



Research article

Nonlinear eigenvalue problems on a half-line and some related topics

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Abstract: The nonlinear eigenvalue problem

$$-\left(|u'(x)|^{p-1} \operatorname{sgn} u'(x)\right)' + (p-1)\omega(x)|u(x)|^{q-1} \operatorname{sgn} u(x) = (p-1)\lambda|u(x)|^{p-1} \operatorname{sgn} u(x), \quad 0 \leq x < \infty,$$

for fixed $p > q > 1$, was studied under some conditions on the positive potential ω . We used various Prüfer angle methods to characterize the eigenvalue properties for problems on the half-line and finite intervals.

Keywords: nonlinear Sturm-Liouville equation; Prüfer angle method

1. Introduction

Our aim is to study the nonlinear half-line eigenvalue problem given by

$$-\left(|u'(x)|^{p-1} \operatorname{sgn} u'(x)\right)' + (p-1)\omega(x)|u(x)|^{q-1} \operatorname{sgn} u(x) = (p-1)\lambda|u(x)|^{p-1} \operatorname{sgn} u(x), \quad 0 \leq x < \infty, \quad (1.1)$$

for fixed $p > q > 1$, with the initial conditions

$$u(0) = \sin_p \alpha, \quad u'(0) = \sin'_p \alpha, \quad \alpha \in [0, \pi_p). \quad (1.2)$$

The differential equation (1.1) is nonlinear and is to be understood in the Carathéodory sense. When $p = q = 2$, it is reduced to the classical Sturm-Liouville equation. Here, $\sin_p$ is the generalized sine function with first positive zero at π_p . We refer the reader to [1–3] for properties of the generalized sine function used here. In particular,

$$|\sin_p x|^p + |\sin'_p x|^p = 1$$

and

$$\left(|\sin'_p x|^{p-1} \operatorname{sgn} \sin'_p x\right)' + (p-1)|\sin_p x|^{p-1} \operatorname{sgn} \sin_p x = 0.$$

For $p = q = 2$, suppose that $\omega(x)$ tends to ∞ as $x \rightarrow \infty$. It is well known that the Sturm-Liouville equation is of limit point type at ∞ , and, given an initial condition

$$u'(0) \sin \alpha = u(0) \cos \alpha, \quad (1.3)$$

the resulting spectrum σ is discrete. Indeed, σ consists of eigenvalues $\lambda_0, \lambda_1, \lambda_2, \dots$ accumulating at ∞ . Moreover, well-known oscillation theory guarantees that, for each $k \geq 0$, any eigenfunction u_k corresponding to λ_k vanishes precisely k times in $(0, \infty)$. For such results we refer to [4], for example. From this it follows that the Prüfer angle $\theta(x, \lambda)$ for any solution u of (1.1) and (1.3) has certain asymptotic properties in x . Indeed, it is well known that $\theta(x, \lambda)$ must increase through values which are multiples of π with

$$k\pi < \theta(x, \lambda) < (k+1)\pi \quad (1.4)$$

for sufficiently large x , whenever $\lambda_{k-1} < \lambda < \lambda_k$ for any $k \geq 0$ (if set $\lambda_{-1} = -\infty$). In [5], Crandall and Reno improved (1.4) to

$$\theta(x, \lambda) \rightarrow k\pi + \text{(respectively, } k\pi - \text{) if } \lambda_{k-1} < \lambda < \lambda_k \text{ (respectively, } \lambda = \lambda_{k-1}) \quad (1.5)$$

for $k > 0$ as $x \rightarrow \infty$ under additional conditions on ω . For the half-linear case ($p = q > 1$), Brown and Reichel [6] established (1.5) under differential additional conditions on ω , which were removed in [7]. More general situations with locally integrable ω were studied in [8] (under Molčanov's conditions [9]) and in [10] (under modifications of Brinck's [11] and Molčanov's conditions). More recently, Binding et al. [12] studied the similar issue on a weighted p -Laplacian problems on the half-line. In [12], λ is replaced by $\lambda r(x)$ and $p = q$. The authors allowed the weight function $r(x)$ to be locally integrable and definite, semidefinite, or indefinite. The authors employed two transformations (Prüfer transformation and $s(x) = r(x)^{1/p}$) to deduce a relatively simple differential inequality. They showed that solutions u are nonoscillatory on $(0, \infty)$ for all λ . For some other results related to nonoscillatory solutions on a half-line, the reader can also refer to [13–16]. Here we mention a work. In [16], the author considered the second order quasilinear ordinary differential equation with two different power nonlinearities

$$(p(x)|y'|^\alpha \operatorname{sgn} y')' + q(x)|y|^\beta \operatorname{sgn} y = 0.$$

Under some conditions on the coefficient functions, the author deduced the existence of a slowly growing positive solution. He also obtained some precise asymptotic forms of slowly growing positive solutions and slowly decaying positive solutions. Note that the methods and main goal of [16] are different from ours, but it is relevant to our result.

Generalized Prüfer methods are efficient to characterize the eigenvalue properties and to investigate the oscillation properties of solutions. On a half-line, modified Prüfer methods have been used in [6, 8, 10, 12, 17]. Here we mention the following two results. In [8], where $p = q = 2$, ω is assumed to be bounded below, while in the relevant part of [6] ($p = q > 1$) ω has to satisfy certain smoothness and growth conditions. We also remark that various authors, for example, of [6, 7, 18], treat “radial” p -Laplacian problems which can be transformed to the form (1.1), and then fall within the above framework.

When (1.1) is considered in any finite interval and is coupled with the Dirichlet boundary conditions, it can be shown that nontrivial solutions exist only when $\lambda > 0$ by multiplying (1.1) by u and integrating over this finite interval. Note that for our case $p \neq q$, the eigenparameter λ shall depend on initial

parameter α . For some relevant result readers can refer to [19]. In this paper, we will consider (1.1)–(1.2) with $\alpha = 0$ ($u(0) = 0$), and the nontrivial solutions satisfying Definition 1.1 (see below) exist only when $\lambda > 0$. Throughout the paper we assume that ω is a positive C^1 potential function on $\mathbb{R}^+$. Also assume the following condition to hold:

(C_0) for every $\epsilon > 0$, $\lim_{x \rightarrow \infty} \int_x^{x+\epsilon} \omega(t)t^{-1}e^{-t}dt = \infty$.

Note that Condition (C_0) is related to an eigenvalue problem considered in the half-line. (C_0) (without the term $t^{-1}e^{-t}$) was first used by Molčanov's work [9] on spectral discreteness in the linear case $p = q = 2$. Molčanov also assumed ω to be bounded below. The Molčanov potential ($\lim_{x \rightarrow \infty} \int_x^{x+\epsilon} \omega(t)dt = \infty$) is also considered in [8, 10, 17]. For our nonlinear case $p \neq q$, the non-homogeneity will appear in the phase equation (see (2.9)). So, we have to set the above modified Molčanov condition (C_0) . As an example, one could take $\omega(t) = t^n e^t$, $n \geq 2$ or $\omega(t) = e^{2t}$.

The generalized Prüfer angle method was applied to 1-d p -Laplacian eigenvalue problems on the half-line in the paper [10] of Binding and Browne. Indeed, this method was developed in a series of papers afterwards. Now motivated by the idea and methods in [10], we plan to employ various Prüfer angle methods to characterize eigenvalue properties of the nonlinear problem (1.1)–(1.2) on the half-line. Since our problem is nonlinear ($p \neq q$), we shall employ suitable analysis to tackle the nonhomogeneity in the phase equation (see (2.9)). We will introduce an energy functional $E[u](x)$ (the first integral) for solutions of (1.1)–(1.2). Use this functional to deduce a priori bounds of u , and then we can derive some technical lemmas which help to analyze of the Prüfer angle θ . To the best of our knowledge, the vast majority of scientific papers have paid attention to linear and half-linear Sturm-Liouville equations in a half-line. This is an attempt to address this type of nonlinear eigenvalue problem. We believe that the related issues have the potential to become interesting and important. As we shall see, the Prüfer angle method on a half-line leads to eigenfunctions that decay exponentially, and belong to L_r for any $r \geq 1$. Also, we shall need to control eigenfunction derivatives as well, and accordingly we shall give the following definition.

Definition 1.1. An eigenvalue for the problem (1.1)–(1.2) is a value of $\lambda \in \mathbb{R}$ for which there exists a $u \in W_p^1$ satisfying the differential equation (1.1) and the initial conditions (1.2). Here and below, L_p and W_p^1 will refer to the half-line $(0, \infty)$.

For the nonlinear problem (1.1)–(1.2), we first derive a nonoscillation property related to (1.4) (see Theorem 2.6 below). Next we can introduce sets related to the spectral parameter λ :

$$\begin{aligned}\Lambda_n &= \{\lambda \in \mathbb{R} : n\pi_p < \theta(x, \lambda) < (n+1)\pi_p \text{ for all } x \text{ sufficiently large}\} \\ \Lambda_n^+ &= \{\lambda \in \Lambda_n : \theta(x, \lambda) \rightarrow (n+1)\pi_p \text{ as } x \rightarrow \infty\} \\ \Lambda_n^- &= \{\lambda \in \Lambda_n : \theta(x, \lambda) \rightarrow n\pi_p \text{ as } x \rightarrow \infty\}.\end{aligned}$$

Then we have the following theorem.

Theorem 1.2. Under Definition 1.1, if $\lambda = \lambda_n \in \Lambda_n^+$ for some n , then λ is an eigenvalue. Any eigenfunction associated with λ_n decays exponentially as $x \rightarrow \infty$ and belongs to W_p^1 .

Next, we will concern the nonlinear eigenvalue problem (1.1) considered in a finite interval. We will deduce the Dirichlet eigenvalue expansion (see Theorem 3.2). For the half-linear case ($p = q$), the

issue of one-dimensional p -Laplacian eigenvalue expansions has been investigated in [1, 20, 21]. Here we plan to employ a modified Prüfer transformation to deal with (1.1).

The paper is organized as follows. In Section 2 we employ an energy identity to derive global existence of solutions. We prepare some technical lemmas to analyze the Prüfer angle. Then we give the proof of Theorem 1.2. In Section 3, we will consider a finite-interval eigenvalue problem. We will derive the eigenvalue expansion as an application.

2. Preliminaries and some technical lemmas

In this section, elementary properties of the solution $u(x)$ will be derived first. For this, we have the following.

Theorem 2.1. *Assume the conditions stated in Section 1 hold. There exists a unique solution of (1.1)–(1.2) in a neighborhood J . Furthermore, this solution $u(x)$ can be extended to the whole positive real axis.*

Proof. The local existence and uniqueness of solutions to the related one-dimensional nonlinear p -Laplacian equation have been proven in [22, 23]. Next, we plan to use an energy identity to show the local solution $u(x)$ can be extended to the whole positive real axis. Now, let a functional $E[u](x)$ be defined by

$$E[u](x) = \frac{1}{p}|u'(x)|^p - \frac{1}{q}\omega(x)|u(x)|^q + \frac{1}{p}\lambda|u(x)|^p. \quad (2.1)$$

In what follows, we will derive a priori bound for $E[u](x)$. For a unique local solution u of (1.1)–(1.2), we denote by $[0, c)$ the maximal right interval of existence for u . Then, in view of (1.1) for $x \in [0, c)$ and $1 < p \leq 2$, we can find

$$\frac{d}{dx}E[u](x) = E[u]'(x) = -\frac{1}{q}\omega'(x)|u(x)|^q. \quad (2.2)$$

For $p > 2$, one shall calculate in the following way

$$\begin{aligned} (|u'|^p)' &= (|u'|^{p-2}u')^{\frac{p}{p-1}})' = \frac{p}{p-1}|u'|^{p-2}u' \left(|u'|^{p-2}u' \right)' \\ &= \frac{p}{p-1}(|u'|^{p-2}u')'u' \end{aligned}$$

since u is not C^2 in this case. Then, we can obtain the same identity as in (2.2). Since ω and ω' are bounded on any finite subinterval of $\mathbb{R}^+$, for fixed p , q , and λ , we can find a positive constant h such that for $x \in [0, c)$

$$\frac{h}{q}[\omega(x) - \omega'(x)] \leq \frac{\lambda}{p}. \quad (2.3)$$

For this h , there exists $M > 0$ such that

$$|u|^q \leq h|u|^p \quad \text{for } |u| \geq M. \quad (2.4)$$

Now, define $I_1 = \{x \in [0, c) : |u| \leq M\}$ and $I_2 = \{x \in [0, c) : |u| > M\}$. Then, from (2.2), there exists $N > 0$ such that $E[u]'(x) \leq N$ for $x \in I_1$, and for $x \in I_2$

$$E[u]'(x) = -\frac{1}{q}\omega(x)|u(x)|^q + \frac{1}{q}[\omega(x) - \omega'(x)]|u(x)|^q$$

$$\leq -\frac{1}{q}\omega(x)|u(x)|^q + \frac{\lambda}{ph} \cdot h|u(x)|^p \leq E[u](x).$$

Thus, for $x \in [0, c)$ we have

$$E[u]'(x) \leq N + E[u](x).$$

Integrating both sides of the above inequality, we get that for $x \in [0, c)$,

$$E[u](x) \leq (E[u](0) + Nc) + \int_0^x E[u](t)dt.$$

By the Gronwall inequality, we find that for $x \in [0, c)$

$$E[u](x) \leq (E[u](0) + Nc) e^x. \quad (2.5)$$

By assumption, we assume that $u(x)$ exists on a maximal right interval $[0, c)$ for some finite $c > 0$. That is, $u(x)$ is unbounded on $[0, c)$ with $\limsup_{x \rightarrow c^-} |u(x)| = \infty$. Otherwise, if $u(x)$ is bounded on $[0, c)$, then $\lim_{x \rightarrow c^-} u'(x)$ exists by integrating (1.1) over $[0, c)$. This implies that $u(x)$ can be extended through c . On the other hand, since $\limsup_{x \rightarrow c^-} |u(x)| = \infty$, there exists a sequence $t_n \rightarrow c^-$ such that $|u(t_n)| \rightarrow \infty$. Hence, by (2.1),

$$\lim_{n \rightarrow \infty} E[u](t_n) = \infty.$$

However this contradicts the arguments in (2.5). Therefore, the solution $u(x)$ exists on the whole positive real axis. $\square$

Remark 2.2. Let $\alpha = 0$. From Theorem 2.1 and its proof, one can assume that the solution $u(x)$ exists on $[0, b]$ for any arbitrary fixed $b > 0$. Here, let $b > 1$. Similarly, for $M > 0$ define $I'_1 = \{x \in [0, b] : |u| \leq M\}$ and $I'_2 = \{x \in [0, b] : |u| > M\}$. Then, one can obtain that for $x \in [0, b]$,

$$E[u](x) \leq (E[u](0) + Nx) e^x = \left(\frac{1}{p} + Nx\right) e^x. \quad (2.6)$$

That is, by (2.6) for $x \in [1, b]$,

$$E[u](x) = \frac{1}{p}|u'(x)|^p + |u(x)|^p \left(\frac{\lambda}{p} - \frac{1}{q}\omega(x)|u(x)|^{q-p} \right) \leq Cxe^x,$$

where the constant C is independent of x and λ and depends on b . Note that $\frac{\lambda}{p} > \frac{1}{q}\omega|u|^{q-p}$ for sufficiently large M as $|u| > M$. Then, for $x \in [1, b]$ one can conclude

$$|u| \leq C_1 x^{\frac{1}{p}} e^{\frac{x}{p}} < C_1 x^{\frac{1}{p-q}} e^{\frac{x}{p-q}}, \quad (2.7)$$

where the constant C_1 is independent of x and λ . Note that C_1 depends on b and can be chosen sufficiently large to satisfy (2.7) when b is large. We remark that the inequality (2.7) is useful to the proofs of Lemmas 2.4–2.5 and Theorem 2.6. It is also related to the condition (C_0) .

Next, for a solution of (1.1)–(1.2), we shall introduce the Prüfer (Elbert) angle θ via

$$u(x, \lambda) = L(x, \lambda) \sin_p \theta(x, \lambda), \quad u'(x, \lambda) = L(x, \lambda) \sin'_p \theta(x, \lambda).$$

This leads to

$$|u(x, \lambda)|^p + |u'(x, \lambda)|^p = L^p(x, \lambda) \quad \text{and} \quad \left(\frac{u'}{u}\right)(x, \lambda) = \frac{\sin'_p \theta(x, \lambda)}{\sin_p \theta(x, \lambda)}. \quad (2.8)$$

Then we can use (1.1)–(1.2) and the properties of $\sin_p$ to derive

$$\theta'(x, \lambda) = |\sin'_p \theta(x, \lambda)|^p + [\lambda - \omega(x)|u(x, \lambda)|^{q-p}] |\sin_p \theta(x, \lambda)|^p \quad (2.9)$$

$$= 1 - (\omega(x)|u(x, \lambda)|^{q-p} - \lambda + 1) |\sin_p \theta(x, \lambda)|^p, \quad (2.10)$$

$$\theta(0, \lambda) = \alpha. \quad (2.11)$$

The phase equation is not autonomous and the term $|u(x, \lambda)|^{q-p} |\sin_p \theta(x, \lambda)|^p$ in (2.9) can be represented by $L^{q-p}(x, \lambda) |\sin_p \theta(x, \lambda)|^q$ from the Prüfer transformation. Since $L(x, \lambda)$ never vanishes by (2.8), the term $|u(x, \lambda)|^{q-p} |\sin_p \theta(x, \lambda)|^p$ shall be well defined. Note that the existence and uniqueness of solutions to the initial value problem (2.9)–(2.11) can be referred to [23, Section 3] for such a coupled and nonlinear problem. Now we derive some elementary properties of θ , which are prepared for the characterization of eigenvalues.

Lemma 2.3. *Let $\alpha = 0$. Suppose the potential function ω satisfies the conditions discussed in Section 1. The following is valid.*

- (i) *The function $\theta(\cdot, \lambda)$ is continuous in λ .*
- (ii) *If $\theta(t_n, \lambda) = n\pi_p$, then $\theta(x, \lambda) > n\pi_p$ for all $x > t_n$.*
- (iii) *For any $x > 0$, $\lim_{\lambda \rightarrow \infty} \theta(x, \lambda) = \infty$.*

Proof. (i) It follows from the continuous dependence of the parameter that $\theta(\cdot, \lambda)$ is continuous with respect to λ .

- (ii) By the phase equation (2.10), we have that

$$\theta'(t_n, \lambda) = 1.$$

This proves (ii).

- (iii) Suppose that φ is chosen absolutely continuous with $|\theta(0, \lambda) - \varphi(0, \lambda)| < \frac{\pi_p}{2}$ and define

$$\lambda^{1/p} u(x) = \rho(x) \sin_p \varphi(x) \quad \text{and} \quad u'(x) = \rho(x) \sin'_p \varphi(x). \quad (2.12)$$

Then,

$$\lambda |u|^p + |u'|^p = \rho^p \quad \text{and} \quad \frac{u'}{u} = \frac{\lambda^{1/p} \sin'_p \varphi}{\sin_p \varphi}, \quad (2.13)$$

with

$$\rho(0) = \sqrt[p]{\lambda |\sin_p \alpha|^p + |\sin'_p \alpha|^p} = 1. \quad (2.14)$$

Applying (1.1) and the above, we can transform (1.1) into the following,

$$\varphi' = \lambda^{\frac{1}{p}} - \lambda^{\frac{1-q}{p}} \omega \rho^{q-p} |\sin_p \varphi|^q. \quad (2.15)$$

$$\rho^{p-q-1} \rho' = \lambda^{\frac{1-q}{p}} \omega \sin'_p \varphi |\sin_p \varphi|^{q-2} \sin_p \varphi. \quad (2.16)$$

Note that the existence and uniqueness of $u(x)$ are valid. By (2.12)–(2.13) and the property of φ , the function $\rho(x)$ is well defined and never vanishes. Integrating (2.16) over $(0, x)$, we can obtain

$$\rho^{p-q}(x) = \rho^{p-q}(0) + (p-q)\lambda^{\frac{1-q}{p}} \int_0^x \omega \sin'_p \varphi |\sin_p \varphi|^{q-2} \sin_p \varphi dt. \quad (2.17)$$

That is, by (2.14) and (2.17),

$$\rho^{q-p}(x) = 1 + O(\lambda^{\frac{1-q}{p}}) \text{ as } \lambda \rightarrow \infty.$$

Next, integrating both sides of (2.15) over $(0, x)$, for $\lambda \rightarrow \infty$, we have that

$$\varphi(x) = \lambda^{\frac{1}{p}} x + O(\lambda^{\frac{1-q}{p}})$$

by using (2.14) and (2.17). Hence,

$$\varphi(x) \rightarrow \infty \text{ as } \lambda \rightarrow \infty.$$

Since the vectors (u, u') and $(u, \lambda^{1/p} u')$ lie in the same quadrant, we see that

$$|\theta(x, \lambda) - \varphi(x, \lambda)| < \frac{\pi_p}{2}$$

for all $x \geq 0$. This completes the proof. $\square$

Next we turn to consider the first-order equation

$$\phi'(x) = 1 - (\omega(x)|u(x)|^{q-p} - \lambda + 1) |\sin_p \phi(x)|^p. \quad (2.18)$$

We plan to derive some technical lemmas to analyze the phase angle.

Lemma 2.4. *Let $\alpha = 0$ and λ be fixed. For given $0 < \gamma < \delta < \pi_p$ and $\eta > 0$, there exists $t_{\gamma, \delta, \eta}$ such that for any solution of (2.18), the conditions*

$$x > t_{\gamma, \delta, \eta}, \quad \phi(x) \in (\gamma, \delta) \text{ and } \phi(t) \leq \delta \text{ for } t \in [x, x + \eta] \quad (2.19)$$

imply that there exists $\epsilon \in (0, \eta)$ satisfying

$$\phi(x + \epsilon) = \gamma. \quad (2.20)$$

Proof. Note that the points $t_{\gamma, \delta, \eta}$, x and $x + \eta$ are assumed in the assumption of this lemma and they can be large if necessary. Then, we take an interval $[0, b]$ such that $t_{\gamma, \delta, \eta} \in [0, b]$ and $[x, x + \eta] \subset [0, b]$. Now let $m = \min\{|\sin_p \phi|^p : \phi \in [\gamma, \delta]\}$. Then $0 < m \leq 1$. By virtue of (C_0) and (2.7) in Remark 2.2, we can select $t_{\gamma, \delta, \eta}$ so that

$$\int_x^{x+\eta} \omega(t)|u(t)|^{q-p} dt > \frac{\delta + (2 + \lambda)\eta - \gamma}{m} \quad (2.21)$$

as $x > t_{\gamma, \delta, \eta}$. Suppose that $x > t_{\gamma, \delta, \eta}$ satisfies (2.19) but that no $\epsilon \in (0, \eta)$ can be found to satisfy (2.20). Then, $\phi(t) \in (\gamma, \delta]$ for all $t \in [x, x + \eta]$, and by (2.18), we have

$$\begin{aligned} \phi(x + \eta) &= \phi(x) + \int_x^{x+\eta} [1 - (\omega(t)|u(t)|^{q-p} - \lambda + 1) |\sin_p \phi(t)|^p] dt \\ &\leq \delta + \eta + (1 + \lambda)\eta - m \int_x^{x+\eta} \omega(t)|u(t)|^{q-p} dt. \end{aligned} \quad (2.22)$$

Thus, $\phi(x + \eta) < \gamma$ by (2.21). This reaches a contradiction. $\square$

Lemma 2.5. *Let $\alpha = 0$ and λ be fixed. For given $0 < \gamma < \delta < \pi_p$, there exists $\bar{t}_{\gamma,\delta}$ such that for any solution of (2.18), $x > \bar{t}_{\gamma,\delta}$, $\phi(x) \leq \gamma$ implies that $\phi(x+t) < \delta$ for all $t > 0$.*

Proof. Set $\eta = \frac{\delta-\gamma}{2}$ and take $\bar{t}_{\gamma,\delta} = t_{\gamma,\delta,\eta}$ from Lemma 2.4. Suppose that $x > \bar{t}_{\gamma,\delta}$ has that $\phi(x) \leq \gamma$ and that $z > x$ has $\phi(z) = \delta$. We can assume that z is the minimum of all points $r > x$, where $\phi(r) = \delta$. Now take $y \in [x, z]$ so that $\phi(y) = \gamma$ and $\phi(\sigma) \in (\gamma, \delta]$ for all $\sigma \in (y, z]$. Then, we have

$$\begin{aligned} \delta - \gamma &= \phi(z) - \phi(y) \\ &= \int_y^z [1 - (\omega(t)|u(t)|^{q-p} - \lambda + 1)|\sin_p \phi(t)|^p] dt \\ &\leq (z - y) + (\lambda + 1)(z - y) - m \int_y^z \omega(t)|u(t)|^{q-p} dt, \end{aligned}$$

where $m = \min\{|\sin_p \phi|^p : \phi \in [\gamma, \delta]\}$. By virtue of (C_0) and (2.7), we can get

$$\int_y^z \omega(t)|u(t)|^{q-p} dt > \frac{\lambda(z - y)}{m}$$

as $z > y > \bar{t}_{\gamma,\delta}$. So we have $z - y > \frac{\delta-\gamma}{2} = \eta$. Thus, we can apply Lemma 2.4 over the interval $(z - \eta, z)$ to obtain a point $\sigma \in (z - \eta, z)$ such that $\phi(\sigma) = \gamma$. This gives a contradiction. $\square$

The following property of the Prüfer angle θ is crucial to the main result.

Theorem 2.6. (nonoscillation) *Let $\alpha = 0$ and λ be fixed. There exists an integer $n = n(\lambda) \geq 0$ such that*

$$n\pi_p < \theta(x, \lambda) < (n + 1)\pi_p$$

for all x sufficiently large.

Proof. Suppose that $\theta(x, \lambda)$ tends to infinity as $x \rightarrow \infty$. Let $0 < \gamma < \delta < \pi_p$ with $m = \min\{|\sin_p \phi|^p : \phi \in [\gamma, \delta]\}$. Also set $x_n = \min\{x : \theta(x, \lambda) = n\pi_p + \gamma\}$, $n \in \mathbb{N}$. From the phase equation (2.10), we have

$$\begin{aligned} \pi_p + \gamma - \alpha &= \theta(x_n, \lambda) - \theta(0, \lambda) \\ &\leq x_n + (\lambda + 1)x_n - m \int_0^{x_n} \omega(t)|u(t)|^{q-p} dt \\ &\leq (\lambda + 2)x_n. \end{aligned}$$

This shows that $x_n \rightarrow \infty$. Now use Lemma 2.5 to find $\bar{t}_{\gamma,\delta}$ and fix N so that $x_N > \bar{t}_{\gamma,\delta}$. Note that $\phi(x) = \theta(x, \lambda) - N\pi_p$ satisfies (2.18) and also that $\phi(x_N) = \gamma$. Then, Lemma 2.5 shows that $\phi(x) < \delta$ for all $x > x_N$. That is, $\theta(x, \lambda) < N\pi_p + \delta$ for all $x > x_N$. Also note that θ increases through integer multiples of π_p by (2.9). Thus, the above gives a contradiction. Therefore, the proof is complete. $\square$

From Theorem 2.6 and for each $n \geq 0$, we recall the spectral sets Λ_n , Λ_n^+ , and Λ_n^- , which are defined in Section 1. Next, we can obtain Lemma 2.7 and the proof of Theorem 1.2. We have to mention that both of them also follow the general line of arguments in [10] and fortunately can work for our case.

Lemma 2.7. *Suppose that $\lambda = \lambda_n \in \Lambda_n^+$ and that u satisfies (1.1)–(1.2). Then u' is bounded on $[0, \infty)$.*

Proof. Since $\lambda \in \Lambda_n^+$, for sufficiently large x and the even n , we can assume

$$u(x) > 0 \text{ and } u'(x) < 0$$

and that $\frac{u'(x)}{u(x)} < -1$. So $u(x) < c_1 e^{-x} < B$ for constants c_1, B , and $x \geq X_1$. On the contrary, assume that for a sequence $x_j \rightarrow \infty$ we have $u'(x_j) \rightarrow -\infty$. Recall that

$$\left(|u'|^{p-2}u'\right)' = -(p-1)\lambda|u|^{p-2}u + (p-1)\omega|u|^{q-2}u$$

from (1.1). Then, for $x \geq X_1$ with $0 \leq t \leq 1$, we have

$$\frac{1}{p-1} \int_{x-t}^x \left(|u'|^{p-2}u'\right)' ds \geq -\lambda \int_{x-t}^x u^{p-1} ds.$$

This implies that

$$\frac{1}{p-1} \left(|u'(x)|^{p-2}u'(x) - |u'(x-t)|^{p-2}u'(x-t)\right) \geq -\lambda B^{p-1},$$

whence

$$\frac{1}{p-1} \left(|u'(x)|^{p-1} - |u'(x-t)|^{p-1}\right) \leq \lambda B^{p-1}.$$

Now choose j large enough to ensure that

$$|u'(x_j)|^{p-1} > \lambda B^{p-1} + (2B)^{p-1}.$$

Hence, we can see that

$$\begin{aligned} |u'(x_j - t)| &\geq 2B, \\ u'(x_j - t) &\leq -2B, \\ \int_0^1 u'(x_j - t) dt &\leq -2B, \\ u(x_j) - u(x_j - 1) &\leq -2B, \end{aligned}$$

whence

$$u(x_j) \leq -2B + u(x_j - 1) \leq -B < 0.$$

This contradiction establishes the result. $\square$

Now we are ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose $\lambda = \lambda_n \in \Lambda_n^+$. Then $\theta(x, \lambda)$ tends to $(n+1)\pi_p$ from below. Thus, for any solution u of (1.1)–(1.2) and for sufficiently large x ,

$$\frac{u'(x)}{u(x)} < -1,$$

and so $u(x) < Ae^{-x}$ for some constant A . Thus, we have $u \in L_p$. Note that if $\lambda \in \Lambda_n^-$, then $\theta(x, \lambda)$ tends to $n\pi_p$ from above. Thus, for sufficiently large x ,

$$\frac{u'(x)}{u(x)} > 1$$

and so $u(x) > Ae^{-x}$. Hence, u is not in L_p and λ cannot be an eigenvalue for the definition considered. Now it will suffice to show that $u' \in L_p$. From (1.1) multiplied by u and by integration over $[0, b]$, we can have

$$\frac{1}{p-1} \int_0^b |u'|^p + \int_0^b \omega |u|^q = \lambda \int_0^b |u|^p + \frac{1}{p-1} |u'|^{p-2} u' u|_0^b.$$

Since $u(x) < Ae^{-x}$ and Lemma 2.7, the terms on the righthand side have finite limits as $b \rightarrow \infty$. Hence, the same is true for the terms on the left-hand side, which are nonnegative. This implies $u' \in L_p$. Therefore, the proof is complete. $\square$

3. A Finite-interval problem

In this section we consider a finite-interval problem and its asymptotic behavior of eigenvalue. We plan to employ another Prüfer-type transformation to deduce the eigenvalue expansion. Now define

$$u(x) = R(x) \sin_p(\lambda^{1/p} \psi(x)) \quad \text{and} \quad u'(x) = \lambda^{1/p} R(x) \sin'_p(\lambda^{1/p} \psi(x)). \quad (3.1)$$

Then,

$$\lambda |u(x)|^p + |u'(x)|^p = \lambda R^p(x). \quad (3.2)$$

Using (1.1) and (3.1), we can obtain

$$\psi' = 1 - \frac{1}{\lambda} \omega R^{q-p} |\sin_p(\lambda^{1/p} \psi)|^q. \quad (3.3)$$

$$R^{p-q-1} R' = \lambda^{\frac{1-p}{p}} \omega \sin'_p(\lambda^{1/p} \psi) |\sin_p(\lambda^{1/p} \psi)|^{q-2} \sin_p(\lambda^{1/p} \psi). \quad (3.4)$$

Next, the following lemma gives some special cases of a generalized Riemann-Lebesgue lemma proved in [20]. Now we state the result as follows for the sake of convenience to readers.

Lemma 3.1. ([20]) *Take any $f \in L^1(0, 1)$, then as $n \rightarrow \infty$, $\int_0^1 f \phi_n \rightarrow 0$ if $\{\phi_n\}$ is a sequence of uniformly bounded and integrable functions such that for some $k = 0, 1, 2, 3, \dots, n-1$,*

$$\int_{\frac{k}{n}}^x \phi_n(t) dt = \begin{cases} O(\frac{1}{n}) & \text{if } x \in (\frac{k}{n}, \frac{k+1}{n}) \\ o(\frac{1}{n}) & \text{if } x = \frac{k+1}{n} \end{cases}.$$

Denote by $I_q = \int_0^{\pi_p/2} \sin_p(x)^q dx$. It is obvious that $\int_0^1 |\sin_p(n\pi_p x)|^q dx = \frac{2I_q}{\pi_p}$. Notice that the sequence of functions $|\sin_p(n\pi_p t)|^q \sin_p(2n\pi_p t)$, $|\sin_p(n\pi_p t)|^q \sin'_p(n\pi_p t)$, and also $|\sin_p(n\pi_p t)|^q - \frac{2I_q}{\pi_p}$ all satisfy the conditions of ϕ_n stated in Lemma 3.1. Now we are ready to derive the following result related to the Dirichlet eigenvalue expansion.

Theorem 3.2. (eigenvalue expansion) *Suppose that the nonlinear eigenvalue problem (1.1) is considered on $(0, 1)$ and is coupled with the Dirichlet boundary conditions $u(0) = u(1) = 0$ ($\alpha = 0$). Then, the eigenvalues μ_n ($n \geq 0$) of this problem satisfy*

$$\mu_n^{1/p} = (n+1)\pi_p + \frac{2I_q}{\pi_p [(n+1)\pi_p]^{q-1}} \int_0^1 \omega(x) dx + o(n^{1-q}) \quad (3.5)$$

for sufficiently large n .

Proof. By Lemma 2.3 (iii), (1.2), and the modified Prüfer transformation (3.1), we have that

$$\mu_n^{1/p}\psi(1) = (n+1)\pi_p \quad \text{and} \quad R(0) = \mu_n^{-1/p}. \quad (3.6)$$

Taking the integral on the radius equation (3.4), for sufficiently large n , we can obtain

$$\begin{aligned} R^{p-q}(x) &= R^{p-q}(0) + (p-q)\mu_n^{\frac{1-p}{p}} \int_0^x \omega \sin'_p(\mu_n^{1/p}\psi) |\sin_p(\mu_n^{1/p}\psi)|^{q-2} \sin_p(\mu_n^{1/p}\psi) dt \\ &= \mu_n^{\frac{q-p}{p}} + O(\mu_n^{\frac{1-p}{p}}). \end{aligned} \quad (3.7)$$

Note that the integrand $\omega \sin'_p(\mu_n^{1/p}\psi) |\sin_p(\mu_n^{1/p}\psi)|^{q-2} \sin_p(\mu_n^{1/p}\psi)$ is uniformly bounded in $[0, 1]$ and $\mu_n^{\frac{q-p}{p}}$ is large. That is,

$$R^{q-p}(x) = \mu_n^{\frac{p-q}{p}} + O(\mu_n^{\frac{1+p-2q}{p}}) \quad (3.8)$$

for sufficiently large n . Substituting (3.8) into (3.3), for sufficiently large n we can get

$$\begin{aligned} \psi' &= 1 - \frac{1}{\mu_n} \omega \left(\mu_n^{\frac{p-q}{p}} + O(\mu_n^{\frac{1+p-2q}{p}}) \right) |\sin_p(\mu_n^{1/p}\psi)|^q \\ &= 1 - \mu_n^{-q/p} \omega |\sin_p(\mu_n^{1/p}\psi)|^q + O(\mu_n^{\frac{1-2q}{p}}). \end{aligned} \quad (3.9)$$

Integrating (3.9) on $(0, 1)$ and applying (3.6), for sufficiently large n , we have

$$\mu_n^{-1/p}(n+1)\pi_p = 1 + O(\mu_n^{-q/p}).$$

So, we have

$$\mu_n^{1/p} = (n+1)\pi_p + O(n^{1-q}) \quad (3.10)$$

for sufficiently large n . Hence, from (3.9) and (3.10), we can get

$$\mu_n^{1/p}\psi(x) = (n+1)\pi_p x + O(n^{1-q}) \quad (3.11)$$

for sufficiently large n . Again, integrating (3.9) on $(0, 1)$ and applying (3.10)–(3.11), we can obtain that

$$\begin{aligned} \mu_n^{1/p} &= (n+1)\pi_p + \frac{2I_q}{\pi_p[(n+1)\pi_p]^{q-1}} \int_0^1 \omega(x) dx \\ &\quad + \frac{1}{[(n+1)\pi_p]^{q-1}} \int_0^1 \omega(x) \left(|\sin_p[(n+1)\pi_p x]|^q - \frac{2I_q}{\pi_p} \right) dx + o(n^{1-q}) \end{aligned}$$

for sufficiently large n . By Lemma 3.1, we can obtain the Dirichlet eigenvalue expansion (3.5). $\square$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The work is partially supported by National Science and Technology Council of Taiwan, NSTC 113-2115-M-507-001.

Conflict of interest

The authors declare there is no conflicts of interest.

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