



Research article

On solution set associated with a class of fractional variational control problems with semi-infinite constraints

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Abstract: In this paper, we extend semi-infinite variational programming problems to a class of fractional variational control problems using Caputo-Fabrizio fractional derivatives. In this context, by using integration by parts, the necessary optimality conditions for these problems are obtained, incorporating complete Karush-Kuhn-Tucker (KKT) criteria which include both state and control stationarity conditions. Then, we build a Lagrange functional, where duality properties are derived under suitable assumptions of the involved functions. Furthermore, saddle point criteria connected to the considered Lagrange functional are developed. The main contribution of the study is the extension of semi-infinite fractional variational programming to a controlled setting, leading to a unified framework that combines state and control stationarity conditions, Lagrange duality, and saddle-point characterization. An illustrative numerical example is also provided to demonstrate the applicability of the developed framework to a specific state-control pair.

Keywords: Caputo-Fabrizio fractional derivative; optimal solution; semi-infinite constraints; Lagrange dual; saddle point criteria; KKT conditions

1. Introduction

Fractional variational control problems provide a useful mathematical framework for optimization problems in which the evolution of the state depends on memory and hereditary effects. In such problems, fractional derivatives extend classical differential operators by allowing past states of the system to influence its current behavior. Among the most widely used fractional operators are the classical Caputo and Riemann–Liouville derivatives, which have been extensively employed in fractional

variational and optimal control problems (see, for example, [1–4]). These classical fractional operators involve singular power-law kernels, which may present certain limitations in the modeling and analysis of some systems. The fractional formulation is considered because fractional-order models provide a natural framework for incorporating nonlocal and memory-dependent effects into variational control problems. Unlike integer-order formulations, in which the evolution is primarily determined by local information, fractional operators allow the current behavior of the system to depend on its previous states. This additional flexibility motivates the study of the fractional form of the variational control model, particularly when memory and hereditary effects are relevant to the underlying optimization process.

To overcome the singularity associated with classical fractional kernels, Caputo and Fabrizio introduced the Caputo–Fabrizio (CF) fractional derivative, which is characterized by a non-singular exponential kernel (see [5, 6]). In contrast to the classical Caputo derivative, the CF derivative avoids the singular power-law kernel while retaining the nonlocal character that enables the description of memory effects. This feature has motivated its application in fractional variational and optimization problems (see, for example, [7, 8]). The Caputo–Fabrizio operator is adopted in the present study because its non-singular exponential kernel provides an alternative to fractional operators involving singular power-law kernels, while preserving the nonlocal character required to represent memory effects. Moreover, the corresponding integration-by-parts formula is suitable for deriving the variational optimality conditions considered in this work. If another fractional operator, such as the classical Caputo or Riemann–Liouville operator, were employed, the associated kernel and integration-by-parts relations would change. Consequently, the derivation and explicit form of the resulting optimality conditions would generally require corresponding modifications. A systematic comparison with other fractional operators is therefore beyond the scope of the present study. From an application perspective, fractional variational control models may provide a useful framework for systems in physics, mechanics, and engineering that exhibit memory and hereditary effects. In mechanics, fractional operators can be used to describe materials and dynamical processes whose present response depends on their previous states. In physics and engineering, the control variable may represent an external input or intervention, while the semi-infinite constraints can represent requirements imposed over a continuum of operating conditions or parameter values. These considerations provide additional motivation for studying fractional variational control problems within a semi-infinite optimization framework. An additional challenge arises when fractional variational control problems are subject to semi-infinite constraints, that is, when the optimization problem involves an infinite family of constraints indexed by a prescribed set. Semi-infinite variational programming provides an appropriate framework for studying such problems (see, for example, [9, 10]). Convexity assumptions and Slater-type constraint qualifications play an important role in establishing optimality and duality results for these optimization problems [11]. Recent studies by Jayswal and Uniyal (see [7, 12]) investigated optimality conditions, Lagrange duality, and saddle point criteria for semi-infinite variational programming problems involving the Caputo–Fabrizio fractional derivative. Rayanki et al. [13] also studied optimality conditions for interval-valued variational problems involving the same fractional operator. Fractional optimal control problems involving fractional derivatives have likewise been investigated in the literature (see, for example, [14–17]). These research directions provide the basis for further investigating of fractional variational control problems involving both control variables and semi-infinite constraints.

Recent developments further demonstrate the continuing interest in fractional differential systems with memory, delay, and control features. Rahman et al. [18] investigated multi-term fractional

differential equations with variable-type delay, while Rahman et al. [19] studied integral-type implicit multi-term fractional delay differential equations with multi-strip conditions. More recently, Dhayal et al. [20] established controllability results for impulsive Atangana–Baleanu fractional stochastic differential systems involving a second-order Hermite process. Although these studies concern different classes of fractional systems, they illustrate recent developments in the analysis of fractional models involving memory, delay, and control characteristics. Despite the above advances in Caputo–Fabrizio fractional variational programming, semi-infinite optimization, and fractional optimal control, these research directions have largely been developed separately. In particular, there remains a need for a unified framework that explicitly incorporates control variables into semi-infinite fractional variational problems governed by the Caputo–Fabrizio derivative.

In comparison with the existing literature, the present study brings these directions together within a single framework. The works of Jayswal and Uniyal [7, 12] address semi-infinite variational programming with the Caputo–Fabrizio derivative and establish optimality and duality results, whereas Rayanki et al. [13] consider interval-valued variational programming involving the same fractional operator. Other studies on fractional optimal control focus on the incorporation of control variables in fractional dynamical systems. The present work differs by considering state and control variables simultaneously within a semi-infinite Caputo–Fabrizio variational framework and by deriving the corresponding KKT necessary optimality conditions, Lagrange duality results, and saddle-point criteria.

Table 1. Comparison of the present study with closely related works.

Study	Fractional operator	Problem setting	Control	KKT	Duality	Saddle-point
Jayswal and Uniyal [7]	Caputo-Fabrizio	Semi-infinite variational programming	–	–	✓	✓
Jayswal and Uniyal [12]	Caputo-Fabrizio	Semi-infinite variational programming	–	✓	✓	–
Rayanki et al. [13]	Caputo-Fabrizio	Interval-valued variational programming	–	✓	–	–
Present study	Caputo-Fabrizio	Semi-infinite fractional variational control problem	✓	✓	✓	✓

Note: ✓ indicates that the corresponding feature is considered in the study, whereas “–” indicates that it is not explicitly incorporated.

The primary motivation of the present study is to integrate these research directions within a unified controlled semi-infinite variational framework. Specifically, the proposed model incorporates control variables into a semi-infinite variational problem governed by the Caputo–Fabrizio fractional derivative. Using the corresponding integration-by-parts formula, we derive KKT necessary optimality conditions that account for both state and control stationarity. The analysis is further extended through a Lagrange-type dual formulation, associated duality results, and saddle-point criteria. In this way, the present work provides a unified theoretical treatment of control variables, semi-infinite constraints, and Caputo–Fabrizio fractional dynamics within the same variational framework.

The paper continues with some preliminary results and notations in Section 2. Section 3 introduces the KKT necessary optimality conditions for the considered controlled semi-infinite variational problem. The subsequent part of Section 3 establishes a Lagrange-type dual model and formulates the corresponding duality theorems. In the end of this section, we investigate the saddle point criteria for the problem under study. Finally, Section 4 provides concluding remarks and suggests future research directions.

2. Preliminaries and notations

This section outlines the essential notations, definitions, and mathematical operators required for the study of controlled semi-infinite variational problems.

2.1. Sobolev space

The Sobolev space of order 1 on (a, b) is defined by

$$[H^1(a, b) = \{y \in L^2(a, b) : y' \in L^2(a, b)\}]. \quad (2.1)$$

where y' denotes the weak derivative of y and $L^2(a, b)$ is the space of square-integrable functions on (a, b) .

2.2. Caputo-Fabrizio fractional operators

Following the recent developments in fractional calculus to avoid singular kernels, we use the Caputo-Fabrizio (CF) derivative.

Definition 2.1. The left Caputo-Fabrizio (CF) fractional derivative of order $\alpha \in (0, 1)$ for a function $\epsilon \in H^1(a, b)$ is defined as:

$${}^{CF}D_{a+}^{\alpha}\epsilon(\theta) = \frac{1}{1-\alpha} \int_a^{\theta} \exp\left(-\frac{\alpha(\theta-\tau)}{1-\alpha}\right) \epsilon'(\tau) d\tau, \quad \theta \geq a. \quad (2.2)$$

Definition 2.2. In the Caputo-Fabrizio sense, the right Riemann fractional derivative of order $\alpha \in (0, 1)$ is expressed as:

$${}^{CFR}D_{b-}^{\alpha}\epsilon(\theta) = -\frac{1}{1-\alpha} \frac{d}{d\theta} \int_{\theta}^b \exp\left(-\frac{\alpha(\tau-\theta)}{1-\alpha}\right) \epsilon(\tau) d\tau, \quad \theta \leq b. \quad (2.3)$$

Proposition 2.1. (Integration by Parts) Let $\alpha \in (0, 1)$ and $\epsilon, z : [a, b] \rightarrow \mathbb{R}$ be two functions of $C^1[a, b]$ -class. Then the following formula holds:

$$\int_a^b \epsilon(\theta) {}^{CF}D_{a+}^{\alpha} z(\theta) d\theta = [z(\theta) I_{b-}^{1-\alpha} \epsilon(\theta)]_{\theta=a}^{\theta=b} + \int_a^b z(\theta) {}^{CFR}D_{b-}^{\alpha} \epsilon(\theta) d\theta, \quad (2.4)$$

where

$$I_{b-}^{1-\alpha} \epsilon(\theta) = \frac{1}{1-\alpha} \int_{\theta}^b \exp\left(-\frac{\alpha}{1-\alpha}(\tau-\theta)\right) \epsilon(\tau) d\tau. \quad (2.5)$$

2.3. The cost functional and its partial derivatives

Let $\mathfrak{I} = [a, b]$ be a real-valued interval and $\psi : \mathfrak{I} \times \mathbb{R}^{2n} \times \mathbb{R}^p \rightarrow \mathbb{R}$ be a continuously differentiable functional with respect to each argument. We consider the functional $\psi(\theta, \epsilon(\theta), {}^{CF}D_{a+}^{\alpha}\epsilon(\theta), u(\theta))$, where $\epsilon : \mathfrak{I} \rightarrow \mathbb{R}^n$ is a $C^1[a, b]$ -class function representing the *state* variable, ${}^{CF}D_{a+}^{\alpha}\epsilon$ represents the Caputo-Fabrizio fractional derivative of order $\alpha \in (0, 1)$ associated with ϵ , and $u : \mathfrak{I} \rightarrow \mathbb{R}^p$ is a $C^1[a, b]$ -class function representing the control variable.

The partial derivatives (gradients) of ψ are represented by:

$$\psi_{\theta} = \frac{\partial \psi}{\partial \theta}, \quad (2.6)$$

$$\psi_{\epsilon} = \left(\frac{\partial \psi}{\partial \epsilon_1}, \frac{\partial \psi}{\partial \epsilon_2}, \dots, \frac{\partial \psi}{\partial \epsilon_n} \right), \quad (2.7)$$

$$\psi_u = \left(\frac{\partial \psi}{\partial u_1}, \frac{\partial \psi}{\partial u_2}, \dots, \frac{\partial \psi}{\partial u_p} \right), \quad (2.8)$$

$$\psi_{CF D_{a+}^\alpha \epsilon} = \left(\frac{\partial \psi}{\partial (CF D_{a+}^\alpha \epsilon_1)}, \dots, \frac{\partial \psi}{\partial (CF D_{a+}^\alpha \epsilon_n)} \right), \quad (2.9)$$

where $\epsilon_1, \dots, \epsilon_n$ are the components of the state variable ϵ . Consider E the space of all state functions and U the space of all control functions, equipped with the uniform norm:

$$\|x\| = \|x\|_\infty + \|CF D_{a+}^\alpha x\|_\infty. \quad (2.10)$$

Let $\mathcal{J} : E \times U \rightarrow \mathbb{R}$ be a real-valued functional defined by

$$\mathcal{J}(\epsilon, u) = \int_a^b \psi(\theta, \epsilon, CF D_{a+}^\alpha \epsilon, u) d\theta. \quad (2.11)$$

For notational convenience, the integrand is written as $\psi(\theta, \epsilon, CF D_{a+}^\alpha \epsilon, u)$.

2.4. Controlled convexity

The functional

$$\int_a^b \psi(\theta, \epsilon(\theta), CF D_{a+}^\alpha \epsilon(\theta), u(\theta)) d\theta \quad (2.12)$$

is said to be convex (respectively strictly convex) at $(\bar{\epsilon}, \bar{u}) \in E \times U$ if, for all $(\epsilon, u) \in E \times U$, we have

$$\begin{aligned} & \int_a^b \psi(\theta, \epsilon(\theta), CF D_{a+}^\alpha \epsilon(\theta), u(\theta)) d\theta - \int_a^b \psi(\theta, \bar{\epsilon}(\theta), CF D_{a+}^\alpha \bar{\epsilon}(\theta), \bar{u}(\theta)) d\theta \\ & \geq (>) \int_a^b \left[(\epsilon(\theta) - \bar{\epsilon}(\theta))^T \psi_\epsilon(\theta, \bar{\epsilon}(\theta), CF D_{a+}^\alpha \bar{\epsilon}(\theta), \bar{u}(\theta)) \right. \\ & \quad + (CF D_{a+}^\alpha (\epsilon(\theta) - \bar{\epsilon}(\theta)))^T \psi_{CF D_{a+}^\alpha \epsilon}(\theta, \bar{\epsilon}(\theta), CF D_{a+}^\alpha \bar{\epsilon}(\theta), \bar{u}(\theta)) \\ & \quad \left. + (u(\theta) - \bar{u}(\theta))^T \psi_u(\theta, \bar{\epsilon}(\theta), CF D_{a+}^\alpha \bar{\epsilon}(\theta), \bar{u}(\theta)) \right] d\theta. \end{aligned} \quad (2.13)$$

2.5. Problem formulation

We now formulate the following semi-infinite controlled variational model (SCVM) involving the fractional derivative of Caputo-Fabrizio type:

$$(SCVM) \min_{(\epsilon, u) \in E \times U} \int_a^b \psi(\theta, \epsilon(\theta), CF D_{a+}^\alpha \epsilon(\theta), u(\theta)) d\theta \quad (2.14)$$

subject to

$$H_t(\theta, \epsilon(\theta), CF D_{a+}^\alpha \epsilon(\theta), u(\theta)) \leq 0_{\mathbb{R}^m}, \quad \forall t \in A, \quad \theta \in [a, b], \quad (2.15)$$

the equality constraint

$$G(\theta, \epsilon(\theta), CF D_{a+}^\alpha \epsilon(\theta), u(\theta)) := CF D_{a+}^\alpha \epsilon(\theta) - P(\theta, \epsilon(\theta), u(\theta)) = 0_{\mathbb{R}^q}, \quad \theta \in [a, b], \quad (2.16)$$

and the boundary conditions

$$\epsilon(a) = \xi, \quad \epsilon(b) = \gamma, \quad (2.17)$$

where A denotes an index set which may be finite or infinite.

Remark 2.1. (i) If we remove the control variables u (and, implicitly, the equality constraints), then the above-considered class of problems reduces to the problems investigated in Jayswal and Uniyal [7].

(ii) If we remove the control variables u (and, implicitly, the equality constraints) and take A to be a finite set, we get the model considered by Dubey et al. [21].

(iii) If we remove the control variables u (and, implicitly, the equality constraints), take A to be a finite set, $m = 1$, and consider the ordinary derivative (that is, $\dot{\epsilon}$ instead of ${}^{CF}D_{a+}^{\alpha}\epsilon$), then we obtain the problem studied in Mond and Hussain [22].

(iv) If we remove the control variables u (and, implicitly, the equality constraints), take A to be a finite set, and consider the objective/cost functional is interval-valued, then we get the variational model studied in Rayanki et al. [13].

(v) If we remove the control variables u (and, implicitly, the equality constraints), take A to be a finite set, assume the objective/cost functional is interval-valued, and consider the ordinary derivative (that is, $\dot{\epsilon}$ instead of ${}^{CF}D_{a+}^{\alpha}\epsilon$), then we obtain the problem investigated in Ahmad et al. [23].

The feasible set associated with (SCVM) is defined by

$$K_c = \left\{ (\epsilon, u) \in E \times U : \begin{array}{l} H_t(\theta, \epsilon, {}^{CF}D_{a+}^{\alpha}\epsilon, u) \leq 0, \quad \forall t \in A \\ G(\theta, \epsilon, {}^{CF}D_{a+}^{\alpha}\epsilon, u) = 0, \\ \epsilon(a) = \xi, \quad \epsilon(b) = \gamma \end{array} \right\}. \quad (2.18)$$

Definition 2.3. A feasible pair $(\bar{\epsilon}, \bar{u}) \in K_c$ is said to be an optimal solution of (SCVM) if there exists no feasible pair $(\epsilon, u) \in K_c$ such that

$$\int_a^b \psi(\theta, \epsilon, {}^{CF}D_{a+}^{\alpha}\epsilon, u) d\theta < \int_a^b \psi(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha}\bar{\epsilon}, \bar{u}) d\theta. \quad (2.19)$$

Definition 2.4. For problem (SCVM), we say that the Slater Constraint Qualification (SCQ) is satisfied at $(\bar{\epsilon}, \bar{u}) \in K_c$ if the constraint functionals $H_t(\theta, \epsilon, {}^{CF}D_{a+}^{\alpha}\epsilon, u), \forall t \in A$, are convex at $(\bar{\epsilon}, \bar{u})$ and the following relations hold:

$$\begin{aligned} & H_t(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha}\bar{\epsilon}, \bar{u}) + (H_t)_{\epsilon}(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha}\bar{\epsilon}, \bar{u})w(\theta) \\ & + (H_t)_{{}^{CF}D_{a+}^{\alpha}\epsilon}(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha}\bar{\epsilon}, \bar{u}){}^{CF}D_{a+}^{\alpha}w(\theta) \\ & + (H_t)_u(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha}\bar{\epsilon}, \bar{u})v(\theta) < 0, \end{aligned} \quad (2.20)$$

for some $w(\theta) \in E$, $v(\theta) \in U$, and for all $t \in A$, $\theta \in [a, b]$.

Definition 2.5. Let $\mathbb{R}_+^{|A|}$ denote the collection of all functions $\sigma_t : A \rightarrow \mathbb{R}^m$ taking positive values only at finitely many points of A and equal to zero at all other points.

For a given feasible pair $(\bar{\epsilon}, \bar{u}) \in K_c$, the index set associated with active constraints is formulated as

$$W(\bar{\epsilon}, \bar{u}) = \left\{ t \in A \mid H_t(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha}\bar{\epsilon}, \bar{u}) = 0 \right\}. \quad (2.21)$$

The set of active constraint multipliers at $(\bar{\epsilon}, \bar{u}) \in K_c$ is defined as

$$\Gamma(\bar{\epsilon}, \bar{u}) = \left\{ \sigma_t \in \mathbb{R}_+^{|A|} \mid \sigma_t H_t(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha}\bar{\epsilon}, \bar{u}) = 0, \quad \forall t \in A \right\}. \quad (2.22)$$

We can notice that $\sigma_t \in \Gamma(\bar{\epsilon}, \bar{u})$ if there exists a finite index set $Q \subset W(\bar{\epsilon}, \bar{u})$ satisfying $\sigma_t > 0, \forall t \in Q$, and $\sigma_t = 0, \forall t \in W(\bar{\epsilon}, \bar{u}) \setminus Q$.

Definition 2.6. A feasible pair $(\bar{\epsilon}, \bar{u}) \in K_c$ is said to be a stationary point of the problem (SCVM) if there exist multipliers $\sigma_t \in \Gamma(\bar{\epsilon}, \bar{u})$, $t \in A$, (corresponding to the inequality constraints) and $\mu \in \mathbb{R}^q$ (corresponding to the equality constraints) such that

$$\begin{aligned} & \psi_{\epsilon}(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) + \sum_{t \in A} \sigma_t (H_t)_{\epsilon}(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) \\ & + \mu G_{\epsilon}(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) \\ & + {}^{CFR}D_{b-}^{\alpha} \left[\psi_{{}^{CF}D_{a+}^{\alpha} \bar{\epsilon}}(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) \right. \\ & + \sum_{t \in A} \sigma_t (H_t)_{CF D_{a+}^{\alpha} \bar{\epsilon}}(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) \\ & \left. + \mu G_{{}^{CF}D_{a+}^{\alpha} \bar{\epsilon}}(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) \right] = 0 \end{aligned} \quad (2.23)$$

and

$$\psi_u(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) + \sum_{t \in A} \sigma_t (H_t)_u(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) + \mu G_u(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) = 0. \quad (2.24)$$

3. Main results

In this section, we derive the Karush-Kuhn-Tucker necessary optimality conditions for the controlled semi-infinite variational problem (SCVM). The proof is based on the Caputo-Fabrizio integration-by-parts formula stated in Proposition 2.1 and follows the variational technique used for semi-infinite variational programming problems. The consistency of the proposed approach follows from the fact that the optimality framework is developed directly from the feasible structure of the semi-infinite controlled variational model (SCVM). Under the stated differentiability and Slater constraint qualification assumptions, the first-order variational conditions lead to the KKT system presented below. The subsequent duality and saddle-point results are derived using the same constraint structure and multiplier conditions, thereby maintaining a consistent theoretical framework throughout the analysis.

Theorem 3.1. (KKT necessary optimality conditions) Let $(\bar{\epsilon}, \bar{u}) \in K_c$ be an optimal solution of problem (SCVM). Assume that the Slater constraint qualification holds at $(\bar{\epsilon}, \bar{u})$ and that all involved functionals are continuously differentiable. Then there exist multipliers $\sigma_t \geq 0$, $t \in A$, and $\mu \in \mathbb{R}^q$ such that the following conditions hold:

$$\begin{aligned} & \psi_{\epsilon}(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) + \sum_{t \in A} \sigma_t (H_t)_{\epsilon}(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) + \mu G_{\epsilon}(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) \\ & + {}^{CFR}D_{b-}^{\alpha} \left[\psi_{{}^{CF}D_{a+}^{\alpha} \bar{\epsilon}}(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) + \sum_{t \in A} \sigma_t (H_t)_{CF D_{a+}^{\alpha} \bar{\epsilon}}(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) \right. \\ & \left. + \mu G_{{}^{CF}D_{a+}^{\alpha} \bar{\epsilon}}(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) \right] = 0, \end{aligned} \quad (3.1)$$

$$\psi_u(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) + \sum_{t \in A} \sigma_t (H_t)_u(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) + \mu G_u(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) = 0, \quad (3.2)$$

$$\sigma_t H_t(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^{\alpha} \bar{\epsilon}, \bar{u}) = 0, \quad \forall t \in A. \quad (3.3)$$

Proof. Since $(\bar{\epsilon}, \bar{u})$ is an optimal solution of (SCVM) and the Slater constraint qualification holds, the classical multiplier rule can be applied to the considered constrained variational problem. Therefore, there exist multipliers $\sigma_t \geq 0$, $t \in A$, associated with the semi-infinite inequality constraints, and a multiplier $\mu \in \mathbb{R}^q$ associated with the equality constraint.

Define the Lagrangian

$$\mathcal{L} = \psi + \sum_{t \in A} \sigma_t H_t + \mu G. \quad (3.4)$$

The first variation of the corresponding Lagrangian at $(\bar{\epsilon}, \bar{u})$ must vanish for every admissible perturbation. Hence,

$$\delta \mathcal{L}(\bar{\epsilon}, \bar{u}) = 0. \quad (3.5)$$

Taking variations with respect to the state variable ϵ , the control variable u , and the fractional derivative ${}^{CF}D_{a+}^\alpha \epsilon$, we obtain

$$\begin{aligned} 0 = \int_a^b & \left[\left(\psi_\epsilon + \sum_{t \in A} \sigma_t (H_t)_\epsilon + \mu G_\epsilon \right) \eta \right. \\ & + \left(\psi_{{}^{CF}D_{a+}^\alpha \epsilon} + \sum_{t \in A} \sigma_t (H_t)_{{}^{CF}D_{a+}^\alpha \epsilon} + \mu G_{{}^{CF}D_{a+}^\alpha \epsilon} \right) {}^{CF}D_{a+}^\alpha \eta \\ & \left. + \left(\psi_u + \sum_{t \in A} \sigma_t (H_t)_u + \mu G_u \right) v \right] d\theta, \end{aligned} \quad (3.6)$$

where η and v are admissible variations of ϵ and u , respectively.

Using the Caputo-Fabrizio integration-by-parts formula from Proposition 2.1, the term containing ${}^{CF}D_{a+}^\alpha \eta$ becomes

$$\int_a^b W(\theta) {}^{CF}D_{a+}^\alpha \eta(\theta) d\theta = \left[\eta(\theta) I_{b-}^{1-\alpha} W(\theta) \right]_a^b + \int_a^b \eta(\theta) {}^{CFR}D_{b-}^\alpha W(\theta) d\theta, \quad (3.7)$$

where

$$W(\theta) = \psi_{{}^{CF}D_{a+}^\alpha \epsilon} + \sum_{t \in A} \sigma_t (H_t)_{{}^{CF}D_{a+}^\alpha \epsilon} + \mu G_{{}^{CF}D_{a+}^\alpha \epsilon}. \quad (3.8)$$

Since the admissible variations satisfy the boundary conditions, we have

$$\eta(a) = \eta(b) = 0. \quad (3.9)$$

Thus, the boundary term vanishes.

Consequently, by the arbitrariness of η and v , we obtain the state stationarity condition

$$\psi_\epsilon + \sum_{t \in A} \sigma_t (H_t)_\epsilon + \mu G_\epsilon + {}^{CFR}D_{b-}^\alpha \left[\psi_{{}^{CF}D_{a+}^\alpha \epsilon} + \sum_{t \in A} \sigma_t (H_t)_{{}^{CF}D_{a+}^\alpha \epsilon} + \mu G_{{}^{CF}D_{a+}^\alpha \epsilon} \right] = 0, \quad (3.10)$$

and the control stationarity condition

$$\psi_u + \sum_{t \in A} \sigma_t (H_t)_u + \mu G_u = 0. \quad (3.11)$$

Finally, the multiplier rule gives the dual feasibility condition $\sigma_t \geq 0$, $t \in A$, and the complementary slackness condition $\sigma_t H_t(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^\alpha \bar{\epsilon}, \bar{u}) = 0$, $t \in A$. Hence, the proof is complete. $\square$

Corollary 3.1. Let $(\bar{\epsilon}, \bar{u})$ be an optimal solution of (SCVM) and the Slater constraint qualification holds. Then $(\bar{\epsilon}, \bar{u})$ is a stationary point in (SCVM).

Proof. By considering Definitions 2.4 and 2.5, in accordance with Theorem 3.1, the proof is complete. $\square$

3.1. Lagrange dual model

In this subsection, we formulate the Lagrange dual problem associated with the controlled semi-infinite variational problem (SCVM) and establish the corresponding duality results.

The Lagrange functional associated with problem (SCVM) is defined by

$$L(\epsilon, u, \sigma_t, \mu) = \int_a^b \left[\psi(\theta, \epsilon, {}^{CF}D_{a+}^\alpha \epsilon, u) + \sum_{t \in A} \sigma_t H_t(\theta, \epsilon, {}^{CF}D_{a+}^\alpha \epsilon, u) + \mu G(\theta, \epsilon, {}^{CF}D_{a+}^\alpha \epsilon, u) \right] d\theta, \quad (3.12)$$

where $\sigma_t \geq 0, \forall t \in A, \mu \in \mathbb{R}^q$.

The dual functional associated with (SCVM) is defined by

$$\Theta(\sigma_t, \mu) = \min_{(\epsilon, u) \in K_c} L(\epsilon, u, \sigma_t, \mu). \quad (3.13)$$

The corresponding Lagrange dual problem is given by

$$(DSVM) \quad \max_{\sigma_t \geq 0, \mu} \Theta(\sigma_t, \mu). \quad (3.14)$$

Theorem 3.2. (Weak duality) Let (ϵ, u) be any feasible solution of (SCVM) and let (σ_t, μ) be any feasible solution of (DSVM). Then

$$\Theta(\sigma_t, \mu) \leq \int_a^b \psi(\theta, \epsilon, {}^{CF}D_{a+}^\alpha \epsilon, u) d\theta. \quad (3.15)$$

Proof. Let (ϵ, u) be feasible for (SCVM). Then

$$H_t(\theta, \epsilon, {}^{CF}D_{a+}^\alpha \epsilon, u) \leq 0, \quad t \in A, \quad (3.16)$$

$$G(\theta, \epsilon, {}^{CF}D_{a+}^\alpha \epsilon, u) = 0. \quad (3.17)$$

Since $\sigma_t \geq 0$, we obtain $\sigma_t H_t(\theta, \epsilon, {}^{CF}D_{a+}^\alpha \epsilon, u) \leq 0$. Hence, we obtain

$$L(\epsilon, u, \sigma_t, \mu) \leq \int_a^b \psi(\theta, \epsilon, {}^{CF}D_{a+}^\alpha \epsilon, u) d\theta. \quad (3.18)$$

By considering the dual functional definition, we get

$$\Theta(\sigma_t, \mu) = \min_{(\epsilon, u) \in K_c} L(\epsilon, u, \sigma_t, \mu) \leq L(\epsilon, u, \sigma_t, \mu). \quad (3.19)$$

Therefore, by using the above two mentioned inequalities, we obtain

$$\Theta(\sigma_t, \mu) \leq \int_a^b \psi(\theta, \epsilon, {}^{CF}D_{a+}^\alpha \epsilon, u) d\theta. \quad (3.20)$$

Thus, the proof is complete. $\square$

Theorem 3.3. (Strong duality) Let $(\bar{\epsilon}, \bar{u})$ be an optimal solution of the problem (SCVM). Assume that the Slater constraint qualification holds and that the controlled convexity assumptions are satisfied for the involved functionals. Then there exist multipliers $\bar{\sigma}_t \geq 0$, $t \in A$, and $\bar{\mu}$, such that $(\bar{\sigma}_t, \bar{\mu})$ is an optimal solution of (DSCVM) and

$$\Theta(\bar{\sigma}_t, \bar{\mu}) = \int_a^b \psi(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^\alpha \bar{\epsilon}, \bar{u}) d\theta. \quad (3.21)$$

Proof. Since $(\bar{\epsilon}, \bar{u})$ is an optimal solution of (SCVM) and the Slater constraint qualification holds, the KKT necessary optimality conditions are satisfied (see Theorem 3.1). Hence, there exist multipliers $\bar{\sigma}_t \geq 0$, $t \in A$, and $\bar{\mu}$, satisfying the stationarity and complementary slackness conditions.

Using the complementary slackness condition and feasibility of $(\bar{\epsilon}, \bar{u})$, we obtain

$$L(\bar{\epsilon}, \bar{u}, \bar{\sigma}_t, \bar{\mu}) = \int_a^b \psi(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^\alpha \bar{\epsilon}, \bar{u}) d\theta. \quad (3.22)$$

Moreover, by considering the controlled convexity, we get

$$L(\bar{\epsilon}, \bar{u}, \bar{\sigma}_t, \bar{\mu}) = \min_{(\epsilon, u) \in K_c} L(\epsilon, u, \bar{\sigma}_t, \bar{\mu}). \quad (3.23)$$

Therefore, using the dual functional associated with (SCVM), we obtain

$$\Theta(\bar{\sigma}_t, \bar{\mu}) = \int_a^b \psi(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^\alpha \bar{\epsilon}, \bar{u}) d\theta. \quad (3.24)$$

Hence, the proof is complete. $\square$

3.2. Saddle point criteria

In this subsection, we establish saddle point criteria associated with the controlled semi-infinite variational problem (SCVM).

A point $(\bar{\epsilon}, \bar{u}, \bar{\sigma}_t, \bar{\mu})$ is called a saddle point of the Lagrange functional if

$$L(\bar{\epsilon}, \bar{u}, \sigma_t, \mu) \leq L(\bar{\epsilon}, \bar{u}, \bar{\sigma}_t, \bar{\mu}) \leq L(\epsilon, u, \bar{\sigma}_t, \bar{\mu}), \quad (3.25)$$

for all admissible $(\epsilon, u) \in K_\epsilon$, $\sigma_t \geq 0$, $t \in A$, and $\mu \in \mathbb{R}^k$.

Theorem 3.4. Let $(\bar{\epsilon}, \bar{u})$ be an optimal solution of (SCVM). Assume that the Slater constraint qualification and controlled convexity assumptions hold for the involved functionals. Then there exist multipliers $\bar{\sigma}_t \geq 0$, $t \in A$, and $\bar{\mu} \in \mathbb{R}^k$ such that $(\bar{\epsilon}, \bar{u}, \bar{\sigma}_t, \bar{\mu})$ is a saddle point of the Lagrange functional.

Proof. Since $(\bar{\epsilon}, \bar{u})$ is an optimal solution of (SCVM), the KKT necessary optimality conditions hold (see Theorem 3.1). Hence, there exist multipliers $\bar{\sigma}_t \geq 0$, $t \in A$, and $\bar{\mu}$ satisfying the stationarity conditions.

Because $(\bar{\epsilon}, \bar{u})$ is feasible, we have

$$H_t(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^\alpha \bar{\epsilon}, \bar{u}) \leq 0, \quad (3.26)$$

and

$$G(\theta, \bar{\epsilon}, {}^{CF}D_{a+}^\alpha \bar{\epsilon}, \bar{u}) = 0. \quad (3.27)$$

Using complementary slackness, we have

$$\bar{\sigma}_t H_t(\theta, \bar{\varepsilon}, {}^{CF}D_{a+}^\alpha \bar{\varepsilon}, \bar{u}) = 0. \quad (3.28)$$

Therefore, it results

$$L(\bar{\varepsilon}, \bar{u}, \bar{\sigma}_t, \bar{\mu}) \leq L(\bar{\varepsilon}, \bar{u}, \bar{\sigma}_t, \bar{\mu}). \quad (3.29)$$

On the other hand, the controlled convexity associated with the involved functionals implies

$$L(\bar{\varepsilon}, \bar{u}, \bar{\sigma}_t, \bar{\mu}) \leq L(\varepsilon, u, \bar{\sigma}_t, \bar{\mu}), \quad (3.30)$$

for all admissible $(\varepsilon, u) \in K_\varepsilon$. Hence, $(\bar{\varepsilon}, \bar{u}, \bar{\sigma}_t, \bar{\mu})$ is a saddle point of the Lagrange functional. $\square$

Remark 3.1. If $(\bar{\varepsilon}, \bar{u}, \bar{\sigma}_t, \bar{\mu})$ is a saddle point of the Lagrange functional, then $(\bar{\varepsilon}, \bar{u})$ is an optimal solution of (SCVM).

Indeed, assume that $(\bar{\varepsilon}, \bar{u}, \bar{\sigma}_t, \bar{\mu})$ is a saddle point. Then,

$$L(\bar{\varepsilon}, \bar{u}, \bar{\sigma}_t, \bar{\mu}) \leq L(\varepsilon, u, \bar{\sigma}_t, \bar{\mu}), \quad (3.31)$$

for all admissible $(\varepsilon, u) \in K_\varepsilon$. Since (ε, u) is feasible solution, we have

$$H_t \leq 0, \quad G = 0. \quad (3.32)$$

Using $\bar{\sigma}_t \geq 0$, we obtain

$$L(\varepsilon, u, \bar{\sigma}_t, \bar{\mu}) \leq \int_a^b \psi(\theta, \varepsilon, {}^{CF}D_{a+}^\alpha \varepsilon, u) d\theta. \quad (3.33)$$

Similarly, we have

$$L(\bar{\varepsilon}, \bar{u}, \bar{\sigma}_t, \bar{\mu}) = \int_a^b \psi(\theta, \bar{\varepsilon}, {}^{CF}D_{a+}^\alpha \bar{\varepsilon}, \bar{u}) d\theta. \quad (3.34)$$

Therefore, it follows

$$\int_a^b \psi(\theta, \bar{\varepsilon}, {}^{CF}D_{a+}^\alpha \bar{\varepsilon}, \bar{u}) d\theta \leq \int_a^b \psi(\theta, \varepsilon, {}^{CF}D_{a+}^\alpha \varepsilon, u) d\theta. \quad (3.35)$$

Hence, $(\bar{\varepsilon}, \bar{u})$ is an optimal solution of (SCVM).

Illustrative application. Fix $\alpha \in (0, 1)$ and consider the interval $[a, b] = [0, 1]$. Consider the state and control functions, respectively, as

$$\varepsilon(\theta) = \theta, \quad u(\theta) = \theta^2. \quad (3.36)$$

Figure 1 illustrates the selected state function $\varepsilon(\theta) = \theta$ and control function $u(\theta) = \theta^2$ over the interval $[0, 1]$. The figure provides a visual representation of the state–control pair used in the numerical example and shows their respective behavior over the considered domain. This pair is subsequently used to evaluate the objective functional and verify the constraints of the controlled semi-infinite variational problem.

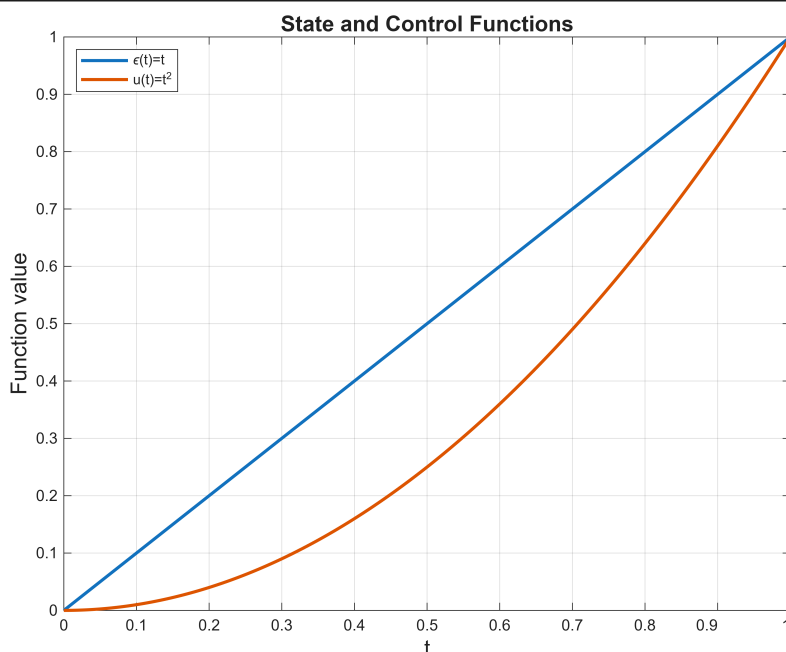


Figure 1. Graphs of the state function $\varepsilon(t) = t$ and the control function $u(t) = t^2$ on $[0, 1]$.

Clearly, the boundary conditions are satisfied since $\varepsilon(0) = 0$, $\varepsilon(1) = 1$. Using the Caputo-Fabrizio fractional derivative, we obtain

$${}^{CF}D_{0+}^{\alpha} \varepsilon(\theta) = \frac{1}{1-\alpha} \int_0^{\theta} \exp\left(-\frac{\alpha(\theta-\tau)}{1-\alpha}\right) d\tau. \quad (3.37)$$

For $k = \frac{\alpha}{1-\alpha}$, we have

$${}^{CF}D_{0+}^{\alpha} \varepsilon(\theta) = \frac{1}{1-\alpha} \int_0^{\theta} e^{-k(\theta-\tau)} d\tau. \quad (3.38)$$

Using the identity $e^{-k(\theta-\tau)} = e^{-k\theta} e^{k\tau}$, we get

$${}^{CF}D_{0+}^{\alpha} \varepsilon(\theta) = \frac{e^{-k\theta}}{1-\alpha} \int_0^{\theta} e^{k\tau} d\tau. \quad (3.39)$$

Since $\int e^{k\tau} d\tau = \frac{1}{k} e^{k\tau}$, it follows that $\int_0^{\theta} e^{k\tau} d\tau = \left[\frac{1}{k} e^{k\tau}\right]_0^{\theta}$. Hence, we get $\int_0^{\theta} e^{k\tau} d\tau = \frac{1}{k} (e^{k\theta} - 1)$. Substituting into the fractional derivative expression it gives

$${}^{CF}D_{0+}^{\alpha} \varepsilon(\theta) = \frac{e^{-k\theta}}{1-\alpha} \cdot \frac{1}{k} (e^{k\theta} - 1) \quad (3.40)$$

or, equivalently,

$${}^{CF}D_{0+}^{\alpha} \varepsilon(\theta) = \frac{1}{k(1-\alpha)} (1 - e^{-k\theta}). \quad (3.41)$$

Since $k = \frac{\alpha}{1-\alpha}$, we obtain $k(1-\alpha) = \alpha$, hence, $\frac{1}{k(1-\alpha)} = \frac{1}{\alpha}$. Therefore, we get

$${}^{CF}D_{0+}^{\alpha} \varepsilon(\theta) = \frac{1}{\alpha} (1 - e^{-k\theta}) \quad (3.42)$$

and substituting back $k = \frac{\alpha}{1-\alpha}$, we finally obtain

$${}^{CF}D_{0+}^{\alpha}\varepsilon(\theta) = \frac{1}{\alpha} \left(1 - e^{-\frac{\alpha\theta}{1-\alpha}}\right). \quad (3.43)$$

Now, we define the objective/cost functional by using the function

$$\psi(\theta, \varepsilon, {}^{CF}D_{0+}^{\alpha}\varepsilon, u) = \theta^2 + \theta^4, \quad (3.44)$$

so, the primal objective functional becomes

$$J(\varepsilon, u) = \int_0^1 (\theta^2 + \theta^4) d\theta. \quad (3.45)$$

Since $\int_0^1 \theta^2 d\theta = \frac{1}{3}$ and $\int_0^1 \theta^4 d\theta = \frac{1}{5}$, we obtain $J(\varepsilon, u) = \frac{1}{3} + \frac{1}{5} = \frac{8}{15} \approx 0.5333$.

Next, define the semi-infinite inequality constraint by

$$H_t(\theta, \varepsilon, {}^{CF}D_{0+}^{\alpha}\varepsilon, u) = tu(\theta) - \varepsilon(\theta), \quad t \in [0, 1]. \quad (3.46)$$

Substituting the selected functions, we get

$$H_t(\theta) = t\theta^2 - \theta = \theta(t\theta - 1). \quad (3.47)$$

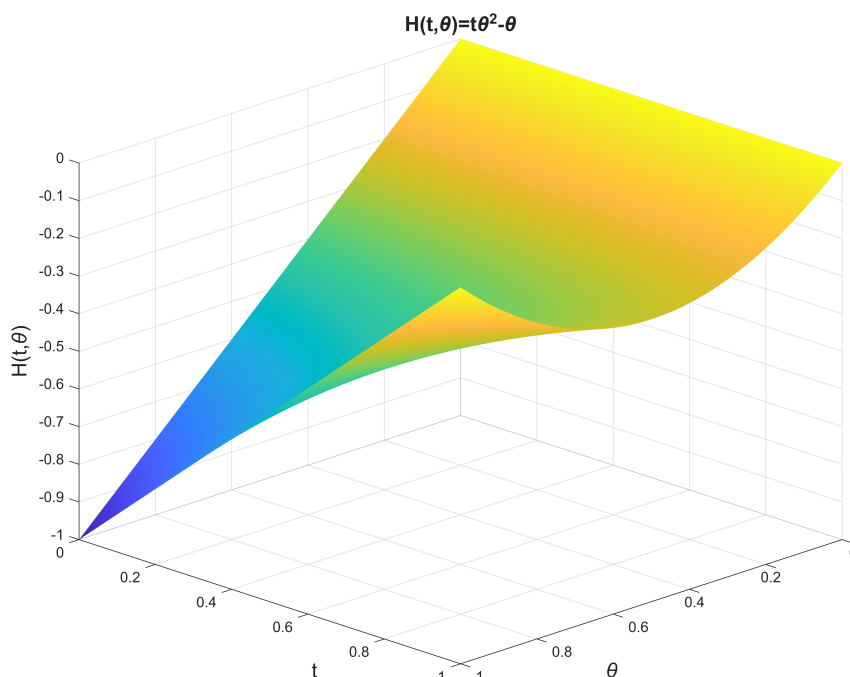


Figure 2. Surface plot of the semi-infinite constraint function $H(t, \theta) = t\theta^2 - \theta$ on $[0, 1] \times [0, 1]$.

Figure 2 illustrates the semi-infinite constraint function $H_t(\theta) = t\theta^2 - \theta$ over $(t, \theta) \in [0, 1] \times [0, 1]$. As shown by the preceding calculation, $H_t(\theta) \leq 0$ throughout the considered domain. The surface plot provides a graphical representation of this result and illustrates that the selected state-control pair satisfies the semi-infinite inequality constraint for all $t \in [0, 1]$.

Since $0 \leq t \leq 1$ and $0 \leq \theta \leq 1$, we have $t\theta \leq 1$, hence,

$$H_t(\theta) \leq 0, \quad \forall t \in [0, 1]. \quad (3.48)$$

Further, define the equality constraint by

$$G(\theta, \varepsilon, {}^{CF}D_{0+}^\alpha \varepsilon, u) = {}^{CF}D_{0+}^\alpha \varepsilon(\theta) - \frac{1}{\alpha} \left(1 - e^{-\frac{\alpha\theta}{1-\alpha}}\right). \quad (3.49)$$

Using the previous computation, we obtain $G = 0$. Thus, the pair

$$(\varepsilon, u) = (\theta, \theta^2) \quad (3.50)$$

is feasible for the considered controlled semi-infinite variational problem.

Now, if we choose the multipliers $\sigma_t = 0$, $\mu = 1$. Clearly, $\sigma_t \geq 0$, $\sigma_t H_t = 0$. Therefore, the complementary slackness condition is satisfied.

The Lagrangian associated with this problem is

$$L = \psi + \sum_{t \in A} \sigma_t H_t + \mu G. \quad (3.51)$$

Since $\sigma_t = 0$ and $\mu = 1$, we obtain $L = \psi = \theta^2 + \theta^4$. Thus, the associated Lagrange functional value becomes

$$\int_0^1 L d\theta = J(\varepsilon, u) = \int_0^1 (\theta^2 + \theta^4) d\theta = \frac{1}{3} + \frac{1}{5} = \frac{8}{15} = 0.5333. \quad (3.52)$$

Moreover, the dual functional at $(\sigma_t, \mu) = (0, 0)$ becomes

$$\Theta(0, 0) = \min_{(\varepsilon, u) \in K_c} L(\varepsilon, u, 0, 0) = \frac{8}{15}, \quad (3.53)$$

thus, we get $\Theta(0, 0) = J(\varepsilon, u) = \frac{8}{15} = 0.5333$.

Now, we verify the controlled convexity condition introduced in Definition 2.4. Since $\psi(\theta, \varepsilon, {}^{CF}D_{0+}^\alpha \varepsilon, u) = \theta^2 + \theta^4$ does not explicitly depend on ε, u , or ${}^{CF}D_{0+}^\alpha \varepsilon$, we have

$$\psi_\varepsilon = 0, \quad \psi_u = 0, \quad \psi_{{}^{CF}D_{0+}^\alpha \varepsilon} = 0. \quad (3.54)$$

Therefore, for any feasible pair (ε, u) and reference pair $(\bar{\varepsilon}, \bar{u})$, the right-hand side of the controlled convexity inequality becomes

$$\int_0^1 \left[(\varepsilon - \bar{\varepsilon})^T \psi_\varepsilon + {}^{CF}D_{0+}^\alpha (\varepsilon - \bar{\varepsilon})^T \psi_{{}^{CF}D_{0+}^\alpha \varepsilon} + (u - \bar{u})^T \psi_u \right] d\theta = 0. \quad (3.55)$$

Also, we have

$$\int_0^1 \psi(\theta, \varepsilon, {}^{CF}D_{0+}^\alpha \varepsilon, u) d\theta - \int_0^1 \psi(\theta, \bar{\varepsilon}, {}^{CF}D_{0+}^\alpha \bar{\varepsilon}, \bar{u}) d\theta = 0. \quad (3.56)$$

Hence, $0 \geq 0$. Thus, the controlled convexity condition is satisfied. Therefore, the proposed numerical example verifies the feasibility of the controlled semi-infinite variational problem together with the equivalence between the primal and dual values established by the KKT optimality conditions and duality theory developed in this paper.

4. Conclusions

In this study, we investigated a class of controlled semi-infinite variational problems governed by the Caputo–Fabrizio fractional derivative. Using the corresponding integration-by-parts formula and under the stated assumptions, we derived KKT-type necessary optimality conditions that incorporate stationarity with respect to both the state and control variables. We also formulated a Lagrange-type dual model and established the associated weak and strong duality results under the assumptions specified in the corresponding theorems. In addition, saddle-point criteria associated with the Lagrange functional were obtained. These results extended the considered semi-infinite fractional variational framework by explicitly incorporating control variables. Furthermore, the theoretical results have been supported by an illustrative numerical application, which demonstrated the feasibility of the proposed framework and provided a basis for further investigations of controlled semi-infinite fractional variational problems. Although the present model is theoretical and is not formulated for a specific biological system, the proposed fractional variational control framework may have potential relevance to biological processes involving memory or hereditary effects. In such an interpretation, the state variable could represent a biological quantity whose evolution depends on its past behavior, while the control variable could represent an external intervention. The semi-infinite constraints may represent requirements that must be satisfied over a continuum of parameter values or conditions. However, a specific biological application would require an appropriate biological model, parameter identification, and experimental validation, which are beyond the scope of the present study. For other connected studies on interval-valued or fuzzy optimization models, the reader can consult [24–28].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there are no conflicts of interest.

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