



Research article

Normalized solutions to a Choquard equation with potential and mixed local-nonlocal nonlinearities

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Abstract: This work investigated normalized solutions for the Choquard equation with external potential and mixed local-nonlocal nonlinearities

$$\begin{cases} -\Delta u + (V(x) + \lambda)u = \gamma(I_\alpha * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{\alpha}{N}-1}u + \mu|u|^{p-2}u, & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = \rho^2, \end{cases}$$

where $N \geq 3$, $\gamma, \mu > 0$, $\lambda \in \mathbb{R}$, $p \in (2 + \frac{4}{N}, 2^*)$, $2^* = \frac{2N}{N-2}$, and I_α stood for the Riesz potential when $\alpha \in (0, N)$. Under finite potential conditions, normalized ground states were verified. For trapping potentials, normalized ground states paired with bound states were obtained.

Keywords: Choquard equation; lower critical exponent; normalized solutions; external potential; variational methods

1. Introduction

This paper addresses a class of Choquard-type equations involving an external potential

$$\begin{cases} -\Delta u + (V(x) + \lambda)u = \gamma(I_\alpha * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{\alpha}{N}-1}u + \mu|u|^{p-2}u, & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = \rho^2, \end{cases} \quad (1.1)$$

where the symbol $*$ acts as convolution, $N \geq 3$, $\gamma, \mu, \rho > 0$, $\lambda \in \mathbb{R}$, $p \in (2 + \frac{4}{N}, 2^*)$ with $2^* = \frac{2N}{N-2}$, and V is an external potential. Rooted in the Hardy-Littlewood-Sobolev inequality, the power $\frac{N+\alpha}{N}$ is commonly regarded as the lower critical exponent. The map $I_\alpha : \mathbb{R}^N \setminus \{0\} \mapsto \mathbb{R}$ corresponds to the Riesz potential formulated as

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \quad A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{\frac{N}{2}} \Gamma(\frac{\alpha}{2})},$$

where $\alpha \in (0, N)$.

The standard Choquard-Pekar equation

$$-\Delta u + \lambda u = (I_\alpha * |u|^q)|u|^{q-2}u, \quad x \in \mathbb{R}^N, \quad (1.2)$$

which is encountered across numerous theoretical settings (see, e.g. [1–3]), is a prototype of (1.1). Lieb [1] stands as the first to discuss solutions of Eq (1.2) via variational arguments. Since then equations of this type have been widely investigated (see, e.g. [4, 5]). Many existing contributions explore the fixed-mass problem concerning (1.2), namely, the investigation of normalized solutions with a prescribed L^2 -norm. Problems of this type arise naturally in physics, where the prescribed L^2 -norm often carries concrete physical meaning. A representative illustration is the atom population inside Bose-Einstein condensates. However, their mathematical treatment is challenging due to the presence of the normalization constraint.

A few studies examine whether normalized solutions exist for the base model (1.2). In the mass-subcritical regime $q \in ((N + \alpha)/N, 2_\alpha)$, with $2_\alpha = (N + \alpha + 2)/N$ denoting the mass critical exponent, we refer to [1, 6, 7]. When considering the mass-supercritical scenario $q \in (2_\alpha, 2_\alpha^*)$, where $2_\alpha^* = \frac{N+\alpha}{N-2}$, see [8, 9]. Studies concerning (1.2) with a more general nonlinearity are available in [10, 11]. After a local perturbation is added, the Choquard equation (1.2) becomes

$$-\Delta u + \lambda u = \gamma(I_\alpha * |u|^q)|u|^{q-2}u + \mu|u|^{p-2}u, \quad x \in \mathbb{R}^N. \quad (1.3)$$

In this setting, various regimes have been investigated. In particular, when $q = (N + \alpha)/N$, Eq (1.3) has been explored in [12–14], with $p \in (2, 2^*]$ in [14], $p \in (2, 2 + \frac{4}{N}]$ in [12], and $p \in (2 + \frac{4}{N}, 2^*)$ in [13]. Related works for Choquard equations with composite nonlinearities appear in [15, 16] and cited references. For the Choquard-type equation with a potential, namely,

$$-\Delta u + (V(x) + \lambda)u = \gamma(I_\alpha * |u|^q)|u|^{q-2}u, \quad x \in \mathbb{R}^N, \quad (1.4)$$

the trapping potential case has been studied by Cao et al. [17], Li and Ye [7], and Luo [8], with the mass-subcritical regime treated in [7, 17] and the mass-supercritical regime addressed in [8]. The non-trapping potential case was investigated for Eq (1.4) in [6, 18], and for its perturbed versions in [19, 20] along with the sources listed there.

Motivated by [12–14, 20], we intend to study normalized solutions of (1.1) with potential $V(x) \not\equiv 0$ for two regimes: the trapping case $V_\infty = \infty$ and the non-trapping case $V_\infty < \infty$, where

$$V_\infty = \lim_{|x| \rightarrow \infty} V(x).$$

1.1. The case $V_\infty < \infty$

We may set $V_\infty = 0$ without loss of generality by substituting $(V(x) - V_\infty, \lambda + V_\infty)$ for $(V(x), \lambda)$. For simplicity, we write H for $H^1(\mathbb{R}^N)$ and $\int \cdot$ for $\int_{\mathbb{R}^N} \cdot dx$. The following assumptions are required in this section.

(V1) $\lim_{|x| \rightarrow +\infty} V(x) = \sup_{x \in \mathbb{R}^N} V(x) = 0$. There exists a constant $\sigma_1 \in [0, \frac{(p-2)N-4}{(p-2)N})$ such that

$$\left| \int V(x)u^2 \right| \leq \sigma_1 \|\nabla u\|_2^2, \quad \forall u \in H.$$

(V2) $V(x) \in W_{loc}^{1,\infty}(\mathbb{R}^N)$. Let $W(x) := \frac{1}{2}\langle \nabla V(x), x \rangle$, $\lim_{|x| \rightarrow \infty} W(x) = 0$, and there exists $\sigma_2 \in [0, \min\{\frac{2N-(N-2)p}{2p}, \frac{(1-\sigma_1)(p-2)N-4}{4}\})$ such that

$$\left| \int W(x)u^2 \right| \leq \sigma_2 \|\nabla u\|_2^2, \quad \forall u \in H.$$

(V3) $W(x)$ possesses a weak derivative. Let $Y(x) = (\frac{Np}{2} - N)W(x) + \langle \nabla W(x), x \rangle$ and $Y_+(x) = \max\{Y(x), 0\}$. There exists $\sigma_3 \in [0, \frac{(p-2)N-4}{2})$ such that

$$\int Y_+(x)u^2 \leq \sigma_3 \|\nabla u\|_2^2, \quad \forall u \in H.$$

Remark 1.1. Numerous functions can be chosen to satisfy (V1)–(V3), for instance,

- 1) $V(x) = -\frac{\kappa}{1+|x|^2}$ with $\kappa > 0$ suitably small.
- 2) $V(x) = -\frac{\kappa}{|x|^2 + \ln(2+|x|)}$ with $\kappa > 0$ suitably small.

Under the previous assumptions, the map $I : H \mapsto \mathbb{R}$ given by

$$I(u) = \frac{1}{2} \int (|\nabla u|^2 + V(x)u^2) - \frac{N\gamma}{2(N+\alpha)} \int (I_\alpha * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}} - \frac{1}{p}\mu \int |u|^p$$

is an energy functional well defined on H . To derive a normalized solution of Eq (1.1), we need to seek a critical point of I , constrained by

$$S(\rho) := \{u \in H : \int |u|^2 = \rho^2\}.$$

We summarize our main findings as follows.

Theorem 1.2. Under hypotheses (V1)–(V3), if $N \geq 3$, $\alpha \in (0, N)$, $2 + \frac{4}{N} < p < 2^*$, then there exists a positive constant ρ^* such that for any $\rho \in (0, \rho^*)$, problem (1.1) has a normalized ground state (u_ρ, λ_ρ) with $\lambda_\rho > 0$.

Due to the external potential V , we are unable to restrict our analysis to the radial subspace $H_r^1(\mathbb{R}^N)$ of H . For this reason, the strategies developed in [13, 14] for the case $V = 0$ are not applicable. To compensate for the loss of compactness originating from V as well as the lower critical exponent, we rely on several properties of the limiting functional (see Remark 2.2), combined with a subtle comparison between the least energy value for problem (1.1) and that associated with the limiting functional. Besides, since p is mass-supercritical, the energy functional lacks lower boundedness. To address this issue, we adopt a minimization approach constrained by the Pohozaev-type condition.

1.2. The case $V_\infty = +\infty$

In this case, a local minimizer of I on the constraint $S(\rho)$ alongside a normalized solution of mountain-pass type is constructed. To this end, the following assumptions are set forth.

($\tilde{V}1$) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $\lim_{|x| \rightarrow +\infty} V(x) = +\infty$.

($\tilde{V}2$) $V(x) \in W_{loc}^{1,\infty}(\mathbb{R}^N)$ and let $W(x) := \frac{1}{2}\langle \nabla V(x), x \rangle$. There exists $\beta \in (0, \frac{(p-2)N-4}{2(p-2)N}]$ such that

$$\left(\frac{1}{2} - \beta\right)V(x) + \frac{2}{(p-2)N}W(x) \geq 0.$$

($\tilde{V}3$) There exists $\beta_1 \geq 0$, $\beta_2 \geq 0$, $b \geq 0$, $R_0 > 0$ such that $V(tx) \leq t^{\beta_1}V(x) + bt^{\beta_2}$ for $t|x| \geq R_0$.

Remark 1.3. Similar conditions can be found in [21]. By $(\tilde{V}1)$, we know that $V_m := \inf_{x \in \mathbb{R}^N} V(x) > -\infty$. After replacing $(V(x), \lambda)$ by $(V(x) - V_m + 1, \lambda + V_m - 1)$, it is reasonable to suppose $\inf_{x \in \mathbb{R}^N} V(x) > 0$. There are several examples satisfying $(\tilde{V}1)$ – $(\tilde{V}3)$, such as the following functions.

- 1) $V(x) = |x|^s$ for $s > 1$.
- 2) $V(x) = |x|^s + \frac{\kappa}{1+|x|^2}$ for $s > 1$ and κ suitably small.

Let E be the Hilbert space defined by

$$E := \{u \in H : \int |\nabla u|^2 + V(x)u^2 < +\infty\},$$

equipped with the norm

$$\|u\|_E := \left(\int |\nabla u|^2 + V(x)u^2 \right)^{\frac{1}{2}}.$$

For $r > 0$, we set $B_r := \{u \in E : \|u\|_E \leq r\}$. Then, we will prove the following result.

Theorem 1.4. Under the assumptions $(\tilde{V}1)$, $(\tilde{V}2)$, $(\tilde{V}3)$, if $N \geq 3$, $\alpha \in (0, N)$, $2 + \frac{4}{N} < p < 2^*$, then there exists a positive constant $\hat{\rho}_1$ such that for any $\rho \in (0, \hat{\rho}_1)$, Eq (1.1) admits both a normalized ground state (u_ρ, λ_ρ) and a normalized bound state $(\bar{u}_\rho, \bar{\lambda}_\rho)$ satisfying $I(\bar{u}_\rho) > I(u_\rho)$.

We examine the minimization problem on the constraint $S(\rho) \cap B_r$ and obtain a local minimizer. With this critical point at hand, we construct the mountain-pass geometry of I over the manifold $S(\rho)$ and further derive a bound state solution. In contrast to the case $V = 0$, in addition to obtaining a normalized ground state solution, we further establish the existence of a normalized higher-energy solution.

2. The case where V_∞ is finite

Throughout this section, all the assumptions in Theorem 1.2 are assumed to hold. To begin with, we establish a lemma on the Pohozaev identity and give some definitions.

2.1. Preliminaries

Lemma 2.1. If $u \in H$ solves Eq (1.1), then u satisfies

$$\begin{aligned} \frac{N-2}{2} \int |\nabla u|^2 + \frac{N}{2} \int (V(x) + \lambda)u^2 + \frac{1}{2} \int \langle \nabla V(x), x \rangle u^2 \\ = \frac{N\gamma}{2} \int (I_\alpha * |u|^{\frac{N+\alpha}{N}}) |u|^{\frac{N+\alpha}{N}} + \frac{N\mu}{p} \int |u|^p. \end{aligned} \quad (2.1)$$

For $(\tau, u) \in \mathbb{R}^+ \times H^1(\mathbb{R}^N)$, define $u_\tau(x) = \tau^{\frac{N}{2}} u(\tau x)$,

$$J_u(\tau) := I(u_\tau) = \frac{\tau^2}{2} \int |\nabla u|^2 + \frac{1}{2} \int V\left(\frac{x}{\tau}\right) u^2 - \frac{N\gamma}{2(N+\alpha)} \int (I_\alpha * |u|^{\frac{N+\alpha}{N}}) |u|^{\frac{N+\alpha}{N}} - \frac{\mu}{p} \tau^{\frac{Np}{2}-N} \int |u|^p, \quad (2.2)$$

and

$$P(u) := \int |\nabla u|^2 - \frac{1}{2} \int \langle \nabla V(x), x \rangle u^2 - \frac{(p-2)N}{2p} \mu \int |u|^p.$$

Then,

$$J'_u(\tau) = \tau \int |\nabla u|^2 - \frac{1}{2\tau} \int \langle \nabla V(\frac{x}{\tau}), \frac{x}{\tau} \rangle u^2 - \frac{(p-2)N}{2p} \mu \tau^{\frac{Np}{2}-N-1} \int |u|^p = \frac{1}{\tau} P(u_\tau). \quad (2.3)$$

By Lemma 2.1 and direct computation, any solution u to (1.1) necessarily fulfills $P(u) = 0$. We now define

$$\mathcal{P} := \{u \in H : P(u) = 0\}, \mathcal{P}_\rho = S(\rho) \cap \mathcal{P},$$

and

$$\mathcal{P}_\rho^+ = \{u \in \mathcal{P}_\rho : J''_u(1) > 0\}, \mathcal{P}_\rho^0 = \{u \in \mathcal{P}_\rho : J''_u(1) = 0\}, \mathcal{P}_\rho^- = \{u \in \mathcal{P}_\rho : J''_u(1) < 0\}.$$

Given the failure of compactness, we resort to the conclusion associated with $V(x) = 0$. Set

$$\begin{aligned} I_\infty(u) &:= \frac{1}{2} \int |\nabla u|^2 - \frac{N}{2(N+\alpha)} \gamma \int (I_\alpha * |u|^{\frac{N+\alpha}{N}}) |u|^{\frac{N+\alpha}{N}} - \frac{1}{p} \mu \int |u|^p, \\ J_{\infty,u}(\tau) &:= I_\infty(u_\tau) = \frac{\tau^2}{2} \int |\nabla u|^2 - \frac{N\gamma}{2(N+\alpha)} \int (I_\alpha * |u|^{\frac{N+\alpha}{N}}) |u|^{\frac{N+\alpha}{N}} - \frac{\mu}{p} \tau^{\frac{Np}{2}-N} \int |u|^p, \\ P_\infty(u) &:= \int |\nabla u|^2 - \frac{(p-2)N}{2p} \mu \int |u|^p, \end{aligned}$$

and

$$\mathcal{P}_\infty := \{u \in H : P_\infty(u) = 0\}, \mathcal{P}_{\infty,\rho} = S(\rho) \cap \mathcal{P}_\infty.$$

Similarly, we have $J'_{\infty,u}(\tau) = \frac{1}{\tau} P_\infty(u_\tau)$. Denote

$$S_\alpha := \inf\{ \int |u|^2 \mid u \in H \setminus \{0\} \text{ and } \int (I_\alpha * |u|^{\frac{N+\alpha}{N}}) |u|^{\frac{N+\alpha}{N}} = 1 \}.$$

Remark 2.2. Let $l_\rho := \inf_{u \in \mathcal{P}_{\infty,\rho}} I_\infty(u)$. According to [13], there is a constant $\rho_0^* > 0$ for which l_ρ is attained for any $\rho < \rho_0^*$, and from the proof of [14, Theorem 1.10], l_ρ is a continuous and decreasing function of ρ .

2.2. Proof of Theorem 1.2

Lemma 2.3. If u is a critical point of $I|_{\mathcal{P}_\rho}$ and $J''_u(1) \neq 0$, then $I'(u) + \lambda u = 0$ holds for some $\lambda \in \mathbb{R}$.

Proof. Direct calculation gives

$$P'(u)u = 2 \int |\nabla u|^2 - 2 \int W(x)u^2 - \frac{(p-2)N\mu}{2} \int |u|^p. \quad (2.4)$$

Substituting $P(u) = 0$ into (2.4) yields

$$P'(u)u = -\frac{(p-2)^2}{2p} N\mu \int |u|^p < 0.$$

Therefore, $\mathcal{P}_\rho$ is a C^1 manifold.

Since u is a critical point for $I|_{\mathcal{P}_\rho}$, we obtain real scalars $\lambda, \nu \in \mathbb{R}$ satisfying

$$I'(u) + \lambda u + \nu P'(u) = 0. \quad (2.5)$$

It remains to prove $\nu = 0$. Given that u satisfies (2.5), the associated Pohozaev identity

$$\frac{d}{d\tau} E(u_\tau)|_{\tau=1} = 0$$

holds, where $E(u) := I(u) + \frac{1}{2}\lambda\|u\|_2^2 + \nu P(u)$. Observe that

$$E(u_\tau) = I(u_\tau) + \frac{1}{2}\lambda\|u\|_2^2 + \nu P(u_\tau) = J_u(\tau) + \frac{1}{2}\lambda\|u\|_2^2 + \nu\tau J'_u(\tau),$$

and

$$\frac{d}{d\tau} E(u_\tau)|_{\tau=1} = ((1 + \nu) J'_u(\tau) + \nu\tau J''_u(\tau))|_{\tau=1} = (1 + \nu) J'_u(1) + \nu J''_u(1) = (1 + \nu) P(u) + \nu J''_u(1).$$

From $P(u) = 0$ and $J''_u(1) \neq 0$, we deduce that $\nu = 0$, and the result follows. $\square$

Lemma 2.4.

$$\inf\{\tau > 0 : \exists u \in S(\rho), \|\nabla u\|_2^2 = 1, u_\tau \in \mathcal{P}_\rho\} \geq \delta_\rho$$

with $\delta_\rho > 0$. Accordingly, $\inf_{u \in \mathcal{P}_\rho} \|\nabla u\|_2^2 \geq \delta_\rho^2$.

Proof. By (2.3) and (V2), we obtain

$$\begin{aligned} J'_u(\tau) &= \tau \int |\nabla u|^2 - \frac{1}{2\tau} \int \langle \nabla V(\frac{x}{\tau}), \frac{x}{\tau} \rangle u^2 - \frac{(p-2)N}{2p} \mu \tau^{\frac{Np}{2}-N-1} \int |u|^p \\ &= \tau \int |\nabla u|^2 - \frac{1}{\tau} \int W(\frac{x}{\tau}) u^2 - \frac{(p-2)N}{2p} \mu \tau^{\frac{Np}{2}-N-1} \int |u|^p \\ &= \tau \int |\nabla u|^2 - \frac{1}{\tau} \int W(x)(u_\tau)^2 - \frac{(p-2)N}{2p} \mu \tau^{\frac{Np}{2}-N-1} \int |u|^p \\ &\geq \tau \|\nabla u\|_2^2 - \frac{1}{\tau} \sigma_2 \|\nabla(u_\tau)\|_2^2 - \frac{(p-2)N}{2p} \mu \tau^{\frac{Np}{2}-N-1} \int |u|^p \\ &= \tau(1 - \sigma_2) \|\nabla u\|_2^2 - \frac{(p-2)N}{2p} \mu \tau^{\frac{Np}{2}-N-1} \int |u|^p. \end{aligned} \quad (2.6)$$

For any $u \in S(\rho)$ with $\|\nabla u\|_2^2 = 1$, if $u_\tau \in \mathcal{P}_\rho$, then $J'_u(\tau) = 0$. Exploiting (2.6), together with the standard Gagliardo–Nirenberg interpolation inequality, we deduce

$$1 - \sigma_2 \leq \frac{(p-2)N}{2p} \mu \tau^{\frac{Np}{2}-N-2} \int |u|^p \leq \frac{(p-2)N}{2p} \mu C_{N,p}^p \|u\|_2^{\frac{2p-(p-2)N}{2}} \|\nabla u\|_2^{\frac{(p-2)N}{2}} \tau^{\frac{Np}{2}-N-2} := C_1 \mu \tau^{\frac{Np}{2}-N-2},$$

where $C_1 := \frac{(p-2)N}{2p} C_{N,p}^p \rho^{\frac{2p-(p-2)N}{2}}$. This implies $\tau \geq \left(\frac{1 - \sigma_2}{C_1 \mu} \right)^{\frac{2}{N(p-2)-4}}$, and the conclusion follows. $\square$

Lemma 2.5. $\mathcal{P}_\rho^- = \mathcal{P}_\rho$, constitutes a natural constraint of $I|_{S(\rho)}$.

Proof. For any $u \in \mathcal{P}_\rho$, using $P(u) = 0$, (V3), and (2.3), we obtain

$$\begin{aligned} J''_u(1) &= \int |\nabla u|^2 + \int W(x) u^2 + \int \langle \nabla W(x), x \rangle u^2 - \frac{\mu}{p} \left(\frac{Np}{2} - N - 1 \right) \left(\frac{Np}{2} - N \right) \int |u|^p \\ &= \int |\nabla u|^2 + \int W(x) u^2 + \int \langle \nabla W(x), x \rangle u^2 - \left(\frac{Np}{2} - N - 1 \right) \left(\int |\nabla u|^2 - \int W(x) u^2 \right) \\ &= \left(N + 2 - \frac{Np}{2} \right) \int |\nabla u|^2 + \left(\frac{Np}{2} - N \right) \int W(x) u^2 + \int \langle \nabla W(x), x \rangle u^2 \\ &\leq - \left(\frac{Np}{2} - N - 2 - \sigma_3 \right) \|\nabla u\|_2^2 \\ &< 0. \end{aligned} \quad (2.7)$$

Thus, $\mathcal{P}_\rho^- = \mathcal{P}_\rho$ forms a closed subset of H , and Lemma 2.3 ensures it is a natural constraint of $I|_{S(\rho)}$. $\square$

Lemma 2.6. *Each $u \in S(\rho)$ admits a unique $\tau_u > 0$ with $u_{\tau_u} \in \mathcal{P}_\rho$ and $I(u_{\tau_u}) = \max_{\tau>0} I(u_\tau)$.*

Proof. Exploiting (2.2) and the Hardy-Littlewood-Sobolev inequality yields

$$J_u(\tau) \rightarrow -\frac{N\gamma}{2(N+\alpha)} \int (I_\alpha * |u|^{\frac{N+\alpha}{N}}) |u|^{\frac{N+\alpha}{N}} \geq -\frac{N\gamma}{2(N+\alpha)} S_\alpha^{-\frac{N+\alpha}{N}} \rho^{\frac{2(N+\alpha)}{N}},$$

as $\tau \rightarrow 0$. From (2.6) and $p > 2 + \frac{4}{N}$, we know that $J'_u(\tau) > 0$ for $\tau > 0$ small enough. Consequently, one finds $\tau^* > 0$ with $J_u(\tau)$ increasing on $(0, \tau^*)$. Since $\lim_{\tau \rightarrow +\infty} J_u(\tau) = \lim_{\tau \rightarrow +\infty} I(u_\tau) = -\infty$, there exists $\tau_u \geq \tau^*$ satisfying

$$I(u_{\tau_u}) = \max_{\tau>0} I(u_\tau),$$

which implies $J'_u(\tau_u) = 0$ and $u_{\tau_u} \in \mathcal{P}_\rho$. It remains to establish the uniqueness of τ_u . Suppose otherwise that there is a $\tilde{\tau} > \tau_u$ with $u_{\tilde{\tau}} \in \mathcal{P}_\rho$. Lemma 2.5 implies that $J_u(\tau)$ attains strict local maxima at τ_u and $\tilde{\tau}$. This guarantees a $\bar{\tau} \in (\tau_u, \tilde{\tau})$ such that

$$J_u(\bar{\tau}) = \min_{\tau \in [\tau_u, \tilde{\tau}]} J_u(\tau).$$

At this point, we have $J'_u(\bar{\tau}) = 0$, $J''_u(\bar{\tau}) \geq 0$, so $u_{\bar{\tau}} \in \mathcal{P}_\rho$. However, the fact $J''_u(\bar{\tau}) = J''_{u_{\bar{\tau}}}(1) \geq 0$ contradicts Lemma 2.5. $\square$

Lemma 2.6 immediately yields the equality

$$\inf_{u \in \mathcal{P}_\rho} I(u) = \inf_{u \in S(\rho)} \max_{\tau>0} I(u_\tau).$$

We thus define

$$m_\rho := \inf_{u \in \mathcal{P}_\rho} I(u) = \inf_{u \in S(\rho)} \max_{\tau>0} I(u_\tau).$$

Lemma 2.7. *There exists $\rho_1^* > 0$ such that $I|_{\mathcal{P}_\rho}$ is positive and coercive for $\rho \in (0, \rho_1^*)$.*

Proof. For any $u \in \mathcal{P}_\rho$, by $P(u) = 0$ and (V2), we obtain

$$\frac{(p-2)N\mu}{2p} \int |u|^p = \left(\int |\nabla u|^2 - \int W(x)u^2 \right) \leq (1 + \sigma_2) \|\nabla u\|_2^2, \quad (2.8)$$

and

$$\frac{(p-2)N\mu}{2p} \int |u|^p = \left(\int |\nabla u|^2 - \int W(x)u^2 \right) \geq (1 - \sigma_2) \|\nabla u\|_2^2. \quad (2.9)$$

Then, by the Gagliardo-Nirenberg inequality, there is a constant $C_{N,p} > 0$ verifying

$$(1 - \sigma_2) \|\nabla u\|_2^2 \leq \frac{(p-2)N\mu}{2p} C_{N,p}^p \|u\|_2^{\frac{2p-(p-2)N}{2}} \|\nabla u\|_2^{\frac{(p-2)N}{2}}. \quad (2.10)$$

Thus,

$$\|\nabla u\|_2 \geq \left(\frac{2p(1 - \sigma_2)}{(p-2)N\mu C_{N,p}^p} \right)^{\frac{2}{(p-2)N-4}} \rho^{-\frac{2p-(p-2)N}{(p-2)N-4}}. \quad (2.11)$$

Taking (2.8) and (2.11) into account, we obtain

$$\begin{aligned}
 I(u) &= \frac{1}{2} \int (|\nabla u|^2 + V(x)u^2) - \frac{N}{2(N+\alpha)} \gamma \int (I_\alpha * |u|^{\frac{N+\alpha}{N}}) |u|^{\frac{N+\alpha}{N}} - \frac{1}{p} \mu \int |u|^p \\
 &\geq \frac{1}{2} (1 - \sigma_1) \|\nabla u\|_2^2 - \frac{N}{2(N+\alpha)} \gamma S_\alpha^{-\frac{N+\alpha}{N}} \|u\|_2^{\frac{2(N+\alpha)}{N}} - \frac{1}{p} \mu \int |u|^p \\
 &\geq \left(\frac{1}{2} (1 - \sigma_1) - \frac{2}{(p-2)N} (1 + \sigma_2) \right) \left(\frac{2p(1 - \sigma_2)}{(p-2)N \mu C_{N,p}^p} \right)^{\frac{4}{(p-2)N-4}} \rho^{-\frac{4p-2(p-2)N}{(p-2)N-4}} \\
 &\quad - \frac{N}{2(N+\alpha)} \gamma S_\alpha^{-\frac{N+\alpha}{N}} \rho^{\frac{2(N+\alpha)}{N}}.
 \end{aligned} \tag{2.12}$$

Hence, there exists $\rho_1^* > 0$ with $I|_{\mathcal{P}_\rho} > 0$ for $\rho \in (0, \rho_1^*)$. Moreover, from (2.12), we deduce

$$I(u) \geq \left(\frac{1}{2} (1 - \sigma_1) - \frac{2}{(p-2)N} (1 + \sigma_2) \right) \|\nabla u\|_2^2 - \frac{N}{2(N+\alpha)} \gamma \rho^{\frac{2(N+\alpha)}{N}} S_\alpha^{-\frac{N+\alpha}{N}}, \tag{2.13}$$

implying $I(u) \rightarrow +\infty$ as $\|\nabla u\|_2 \rightarrow +\infty$. $\square$

Remark 2.8. Let $\{u_n\} \subset \mathcal{P}_\rho$ be such that $I(u_n) \rightarrow m_\rho$ as $n \rightarrow +\infty$. By Ekeland's variational principle, $(I|_{\mathcal{P}_\rho})'(u_n) \rightarrow 0$. Since the constraint $J''_{u_n}(1) < 0$ is open, there exist two sequences $\{\lambda_n\}, \{\nu_n\} \subset \mathbb{R}$ with

$$I'(u_n) + \lambda_n u_n + \nu_n P'(u_n) \rightarrow 0.$$

By an argument analogous to Lemma 2.1, we have

$$\nu_n J''_{u_n}(1) \rightarrow 0.$$

Then from (2.7), we obtain $\nu_n \|\nabla u_n\|_2^2 \rightarrow 0$. However, according to Lemma 2.4,

$$\|\nabla u_n\|_2^2 \geq \delta_\rho^2.$$

Thus, $\nu_n \rightarrow 0$. Hence, we obtain

$$I'(u_n) + \lambda_n u_n \rightarrow 0.$$

Proof of Theorem 1.2. Fix $\rho^* = \min\{\rho_0^*, \rho_1^*\}$, if $\{u_n\} \subset \mathcal{P}_\rho$ is a minimizing sequence with

$$I(u_n) \rightarrow m_\rho.$$

From Remark 2.8, there exists $\{\lambda_n\} \subset \mathbb{R}$ satisfying $I'(u_n) + \lambda_n u_n \rightarrow 0$. In view of Lemma 2.7, we obtain the boundedness of $\{u_n\}$ in H . Then the proof is divided into four steps for clarity.

Step 1. We prove that $m_\rho < l_\rho$. Actually, by Remark 2.2, we can choose $w \in S(\rho)$ as a positive solution attaining l_ρ . Then, by $V(x) \not\equiv 0$ and $\sup_{x \in \mathbb{R}^N} V(x) = 0$, we have

$$\begin{aligned}
 m_\rho &\leq \max_{\tau > 0} I(w_\tau) = I(w_{\tau_w}) = I_\infty(w_{\tau_w}) + \frac{1}{2} \int V(x) |w_{\tau_w}|^2 \\
 &< I_\infty(w_{\tau_w}) \leq I_\infty(w) = l_\rho.
 \end{aligned}$$

Step 2. We assert that $u_n \rightarrow u$ weakly in H for some nontrivial u . The boundedness of $\{u_n\}$ justifies this weak convergence. Now assume for contradiction that $u = 0$. Then, $u_n \rightarrow 0$ in $L^2_{loc}(\mathbb{R}^N)$. From (V1) and (V2), there exists $R > 0$ sufficiently large and $C_2 > 0$ such that

$$\left| \int_{|x|>R} V(x)u_n^2 dx \right| = o(1),$$

and

$$\left| \int_{|x|\leq R} V(x)u_n^2 dx \right| \leq C_2 \int_{|x|\leq R} u_n^2 dx = o(1).$$

Thus, $\int V(x)u_n^2 = o(1)$. Similarly, we have $\int W(x)u_n^2 = o(1)$. Applying Brezis-Lieb Lemma, we obtain

$$m_\rho + o(1) = I(u_n) = I_\infty(u_n), \quad J'_{\infty, u_n}(1) = J'_{u_n}(1) + o(1) = o(1).$$

Arguing similarly to Lemmas 2.5 and 2.6, we can pick τ_n so that $(u_n)_{\tau_n} \in \mathcal{P}_{\infty, \rho}$. Moreover, since $J'_{\infty, u_n}(1) = o(1)$, we have $\tau_n = 1 + o(1)$. Then, it follows

$$l_\rho \leq I_\infty((u_n)_{\tau_n}) = I_\infty(u_n) + o(1) = m_\rho.$$

This contradicts the strict inequality $m_\rho < l_\rho$.

Step 3. We prove that $\lambda_n \rightarrow \lambda$ for some $\lambda > 0$. From the fact that $I'(u_n) + \lambda_n u_n \rightarrow 0$, we deduce

$$\int (|\nabla u_n|^2 + V(x)u_n^2 + \lambda_n u_n^2) - \int (I_\alpha * |u_n|^{\frac{N+\alpha}{N}})|u_n|^{\frac{N+\alpha}{N}} - \mu \int |u_n|^p = o(1). \quad (2.14)$$

It follows that

$$\lambda_n \rho^2 = \int (I_\alpha * |u_n|^{\frac{N+\alpha}{N}})|u_n|^{\frac{N+\alpha}{N}} + \mu \int |u_n|^p + o(1) - \int (|\nabla u_n|^2 + V(x)u_n^2).$$

Then, λ_n is bounded. By taking a subsequence, we may suppose that $\lambda_n \rightarrow \lambda \in \mathbb{R}$. By (V1), $P(u_n) = 0$, and (2.9), we have

$$\begin{aligned} \lambda_n \rho^2 &\geq \int (I_\alpha * |u_n|^{\frac{N+\alpha}{N}})|u_n|^{\frac{N+\alpha}{N}} + \mu \int |u_n|^p + o(1) - \int |\nabla u_n|^2 \\ &= \int (I_\alpha * |u_n|^{\frac{N+\alpha}{N}})|u_n|^{\frac{N+\alpha}{N}} + \mu \int |u_n|^p + o(1) - \int W(x)u_n^2 - \frac{(p-2)N\mu}{2p} \int |u_n|^p \\ &\geq \int (I_\alpha * |u_n|^{\frac{N+\alpha}{N}})|u_n|^{\frac{N+\alpha}{N}} + \frac{2p-(p-2)N}{2p} \mu \int |u_n|^p - \sigma_2 \|\nabla u_n\|_2^2 \\ &\geq \left(\frac{2p-(p-2)N}{(p-2)N} - \frac{2p}{(p-2)N} \sigma_2 \right) \|\nabla u_n\|_2^2 + o(1). \end{aligned}$$

Then by Lemma 2.4, $\lambda_n \rho^2 \geq \xi > 0$ for some $\xi > 0$, which implies $\lambda_n \rightarrow \lambda > 0$. Combining this with the result obtained in Step 2, we conclude that u solves (1.1) and $P(u) = 0$.

Step 4. $u_n \rightarrow u$ strongly in H . Set $\|u\|_2 = d$. We first aim to show that $d = \rho$. To this end, we proceed by contradiction. Assuming that $d < \rho$, we set $c := \sqrt{\rho^2 - d^2} > 0$ and $v_n := u_n - u$. Then, $\|v_n\|_2 = c + o(1) > 0$, and $v_n \rightarrow 0$ in H . As in Step 2, we obtain

$$\int V(x)v_n^2 = o(1), \quad \int W(x)v_n^2 = o(1).$$

So we have

$$m_\rho = I(u_n) + o(1) = I(u) + I_\infty(v_n) + o(1), \quad (2.15)$$

$$o(1) = P(u_n) = P(u) + P_\infty(v_n) = P_\infty(v_n) = J'_{\infty, v_n}(1). \quad (2.16)$$

Then we can pick $\tau_n = 1 + o(1)$ so that $(v_n)_{\tau_n} \in \mathcal{P}_{\infty, \|v_n\|_2^2}$. Consequently,

$$l_{\|v_n\|_2^2} \leq I_\infty((v_n)_{\tau_n}) = I_\infty(v_n) + o(1) = m_\rho - I(u) + o(1).$$

Taking into account that l_ρ is decreasing in ρ , we obtain

$$l_\rho < l_c = \lim_{n \rightarrow \infty} l_{\|v_n\|_2^2} \leq \lim_{n \rightarrow \infty} I_\infty((v_n)_{\tau_n}) = m_\rho - I(u) < m_\rho,$$

which contradicts $m_\rho < l_\rho$. Thus, $u_n \rightarrow u$ in $L^2(\mathbb{R}^N)$, $u \in \mathcal{P}_\rho$, and $I(u) \geq m_\rho$.

It is sufficient to show that $\|\nabla v_n\|_2 \rightarrow 0$. From the fact $\|v_n\|_2 = o(1)$, the Gagliardo-Nirenberg and Hardy-Littlewood-Sobolev inequality, we obtain

$$\int |v_n|^p = o(1), \quad \int (I_\alpha * |v_n|^{\frac{N+\alpha}{N}}) |v_n|^{\frac{N+\alpha}{N}} = o(1).$$

Combining this with (2.15) yields

$$0 \geq m_\rho - I(u) = I_\infty(v_n) + o(1) = \frac{1}{2} \|\nabla v_n\|_2^2 + o(1).$$

Therefore, $\|\nabla v_n\|_2 \rightarrow 0$, which implies that u_n converges strongly to u in H and $I(u) = m_\rho$. $\square$

3. The confining potential case

3.1. Existence of normalized ground states

We shall work under the full set of assumptions from Theorem 1.4 throughout this section. Introduce the following infimum

$$m'_\rho := \inf_{u \in B_r \cap S(\rho)} I(u).$$

First, we establish some auxiliary lemmas.

Lemma 3.1. (i) Given any $r > 0$, there is a $\bar{\rho}_r > 0$ satisfying $S(\rho) \cap B_r \neq \emptyset$ for all $\rho < \bar{\rho}_r$.
(ii) Provided that $S(\rho) \cap B_r \neq \emptyset$, then m'_ρ is achieved.

Proof. (i) By $(\tilde{V}1)$, $-\Delta + V$ is compact in $L^2(\mathbb{R}^N)$. We denote by η_1 its first eigenvalue and by ψ_1 the corresponding eigenfunction, normalized such that $\|\psi_1\|_E = 1$ and $\|\psi_1\|_2 = \eta_1^{-\frac{1}{2}}$. For any $r > 0$, set $\bar{\rho}_r := |r\psi_1|_2$. Then, it is clear that for every $\rho < \bar{\rho}_r$, $\psi_\rho := \frac{\rho}{\bar{\rho}_r} \psi_1 \in S(\rho) \cap B_r$. This completes the proof.
(ii) The result follows directly from the compact embedding of E into $L^p(\mathbb{R}^N)$, $2 \leq p < 2^*$. $\square$

Lemma 3.2. Given any $r > 0$, there is a constant $\tilde{\rho}_r > 0$ such that if $\rho < \tilde{\rho}_r$ and $S(\rho) \cap (B_r \setminus B_{\frac{r}{2}}) \neq \emptyset$, then

$$\inf_{u \in S(\rho) \cap B_{\frac{r}{4}}} I(u) < \inf_{u \in S(\rho) \cap (B_r \setminus B_{\frac{r}{2}})} I(u).$$

Proof. Given any $r > 0$ and $u \in S(\rho) \cap (B_r \setminus B_{\frac{r}{2}})$, it holds

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|_E^2 - \frac{\mu}{p} \int |u|^p - \frac{N\gamma}{2(N+\alpha)} \int (I_\alpha * |u|^{\frac{N+\alpha}{N}}) |u|^{\frac{N+\alpha}{N}} \\ &\geq \frac{1}{2} \|u\|_E^2 - \frac{\mu}{p} C_{N,p}^p \|u\|_2^{\frac{2p-(p-2)N}{2}} \|\nabla u\|_2^{\frac{(p-2)N}{2}} - \frac{N\gamma}{2(N+\alpha)} S_\alpha^{-\frac{N+\alpha}{N}} \|u\|_2^{\frac{2(N+\alpha)}{N}} \\ &\geq \frac{r^2}{4} \left(\frac{1}{2} - \frac{\mu}{p} C_{N,p}^p \rho^{\frac{2p-(p-2)N}{2}} r^{\frac{(p-2)N}{2}-2} - \frac{N\gamma}{2(N+\alpha)\eta_1} S_\alpha^{-\frac{N+\alpha}{N}} \rho^{\frac{2\alpha}{N}} \right). \end{aligned} \quad (3.1)$$

Choose ρ_r sufficiently small such that for $\rho < \rho_r$,

$$\frac{N\gamma}{2(N+\alpha)\eta_1} S_\alpha^{-\frac{N+\alpha}{N}} \rho^{\frac{2\alpha}{N}} < \frac{1}{16},$$

and

$$\frac{\mu}{p} C_{N,p}^p \rho^{\frac{2p-(p-2)N}{2}} r^{\frac{(p-2)N}{2}-2} < \frac{1}{16}$$

hold. Then it follows that

$$I(u) \geq \frac{3r^2}{32}. \quad (3.2)$$

Let $\tilde{\rho}_r := \min\{\tilde{\rho}_{\frac{r}{4}}, \rho_r\}$, then according to Lemma 3.1, we have $S(\rho) \cap B_{\frac{r}{4}} \neq \emptyset$. Moreover, a direct computation shows that

$$I(u) \leq \frac{1}{2} \|u\|_E^2 \leq \frac{r^2}{32}, \quad \forall u \in S(\rho) \cap B_{\frac{r}{4}}. \quad (3.3)$$

Thus, the desired result follows from (3.2) and (3.3). $\square$

Lemma 3.3. For some $\hat{\rho}_0 > 0$, Eq (1.1) admits a normalized solution u_ρ for $\rho < \hat{\rho}_0$, and u_ρ attains a local minimum of the functional I on $S(\rho) \cap B_1$. In addition, if $(\tilde{V}2)$ is satisfied, then there is some $\hat{\rho}_1 \in (0, \hat{\rho}_0]$ such that u_ρ is a normalized ground state of (1.1) for any $\rho < \hat{\rho}_1$.

Proof. Take $r = 1$ for simplicity and set $\hat{\rho}_0 := \tilde{\rho}_1$ with $\tilde{\rho}_1$ defined as in Lemma 3.2. For $\rho < \hat{\rho}_0$, Lemma 3.1 yields a minimizer $u_\rho \in S(\rho) \cap B_1$ such that $I(u_\rho) = m_\rho^1$. Lemma 3.2 further gives $u_\rho \in S(\rho) \cap B_{\frac{1}{2}}$. Thus, u_ρ is a critical point of $I|_{S(\rho)}$, i.e, for some $\lambda \in \mathbb{R}$, $I'(u_\rho) + \lambda u_\rho = 0$.

It remains to show there exists $\hat{\rho}_1 > 0$ such that u_ρ is a ground state for any $\rho < \hat{\rho}_1$. We begin by proving $m_\rho^1 \rightarrow 0$ as $\rho \rightarrow 0$. As in (3.1), for $u \in S(\rho) \cap B_1$, we have

$$I(u) \geq \|u\|_E^2 \left(\frac{1}{2} - \frac{\mu}{p} \rho^{\frac{2p-(p-2)N}{2}} \|\nabla u\|_2^{\frac{(p-2)N}{2}-2} - \frac{N\gamma}{2(N+\alpha)\eta_1} S_\alpha^{-\frac{N+\alpha}{N}} \rho^{\frac{2\alpha}{N}} \right) \geq \frac{3}{8} \|u\|_E^2, \quad \text{for } \rho < \hat{\rho}_0.$$

Combining the above result with Lemma 3.1(ii), we deduce $m_\rho^1 > 0$. To proceed, take $w \in S(\hat{\rho}_0) \cap B_1$ and set $w_\rho := \frac{\rho}{\hat{\rho}_0} w$. A direct computation shows that $w_\rho \in S(\rho) \cap B_1$ and

$$m_\rho^1 \leq I(w_\rho) \leq \frac{\rho^2}{\hat{\rho}_0^2} \int |\nabla w|^2 + V(x)w^2 dx,$$

which yields $m_\rho^1 \rightarrow 0$ as $\rho \rightarrow 0$.

Next, we use an argument by contradiction. Suppose there are $\rho_n \rightarrow 0$ and $w_n \in S(\rho_n)$ with

$$(I|_{S(\rho_n)})'(w_n) = 0, \quad I(w_n) < m_{\rho_n}^1. \quad (3.4)$$

Then we have $P(w_n) = 0$. Taking $(\tilde{V}2)$ into account, we obtain

$$\begin{aligned}
 m_{\rho_n}^1 &> I(w_n) - \frac{2}{(p-2)N} P(w_n) \\
 &= \frac{(p-2)N-4}{2(p-2)N} \int |\nabla w_n|^2 + \int \left(\frac{1}{2} V(x) + \frac{1}{(p-2)N} \langle \nabla V(x), x \rangle \right) w_n^2 \\
 &\quad - \frac{N\gamma}{2(N+\alpha)} \int (I_\alpha * |w_n|^{\frac{N+\alpha}{N}}) |w_n|^{\frac{N+\alpha}{N}} \\
 &\geq \beta \int (|\nabla w_n|^2 + V(x) w_n^2) - \frac{N\gamma}{2(N+\alpha)} S_\alpha^{-\frac{N+\alpha}{N}} \|w_n\|_2^{\frac{2(N+\alpha)}{N}} \\
 &= \beta \|w_n\|_E^2 - \frac{N\gamma}{2(N+\alpha)} S_\alpha^{-\frac{N+\alpha}{N}} \rho_n^{\frac{2(N+\alpha)}{N}}.
 \end{aligned} \tag{3.5}$$

Since $\rho_n \rightarrow 0$, we have $m_{\rho_n}^1 \rightarrow 0$ and $\frac{N\gamma}{2(N+\alpha)} S_\alpha^{-\frac{N+\alpha}{N}} \rho_n^{\frac{2(N+\alpha)}{N}} \rightarrow 0$. Then from (3.5), we deduce $\|w_n\|_E \leq 1$ for $\rho_n > 0$ sufficiently small, which implies $w_n \in B_1 \cap S(\rho_n)$, and so $I(w_n) \geq m_{\rho_n}^1$. This contradicts (3.4) and establishes the desired result. $\square$

3.2. Existence of normalized bound state solutions

For $r > 0$, $\rho < \bar{\rho}_r$ with $\bar{\rho}_r$ defined as in Lemma 3.1, set

$$A_\rho^r := \{u \in S(\rho) \cap B_r : I(u) = m_\rho^r\}.$$

As proved in Lemma 3.1, we have $A_\rho^r \neq \emptyset$. Let $\tau \star u := e^{\frac{N}{2}\tau} u(e^\tau x)$. Fix $v_\rho \in A_\rho^r$ and set $w_\rho := \tau_0 \star v_\rho$ for some $\tau_0 > 0$. Since

$$\|\tau \star u\|_E^2 = e^{2\tau} \int |\nabla u|^2 + \int V\left(\frac{x}{e^\tau}\right) u^2 \geq e^{2\tau} \int |\nabla u|^2 + \inf_{x \in \mathbb{R}^N} V(x) \int u^2 \rightarrow +\infty, \text{ as } \tau \rightarrow +\infty,$$

and

$$I(\tau \star u) \rightarrow -\infty \text{ as } \tau \rightarrow +\infty,$$

we can select $\tau_0 > 0$ sufficiently large such that $w_\rho \in S(\rho) \setminus B_1$ and $I(w_\rho) < 0$. Define

$$\Gamma_\rho := \{h \in C([0, 1], S(\rho)) : h(0) = v_\rho, h(1) = w_\rho\},$$

and

$$M_\rho := \inf_{h \in \Gamma_\rho} \max_{t \in [0, 1]} I(h(t)).$$

Lemma 3.4. $\Gamma_\rho \neq \emptyset$.

Proof. Define $h_1(t) := (t\tau_0) \star v_\rho$. It is sufficient to prove the continuity of $h_1(t)$ with respect to t . That is, for any $t_n \rightarrow t_0$, we need to show

$$\|h_1(t_n) - h_1(t_0)\|_E = \|\nabla((t_n\tau_0) \star v_\rho) - \nabla((t_0\tau_0) \star v_\rho)\|_2^2 + \int V(x)[(t_n\tau_0) \star v_\rho - (t_0\tau_0) \star v_\rho]^2 \rightarrow 0.$$

It follows from [22] that $h_1(t)$ is a continuous function in H . Hence, it remains to show

$$\int V(x)[(t_n\tau_0) \star v_\rho - (t_0\tau_0) \star v_\rho]^2 \rightarrow 0. \tag{3.6}$$

Actually,

$$\int V(x)[(t_n\tau_0) \star v_\rho]^2 = \int V\left(\frac{x}{e^{t_n\tau_0}}\right)v_\rho^2,$$

and by (V3), it holds

$$\begin{aligned} V\left(\frac{x}{e^{t_n\tau_0}}\right)v_\rho^2 &\leq \left(\max\{C_1, e^{-t_n\tau_0\beta_1}\}V(x) + be^{-t_n\tau_0\beta_2}\right)v_\rho^2 \\ &\leq \left(\max\{C_1, 2e^{-t_0\tau_0\beta_1}\}V(x) + 2be^{-t_0\tau_0\beta_2}\right)v_\rho^2. \end{aligned}$$

This together with Lebesgue's dominated convergence theorem gives

$$\int V(x)[(t_n\tau_0) \star v_\rho]^2 \rightarrow \int V(x)[(t_0\tau_0) \star v_\rho]^2. \quad (3.7)$$

Using (3.7) and Hölder's inequality, we deduce

$$\int V(x)(t_n\tau_0) \star v_\rho \cdot (t_0\tau_0) \star v_\rho \rightarrow \int V(x)[t_0\tau_0 \star v_\rho]^2. \quad (3.8)$$

Combining (3.7) and (3.8) yields (3.6). Hence, $h_1(t) \in \Gamma_\rho$, and the conclusion follows. $\square$

Lemma 3.5. *Given $\hat{\rho}_0 > 0$ as defined in Lemma 3.3, for $\rho < \hat{\rho}_0$, a sequence $\{u_n\} \subset S(\rho)$ exists such that*

$$I(u_n) \rightarrow M_\rho, (I|_{S(\rho)})'(u_n) \rightarrow 0, P(u_n) \rightarrow 0. \quad (3.9)$$

Proof. For any $h \in \Gamma_\rho$, we observe that $h(0) = v_0$. Then from Lemma 3.2, we know that $h(0) \in S(\rho) \cap B_{\frac{1}{2}}$. Taking account of the continuity of $\|h(t)\|_E$, we deduce that $h(t_1) \in S(\rho) \cap (B_1 \setminus B_{\frac{1}{2}})$ for some $t_1 \in [0, 1]$. Thus,

$$\max_{t \in [0,1]} I(h(t)) \geq \inf_{u \in S(\rho) \cap (B_1 \setminus B_{\frac{1}{2}})} I(u) > \inf_{u \in S(\rho) \cap B_{\frac{1}{4}}} I(u) \geq \inf_{u \in S(\rho) \cap B_1} I(u) = m_\rho^1.$$

Then,

$$M_\rho = \inf_{h \in \Gamma_\rho} \max_{t \in [0,1]} I(h(t)) > m_\rho^1. \quad (3.10)$$

As in [22], we consider the auxiliary function given by

$$\hat{I} : S(\rho) \times \mathbb{R} \rightarrow \mathbb{R}, (u, \tau) \mapsto I(\tau \star u).$$

Setting

$$\hat{\Gamma}_\rho := \{\hat{h} \in C([0, 1], S(\rho) \times \mathbb{R}) : \hat{h}(0) = (u_0, 0), \hat{h}(1) = (w_0, 0)\},$$

and

$$\hat{M}_\rho := \inf_{\hat{h} \in \hat{\Gamma}_\rho} \max_{\tau \in [0,1]} \hat{I}(\hat{h}(\tau)),$$

we have $\hat{M}_\rho = M_\rho$ as in [21]. By Ekeland's variational principle, there exists $\{(u_n, \tau_n)\} \in S(\rho) \times \mathbb{R}$ with

$$\tau_n \rightarrow 0, \hat{I}(u_n, \tau_n) \rightarrow M_\rho, (\hat{I}|_{S(\rho) \times \mathbb{R}})'(u_n, \tau_n) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Since $\hat{I}(u_n, \tau_n) = \hat{I}(\tau_n \star u_n, 0)$ and $(\hat{I}|_{S(\rho) \times \mathbb{R}})'(u_n, \tau_n)(\phi, s) = (I|_{S(\rho) \times \mathbb{R}})'(\tau_n \star u_n, 0)(\tau_n \star \phi, s)$, we may assume that $\tau_n = 0$, which implies

$$I(u_n) \rightarrow M_\rho, (I|_{S(\rho)})'(u_n) \rightarrow 0, P(u_n) \rightarrow 0.$$

$\square$

Lemma 3.6. *The sequence $\{u_n\}$ in (3.9) is bounded in E .*

Proof. By (3.9), ($\tilde{V}2$), and simple calculation, we deduce

$$M_\rho + o(1) = I(u_n) - \frac{2}{(p-2)N} P(u_n) \geq \beta \|u_n\|_E^2 - \frac{N\gamma}{2(N+\alpha)} S_\alpha^{-\frac{N+\alpha}{N}} \rho^{\frac{2(N+\alpha)}{N}}.$$

The conclusion follows. $\square$

Proof of Theorem 1.4. As indicated in Lemma 3.3, a normalized ground state exists whenever $\rho < \hat{\rho}_1$. Applying Lemmas 3.5 and 3.6, we derive a bounded sequence $\{u_n\} \subset S(\rho)$ satisfying

$$I(u_n) \rightarrow M_\rho, (I|_{S(\rho)})'(u_n) \rightarrow 0, P(u_n) \rightarrow 0.$$

We can take a subsequence and assume that

$$u_n \rightharpoonup \bar{u}_\rho \text{ in } E, \quad u_n \rightarrow \bar{u}_\rho \text{ in } L^p(\mathbb{R}^N), \quad 2 \leq p < 2^*, \quad u_n \rightarrow \bar{u}_\rho \text{ a.e. in } \mathbb{R}^N. \quad (3.11)$$

From the method of Lagrange multiplier, there is some λ_n verifying

$$I'(u_n) + \lambda_n u_n \rightarrow 0. \quad (3.12)$$

Then, it is easy to see that

$$\lambda_n = -\frac{1}{\rho^2} \langle I'(u_n), u_n \rangle + o(1),$$

is bounded. Thus, taking an appropriate subsequence, we obtain $\lambda_n \rightarrow \bar{\lambda}_\rho$ for some $\bar{\lambda}_\rho \in \mathbb{R}$. This combined with (3.11), (3.12) implies

$$I'(\bar{u}_\rho) + \bar{\lambda}_\rho \bar{u}_\rho = 0, \text{ in } E^{-1}. \quad (3.13)$$

Then, by the fact that

$$\langle I'(\bar{u}_\rho) + \bar{\lambda}_\rho \bar{u}_\rho, \bar{u}_\rho \rangle_E = 0, \quad \langle I'(u_n) + \lambda_n u_n, u_n \rangle_E = o(1),$$

and the compact embedding, we have $u_n \rightarrow \bar{u}_\rho$ in E , that is, there exists $\bar{u}_\rho \in S(\rho)$ with $I(\bar{u}_\rho) = M_\rho$. Moreover, by (3.10), we have $M_\rho > m_\rho^1$. $\square$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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