum Principle with delay in state (see [34, 35]), there exist adjoint variables $\lambda_i(t)$ ($i = 1, 2, 3, 4, 5$) that satisfy the following canonical equations:

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -\left[\frac{\partial H}{\partial S}(t) + \chi_{[0, T-\tau]}(t) \frac{\partial H}{\partial S(t-\tau)}(t)\right], \quad \frac{d\lambda_2}{dt} = -\left[\frac{\partial H}{\partial V}(t) + \chi_{[0, T-\tau]}(t) \frac{\partial H}{\partial V(t-\tau)}(t)\right], \\ \frac{d\lambda_3}{dt} &= -\left[\frac{\partial H}{\partial E}(t) + \chi_{[0, T-\tau]}(t) \frac{\partial H}{\partial E(t-\tau)}(t)\right], \quad \frac{d\lambda_4}{dt} = -\left[\frac{\partial H}{\partial I}(t) + \chi_{[0, T-\tau]}(t) \frac{\partial H}{\partial I(t-\tau)}(t)\right], \\ \frac{d\lambda_5}{dt} &= -\left[\frac{\partial H}{\partial B}(t) + \chi_{[0, T-\tau]}(t) \frac{\partial H}{\partial B(t-\tau)}(t)\right], \end{aligned} \quad (4.7)$$

with transversality conditions $\lambda_i(T) = 0$ ($i = 1, 2, 3, 4, 5$). Substituting the corresponding derivatives in the above inequalities and rearranging them, we get the adjoint equation (4.4). From the optimality necessary condition, we have that when $(u_1, u_2, u_3, u_4, u_5) = (u_1^*, u_2^*, u_3^*, u_4^*, u_5^*)$,

$$\frac{\partial H}{\partial u_1} = 0, \quad \frac{\partial H}{\partial u_2} = 0, \quad \frac{\partial H}{\partial u_3} = 0, \quad \frac{\partial H}{\partial u_4} = 0, \quad \frac{\partial H}{\partial u_5} = 0.$$

Solving these equations, we can obtain the solution:

$$\begin{aligned}
 u_1^* &= \frac{\lambda_3(t) \int_0^\infty \beta[S^*(t-\tau) + \epsilon V^*(t-\tau)][E^*(t-\tau)f_1(\tau) + I^*(t-\tau)f_2(\tau)]e^{-\mu\tau}d\tau}{B_1} \\
 &\quad - \frac{(\lambda_1(t)\beta S^* + \lambda_2(t)\epsilon\beta V^*)(E^* + I^*)}{B_1} := k_1, \\
 u_2^* &= \frac{(\lambda_3(t) - \lambda_4(t))\sigma E^*}{B_2} := k_2, \quad u_3^* = \frac{(\lambda_1(t) - \lambda_2(t))\nu S^*}{B_3} := k_3, \\
 u_4^* &= \frac{\lambda_5(t)nB^*}{B_4} := k_4, \quad u_5^* = \frac{\lambda_4(t)cI^*}{B_5} := k_5.
 \end{aligned}$$

Using the property of the control space, we further obtain

$$u_1^* = \begin{cases} 0, & \text{if } k_1 < 0, \\ k_1, & 0 \leq k_1 \leq 1, \\ 1, & k_1 > 1, \end{cases} \quad u_2^* = \begin{cases} 1, & \text{if } k_2 < 1, \\ k_2, & 1 \leq k_2 \leq \bar{u}_2, \\ \bar{u}_2, & k_2 > \bar{u}_2, \end{cases}$$

$$u_3^* = \begin{cases} 1, & \text{if } k_3 < 1, \\ k_3, & 1 \leq k_3 \leq \bar{u}_3, \\ \bar{u}_3, & k_3 > \bar{u}_3, \end{cases} \quad u_4^* = \begin{cases} 1, & \text{if } k_4 < 1, \\ k_4, & 1 \leq k_4 \leq \bar{u}_4, \\ \bar{u}_4, & k_4 > \bar{u}_4, \end{cases} \quad u_5^* = \begin{cases} 1, & \text{if } k_5 < 1, \\ k_5, & 1 \leq k_5 \leq \bar{u}_5, \\ \bar{u}_5, & k_5 > \bar{u}_5, \end{cases}$$

Substituting u_1^* , u_2^* , u_3^* , u_4^* and u_5^* into Eqs (4.1) and (4.4), the optimality system with minimized Hamiltonian H^* at $(S^*, V^*, E^*, I^*, B^*)$, $(u_1^*, u_2^*, u_3^*, u_4^*, u_5^*)$, and $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ satisfy

$$\begin{cases} \frac{dS^*}{dt} = A - (1 - u_1^*)\beta S^*(E^* + I^*) - \phi S^* B^* - (\mu + \nu u_3^*)S^* + \delta V^*, \\ \frac{dV^*}{dt} = \nu u_3^* S^* - (\mu + \delta)V^* - (1 - u_1^*)\epsilon\beta V^*(E^* + I^*) - \epsilon\phi V^* B^*, \\ \frac{dE^*}{dt} = \int_0^\infty (1 - u_1^*)\beta[S^*(t-\tau) + \epsilon V^*(t-\tau)][E^*(t-\tau)f_1(\tau) + I^*(t-\tau)f_2(\tau)]e^{-\mu\tau}d\tau \\ \quad + \int_0^\infty \phi[S^*(t-\tau) + \epsilon V^*(t-\tau)]B^*(t-\tau)f_3(\tau)e^{-\mu\tau}d\tau - (\mu + \sigma u_2^*)E^*, \\ \frac{dI^*}{dt} = \sigma u_2^* E^* - (\mu + c u_5^*)I^*, \\ \frac{dB^*}{dt} = k(E^* + I^*) - (d + n u_4^*)B^*, \end{cases} \quad (4.8)$$

$$\left\{ \begin{aligned}
\frac{d\lambda_1}{dt} &= \lambda_1(t)[(1 - u_1^*)\beta(E^* + I^*) + \phi B^* + \mu + \nu u_3^*] - \lambda_2(t)\nu u_3^* \\
&\quad - \chi_{[0, T-\tau]} \lambda_3(t + \tau) \left[\int_0^\infty (1 - u_1^*)\beta[E^*(t - \tau)f_1(\tau) + I^*(t - \tau)f_2(\tau)]e^{-\mu\tau} d\tau \right. \\
&\quad \left. + \int_0^\infty \phi B^*(t - \tau)f_3(\tau)e^{-\mu\tau} d\tau \right], \\
\frac{d\lambda_2}{dt} &= -\lambda_1(t)\delta + \lambda_2(t)[\delta + \mu + \epsilon\phi B^* + (1 - u_1^*)\epsilon\beta(E^* + I^*)] \\
&\quad - \chi_{[0, T-\tau]} \lambda_3(t + \tau) \left[\int_0^\infty (1 - u_1^*)\epsilon\beta[E^*(t - \tau)f_1(\tau) + I^*(t - \tau)f_2(\tau)]e^{-\mu\tau} d\tau \right. \\
&\quad \left. + \int_0^\infty \epsilon\phi B^*(t - \tau)f_3(\tau)e^{-\mu\tau} d\tau \right], \\
\frac{d\lambda_3}{dt} &= \lambda_1(t)(1 - u_1^*)\beta S^* + \lambda_2(t)(1 - u_1^*)\epsilon\beta V^* + \lambda_3(t)(\mu + \sigma * u_2^*) - \lambda_4(t)\sigma * u_2^* - \lambda_5(t)k \\
&\quad - \chi_{[0, T-\tau]} \lambda_3(t + \tau) \int_0^\infty (1 - u_1^*)\beta[S^*(t - \tau) + \epsilon V^*(t - \tau)]e^{-\mu\tau} f_1(\tau) d\tau - A_1, \\
\frac{d\lambda_4}{dt} &= \lambda_1(t)(1 - u_1^*)\beta S^* + \lambda_2(t)(1 - u_1^*) * \epsilon\beta V^* - \chi_{[0, T-\tau]} \lambda_3(t + \tau) \\
&\quad \times \int_0^\infty (1 - u_1^*)\beta[S^*(t - \tau) + \epsilon V^*(t - \tau)]e^{-\mu\tau} f_2(\tau) d\tau + \lambda_4(\mu + u_5^*c) - \lambda_5k - A_2, \\
\frac{d\lambda_5}{dt} &= \lambda_1(t)\phi B^* + \lambda_2(t)\epsilon\phi V^* - \chi_{[0, T-\tau]} \lambda_3(t + \tau) \\
&\quad \times \int_0^\infty \phi[S^*(t - \tau) + \epsilon V^*(t - \tau)]e^{-\mu\tau} f_3(\tau) d\tau + \lambda_5(d + nu_4^*) - A_3 \\
\lambda_i(T) &= 0 \quad (i = 1, 2, 3, 4, 5),
\end{aligned} \right. \quad (4.9)$$

and

$$\begin{aligned}
u_1^*(t) &= \min\{\max\{k_1, 0\}, 1\}, \quad u_2^*(t) = \min\{\max\{k_2, 1\}, \bar{u}_2\}, \quad u_3^*(t) = \min\{\max\{k_3, 1\}, \bar{u}_3\}, \\
u_4^*(t) &= \min\{\max\{k_4, 1\}, \bar{u}_4\}, \quad u_5^*(t) = \min\{\max\{k_5, 1\}, \bar{u}_5\}.
\end{aligned} \quad (4.10)$$

This completes the proof. $\square$

However, when funds are insufficient, it is impossible to implement all control measures: $u_i(t)$ ($i = 1, 2, 3, 4, 5$). At this point, we can adopt only some control measures that are highly effective and easy to implement. This means that only a part of the control variables $u_i(t)$ appear in system (4.1). To this end, let us consider the following several special cases:

(1) Single optimal control strategy.

(i) $u_1(t)$ appears, and $u_i(t)$ ($i = 2, 3, 4, 5$) do not appear in system (4.1), that is, $0 \leq u_1(t) \leq 1$, and $u_i(t) \equiv 1$ ($i = 2, 3, 4, 5$). Similar to Theorem 4.2, we can derive that the optimal control strategy is $u_1^*(t) = \min\{\max\{k_1, 0\}, 1\}$, where k_1 is defined in (4.5) in which $\lambda_i(t)$ ($i = 1, 2, 3, 4, 5$) are presented in (4.4) with $u_i^*(t) \equiv 1$ for $i = 2, 3, 4, 5$, and $(S^*, V^*, E^*, I^*, B^*)$ is the corresponding optimal solution of control system (4.1).

(ii) $u_i(t)$ appears for some $i \in \{2, 3, 4, 5\}$, and $u_j(t)$ ($j = 1, 2, 3, 4, 5, j \neq i$) do not appear in system (4.1), that is, $l \leq u_i(t) \leq \bar{u}_i$, $u_1(t) \equiv 0$ and $u_j(t) \equiv 1$ ($j = 2, 3, 4, 5, j \neq i$). Similar to

Theorem 4.2, we can derive that the optimal control strategy is $u_i^*(t) = \min\{\max\{k_i, 1\}, \bar{u}_i\}$, where k_i is defined in (4.5) in which $\lambda_i(t)$ ($i = 1, 2, 3, 4, 5$) are presented in (4.4) with $u_1^*(t) \equiv 0$ and $u_j^*(t) \equiv 1$ for $j = 2, 3, 4, 5$ with $j \neq i$, and $(S^*, V^*, E^*, I^*, B^*)$ is the corresponding optimal solution of control system (4.1).

(2) Two measures joint optimal control strategy. Here, we discuss only the optimal control strategy for the combined implementation of vaccination immunization and other control measures.

(i) Only $u_3(t)$ and $u_1(t)$ appear in system (4.1), that is, $0 \leq u_1(t) \leq 1$, $1 \leq u_3(t) \leq \bar{u}_3$, and $u_j(t) \equiv 1$ ($j = 2, 4, 5$). With a similar argument as in Theorem 4.2, we can derive that the optimal control strategy is $u_1^*(t) = \min\{\max\{k_1, 0\}, 1\}$ and $u_3^*(t) = \min\{\max\{k_3, 1\}, \bar{u}_3\}$, where k_1 and k_3 are defined in (4.5) in which $\lambda_i(t)$ ($i = 1, 2, 3, 4, 5$) are presented in (4.4) with $u_j^*(t) \equiv 1$ for $j = 2, 4, 5$, and $(S^*, V^*, E^*, I^*, B^*)$ is the corresponding optimal solution of control system (4.1).

(ii) Only $u_3(t)$ and $u_i(t)$ appear in system (4.1) for some $i \in \{2, 4, 5\}$, that is, $u_1(t) \equiv 0$, $1 \leq u_3(t) \leq \bar{u}_3$, $1 \leq u_i(t) \leq \bar{u}_i$ and $u_j(t) \equiv 1$ for $j = 2, 4, 5$ with $j \neq i$. Similar argument as in Theorem 4.2, we can derive that the optimal control strategy is $u_3^*(t) = \min\{\max\{k_3, 1\}, \bar{u}_3\}$ and $u_i^*(t) = \min\{\max\{k_i, 1\}, \bar{u}_i\}$, where k_3 and k_i are defined in (4.5) in which $\lambda_i(t)$ ($i = 1, 2, 3, 4, 5$) are presented in (4.4) with $u_1^*(t) \equiv 0$ and $u_j^*(t) \equiv 1$ for $j = 2, 4, 5$ with $j \neq i$, and $(S^*, V^*, E^*, I^*, B^*)$ is the corresponding optimal solution of control system (4.1).

5. Sensitivity analysis

To demonstrate the theoretical findings, we conduct numerical simulations for model (1) using specific distribution functions $f_1(s) = \delta(s - \tau_1)$, $f_2(s) = \delta(s - \tau_2)$, $f_3(s) = \delta(s - \tau_3)$, where τ_i ($i = 1, 2, 3$) are positive constants and $\delta(\cdot)$ is the dirac delta function. Then, we can see that

$$\int_0^\infty f_i(s)ds = 1, \quad \eta_i = \int_0^\infty \delta(s - \tau_i)e^{-\mu s}ds = e^{-\mu\tau_i}, \quad i = 1, 2, 3.$$

Based on this, model (1) is reduced to

$$\begin{cases} \frac{dS}{dt} = A - \beta S(E + I) - \phi S B - (\mu + \nu)S + \delta V, \\ \frac{dV}{dt} = \nu S - (\mu + \delta)V - \epsilon \beta V(E + I) - \epsilon \phi V B, \\ \frac{dE}{dt} = \beta[S(t - \tau_1) + \epsilon V(t - \tau_1)]E(t - \tau_1)e^{-\mu\tau_1} + \beta[S(t - \tau_2) + \epsilon V(t - \tau_2)] \\ \quad \times I(t - \tau_2)e^{-\mu\tau_2} + \phi[S(t - \tau_3) + \epsilon V(t - \tau_3)]B(t - \tau_3)e^{-\mu\tau_3} - (\mu + \sigma)E, \\ \frac{dI}{dt} = \sigma E - (\mu + c)I, \\ \frac{dB}{dt} = k(E + I) - (d + n)B. \end{cases} \quad (5.1)$$

From model (5.1), the the basic reproduction number R_0 is calculated as

$$R_0 = \left[\frac{(d + n)\beta e^{-\mu\tau_1} + k\phi e^{-\mu\tau_3}}{(d + n)(\mu + \sigma)} + \frac{((d + n)\beta e^{-\mu\tau_2} + k\phi e^{-\mu\tau_3})\sigma}{(d + n)(\mu + \sigma)(\mu + c)} \right] (S_0 + \epsilon V_0), \quad (5.2)$$

where $S_0 = \frac{A(\mu + \delta)}{\mu(\mu + \delta + \nu)}$ and $V_0 = \frac{A\nu}{\mu(\mu + \delta + \nu)}$.

Next, through sensitivity analysis, we will assess the impact of parameter changes on model output. The core objective is to identify which parameters have the most significant influence on the model results. This analysis helps us understand the crucial role of each parameter in the transmission of disease. Here, we will apply the normalized forward sensitivity index to the basic reproduction number R_0 to determine the sensitivity of related parameters to the spread and prevalence of infection.

The normalized sensitivity index of the basic reproduction number R_0 with respect to parameter q can be calculated using the following formula:

$$\varepsilon_{R_0}^q = \frac{\partial R_0}{\partial q} \times \frac{q}{R_0},$$

where q is one of the parameters associated with R_0 . Thus, the Sensitivity indices of the basic reproduction number R_0 , based on the parameter values provided in Table 2, are presented in Table 3.

Table 2. Parameters and their values (unit: year⁻¹).

Parameter	Value	Biological description	Source
A	138,919,000	The recruitment rate of sheep population	[A]
μ	0.22	The natural elimination rate of sheep	[A]
c	0.15	The clearance rate of infected sheep	[7]
δ	0.4	The failure rate of sheep vaccines	[B]
ν	0.316	The vaccination rate of sheep	[36]
β	1.3×10^{-8}	Sheep-to-sheep infection rate	[37]
ϵ	0.18	The invalid vaccination rate of sheep	[7]
ϕ	1.7×10^{-10}	Brucella-to-susceptible sheep infection rate	[7]
σ	26	The rate of clinical outcome of asymptomatic infected sheep	[C]
k	15	Brucella shedding rate by asymptomatic infected and infected sheep	[7]
d	3.6	The natural clearance rate of brucella in the environment	[7]
n	52	The artificial clearance rate of environmental pathogens	Assumed
τ_1	15, 8.5	The incubation periods after being infected by latent infected individuals	Assumed
τ_2	15, 8.5	The incubation periods after being infected by infected individuals	Assumed
τ_3	15, 8.5	The incubation periods after being infected by environmental pathogens	Assumed

In Table 2, for the sources [A], [B], and [C], we have the following explanation.

[A]: In Xinjiang, sheep life span is about 4–5 years, and according to the Statistic Bureau of Xinjiang Uygur Autonomous Region, the number of sheep remains about 6.3145×10^8 from 2005 to 2020, so we estimate that the average elimination rate of sheep μ is $\frac{1}{4.5} = 0.22$, and the recruitment rate is $6.3145 \times 10^8 \times 0.22 = 138,919,000$.

[B]: In Xinjiang, vaccine B. suis strain 2 has been used to control brucellosis. The immune duration of vaccine B.suis strain 2 is about 2.5 years [7], so we have $\delta = 1/2.5 = 0.4$

[C]: The mean incubation period of brucellosis is almost two weeks [38], so we have $\sigma = 26$.

Table 3. Sensitivity indices of parameters in R_0 .

Parameter	Sensitivity index	Parameter	Sensitivity index
A	1.0000	μ	-3.3498
β	0.9925	τ_2	-1.8113
σ	0.9760	ν	-1.1526
δ	0.7436	c	-0.3957
ϵ	0.0840	τ_1	-0.0315
k	0.0075	τ_3	-0.0082
ϕ	0.0075	n	-0.0070
		d	-0.0005

Seven parameters ($A, \beta, \sigma, \delta, \epsilon, k, \phi$) possess positive sensitivity values, meaning their increments elevate R_0 and serve as dominant risk factors sustaining endemic brucellosis. The sheep recruitment rate A yields the maximum positive sensitivity ($S_A = 1.0000$): Continuous lamb birth and external sheep introduction replenish susceptible host pool persistently, supplying abundant susceptible animals for long-term brucellosis circulation. The direct sheep-to-sheep infection rate β ($S_\beta = 0.9925$) dominates horizontal transmission among animals by determining the infection probability via effective contact. The transition rate σ from latent infection to clinical cases ($S_\sigma = 0.9760$) accelerates accumulation of clinically diseased sheep shedding viable *Brucella*. Vaccine failure rate δ ($S_\delta = 0.7436$) depletes immune protection by converting vaccinated sheep back to susceptible status. In contrast, the vaccinated sheep breakthrough infection coefficient ϵ has marginal positive impact ($S_\epsilon = 0.0840$), while environmental pathogen-related parameters k (bacteria shedding rate) and ϕ (environmental infection rate) present negligible sensitivity ($S_k = S_\phi = 0.0075$), suggesting interventions targeting environmental contamination exert limited effects on reducing R_0 .

The other eight parameters ($\mu, \tau_2, \nu, c, \tau_1, \tau_3, n, d$) have negative elastic sensitivity values, indicating that increasing these parameters reduces R_0 , and these variables are core intervention targets for ovine brucellosis prevention and control. The natural elimination rate of sheep μ exhibits the strongest negative sensitivity ($S_\mu = -3.3498$): A 1% relative increase of μ leads to a 3.35% drop of R_0 . Natural death continuously removes latently infected and clinically diseased sheep from host population and eliminates infectious sources naturally, making it the most efficient parameter to restrain endemic brucellosis. The time delay τ_2 for clinically infected sheep yields $S_{\tau_2} = -1.8113$; prolonging τ_2 delays the onset of *Brucella* shedding and shortens the effective infectious duration of diseased animals, effectively limiting pathogen transmission within flocks. Sheep vaccination rate ν has a sensitivity of -1.1526 : Systematic vaccination expands immune population size and shrinks susceptible reservoir, cutting off the infection chain fundamentally. The clearance rate c of infected sheep ($S_c = -0.3957$) accelerates pathogen clearance in diseased individuals; recovered sheep lose infectious capacity and gradually decrease the number of active infectious animals in the herd. In contrast, τ_1, τ_3, n, d show negligible absolute sensitivity below 0.032. τ_1 and τ_3 denote time delays for asymptomatic infection and environment-mediated infection respectively, whose fine-tuning produces

only marginal reduction on R_0 . The natural environmental clearance rate d and artificial pathogen elimination rate n of environmental *Brucella* contribute weakly to R_0 regulation, meaning environmental disinfection can serve only as an auxiliary control strategy rather than core epidemic intervention.. In Figure 1, the bar plot of the sensitivity indices is illustrated.

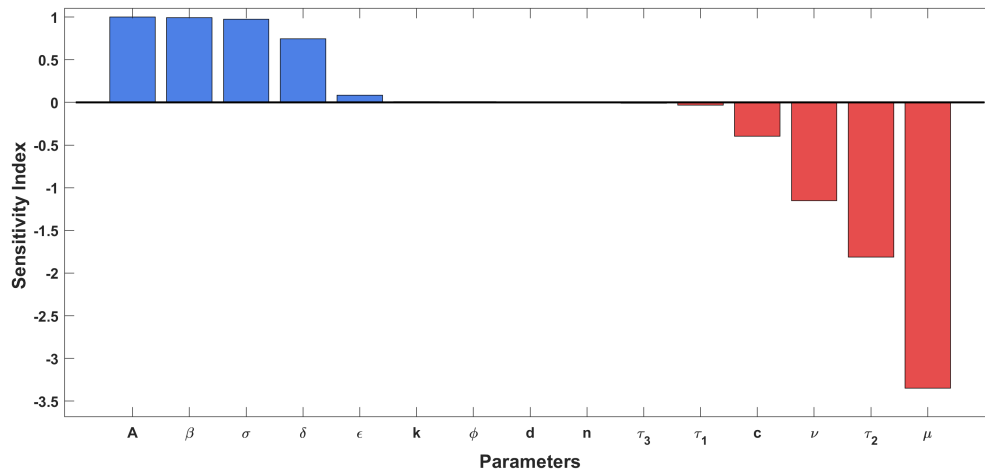


Figure 1. Sensitivity indices of the parameters associated with R_0 .

6. Numerical simulations

Example 1. Verification of Theorems 3.1 and 3.2.

Case 1: Global stability of disease-free equilibrium. In model (5.1), we choose the parameters $\tau_1 = \tau_2 = \tau_3 = 15$, and other parameters are shown in Table 2. From (5.2), by calculating, we have $R_0 = 0.6$. Therefore, the conclusion in Theorem 3.1 holds. The numerical simulation in Figure 2 presents that the disease-free equilibrium $P_0 = (S_0, V_0, 0, 0, 0)$ is globally asymptotically stable.

Case 2: Global stability of endemic equilibrium. In addition, we choose the parameters $\tau_1 = \tau_2 = \tau_3 = 8.5$, and other parameters are shown in Table 2. From (5.2), by calculating, we have $R_0 = 2.5$. Therefore, the conclusion in Theorem 3.2 holds. The numerical simulation in Figure 3 presents that the endemic equilibrium $P^* = (S^*, V^*, E^*, I^*, B^*)$ is globally asymptotically stable.

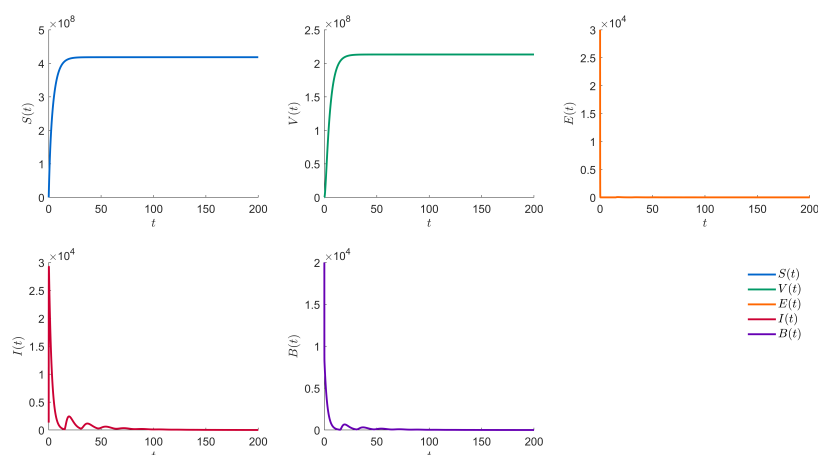


Figure 2. The numerical simulation of solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1) when $R_0 = 0.6$.

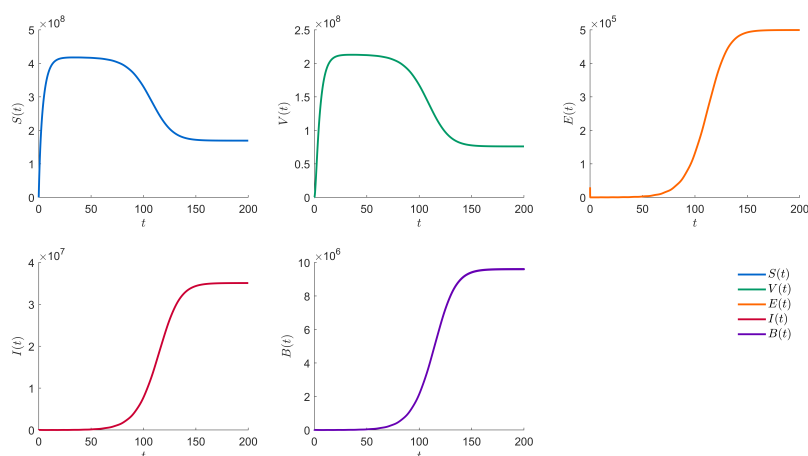


Figure 3. The numerical simulation of solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1) when $R_0 = 2.56$.

Example 2. Discussion of Remark 3.2.

In model (5.1), we first select the parameters $\mu_1 = 0.42$, $\mu_2 = 0.45$, $\mu_3 = 0.35$, $\tau_1 = 12$, $\tau_2 = 6.5$, and $\tau_3 = 10$, with other parameters as shown in Table 2, which shows that $\eta_1 = e^{\mu_1 \tau_1} \neq \eta_2 = e^{\mu_2 \tau_2} \neq \eta_3 = e^{\mu_3 \tau_3}$. Calculating (5.2), we have $R_0 \doteq 0.85$. The numerical simulation in Figure 4 indicates that disease-free equilibrium $P_0 = (S_0, V_0, 0, 0, 0)$ may be globally asymptotically stable. Therefore, the conclusions in Theorem 3.1 may be correct.

In addition, take parameters $\mu_1 = 0.22$, $\mu_2 = 0.12$, $\mu_3 = 0.02$, $\tau_1 = 8.5$, $\tau_2 = 6.5$, and $\tau_3 = 5.5$ with other parameters, as shown in Table 2, which shows that $\eta_1 = e^{\mu_1 \tau_1} \neq \eta_2 = e^{\mu_2 \tau_2} \neq \eta_3 = e^{\mu_3 \tau_3}$, with other parameters, shown in Table 2, which also shows that $\eta_1 = e^{\mu_1 \tau_1} \neq \eta_2 = e^{\mu_2 \tau_2} \neq \eta_3 = e^{\mu_3 \tau_3}$. Calculating (5.2), we have $R_0 \doteq 7.4$. The numerical simulation in Figure 5 indicates that endemic

equilibrium $P^* = (S^*, V^*, E^*, I^*, B^*)$ may be globally asymptotically stable. Therefore, the conclusions in Theorem 3.2 may be correct.

The above discussions show that the conjecture: When $R_0 \leq 1$, the disease-free equilibrium is globally stable, while when $R_0 > 1$, the endemic equilibrium is globally stable, which is proposed in Remark 3.2 and may be rational.

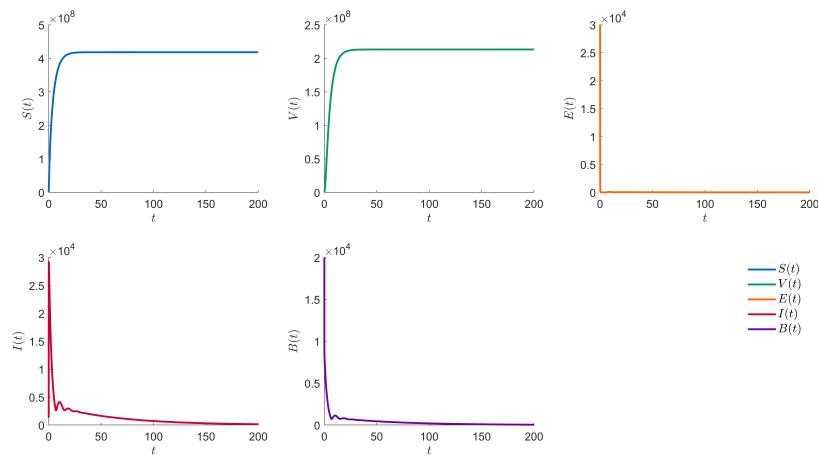


Figure 4. The numerical simulation of solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1) when $R_0 = 0.85$.

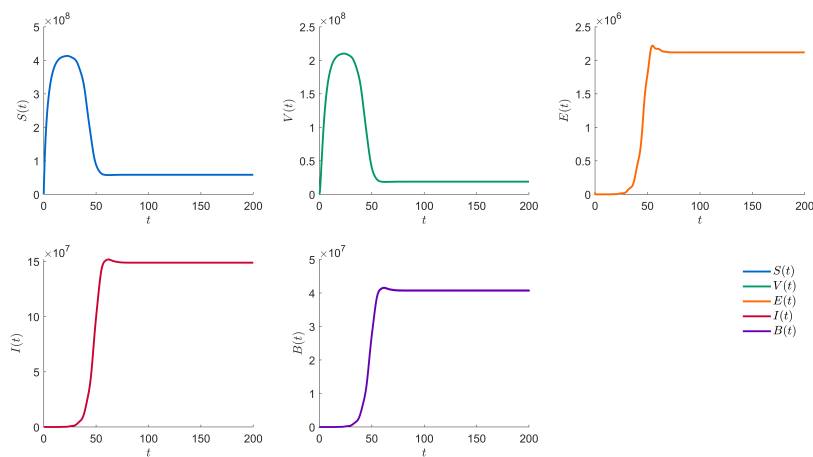


Figure 5. The numerical simulation of solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1) when $R_0 = 7.4$.

Example 3. The impact of model parameters.

According to Figures 6–12, it can be seen that the increase in n , σ , ν , c , τ_1 , τ_2 , and τ_3 can effectively reduce the number of latent sheep (E), the number of infected individuals (I), the amount of brucella in the environment (B), and basic reproduction number (R_0). These parameters are the key focus of

prevention and control strategies by enhancing early case identification, improving herd immunity, or blocking the transmission chain to contain the development of the epidemic. In addition, we find from the numerical simulations in Figures 6–12 that the effects of parameters τ_1 , τ_3 , and n are relatively weak. This indicates that, by enhancing the removal of pathogens in the environment, it seems that the expected effect of controlling the spread of diseases cannot be achieved.

According to Figures 13–17, it can be seen that the increase β , ϕ , δ , k , and ϵ will exacerbate the spread of the epidemic, leading to an increase in E , I , B , and R_0 . Among them, β and δ have the most significant impact, directly determining transmission efficiency and immune protection persistence. In addition, the numerical simulations in Figures 14–16 show that although ϕ , ϵ and k promote transmission of disease, their effects are relatively weak, which is consistent with the conclusion of sensitivity analysis Figure 1. This indicates that prevention and control measures should prioritize efficient transmission routes (such as direct contact) and immunization effectiveness, rather than excessive investment in environmental media management or simply pursuing vaccine coverage.

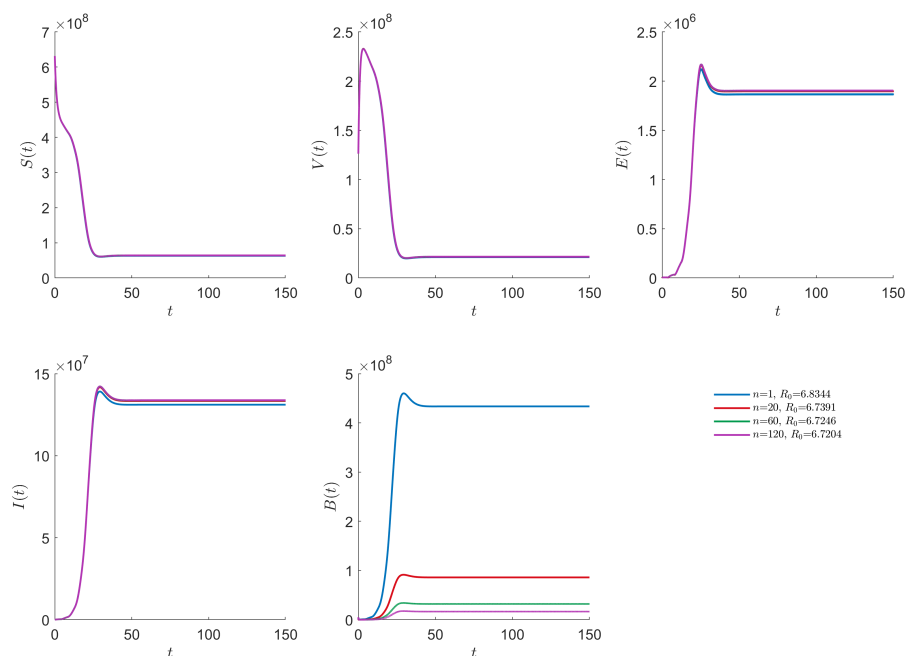


Figure 6. Influence of parameter n on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

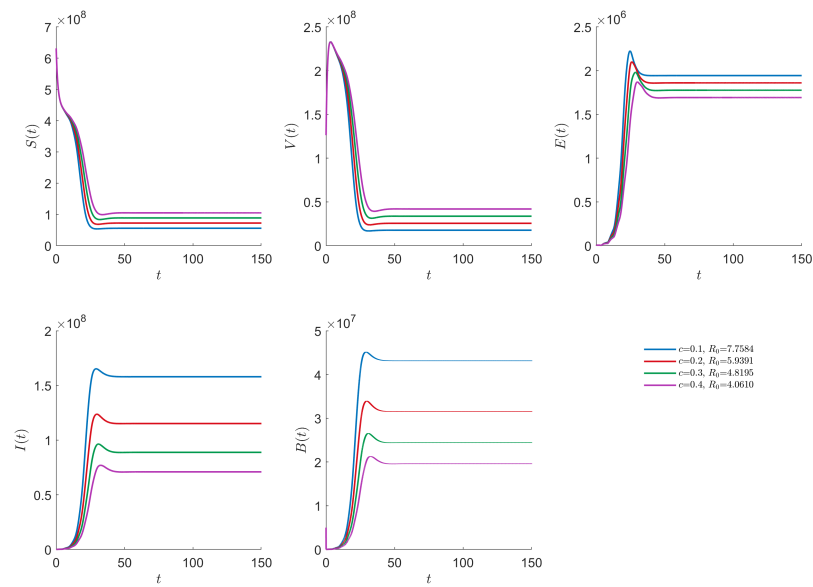


Figure 7. Influence of parameter c on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

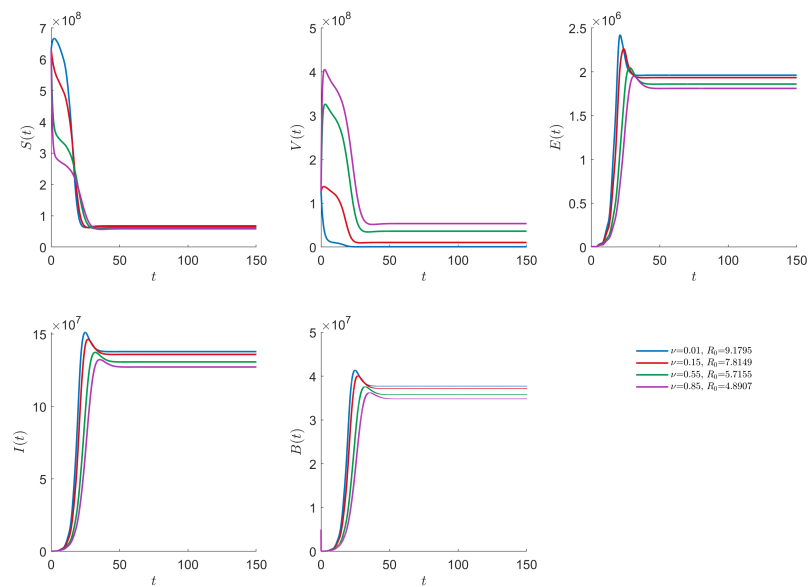


Figure 8. Influence of parameter ν on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

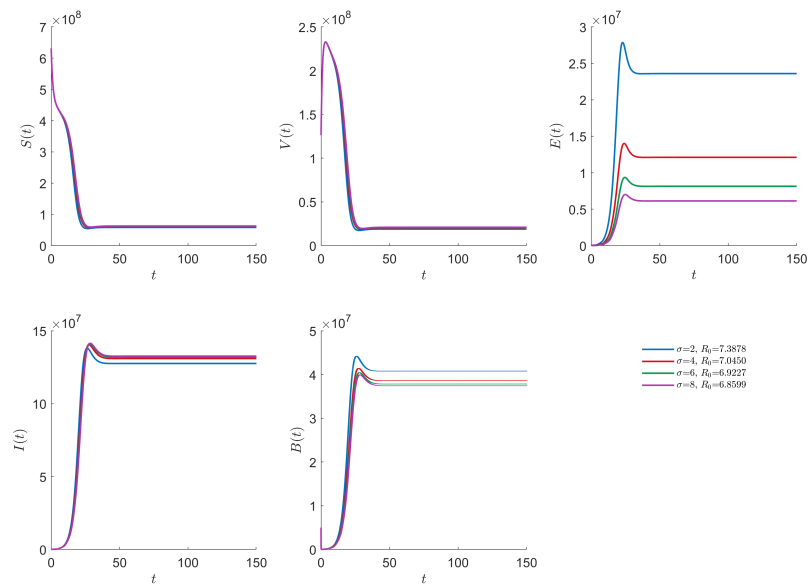


Figure 9. Influence of parameter σ on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1) .

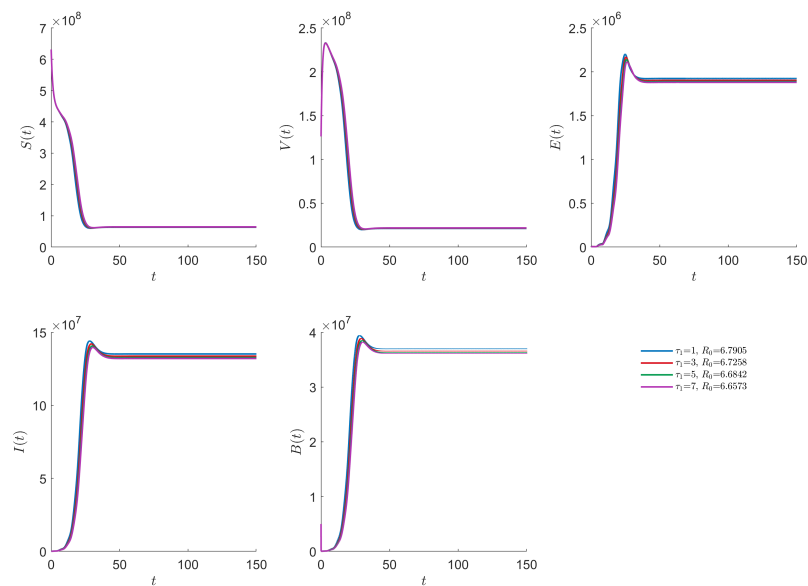


Figure 10. Influence of the parameter τ_1 on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

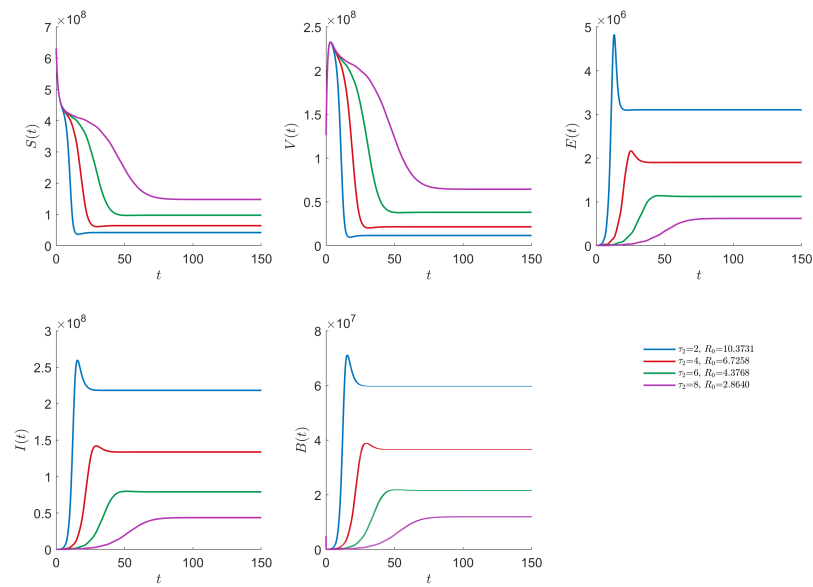


Figure 11. Influence of the parameter τ_2 on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

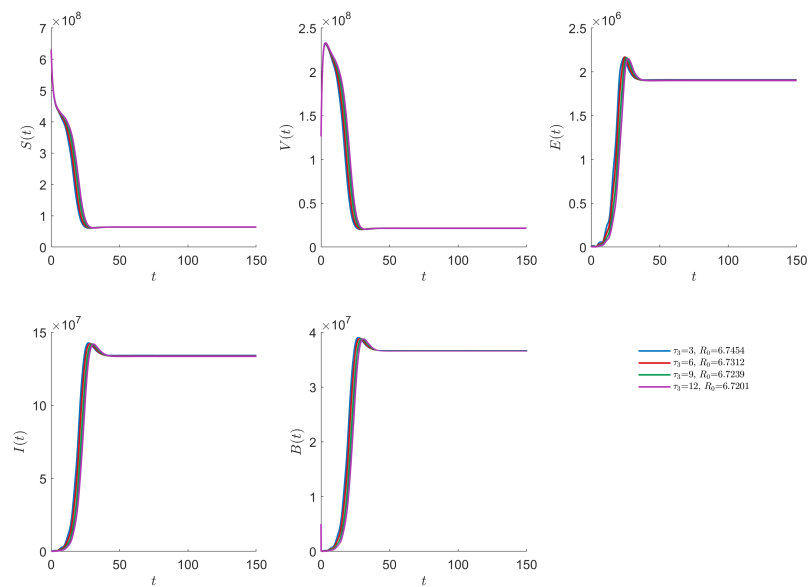


Figure 12. Influence of parameter τ_3 on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

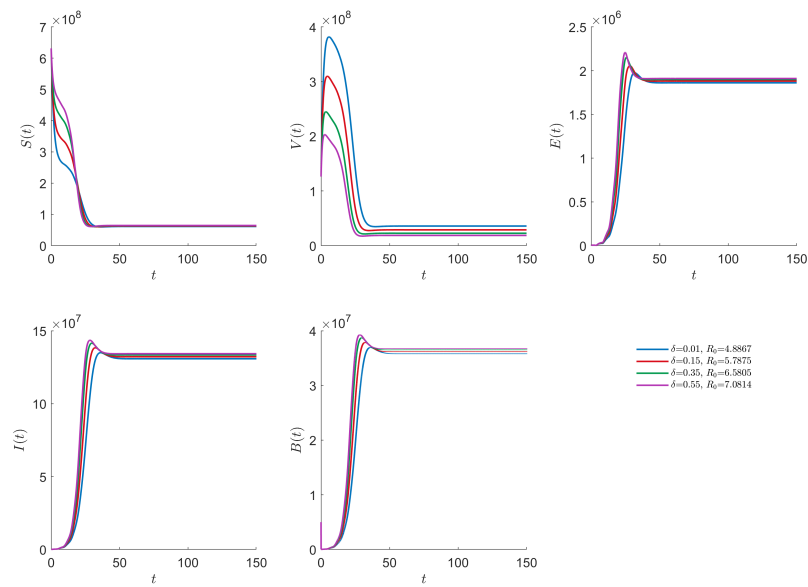


Figure 13. Influence of the parameter δ on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

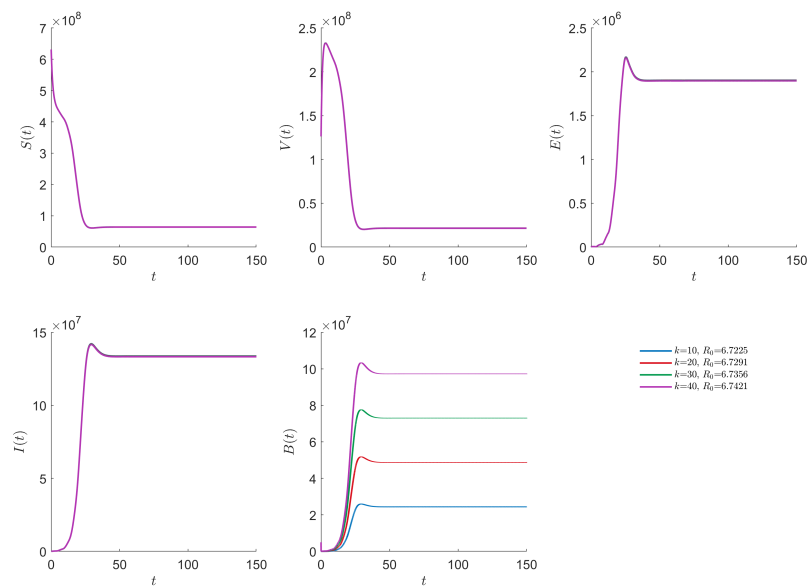


Figure 14. Influence of parameter k on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

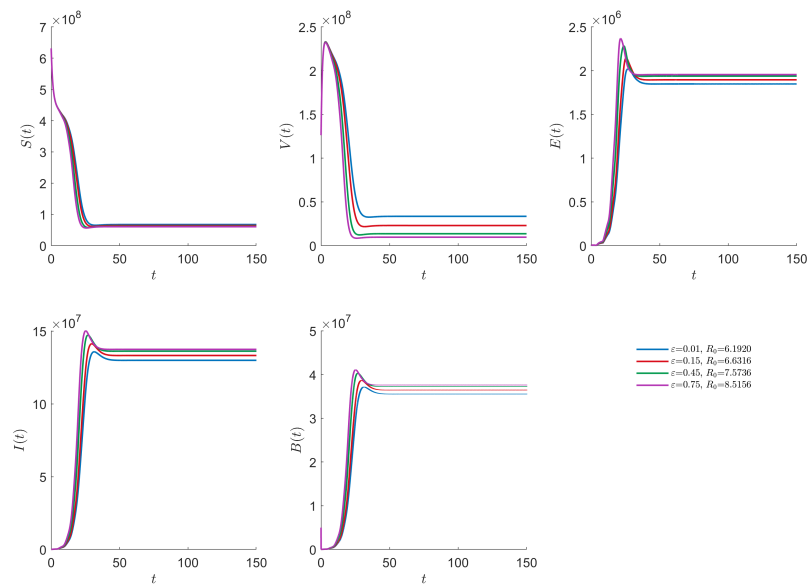


Figure 15. Influence of parameter ϵ on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

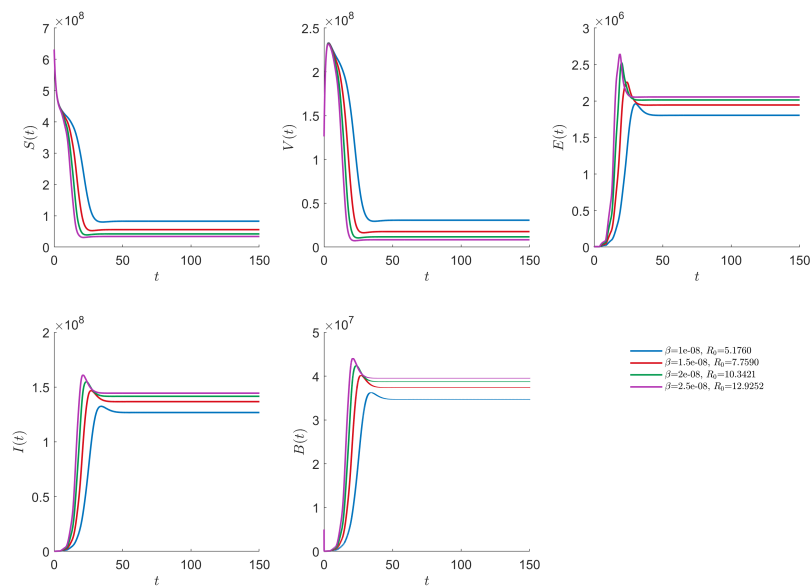


Figure 16. Influence of parameter β on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

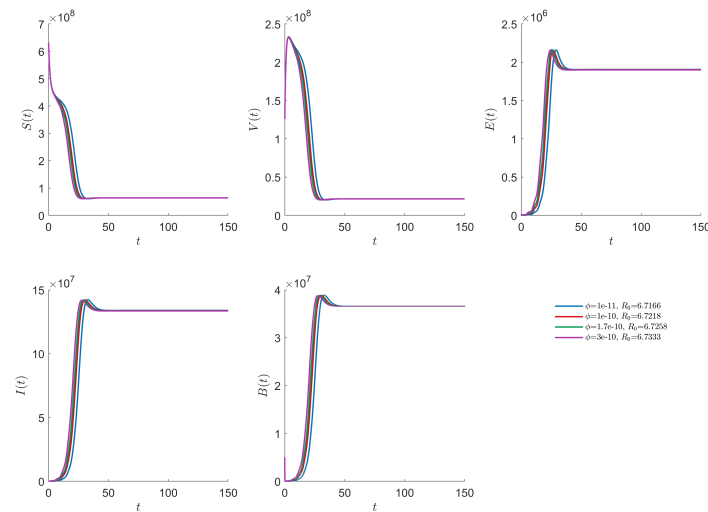


Figure 17. Influence of parameter ϕ on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

To quantitatively compare the control effectiveness of different intervention strategies, we define the range of intervention values as $u_1 \in [0, 1]$ and $u_i \in [1, 2]$, where $i = 2, 3, 4, 5$. Moreover, we choose $A_1 = 0.2, A_2 = 0.3, A_3 = 0.03, B_1 = 0.005, B_2 = 0.01, B_3 = 0.005, B_4 = 0.02, B_5 = 0.005$, and perform numerical simulations of the dynamic changes in the epidemic system under scenarios with no intervention, and a single intervention and Two measures joint optimal control.

Example 4. The impact of the full optimal control strategy.

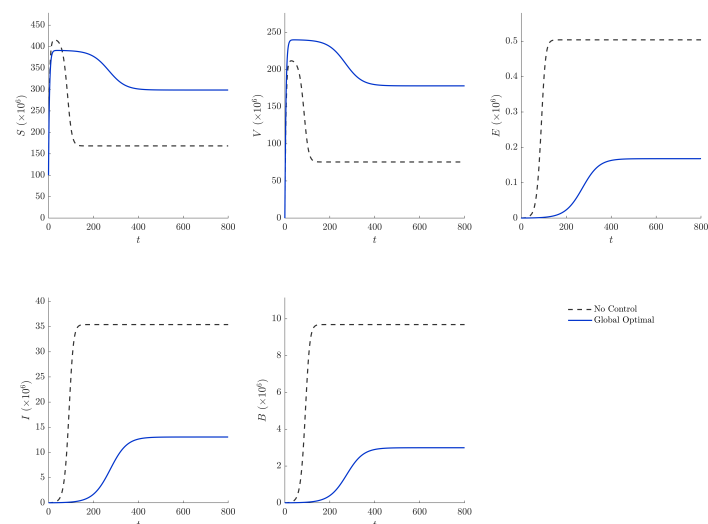


Figure 18. The impact of full optimal control strategy on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

In this example, all five control measures $u_i(t)$ ($i = 1, 2, 3, 4, 5$) will be employed. The numerical simulation given in Figure 18 presents the control effects corresponding to all the optimal control measures $u_i = u_i^*$ ($i = 1, 2, 3, 4, 5$). It can be seen that implementing all control measures to maximize the containment of the final scale of the disease is the optimal prevention and control strategy.

Example 5. The effect of single optimal control strategy.

In this example, five groups of single-variable optimal control simulation experiments are carried out to quantify the independent epidemic suppression effects of various intervention measures. All experimental groups share a unified baseline parameter setting (no-control reference group), i.e., $u_1 \equiv 0, u_2 = u_3 = u_4 = u_5 \equiv 1$. The black dashed lines in Figures 19–23 represent the endemic equilibrium under this baseline, where the susceptible population S and vaccinated population V decay to low steady-state levels, while the exposed population E , infected population I , and environmental bacterial load B converge to persistently high levels, verifying the necessity of optimizing single intervention variables.

Case 1: $u_1(t) \in [0, 1], u_2 = u_3 = u_4 = u_5 = 1$.

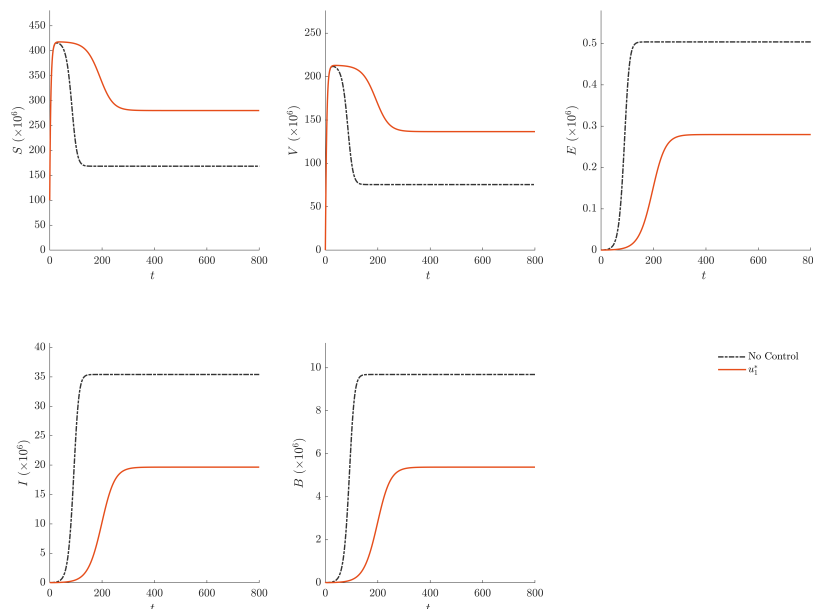


Figure 19. The effect of single optimal control strategy u_1^* on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

The results show that after implementing the optimal control strategy u_1^* , the steady-state quantities of the susceptible S and vaccinated V are significantly higher than those in the no-control scenario (see Figure 19). The optimal control strategy u_1^* restricts cross-infection pathways from the source of transmission, retains a larger-scale susceptible population, and maximizes the protective benefits brought by baseline vaccination. Moreover, $E(t)$, $I(t)$, and $B(t)$ all exhibit a unimodal evolutionary trend of rising first and then decaying, with their terminal steady-state values far lower than the no-control baseline. From the perspective of action mechanism, timely screening and isolating infected sheep,

adding anti-brucellosis drugs to feed and drinking water in a timely manner, and dividing large flocks into smaller ones for decentralized breeding can effectively reduce infection rates, thereby decreasing the number of new latent carriers and subsequently lowering the accumulation of pathogens in the environment. Among all single-control schemes, u_1^* can achieve comprehensive suppression of all epidemic compartments and drive the system to converge to an equilibrium state.

Case 2: $u_2(t) \in [1, 2]$, $u_1 \equiv 0$, $u_3 = u_4 = u_5 = 1$.

Under the effect of the optimized strategy u_2^* , the long-term steady states of S and V are only slightly higher than those in the no-control scenario, indicating that the gain of this strategy in protecting susceptible and expanding vaccinated is extremely weak. Only the quantity of E decreases slightly, while the trajectories of I and B almost overlap with the no-control dashed lines, showing no obvious downward trend (Figure 20). In this scenario, even if measures such as routine random screening of low-risk flocks and comprehensive clinical screening of high-risk flocks are implemented, infected animals in the incubation period will continue to cause secondary infections. Applying these measures to infected animals alone will not be sufficient to curb sustained local transmission.

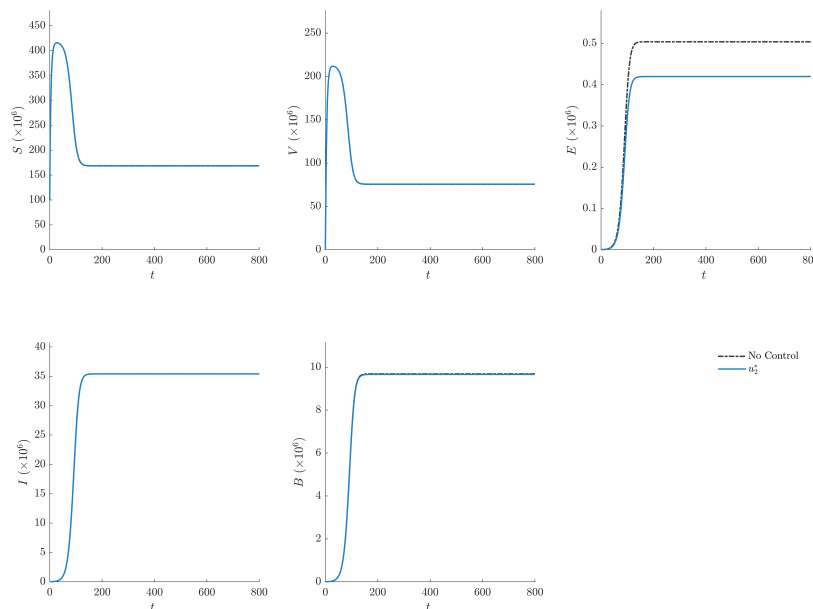


Figure 20. The effect of single optimal control strategy u_2^* on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1) .

Case 3: $u_3(t) \in [1, 2]$, $u_1 = 0$, $u_2 = u_4 = u_5 = 1$.

After dynamic optimization of vaccination coverage, the steady state of the susceptible S is almost identical to that in the no-control scenario, and only V increases slightly, failing to establish an effective protective buffer for susceptible hosts. The curves of E , I , and B are highly consistent with the baseline curves, with no significant decrease in peak values and equilibrium levels (Figure 21). Even with an accelerated and optimized vaccination program, if the infection rate among susceptible and vaccinated individuals caused by asymptomatic or symptomatic cases remains unchecked ($u_1 = 0$), unvaccinated

susceptible individuals will continue to be exposed to risk, leading to a constant emergence of new asymptomatic and confirmed cases. Consequently, without the coordinated efforts of u_1 , optimizing vaccination alone cannot break the core transmission chains, and the effectiveness of epidemic control measures will be limited.

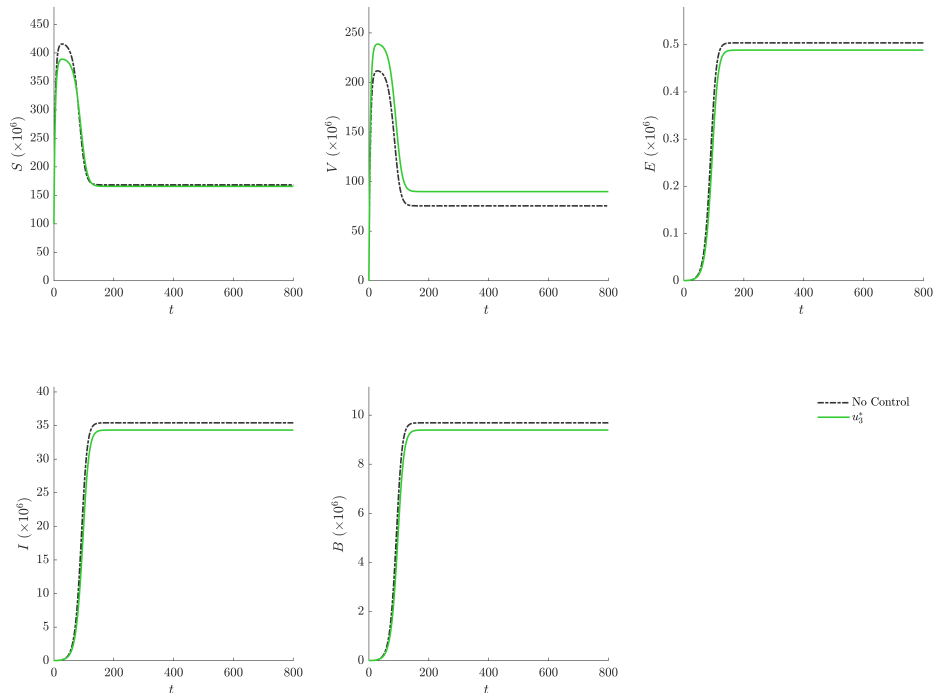


Figure 21. The effect of single optimal control strategy u_3^* on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

Case 4: $u_4(t) \in [1, 2]$, $u_1 = 0$, $u_2 = u_3 = u_5 = 1$.

Compared with the no-control scenario, the steady-state levels of S and V increase only slightly, and only a small number of susceptible hosts are retained. Notably, the time-series trajectories of E and I overlap with the no-control baseline without any decay characteristics, and only B decreases slightly (Figure 22). Environmental disinfection can reduce only the concentration of free pathogens in the medium, and it is not possible to prevent transmission through direct contact. This single-variable scheme has the weakest comprehensive epidemic control performance among the five types of intervention measures.

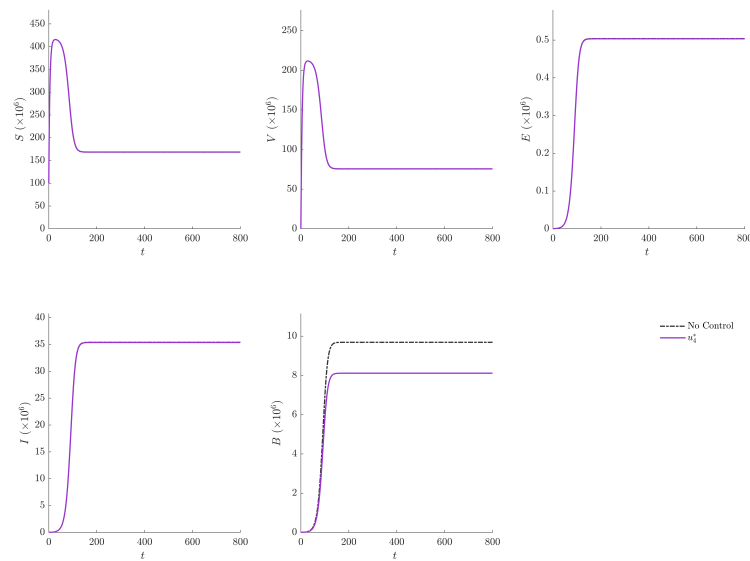


Figure 22. The effect of single optimal control strategy u_4^* on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1) .

Case 5: $u_5(t) \in [1, 2]$, $u_1 = 0$, $u_2 = u_3 = u_4 = 1$.

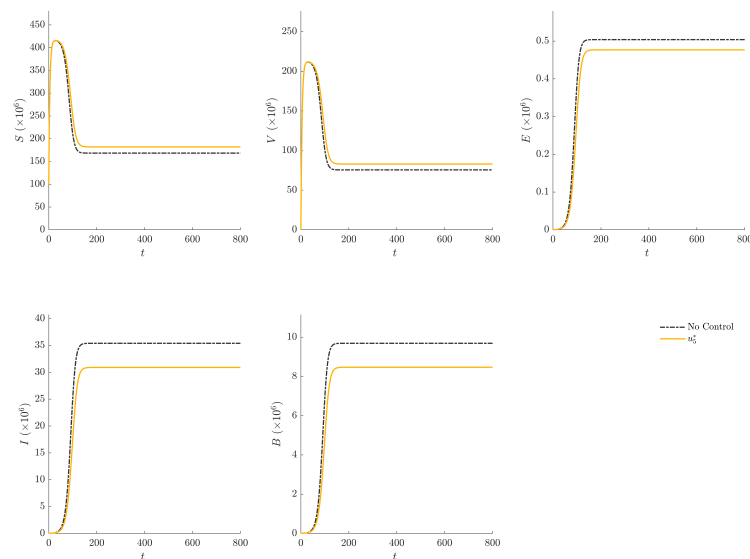


Figure 23. The effect of single optimal control strategy u_5^* on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1) .

The control strategy u_5 can moderately raise the terminal steady states of S and V , protecting the

susceptible to a certain extent and expanding the coverage of vaccination. Populations E , I , and B all follow a pattern of rising first and then falling, and their values are lower than those in the no-control scenario, but the improvement range is far weaker than that of u_1^* (Figures 19 and 23). Timely culling can shorten the infection cycle and the duration of pathogen shedding, but due to the failure of contact blocking, the problem of secondary infections induced by latent infected individuals remains unconstrained. This strategy can only moderately alleviate the epidemic prevalence intensity and cannot fundamentally eliminate endemic transmission.

Based on the above numerical analysis of five cases, we see that the effectiveness of various intervention measures is determined by combining the state trajectories and steady-state differences. The order of their effectiveness is: $u_1^* > u_5^* > u_3^* > u_2^* > u_4^*$. Dynamic optimal strategy u_1^* can simultaneously realize the retention of susceptible hosts, expansion of vaccinated populations, and multiple reductions in the scale of latent/infected populations and environmental pathogen load, making it the highest-priority intervention measure among all intervention means. Under the baseline constraint of $u_1 = 0$, vaccination, infected individual slaughtering, environmental disinfection, and treatment interventions can improve only a single epidemic index independently, and do not have the ability to cut off the core contact transmission pathway.

Example 6. Two measures joint optimal control strategy.

Single prevention and control measures have obvious limitations in epidemic control efficacy and are difficult to effectively contain the full chain of epidemic transmission. In practical epidemic prevention and control work, a combination of multiple measures is often adopted. Based on this, to further explore the epidemic suppression effect under the synergistic action of different prevention and control measures and identify the optimal synergistic prevention and control combination, we conduct numerical simulation research on two-variable combined optimal control on the basis of single-variable optimal control simulations. Specific analysis based on Figures 24–27 is as follows:

Case 1: $u_1(t) \in [0, 1]$, $u_3(t) \in [1, 2]$ $u_2 = 1$, $u_4 = 1$, $u_5 = 1$.

The results show that after the synergistic optimization of $(u_3 + u_1)^*$, the long-term steady-state values of the susceptible S and the vaccinated V are much higher than those of the no-control benchmark(see Figure 24). The two interventions form complementary protection: The optimal control u_1 cuts off interpersonal contact transmission from the source, greatly reducing the exposure risk of susceptible individuals; u_3 synchronously expands the scale of the immune population, and the dual effects maximize the retention of susceptible sheep.

In addition, the latent infected E , infected I , and environmental bacterial load B exhibit a unimodal evolution characteristic of a slight initial rise followed by a continuous sharp decline. Their terminal steady-state values are far lower than those in the no-control scenario, making this the control measure with the largest reduction in the number of infected individuals among the four dual-control schemes.

The results confirm that reducing the infection rate of susceptible and vaccinated individuals and strengthening the vaccination rate of susceptible sheep can comprehensively suppress epidemic transmission, and the dual-intervention combination has significantly better epidemic control efficacy than the other three combinations.

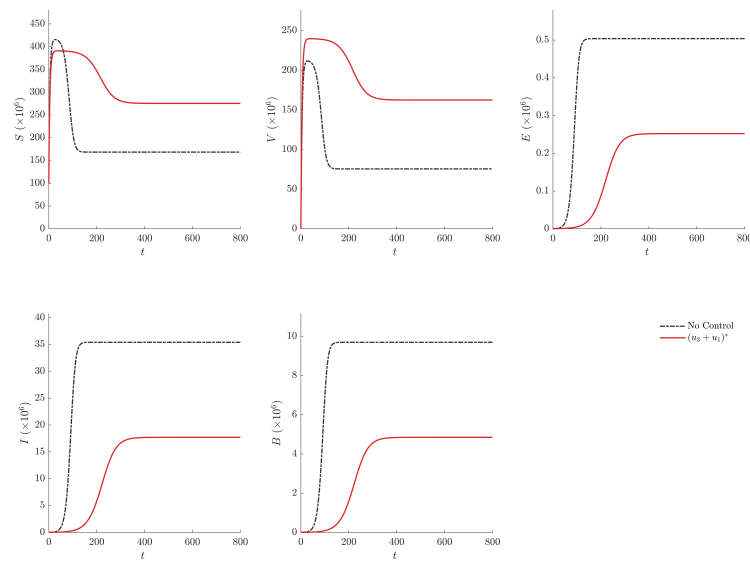


Figure 24. The effect of two measures joint optimal control strategy $(u_3 + u_1)^*$ on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1) .

Case 2: $u_2(t) \in [1, 2]$, $u_3(t) \in [1, 2]$, $u_1 \equiv 0$, $u_4 = 1$, $u_5 = 1$.

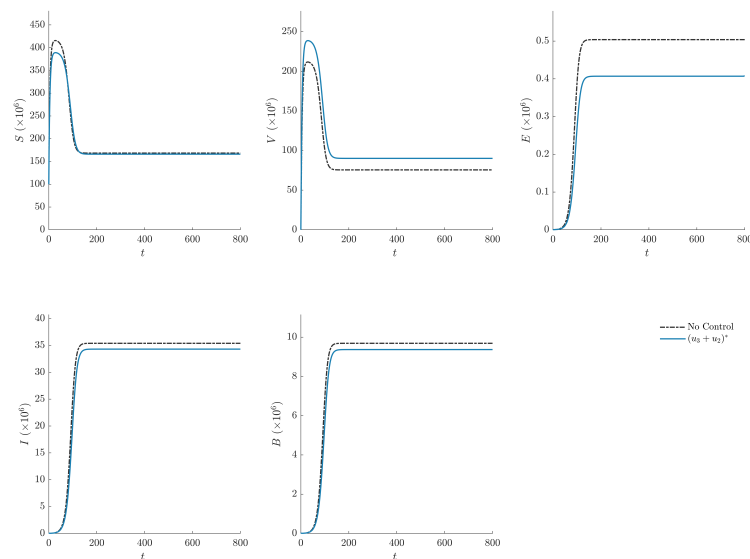


Figure 25. The effect of two measures joint optimal control strategy $(u_3 + u_2)^*$ on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

The results indicate that after the synergistic optimization of strengthening the vaccination rate of

susceptible sheep and strengthening medical screening rate for asymptomatic infected sheep $((u_3 + u_2)^*)$, the steady-state values of S and V are only slightly higher than those of the no-control strategy, and the protective gain for the susceptible population is extremely limited (see Figure 25).

In addition, the latent infected E decreases only slightly, and the time-series trajectories of the infected population I and environmental bacterial load B almost overlap with the black dashed line of the no-control scenario, showing no obvious downward trend. The synergy of the two interventions can only slightly alleviate the scale of exposure, and it is difficult to reduce the total number of prevalent infected individuals and the environmental bacterial load, resulting in a weak overall epidemic control gain.

Case 3: $u_3(t) \in [1, 2]$, $u_4(t) \in [1, 2]$, $u_1 = 0$, $u_2 = 1$, $u_5 = 1$.

The results show that the terminal steady-state values of S and V have only a tiny increase, and only a small number of susceptible individuals are retained. The evolution trajectories of the latent infected E and infected population I coincide with the no-control benchmark without any decay characteristics; only the environmental bacterial load B decreases slightly (Figure 26).

This dual-variable combination can only slightly reduce the stock of environmental pathogens and cannot interrupt the infection process, resulting in weak comprehensive epidemic control performance among the four dual-control schemes.

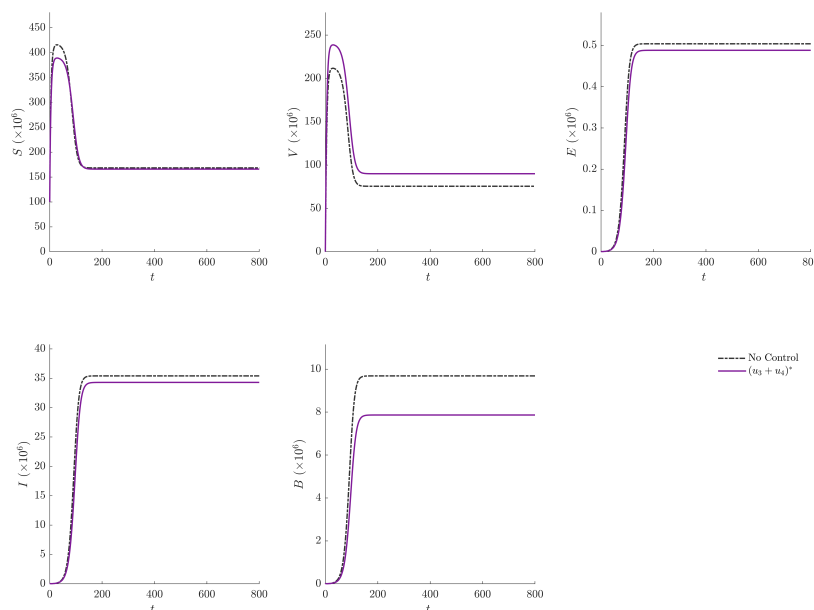


Figure 26. The effect of two measures joint optimal control strategy $(u_3 + u_4)^*$ on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

Case 4: $u_3(t) \in [1, 2]$, $u_5(t) \in [1, 2]$, $u_1 = 0$, $u_2 = 1$, $u_4 = 1$.

The results indicate that after the synergistic optimization of strengthening the vaccination rate of susceptible sheep and strengthening the clearance rate of infected sheep $((u_3 + u_5)^*)$, the terminal steady-state values of S and V are higher than those of the no-control strategy, which can retain the

susceptible population to a certain extent and expand the scope of immune coverage (see Figure 27).

The latent infected E , infected I , and environmental bacterial load B show a trend of rising first and then falling slightly, with their terminal values lower than those of the no-control strategy, but the improvement range of each index is far weaker than that of the $(u_3 + u_1)^*$ combination. This dual-intervention combination has a moderate epidemic control gain, which can moderately reduce the prevalence intensity, but cannot fundamentally curb persistent endemic transmission.

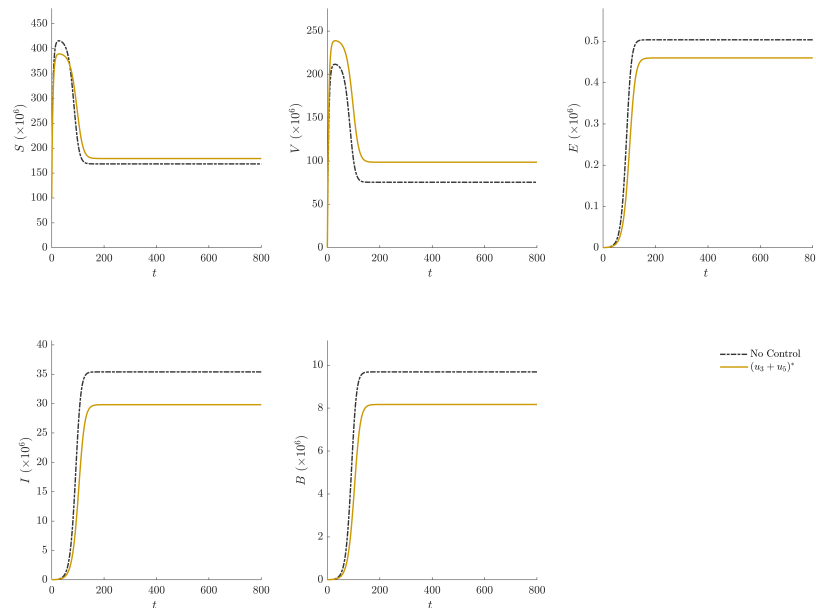


Figure 27. The effect of two measures joint optimal control strategy $(u_3 + u_5)^*$ on solution $(S(t), V(t), E(t), I(t), B(t))$ for model (1).

Based on the above numerical analysis of four cases, we observe that the comprehensive epidemic control efficacy of the four dual-variable synergistic control schemes is determined by combining state trajectories, steady-state differences, and transmission mechanisms. The control effect ranking is as follows: $(u_3 + u_1)^* > (u_3 + u_5)^* > (u_3 + u_4)^* > (u_3 + u_2)^*$.

- (1) $(u_3 + u_1)^*$: Has the strongest synergistic gain and is the optimal joint prevention and control scheme.
- (2) $(u_3 + u_5)^*$: Has a moderate epidemic control effect and can slightly reduce the full-chain prevalence scale, but the improvement range is limited due to the lack of coordination with u_1 .
- (3) $(u_3 + u_4)^*$, $(u_3 + u_2)^*$: They can improve only a single epidemic index (bacterial load or exposed population), and cannot cut off the core contact transmission pathway under the constraint of $u_1 = 0$, resulting in extremely low synergistic intervention benefits.

This result reveals that if vaccination is adopted as the fundamental prevention and control measure, pairing it with a reduced the infection rate can achieve the maximum epidemic suppression effect;

pairing it with medical screening, clearance of infected sheep/environmental pathogens can obtain only limited auxiliary gains and cannot block interpersonal transmission from the source.

7. Conclusions

In this paper, we investigate the global dynamics and optimal prevention control strategies of a brucellosis transmission model with distributed time delays and a latent stage. First, the basic reproduction number of the model is calculated, and threshold conditions for the global asymptotic stability of the disease-free equilibrium and endemic equilibrium are established. Second, various intervention measures capable of blocking disease transmission are introduced, and an optimal control model integrating all preventive control measures is constructed. Numerical simulations are conducted to study the effects of different control measures on preventing and controlling brucellosis, with the main findings as follows:

(1) The results of single-variable optimal control simulations show that there are significant differences in the epidemic control efficacy of the five single intervention measures, and the ranking according to comprehensive suppression effect is: Reduce the infection rate (u_1^*) > strengthen the clearance rate of infected sheep (u_5^*) > vaccination (u_3^*) > strengthen medical screening rate for asymptomatic infected sheep (u_2^*) > strengthen the clearance rate of environmental pathogens (u_4^*). Among them, reduce the infection rate, as the only measure that can cut off interpersonal transmission pathways from the source, can simultaneously increase the steady-state levels of susceptible and vaccinated populations, drastically reduce the scale of exposed and infected populations and environmental bacterial load, and has the most prominent epidemic control effect. In contrast, single measures, such as strengthening the medical screening rate for asymptomatic infected sheep and strengthening the clearance rate of environmental pathogens, can improve only a single epidemic index, failing to interrupt the core transmission chain, and are difficult to effectively contain the epidemic.

(2) Two-variable combined optimal control takes vaccination as the foundation and synergistically optimizes it with the other four measures, respectively. The results show that there are obvious differences in the synergistic epidemic control efficacy of different combinations. The optimal combination is vaccination + reduce the infection rate ($(u_3 + u_1)^*$), followed by vaccination + strengthen the clearance rate of infected sheep ($(u_3 + u_5)^*$), and vaccination + environmental disinfection ($(u_3 + u_4)^*$) and vaccination + strengthen medical screening rate for asymptomatic infected sheep ($(u_3 + u_2)^*$) have the worst effects. Among them, the synergistic effect of vaccination and reducing the infection rate can form complementary protection, realizing all-round suppression of the full epidemic transmission chain and driving the system to converge to a disease-free equilibrium.

(3) Combining the results of single-control and dual-control simulations, it can be seen that reducing the infection rate occupies a core position in epidemic prevention and control, and can play a significant epidemic control role whether used alone or in synergy with vaccination. The epidemic control efficacy of vaccination, as a fundamental prevention and control measure, is highly dependent on the coordinated cooperation of reducing the infection rate; when used alone or in combination with medical screening, clearance and other measures, it cannot give full play to its protective effect. In addition, the three measures of strengthening the medical screening rate for asymptomatic infected sheep, strengthening the clearance rate of environmental pathogens, and strengthening the clearance

rate of infected sheep can serve only as auxiliary prevention and control means, cannot replace the core role of reducing the infection rate, and can only further optimize the epidemic control effect on the basis of reduce the infection rate.

Use of AI tools declaration

The author(s) declare they have used Artificial Intelligence (AI) tools in the creation of this article. The AI tool was used for English grammar revision and language polishing in sensitivity analysis ,the numerical simulation and conclusion sections. The authors independently conducted the numerical simulation, obtained all simulation results, drew the research conclusions, and fully reviewed and modified all AI-polished text. No raw data, figures, or core research findings were generated by the AI tool.

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Conflict of interest

The authors declare that they have no conflict of interest.

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