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*Research article*

## Geometry of Ricci-Yamabe solitons on four-dimensional Lorentzian manifolds

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**Abstract:** We investigate Ricci-Yamabe solitons (RYSs) and gradient Ricci-Yamabe solitons (GRYSs) on four-dimensional Lorentzian manifolds  $(M^4, g)$  admitting a unit timelike torse-forming vector field. We show that when  $V = \zeta$  and  $\rho \neq 0$ , the soliton equation is compatible with a perfect fluid spacetime structure. For the case of a constant scalar curvature, it is shown that if  $(M^4, g)$  admits an RYS or a GRYS, then  $(M^4, g)$  is an Einstein spacetime, the associated soliton vector field is Killing, and consequently, the RYS or GRYS is trivial. In the non-constant scalar curvature case, the scalar curvature is shown to satisfy a screened wave (Klein-Gordon-type) equation in the RYS setting, whereas the potential function of a GRYS satisfies a second-order differential equation governing its convexity and concavity along the integral curves of the unit timelike vector field. Finally, a specific constant-curvature example is presented in which the GRYS becomes trivial under the corresponding parameter relations.

**Keywords:** Ricci-Yamabe solitons; Einstein spacetimes; Lorentzian manifolds; screened wave equation; partial differential equations

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### 1. Introduction

A Lorentzian manifold is an  $n$ -dimensional semi-Riemannian manifold  $(M^n, g)$  whose metric tensor  $g$  has the signature  $(1, n - 1)$  [1]. A relativistic spacetime is a connected four-dimensional smooth manifold  $M^4$  endowed with a Lorentzian metric  $g$  of the signature  $(-, +, +, +)$ . In Einstein's theory of general relativity, spacetime provides the geometric framework for describing the universe, with the

Lorentzian metric encoding its gravitational and causal structure. This theory provides a geometric description of gravitation by relating the curvature of spacetime to the distribution of energy and matter. Owing to their fundamental importance in both mathematics and physics, the geometric properties and physical implications of spacetimes have been widely studied; for related works, the reader is referred to [2–4].

More specifically, the present work is concerned with a four-dimensional Lorentzian manifold  $(M^4, g)$  admitting a globally defined unit timelike vector field  $\zeta$ , satisfying  $g(\zeta, \zeta) = -1$ , and the torse-forming condition

$$\nabla_E \zeta = E + \omega(E)\zeta, \quad (1.1)$$

where  $g(E, \zeta) = \omega(E)$  for all vector fields  $E$  on  $M^4$ , and  $\nabla$  denotes the Levi-Civita connection of  $g$  [5]. Condition (1.1) is the special case  $\alpha = 1$  of the general torse-forming condition  $\nabla_E \zeta = \alpha E + \omega(E)\zeta$ , introduced by Yano [6]. Accordingly, throughout this paper, we refer to  $\zeta$  as a unit timelike torse-forming vector field.

Among the important geometric models arising in relativistic geometry are perfect fluid spacetimes (PFSs). Their geometry is governed by a distinguished timelike direction, representing the fluid velocity field, together with the orthogonal spatial distribution. From a geometric perspective, a PFS is characterized by a Ricci tensor that distinguishes a preferred timelike direction from its orthogonal spatial distribution. In a four-dimensional Lorentzian manifold  $(M^4, g)$ , the Ricci tensor is given by  $S = pg + q\omega \otimes \omega$ , where  $q$  and  $p$  are smooth scalar functions on  $(M^4, g)$ . When  $q = 0$ , the spacetime becomes an Einstein spacetime (ES). It is worth mentioning that Robertson-Walker spacetimes constitute an important class of PFSs [7]. The geometry of PFSs has attracted considerable attention, leading to numerous significant contributions in the literature; for example, see [8–12].

The investigation of geometric evolution equations has attracted considerable attention because of their significant applications across mathematics, physics, and engineering. Motivated by the pioneering introduction of the Ricci flow by [13], given by  $\frac{\partial g}{\partial t} = -2S$ , Hamilton later proposed the Yamabe flow [14] on a Riemannian manifold, defined by  $\frac{\partial g}{\partial t} = -rg$ , with the initial condition  $g(0) = g_0$ . Here,  $r$  denotes the scalar curvature,  $t$  is the flow parameter (time), and  $g$  represents the Riemannian metric. It is worth mentioning that these two flows agree in the two-dimensional case ( $\dim M = 2$ ); however, for manifolds of dimension greater than two, they exhibit fundamentally different behaviors.

A fundamental class of solutions to geometric flows is given by self-similar solutions whose evolution is governed by a vector field and a uniform metric scaling. Such solutions naturally give rise to geometric solitons. In particular, Ricci (Yamabe) solitons arise as self-similar solutions of the Ricci (Yamabe) flow. A Ricci soliton on  $(M^n, g)$  is defined by the existence of a vector field  $V$  satisfying  $\mathcal{L}_V g + 2S + 2\vartheta g = 0$ , where  $S$  denotes the Ricci tensor,  $\mathcal{L}_V$  is the Lie derivative with respect to  $V$ , and  $\vartheta$  is a constant [13]. The sign of  $\vartheta$  determines the character of the soliton: it is called steady, expanding, or shrinking according as  $\vartheta = 0$ ,  $\vartheta > 0$ , or  $\vartheta < 0$ , respectively. The soliton is said to be trivial whenever  $V = 0$  or  $V$  generates isometries of the manifold; in this case,  $(M^n, g)$  is an ES.

If the soliton vector field satisfies  $V = Df = \text{grad} f$ , where  $f \in C^\infty(M^4, g)$  denotes a smooth function on  $(M^4, g)$  and  $D$  is the gradient operator, then  $(g, V, \vartheta)$  is called a gradient Ricci soliton and is denoted by  $(g, V = Df, \vartheta)$ . Consequently, the Ricci soliton equation takes the following form:

$$\text{Hess } f + S + \vartheta g = 0,$$

where  $\text{Hess } f$  stands for the Hessian of  $f$ , and is defined by  $\text{Hess } f = \nabla^2 f = g(\cdot, \nabla Df)$ , where  $\nabla^2$  is the Hessian operator. In this framework,  $V$  and  $f$  are known as the potential vector field and the potential function, respectively. Perelman showed that every compact Ricci soliton possesses a gradient structure [15]. Similarly, a Yamabe soliton is a self-similar solution associated with the Yamabe flow and satisfies the following [16]:

$$\mathcal{L}_V g = 2(\vartheta - r)g.$$

The classification of Yamabe solitons into steady, expanding, and shrinking follows the same criterion as in the Ricci soliton case.

The Ricci and Yamabe flows emphasize two different aspects of curvature: the former is governed by the full Ricci tensor, whereas the latter only depends on the scalar curvature. Therefore, it is natural to consider a unified evolution equation containing both curvature contributions. This motivates the  $(\rho, \sigma)$ -type Ricci-Yamabe flow introduced by [17]. This flow is governed by the following:

$$\frac{\partial}{\partial t} g(t) + \sigma r(t)g(t) + 2\rho S(g(t)) = 0, \quad g(0) = g_0,$$

where  $\rho, \sigma$  are constants.

Motivated by this flow, a pseudo-Riemannian manifold  $(M^n, g)$  is said to admit a Ricci-Yamabe soliton (RYS) if there exist a vector field  $V$  such that

$$(\mathcal{L}_V g)(F, G) + (2\vartheta - \sigma r)g(F, G) + 2\rho S(F, G) = 0, \quad (1.2)$$

for any vector fields  $F$  and  $G$  on  $(M^n, g)$ . Here,  $\rho, \sigma$ , and  $\vartheta$  are real constants. Various geometric and analytical properties of RYSs have been investigated in recent years; interested readers may consult [18–21] and the references therein.

A Ricci-Yamabe soliton reduces to the following: (i) a Ricci soliton [13] for  $(\rho, \sigma) = (1, 0)$ ; (ii) a Yamabe soliton [14] for  $(\rho, \sigma) = (0, 1)$ ; (iii) an Einstein soliton [22] for  $(\rho, \sigma) = (1, -1)$ ; and (iv) a  $\varrho$ -Einstein soliton [23] for  $(\rho, \sigma) = (1, -2\rho)$ . Hence, the Ricci-Yamabe framework provides a common setting in which different curvature-driven soliton geometries can be simultaneously studied. In particular, if  $V = 0$  or Killing, then the RYS reduces to the Einstein equation  $S(F, G) = -\frac{1}{\rho}(\vartheta - \frac{\sigma r}{2})g(F, G)$ ,  $\rho \neq 0$ .

In case  $\vartheta$  is smooth, then (1.2) is named an almost Ricci-Yamabe soliton (ARYS) equation, and  $\vartheta$  is referred to as the soliton function. We symbolize  $(g, V, \vartheta, \rho, \sigma)$  for an ARYS and it will be steady (resp., expanding or shrinking) if  $\vartheta = 0$  (resp.,  $\vartheta > 0$  or  $\vartheta < 0$ ).

If we choose  $V = Df$ , then (1.2) represents the GRYS equation and assumes the following form:

$$\text{Hess } f(F, G) + \rho S(F, G) + \left(\vartheta - \frac{\sigma r}{2}\right)g(F, G) = 0, \quad (1.3)$$

where  $\rho$  and  $\sigma$  are real constants on  $(M^n, g)$ . Throughout this work, all flows and soliton equations are formulated with respect to the Lorentzian metric  $g$ .

Recent studies have further emphasized the geometric role of soliton structures in Lorentzian and relativistic spacetime settings. Thangjam et al. investigated Ricci solitons in relativistic thermodynamical fluid spacetimes and examined their implications for the associated spacetime geometry [24]. Suh and Chaubey studied Ricci solitons on general relativistic spacetimes equipped with a special unit timelike vector field and established several geometric characterizations of such spacetimes [5]. Related solitonic

structures on general relativistic spacetimes have also been investigated in the context of almost Ricci-Bourguignon and gradient almost Ricci-Bourguignon solitons [25]. More recently, generalized gradient flows of curvature have provided further developments in the geometric-flow theory and motivation for studying generalized curvature-driven soliton structures [26]. These works provide relevant geometric background for the present investigation of RYSs and GRYSs on four-dimensional Lorentzian manifolds  $(M^4, g)$  admitting a unit timelike torse-forming vector field  $\zeta$ .

The principal novelty of the present work lies in the geometric characterization of a four-dimensional Lorentzian manifold  $(M^4, g)$  admitting RYS and GRYS structures in the presence of a unit timelike torse-forming vector field  $\zeta$ . While the Ricci soliton and related geometric structures have been extensively investigated in various Lorentzian, perfect-fluid, and general relativistic settings, the specific interaction between Ricci-Yamabe-type soliton equations and the geometry induced by the assumed unit timelike torse-forming vector field has not been fully explored. In this work, we establish several new geometric consequences for  $(M^4, g)$ . For the constant scalar curvature case, we derive the relation  $\vartheta = -3\rho + 6\sigma$ , and show that the corresponding soliton vector field is Killing and that the spacetime is Einstein. When the scalar curvature is non-constant, we derive a screened wave-type (Klein-Gordon-type) equation governing the scalar curvature in the RYS setting, while in the GRYS case, we obtain a second-order differential equation for the potential function. These results establish meaningful connections among the soliton geometry, the structure of PFSs, and the associated differential equations. These results describe geometric relationships among soliton structures, the Lorentzian curvature, and the perfect-fluid spacetime geometry. Finally, a specific constant-curvature example is presented in which the GRYS becomes trivial under the corresponding parameter relations.

Throughout this paper, the analysis is purely geometric rather than phenomenological. Accordingly, no specific physical units are assigned to the metric  $g$ , the curvature tensors  $R$ , the Ricci tensor  $S$ , the scalar curvature  $r$ , the vector fields, or the soliton parameters  $\rho$ ,  $\sigma$ , and  $\vartheta$ . All equations and conclusions are intrinsically understood on the Lorentzian manifold  $(M^4, g)$ , with  $g(\zeta, \zeta) = -1$  and the torse-forming condition  $\nabla_E \zeta = E + \omega(E)\zeta$  regarded as part of the assumed geometric structure.

The geometric and physical characteristics of Perfect fluid spacetimes, Einstein spacetimes, GRW spacetimes, and relativistic spacetimes admitting various soliton structures have attracted substantial interest in recent years. Several studies have contributed to the understanding of geometric solitons and Lorentzian manifolds. Azami and Jafari [27] investigated Riemann solitons in relativistic spacetimes, while Dey et al. [28] studied conformal  $\eta$ -Ricci-Bourguignon solitons in general relativistic spacetimes. Li et al. [29], Haseeb et al. [30], and Prasad et al. [31] studied  $\eta$ -Ricci,  $*$ -Ricci-Yamabe and gradient  $*$ -Ricci-Yamabe, and  $\eta$ -Ricci-Yamabe and  $*$ - $\eta$ -Ricci-Yamabe solitons, respectively, on LP-Kenmotsu manifolds. Furthermore, Chaubey and Suh [32] studied the non-metric case, whereas Chaubey et al. [33] examined the metric case of semi-symmetric  $\rho$ -connections on Lorentzian manifolds.

## 2. Preliminaries

The purpose of this section is to collect the basic geometric identities associated with a unit timelike torse-forming vector field  $\zeta$ , which will be used repeatedly in the characterization of Ricci-Yamabe solitons (RYSs).

The torse-forming condition, together with  $g(\zeta, \zeta) = -1$ , yields the following curvature identities [5]:

$$R(E, F)\zeta = \omega(F)E - \omega(E)F \iff R(\zeta, E)F = g(E, F)\zeta - \omega(F)E, \quad (2.1)$$

where the Riemann curvature tensor is expressed as  $R(E, F)G = \nabla_E \nabla_F G - \nabla_F \nabla_E G - \nabla_{[E, F]}G$ . On contracting (2.1) over  $E$ , we have the following:

$$S(F, \zeta) = 3\omega(F) \iff Q\zeta = 3\zeta, \quad (2.2)$$

where  $Q$  indicates the Ricci operator of  $g$ , and  $S(E, F) = g(QE, F)$ . Thus,  $\zeta$  is an eigenvector of the Ricci operator with an eigenvalue of 3. This observation will provide the key link between the spacetime geometry and the RYS equation.

Next, we examine the ARYS structure when its potential vector field is chosen to coincide with a unit timelike torse-forming vector field  $\zeta$

Let  $(g, \zeta, \vartheta, \rho, \sigma)$  be an ARYS on the four-dimensional Lorentzian manifold  $(M^4, g)$ . Setting  $V = \zeta$  in (1.2), we obtain the following:

$$(\mathbb{L}_\zeta g)(F, G) + 2\rho S(F, G) + (2\vartheta - \sigma r)g(F, G) = 0. \quad (2.3)$$

Thus,  $(\mathbb{L}_\zeta g)(F, G) = 2(\omega(F)\omega(G) + g(F, G))$ . Therefore, (2.3) reduces to the following:

$$\rho S(F, G) + \omega(F)\omega(G) + \left(\vartheta + 1 - \frac{\sigma r}{2}\right)g(F, G) = 0. \quad (2.4)$$

Now, the coefficient  $\rho$  distinguishes the geometric character of (2.4) according to whether the Ricci tensor contributes to the equation. Thus, we separately consider the cases  $\rho = 0$  and  $\rho \neq 0$ , leading to the following theorems.

First, we consider the case  $\rho \neq 0$  and establish the following result.

**Theorem 2.1.** Assume that a four-dimensional Lorentzian manifold  $(M^4, g)$  endowed with a unit timelike torse-forming vector field  $\zeta$  satisfying (1.1) admits an ARYS  $(g, \zeta, \vartheta, \rho, \sigma)$  with  $\rho \neq 0$ . Then,  $(M^4, g)$  is a PFS with the soliton function  $\vartheta = \frac{\sigma r}{2} - 3\rho$ . Moreover, the soliton  $(g, \zeta, \vartheta, \rho, \sigma)$  is expanding (resp., shrinking or steady) if  $\frac{\sigma r}{6} > \rho$  (resp.,  $\frac{\sigma r}{6} < \rho$  or  $\frac{\sigma r}{6} = \rho$ ).

*Proof.* Let  $(M^4, g)$  be a four-dimensional Lorentzian manifold. If  $\rho \neq 0$ , then (2.4) gives  $S(F, G) = pg(F, G) + q\omega(F)\omega(G)$ , where  $p = -\frac{1}{\rho}\left(\vartheta + 1 - \frac{\sigma r}{2}\right)$  and  $q = -\frac{1}{\rho}$ . Thus, the Ricci tensor has the algebraic form of a perfect fluid spacetime. Hence,  $(M^4, g)$  is a PFS. Moreover, setting  $G = \zeta$  in Eq (2.4) and using (2.2) together with the relations  $\omega(\zeta) = -1$  and  $g(F, \zeta) = \omega(F)$ , we obtain the following:

$$\vartheta = \frac{\sigma r}{2} - 3\rho. \quad (2.5)$$

This relation determines the soliton function in terms of the scalar curvature and the soliton parameters. The classification of the soliton follows immediately from the sign of  $\vartheta$ . Therefore, the claim follows.  $\square$

The preceding result imposes a further restriction on the scalar curvature. Indeed, contracting (2.4) and using  $\text{tr}_g(\omega \otimes \omega) = g(\zeta, \zeta) = -1$  together with (2.5), we obtain the following:

$$r = 12 - \frac{3}{\rho}, \quad \rho \neq 0. \quad (2.6)$$

Hence, for  $\rho \neq 0$ , the scalar curvature is uniquely determined by  $\rho$ . Additionally, we shall use the following scalar-curvature identity for a four-dimensional Lorentzian manifold  $(M^4, g)$ .

**Remark 2.2.** [5] Let  $(M^4, g)$  be a four-dimensional Lorentzian manifold endowed with a unit timelike vector field  $\zeta$  such that  $g(\zeta, \zeta) = -1$ ,  $\omega(E) = g(E, \zeta)$ , and  $\nabla_E \zeta = E + \omega(E)\zeta$ ; then, we have the following:

$$\zeta(r) = -2(r - 12). \quad (2.7)$$

This identity, together with (2.6), yields the following non-existence criterion.

**Corollary 2.3.** If the scalar curvature of a four-dimensional Lorentzian manifold  $(M^4, g)$  endowed with a unit timelike torse-forming vector field  $\zeta$  satisfying (1.1) is  $\zeta$ -constant, then the spacetime does not admit an ARYS  $(g, V, \vartheta, \rho, \sigma)$  with  $\rho \neq 0$  and  $V = \zeta$ .

*Proof.* If the scalar curvature  $r$  of  $(M^4, g)$  is  $\zeta$ -constant, then (2.7) gives  $r = 12$ . By using this fact in (2.6), it follows that  $-\frac{3}{\rho} = 0$ , which contradicts  $\rho \neq 0$ . Thus, the assumed ARYS cannot exist. This completes the proof.  $\square$

The following theorem shows that the special case  $\rho = 0$  and  $\sigma = 1$  is incompatible with the almost Yamabe soliton equation.

**Theorem 2.4.** Let  $(M^4, g)$  be a four-dimensional Lorentzian manifold endowed with a unit timelike torse-forming vector field  $\zeta$  satisfying (1.1). If  $(M^4, g)$  admits an almost Yamabe soliton  $(g, V = \zeta, \vartheta, 0, 1)$ , then the corresponding soliton equation has no solution.

*Proof.* If  $\rho = 0$  and  $\sigma = 1$ , then (2.4) reduces to the following:

$$\left(\vartheta + 1 - \frac{r}{2}\right)g(F, G) + \omega(F)\omega(G) = 0. \quad (2.8)$$

Putting  $G = \zeta$  in (2.8) and using  $g(F, \zeta) = \omega(F)$ ,  $g(\zeta, \zeta) = \omega(\zeta) = -1$ , we obtain the following:

$$\vartheta = \frac{r}{2}, \text{ where } \omega(F) \neq 0.$$

On the other hand, for  $F, G \perp \zeta$ , we have  $\omega(F) = \omega(G) = 0$ ; therefore, (2.8) gives

$$\vartheta = \frac{r}{2} - 1, \text{ where } g(F, G) \neq 0.$$

This contradicts the preceding relation. Hence, the soliton equation cannot be satisfied when  $\rho = 0$  and  $\sigma = 1$ . This proves the desired result.  $\square$

We shall use the following identities established in [5].

**Lemma 2.5.** Let  $(M^4, g)$  be a four-dimensional Lorentzian manifold endowed with a unit timelike vector field  $\zeta$  satisfying (1.1); then, we have the following:

$$S = \left(\frac{r}{3} - 1\right)g + \left(\frac{r}{3} - 4\right)\omega \otimes \omega. \quad (2.9)$$

$$F(r) = 2(r - 12)\omega(F). \quad (2.10)$$

Equations (2.9) and (2.10) will be used to simplify the curvature expressions arising in the subsequent analysis.

### 3. RYSs on GRs

The purpose of this section is to examine how the geometric properties of four-dimensional Lorentzian manifolds  $(M^4, g)$  endowed with a unit timelike torse-forming vector field  $\zeta$  influence the behavior of an RYS. In particular, we investigate the restrictions imposed by the RYS equation on the scalar curvature and distinguish two fundamentally different cases according to whether the scalar curvature is constant or non-constant.

For the constant scalar curvature case, we first establish the following result.

**Theorem 3.1.** *Let  $(M^4, g)$  be a four-dimensional Lorentzian manifold endowed with a unit timelike torse-forming vector field  $\zeta$  satisfying (1.1) and all the hypotheses of Remark 2.2. Suppose that  $(M^4, g)$  has a constant scalar curvature and admits an RYS  $(g, V, \vartheta, \rho, \sigma)$ , where  $\vartheta, \rho$ , and  $\sigma$  are real constants. Then, the soliton vector field  $V$  is Killing, and hence the RYS is trivial. Moreover,  $(M^4, g)$  is an Einstein spacetime. The RYS is steady if  $2\sigma = \rho$ , expanding if  $2\sigma > \rho$ , and shrinking if  $2\sigma < \rho$ .*

*Proof.* Since (2.9) holds under the hypotheses of Remark 2.2, the RYS equation (1.2) yields

$$(\mathfrak{L}_V g)(F, G) = -2\left[\vartheta - \rho + \left(\frac{2\rho - 3\sigma}{6}\right)r\right]g(F, G) - 2\rho\left(\frac{r-12}{3}\right)\omega(F)\omega(G), \quad (3.1)$$

for any vector fields  $F$  and  $G$ .

The first term of (3.1) represents an isotropic deformation of the metric, whereas the second term describes the anisotropic contribution associated with the timelike direction  $\zeta$ . To determine its covariant behavior, we covariantly differentiate (3.1) in the direction of  $E$ . Consequently, using  $\nabla g = 0$ , we have the following:

$$\begin{aligned} (\nabla_E \mathfrak{L}_V g)(F, G) &= -\left(\frac{2\rho - 3\sigma}{3}\right)E(r)g(F, G) - \frac{2\rho}{3}E(r)\omega(F)\omega(G) \\ &\quad - 2\rho\left(\frac{r-12}{3}\right)[g(E, F)\omega(G) + g(E, G)\omega(F) + 2\omega(E)\omega(F)\omega(G)]. \end{aligned} \quad (3.2)$$

Since  $g$  is parallel w.r.t. to  $\nabla$ , then we have [35]

$$(\nabla_E \mathfrak{L}_V g)(F, G) = g((\mathfrak{L}_V \nabla)(E, F), G) + g((\mathfrak{L}_V \nabla)(E, G), F),$$

which due to symmetric property of  $\mathfrak{L}_V \nabla$  turns to

$$2g((\mathfrak{L}_V \nabla)(E, F), G) = (\nabla_E \mathfrak{L}_V g)(F, G) + (\nabla_F \mathfrak{L}_V g)(E, G) - (\nabla_G \mathfrak{L}_V g)(E, F). \quad (3.3)$$

Using (3.2) in (3.3), we have

$$\begin{aligned} g((\mathfrak{L}_V \nabla)(E, F), G) &= -\left(\frac{2\rho - 3\sigma}{6}\right)[E(r)g(F, G) + F(r)g(E, G) - G(r)g(E, F)] \\ &\quad - \frac{\rho}{3}[E(r)\omega(F)\omega(G) + F(r)\omega(E)\omega(G) - G(r)\omega(E)\omega(F)] \\ &\quad - 2\rho\left(\frac{r-12}{3}\right)[g(E, F)\omega(G) + \omega(E)\omega(F)\omega(G)], \end{aligned}$$

which infers

$$(\mathfrak{L}_V \nabla)(E, F) = -\left(\frac{2\rho - 3\sigma}{6}\right)[E(r)F + F(r)E - D(r)g(E, F)] \quad (3.4)$$

$$-\frac{\rho}{3}[E(r)\omega(F)\zeta + F(r)\omega(E)\zeta - D(r)\omega(E)\omega(F)] \\ -2\rho\left(\frac{r-12}{3}\right)[g(E, F)\zeta + \omega(E)\omega(F)\zeta].$$

Here,  $Dr$  is the gradient of the scalar curvature  $r$ . Since  $\zeta$  is the timelike direction of the perfect fluid spacetime, it is particularly useful to evaluate the preceding relation when one argument is  $\zeta$ . Interchanging  $E$  and  $F$  in (3.4), then setting  $E = \zeta$  and employing  $\omega(\zeta) = -1$  and  $g(F, \zeta) = \omega(F)$  along with (2.7), we obtain the following:

$$(\mathfrak{L}_V \nabla)(F, \zeta) = \frac{\sigma}{2}F(r)\zeta - \zeta(r)\left[\left(\frac{2\rho - 3\sigma}{6}\right)F + \frac{\rho}{3}\omega(F)\zeta\right] - \frac{\sigma}{2}D(r)\omega(F). \quad (3.5)$$

Using (2.7) and (2.10) in (3.5) leads to the following form:

$$(\mathfrak{L}_V \nabla)(F, \zeta) = \left(\frac{r-12}{3}\right)[(2\rho - 3\sigma)F + (2\rho + 3\sigma)\omega(F)\zeta] - \frac{\sigma}{2}D(r)\omega(F). \quad (3.6)$$

Taking the covariant derivative of (3.6) w.r.t.  $E$ , followed by the use of (1.1), gives the following:

$$(\nabla_E \mathfrak{L}_V \nabla)(F, \zeta) = \left(\frac{2\rho - 3\sigma}{6}\right)(3E(r)F + F(r)E) - \frac{\rho}{3}D(r)g(E, F) \\ + (\rho + \sigma)E(r)\omega(F)\zeta + \frac{\rho}{3}F(r)\omega(E)\zeta - \frac{\rho}{3}D(r)\omega(E)\omega(F) \\ + \left(\frac{r-12}{3}\right)(4\rho + 3\sigma)(g(E, F)\zeta + \omega(E)\omega(F)\zeta) \\ + \left(\frac{r-12}{3}\right)[(2\rho + 3\sigma)\omega(F)E - (2\rho - 3\sigma)\omega(E)F] - \frac{\sigma}{2}(\nabla_E Dr)\omega(F). \quad (3.7)$$

The variation of the curvature tensor is obtained from the covariant derivative of the variation of the connection. Again, from [35], we have the following:

$$(\mathfrak{L}_V R)(E, F)G = (\nabla_E \mathfrak{L}_V \nabla)(F, G) - (\nabla_F \mathfrak{L}_V \nabla)(E, G),$$

which setting  $\zeta = G$  and using (3.7) leads to

$$(\mathfrak{L}_V R)(E, F)\zeta = -\left(\frac{2\rho - 3\sigma}{3}\right)[F(r)E - E(r)F] + \left(\frac{2\rho + 3\sigma}{3}\right)[E(r)\omega(F) - F(r)\omega(E)]\zeta \\ + 4\rho\left(\frac{r-12}{3}\right)[\omega(F)E - \omega(E)F] - \frac{\sigma}{2}(\nabla_E Dr)\omega(F) + \frac{\sigma}{2}(\nabla_F Dr)\omega(E). \quad (3.8)$$

Taking the inner product of (3.8) with  $K$ , we have the following:

$$g((\mathfrak{L}_V R)(E, F)\zeta, K) = -\left(\frac{2\rho - 3\sigma}{3}\right)[F(r)g(E, K) - E(r)g(F, K)] \\ + \left(\frac{2\rho + 3\sigma}{3}\right)[E(r)\omega(F) - F(r)\omega(E)]g(\zeta, K) \\ + 4\rho\left(\frac{r-12}{3}\right)[\omega(F)g(E, K) - \omega(E)g(F, K)] \\ - \frac{\sigma}{2}g(\nabla_E Dr, K)\omega(F) + \frac{\sigma}{2}g(\nabla_F Dr, K)\omega(E). \quad (3.9)$$

Let  $\{l_1, l_2, l_3, l_4\}$  be a local pseudo-orthonormal frame field on  $(M^4, g)$  such that  $l_4 = \zeta$  and  $\varepsilon_i = g(l_i, l_i)$ , where  $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1, \varepsilon_4 = -1$ . If we put  $E = K = l_i$  in (3.9) and sum up with respect to  $i (1 \leq i \leq 4)$ , then we have the following:

$$\begin{aligned} \sum_{i=1}^4 \varepsilon_i g((\mathbb{L}_V R)(l_i, F)\zeta, l_i) &= -\left(\frac{2\rho - 3\sigma}{3}\right) \sum_{i=1}^4 \varepsilon_i [F(r)g(l_i, l_i) - l_i(r)g(F, l_i)] \\ &\quad + \left(\frac{2\rho + 3\sigma}{3}\right) \sum_{i=1}^4 \varepsilon_i [l_i(r)\omega(F) - F(r)\omega(l_i)]g(\zeta, l_i) \\ &\quad + 4\rho\left(\frac{r - 12}{3}\right) \sum_{i=1}^4 \varepsilon_i [\omega(F)g(l_i, l_i) - \omega(l_i)g(F, l_i)] \\ &\quad - \frac{\sigma}{2} \sum_{i=1}^4 \varepsilon_i g(\nabla_{l_i} Dr, l_i)\omega(F) + \frac{\sigma}{2} \sum_{i=1}^4 \varepsilon_i g(\nabla_F Dr, l_i)\omega(l_i). \end{aligned} \quad (3.10)$$

The individual contracted terms are given by the following:

$$\left\{ \begin{aligned} \sum_{i=1}^4 \varepsilon_i g((\mathbb{L}_V R)(l_i, F)\zeta, l_i) &= (\mathbb{L}_V S)(F, \zeta), \\ \sum_{i=1}^4 \varepsilon_i [F(r)g(l_i, l_i) - l_i(r)g(F, l_i)] &= F(r) \sum_{i=1}^4 \varepsilon_i g(l_i, l_i) - \sum_{i=1}^4 \varepsilon_i l_i(r)g(F, l_i) \\ &= 4F(r) - F(r) \\ &= 3F(r), \\ \sum_{i=1}^4 \varepsilon_i [l_i(r)\omega(F) - F(r)\omega(l_i)]g(\zeta, l_i) &= \omega(F) \sum_{i=1}^4 \varepsilon_i l_i(r)g(\zeta, l_i) - F(r) \sum_{i=1}^4 \varepsilon_i \omega(l_i)g(\zeta, l_i) \\ &= \zeta(r)\omega(F) - F(r)g(\zeta, \zeta) \\ &= \zeta(r)\omega(F) + F(r), \\ \sum_{i=1}^4 \varepsilon_i [\omega(F)g(l_i, l_i) - \omega(l_i)g(F, l_i)] &= \omega(F) \sum_{i=1}^4 \varepsilon_i g(l_i, l_i) - \sum_{i=1}^4 \varepsilon_i \omega(l_i)g(F, l_i) \\ &= 4\omega(F) - g(F, \zeta) \\ &= 4\omega(F) - \omega(F) = 3\omega(F), \\ &= 3\omega(F), \\ \sum_{i=1}^4 \varepsilon_i g(\nabla_{l_i} Dr, l_i)\omega(F) &= (\square r)\omega(F), \\ \sum_{i=1}^4 \varepsilon_i g(\nabla_F Dr, l_i)\omega(l_i) &= g(\nabla_F Dr, \sum_{i=1}^4 \varepsilon_i \omega(l_i)l_i) = g(\nabla_F Dr, \zeta) = \omega(\nabla_F Dr). \end{aligned} \right.$$

Using the above equations in (3.10), we obtain the following:

$$\begin{aligned} (\mathbb{L}_V S)(F, \zeta) &= -\left(\frac{4\rho - 12\sigma}{3}\right)F(r) + \left(\frac{2\rho + 3\sigma}{3}\right)\zeta(r)\omega(F) + 4\rho(r - 12)\omega(F) \\ &\quad - \frac{\sigma}{2}(\square r)\omega(F) + \frac{\sigma}{2}\omega(\nabla_F Dr), \end{aligned} \quad (3.11)$$

where  $\square$  denotes the Lorentzian d'Alembert operator (wave operator) and defined by  $\square r = \text{div}(\text{grad } r) = \sum_{i=1}^4 \varepsilon_i g(\nabla_{l_i} Dr, l_i)$ . Using (2.7) and (2.10), (3.11) simplifies to the following:

$$(\mathbb{L}_V S)(F, \zeta) = 6\sigma(r - 12)\omega(F) - \frac{\sigma}{2}(\square r)\omega(F) + \frac{\sigma}{2}\omega(\nabla_F Dr). \quad (3.12)$$

Applying the Lie derivative with respect to  $V$  to (2.2) and using the identity  $g(F, \zeta) = \omega(F)$ , we obtain the following:

$$\mathfrak{L}_V(S(F, \zeta)) = (\mathfrak{L}_V S)(F, \zeta) + S(\mathfrak{L}_V F, \zeta) + S(F, \mathfrak{L}_V \zeta), \quad (3.13)$$

for any vector field  $F$ .

On the other hand, since  $S(F, \zeta) = 3\omega(F)$ , we have the following:

$$\mathfrak{L}_V(S(F, \zeta)) = 3\mathfrak{L}_V(\omega(F)) = 3(\mathfrak{L}_V \omega)(F) + 3\omega(\mathfrak{L}_V F). \quad (3.14)$$

From (3.13) and (3.14), we obtain the following:

$$(\mathfrak{L}_V S)(F, \zeta) = 3(\mathfrak{L}_V \omega)(F) - S(F, \mathfrak{L}_V \zeta). \quad (3.15)$$

Similarly, taking the Lie derivative of  $g(F, \zeta) = \omega(F)$  along  $V$  and applying the Leibniz rule gives the following:

$$(\mathfrak{L}_V \omega)(F) = g(F, \mathfrak{L}_V \zeta) + (\mathfrak{L}_V g)(F, \zeta). \quad (3.16)$$

Upon setting  $G = \zeta$  in (3.1), we have the following:

$$(\mathfrak{L}_V g)(F, \zeta) = -2\left(\vartheta + 3\rho - \frac{\sigma r}{2}\right)\omega(F); \quad (3.17)$$

here,  $\omega(\zeta) = -1$  and  $g(F, \zeta) = \omega(F)$  being used.

From (3.16) and (3.17), we find

$$(\mathfrak{L}_V \omega)(F) = -2\left(\vartheta + 3\rho - \frac{\sigma r}{2}\right)\omega(F) + g(F, \mathfrak{L}_V \zeta). \quad (3.18)$$

Combining (3.12) and (3.15)–(3.18), we deduce the following:

$$\begin{aligned} &6\left[\vartheta + 3\rho - \frac{\sigma r}{2} + \sigma(r - 12)\right]\omega(F) - \frac{\sigma}{2}(\square r)\omega(F) + \frac{\sigma}{2}\omega(\nabla_F Dr) \\ &+ \left(\frac{r - 12}{3}\right)g(F, \mathfrak{L}_V \zeta) + \left(\frac{r - 12}{3}\right)\omega(F)\omega(\mathfrak{L}_V \zeta) = 0. \end{aligned} \quad (3.19)$$

Taking the Lie derivative of  $1 + g(\zeta, \zeta) = 0$  and applying the Leibniz rule, we get  $(\mathfrak{L}_V g)(\zeta, \zeta) + 2g(\mathfrak{L}_V \zeta, \zeta) = 0$ . Since  $g(\mathfrak{L}_V \zeta, \zeta) = \omega(\mathfrak{L}_V \zeta)$ , while (3.17) gives  $(\mathfrak{L}_V g)(\zeta, \zeta) = 2\left(\vartheta + 3\rho - \frac{\sigma r}{2}\right)$ , it follows that

$$\omega(\mathfrak{L}_V \zeta) = -\left(\vartheta + 3\rho - \frac{\sigma r}{2}\right). \quad (3.20)$$

By using (3.20) in (3.19), we obtain the following:

$$\begin{aligned} &\left[\left(\vartheta + 3\rho - \frac{\sigma r}{2}\right)\left(10 - \frac{r}{3}\right) + 6\sigma(r - 12)\right]\omega(F) - \frac{\sigma}{2}(\square r)\omega(F) \\ &+ \frac{\sigma}{2}\omega(\nabla_F Dr) + \left(\frac{r - 12}{3}\right)g(F, \mathfrak{L}_V \zeta) = 0. \end{aligned} \quad (3.21)$$

Equation (3.21) gives a fundamental differential constraint imposed by the RYS structure, relating the scalar curvature to the geometry of a unit timelike torse-forming vector field  $\zeta$ . This relation underlies the analysis of both the constant and non-constant scalar-curvature cases.

Now, we consider the case of a constant scalar curvature. If  $r$  is constant, then Remark 2.2 gives  $r = 12$ . Consequently, (3.21) yields  $\vartheta = -3\rho + 6\sigma$ . Moreover, (2.9) reduces to  $S = 3g$ , and hence  $(M^4, g)$  is an ES. Substituting  $r = 12$  and  $\vartheta = -3\rho + 6\sigma$  in (1.2), we obtain  $\mathbb{L}_V g = 0$ . Thus,  $V$  is a Killing vector field and the RYS is trivial. Therefore,  $(M^4, g)$  is an Einstein spacetime with a Killing soliton vector field. Finally,  $\vartheta = 0 \iff 2\sigma = \rho$ . Hence, the RYS is steady when  $2\sigma = \rho$ , expanding when  $2\sigma > \rho$ , and shrinking when  $2\sigma < \rho$ . This concludes the proof.  $\square$

Next, we consider the non-constant scalar-curvature case and prove the following theorem.

**Theorem 3.2.** *Let  $(M^4, g)$  be a four-dimensional Lorentzian manifold endowed with a unit timelike torse-forming vector field  $\zeta$  satisfying (1.1). Suppose that  $(M^4, g)$  admits a RYS  $(g, V, \vartheta, \rho, \sigma)$ , where  $\rho, \sigma (\neq 0)$ , and  $\vartheta$  are real constants. If the scalar curvature  $r$  is non-constant, then  $r$  satisfies the screened wave equation  $(\square - 2)r = \Psi$ , where  $\Psi = \frac{12}{\sigma}(\vartheta + 3\rho - 8\sigma)$ .*

*Proof.* Replacing  $F$  by  $\zeta$  in (3.21) and using (3.20), we find the following:

$$-6\vartheta - 18\rho + 72\sigma - 3\sigma r + \frac{\sigma}{2}(\square r) + \frac{\sigma}{2}\omega(\nabla_\zeta Dr) = 0. \quad (3.22)$$

Therefore, since  $\omega(\nabla_\zeta Dr) = -2\zeta(r) = 4(r - 12)$ , from (3.22), it follows that

$$(\square - 2)r = \Psi, \quad (3.23)$$

where  $\Psi = \frac{12}{\sigma}[\vartheta + 3\rho - 8\sigma]$ ,  $\sigma \neq 0$ .

Thus, the non-constant scalar curvature  $r$  satisfies the non-homogeneous Klein–Gordon-type equation (3.23). Such equations frequently arise in Lorentzian geometry, general relativity, and the relativistic field theory, where the d'Alembert operator  $\square$  governs wave propagation on spacetime [36]. Therefore, the scalar curvature of the spacetime satisfies a screened wave equation determined by the RYS structure. This concludes the proof.  $\square$

#### 4. GRYSs on GRSSs

In this section, we proceed to study GRYs on a four-dimensional Lorentzian manifold  $(M^4, g)$ . Our aim is to characterize the geometric interplay between the gradient soliton vector field and the unit timelike torse-forming vector field  $\zeta$  associated with the spacetime.

We begin this section by establishing the following theorem.

**Theorem 4.1.** *Let  $(M^4, g)$  be a four-dimensional Lorentzian manifold endowed with a unit timelike torse-forming vector field  $\zeta$  satisfying (1.1). Suppose that  $(M^4, g)$  admits a GRYS  $(g, Df, \rho, \sigma, \vartheta)$ , where  $\rho, \sigma$ , and  $\vartheta$  are real constants. Then, the potential vector field  $Df$  is pointwise collinear with  $\zeta$ . Consequently, the potential function  $f$  is constant along every direction orthogonal to  $\zeta$ .*

*Proof.* Let  $(M^4, g)$  admit a GRYS  $(g, Df, \rho, \sigma, \vartheta)$ . Then, (1.3) takes the following form:

$$\nabla_E Df + \rho QE + \left(\vartheta - \frac{\sigma r}{2}\right)E = 0, \quad (4.1)$$

for all vector fields  $E$  on  $M^4$ .

The covariant differentiation of (4.1) w.r.t.  $F$  gives the following:

$$\nabla_F \nabla_E Df = -\rho\{(\nabla_F Q)E + Q(\nabla_F E)\} - \left(\vartheta - \frac{\sigma r}{2}\right) \nabla_F E + \frac{\sigma F(r)}{2} E, \quad (4.2)$$

where  $E(\vartheta) = 0$  being used.

Interchanging  $F$  and  $E$  in (4.2), we have the following:

$$\nabla_E \nabla_F Df = -\rho\{(\nabla_E Q)F + Q(\nabla_E F)\} - \left(\vartheta - \frac{\sigma r}{2}\right) \nabla_E F + \frac{\sigma E(r)}{2} F. \quad (4.3)$$

By making use of (4.1)–(4.3) in the curvature identity  $R(E, F) + \nabla_{[E, F]} = [\nabla_E, \nabla_F]$ , we obtain the following:

$$R(E, F)Df = -\rho\{(\nabla_E Q)F - (\nabla_F Q)E\} + \frac{\sigma}{2}\{E(r)F - F(r)E\}. \quad (4.4)$$

Taking the inner product of (4.4) with  $\zeta$ , we have the following:

$$\begin{aligned} g(R(E, F)Df, \zeta) &= -\rho\{(\nabla_E S)(F, \zeta) - (\nabla_F S)(E, \zeta)\} \\ &\quad + \frac{\sigma}{2}\{E(r)\omega(F) - F(r)\omega(E)\}. \end{aligned} \quad (4.5)$$

Differentiating (2.2) covariantly along (E), and using (1.1), we obtain the following:

$$(\nabla_E S)(F, \zeta) + S(F, \nabla_E \zeta) = 3g(F, \nabla_E \zeta).$$

Again, using (1.1), the above relation becomes  $(\nabla_E S)(F, \zeta) + S(F, E) + \omega(E)S(F, \zeta) = 3g(E, F) + 3\omega(E)g(F, \zeta)$ . Since  $S(F, \zeta) = 3\omega(F)$  and  $g(F, \zeta) = \omega(F)$ , the terms involving  $\omega(E)\omega(F)$  cancel, yielding the following:

$$(\nabla_E S)(F, \zeta) = 3g(E, F) - S(E, F). \quad (4.6)$$

In view of (4.6), (4.5) reduces to the following:

$$g(R(E, F)Df, \zeta) = -\frac{\sigma}{2}\{F(r)\omega(E) - E(r)\omega(F)\}. \quad (4.7)$$

By virtue of (2.1), (4.7) takes the following form:

$$E(f)\omega(F) - F(f)\omega(E) - \frac{\sigma}{2}\{F(r)\omega(E) - E(r)\omega(F)\} = 0. \quad (4.8)$$

The preceding relation is the key step in identifying the direction of the potential vector field. Setting  $F = \zeta$  in (4.8) and using  $\omega(\zeta) = -1$ , together with (2.10), we obtain the following:

$$-E(f) - \zeta(f)\omega(E) - \frac{\sigma}{2}\{\zeta(r)\omega(E) + E(r)\} = 0.$$

By (2.10), we have  $\zeta(r)\omega(E) + E(r) = 0$ . Consequently, from the above relation, we obtain the following:

$$E(f) = -\zeta(f)\omega(E) \implies Df = -\zeta(f)\zeta. \quad (4.9)$$

Thus,  $Df$  is pointwise collinear with  $\zeta$ . In particular, if  $E$  assumes to be orthogonal to  $\zeta$ , then it follows that  $E(f) = 0$ . Hence,  $f$  is constant along every direction orthogonal to  $\zeta$ . This completes the proof.  $\square$

Now, we establish the following theorem.

**Theorem 4.2.** *Let  $(M^4, g)$  be a four-dimensional Lorentzian manifold endowed with a unit timelike torse-forming vector field  $\zeta$  satisfying (1.1). Suppose that  $(M^4, g)$  admits a GRYS  $(g, Df, \rho, \sigma, \vartheta)$ , where  $\rho (\neq 0)$ ,  $\sigma$ , and  $\vartheta$  are real constants. Then, its Ricci operator satisfies  $QE = \left(\frac{r}{3} - 1\right)E + \left(\frac{r}{3} - 4\right)\omega(E)\zeta$ . Furthermore, the soliton parameters and the scalar curvature satisfy  $\vartheta = r\left(\frac{9\sigma - 4\rho}{6}\right) - 12\sigma + 5\rho$ .*

*Proof.* Differentiating (4.9) along  $E$ , we have

$$\nabla_E Df = -E(\zeta(f))\zeta - \zeta(f)\nabla_E \zeta,$$

which using (1.1) with (4.1) transforms to

$$\rho QE + \left(\vartheta - \frac{\sigma r}{2}\right)E = E(\zeta(f))\zeta + \zeta(f)(\omega(E)\zeta + E). \quad (4.10)$$

Taking the inner product of (4.10) with  $\zeta$  and using (2.2), we have the following:

$$E(\zeta(f)) = -\left(\vartheta + 3\rho - \frac{\sigma r}{2}\right)\omega(E). \quad (4.11)$$

It follows immediately from (4.10) and (4.11) that

$$QE = \frac{1}{\rho}\left\{\zeta(f) - \vartheta + \frac{\sigma r}{2}\right\}E + \frac{1}{\rho}\left\{\zeta(f) - \vartheta - 3\rho + \frac{\sigma r}{2}\right\}\omega(E)\zeta, \quad \rho \neq 0. \quad (4.12)$$

Thus, under the assumption  $\rho \neq 0$ , the Ricci operator is completely determined by the timelike vector field  $\zeta$ .

Contracting (4.12), we obtain the following:

$$r = \frac{3}{\rho}\left\{\zeta(f) - \vartheta + \frac{\sigma r}{2}\right\} + 3, \quad \rho \neq 0. \quad (4.13)$$

Using (4.13), (4.12) turns to the following:

$$QE = \left(\frac{r}{3} - 1\right)E + \left(\frac{r}{3} - 4\right)\omega(E)\zeta. \quad (4.14)$$

Thus, (4.14) coincides with the Ricci structure specified in (2.9). This shows that the GRYS structure determines not only the behavior of the potential function, but also imposes a specific Ricci geometry, namely a perfect-fluid structure with respect to the unit timelike torse-forming vector field  $\zeta$ .

Furthermore, combining (4.11) and (4.13), we obtain the following:

$$(3\sigma - 2\rho)E(r) = 6\left(\vartheta + 3\rho - \frac{\sigma r}{2}\right)\omega(E). \quad (4.15)$$

Using (2.10) in (4.15), we obtain  $(3\sigma - 2\rho)(r - 12) = 3\left(\vartheta + 3\rho - \frac{\sigma r}{2}\right)$ , which simplifies to the following:

$$\vartheta = r\left(\frac{9\sigma - 4\rho}{6}\right) - 12\sigma + 5\rho. \quad (4.16)$$

This proves the desired result.  $\square$

The relation given by (4.16) connects the soliton parameter  $\vartheta$  with the scalar curvature  $r$  and the constants  $\rho$  and  $\sigma$ . Hence, our discussion naturally divides into two cases accordingly as  $r$  is constant or non-constant.

For the case of a constant scalar curvature, we first establish the following result.

**Theorem 4.3.** *Let  $(M^4, g)$  be a four-dimensional Lorentzian manifold endowed with a unit timelike torse-forming vector field  $\zeta$  satisfying (1.1) and all the hypotheses of Remark 2.2. Suppose that  $(M^4, g)$  has a constant scalar curvature and admits a GRYS  $(g, Df, \rho, \sigma, \vartheta)$ , where  $\rho(\neq 0)$ ,  $\sigma$ , and  $\vartheta$  are real constants. Then, the GRYS is steady (resp., expanding or shrinking) if  $2\sigma = \rho$  (resp.,  $2\sigma > \rho$  or  $2\sigma < \rho$ ). Moreover,  $\text{Hess}f = 0$ , then the potential vector field  $V = Df$  is Killing. Consequently, the GRYS is trivial in the sense that its potential function has a vanishing Hessian, and  $(M^4, g)$  is an Einstein spacetime.*

*Proof.* Under the assumption that the scalar curvature  $r$  of  $(M^4, g)$  is constant (i.e.,  $r = 12$ , from Remark 2.2), then (4.16) gives  $\vartheta = -3\rho + 6\sigma$ . Therefore, according to the sign of  $\vartheta$ , the GRYS is steady, expanding, or shrinking according as  $2\sigma = \rho$ ,  $2\sigma > \rho$ , or  $2\sigma < \rho$ , respectively. Moreover, from (2.9), we obtain  $S = 3g$ . Substituting  $r = 12$ ,  $S = 3g$ , and  $\vartheta = -3\rho + 6\sigma$  in (1.3), we get  $\text{Hess}f = 0$ . Therefore,  $\mathbb{L}_{Df}g = 2\text{Hess}f = 0$ , which shows that  $Df$  is Killing. Since the soliton vector field is both gradient and Killing, then  $f$  is locally affine. Moreover,  $(M^4, g)$  becomes an Einstein spacetime [37]. This establishes the stated result.  $\square$

Now, we prove the following corollary.

**Corollary 4.4.** *Let  $(M^4, g)$  be a four-dimensional Lorentzian manifold endowed with a unit timelike torse-forming vector field  $\zeta$  satisfying (1.1). Suppose that  $(M^4, g)$  has a constant scalar curvature and admits a GRYS  $(g, Df, \rho, \sigma, \vartheta)$ , where  $\rho(\neq 0)$ ,  $\sigma$ , and  $\vartheta$  are real constants satisfying  $\vartheta = -3\rho + 6\sigma$ . Then,  $(M^4, g)$  is an Einstein spacetime and the soliton is trivial.*

*Proof.* From (4.13), we have  $\zeta(f) = \vartheta + 3\rho - 6\sigma$ , which, using  $\vartheta = -3\rho + 6\sigma$ , reduces to  $\zeta(f) = 0$ . Hence, from (4.9), we find that  $f = \text{constant}$ . This shows that the GRYS  $(g, Df, \vartheta, \rho, \sigma)$  is trivial. Moreover, Eq (4.12) yields  $S = 3g$ , that is,  $(M^4, g)$  is an Einstein spacetime. Thus, the proof is complete.  $\square$

For the non-constant scalar curvature, we obtain the following result.

**Theorem 4.5.** *Let  $(M^4, g)$  be a four-dimensional Lorentzian manifold endowed with a unit timelike torse-forming vector field  $\zeta$  satisfying (1.1). Suppose that  $(M^4, g)$  admits a gradient Ricci-Yamabe soliton  $(g, Df, \vartheta, \rho, \sigma)$ , where  $\rho(\neq 0)$ ,  $\sigma$ , and  $\vartheta$  are real constants, and  $2\rho \neq 3\sigma$ . If the scalar curvature  $r$  is non-constant, then  $\zeta^2(f) = \left(\sigma - \frac{2\rho}{3}\right)(r - 12)$ , where  $\zeta^2(f) = \zeta(\zeta(f))$ . Consequently, along every integral curve  $\gamma$  of  $\zeta$ , the function  $f \circ \gamma$  satisfies the second-order ordinary differential equation  $\frac{d^2}{dt^2}(f \circ \gamma)(t) = \left(\sigma - \frac{2\rho}{3}\right)((r \circ \gamma)(t) - 12)$ , equivalently, in a local coordinate system for which  $\zeta = \frac{\partial}{\partial t}$ ,  $\frac{\partial^2 f}{\partial t^2} = \left(\sigma - \frac{2\rho}{3}\right)(r - 12)$ .*

*Proof.* Since  $\vartheta$ ,  $\rho$ , and  $\sigma$  are real constants, differentiating Eq (4.13) in the direction of  $E$ , we obtain the following:

$$(2\rho - 3\sigma)E(r) = 6E(\zeta(f)). \quad (4.17)$$

Consequently, the potential function  $f$  satisfies the partial differential equation

$$\zeta^2(f) = \left(\sigma - \frac{2\rho}{3}\right)(r - 12), \quad (4.18)$$

where we took  $E = \zeta$  in (4.17) and used (2.7), with  $\zeta^2(f) = \zeta(\zeta(f))$ .

Equation (4.18) is most naturally interpreted along the integral curves of the unit timelike torse-forming vector field  $\zeta$ . Let  $\gamma : I \rightarrow M^4$  be an integral curve of  $\zeta$ , parametrized so that  $\dot{\gamma}(t) = \zeta_{\gamma(t)}$ . Then,

$$\frac{d^2}{dt^2}(f \circ \gamma)(t) = \zeta(\zeta(f))(\gamma(t)).$$

Using (4.18), we obtain the following:

$$\frac{d^2}{dt^2}(f \circ \gamma)(t) = \left(\sigma - \frac{2\rho}{3}\right)((r \circ \gamma)(t) - 12). \quad (4.19)$$

Furthermore, in a local coordinate neighborhood in which  $\zeta = \frac{\partial}{\partial t}$ , Eq (4.19) takes the following form:

$$\frac{\partial^2 f}{\partial t^2} = \left(\sigma - \frac{2\rho}{3}\right)(r - 12). \quad (4.20)$$

Consequently, along each integral curve of  $\zeta$ , the scalar curvature  $r$  determines the second-order differential behavior of the potential function  $f$ . This establishes the claimed result.  $\square$

The preceding differential equation has a direct geometric interpretation: it governs the local convexity or concavity of the potential function along the integral curves of  $\zeta$ . This observation yields the following corollary.

**Corollary 4.6.** *Let  $(M^4, g)$  be a four-dimensional Lorentzian perfect fluid spacetime endowed with a unit timelike torse-forming vector field  $\zeta$  satisfying (1.1). Suppose that  $(M^4, g)$  admits a gradient Ricci-Yamabe soliton  $(g, Df, \vartheta, \rho, \sigma)$ , where  $\rho (\neq 0)$ ,  $\sigma$ , and  $\vartheta$  are real constants, and  $2\rho \neq 3\sigma$  with  $\kappa = \left(\sigma - \frac{2}{3}\rho\right) \neq 0$ . If the scalar curvature  $r$  is non-constant, then, along any integral curve  $\gamma$  of  $\zeta$  parametrized by proper time  $t$ , with  $\dot{\gamma}(t) = \zeta_{\gamma(t)}$ , the potential function satisfies  $\frac{\partial^2 f}{\partial t^2} = \kappa(r - 12)$ , and hence  $f$  is strictly convex (resp., strictly concave) on every interval where  $\kappa(r - 12) > 0$  (resp.,  $\kappa(r - 12) < 0$ ).*

*Proof.* Assume that  $\kappa = \left(\sigma - \frac{2}{3}\rho\right) \neq 0$ . Then, (4.20) takes the following form:

$$\frac{\partial^2 f}{\partial t^2} = \kappa(r - 12). \quad (4.21)$$

For an integral curve  $\gamma$ , parametrized by  $t$  satisfying  $\dot{\gamma} = \zeta$ , we have  $\frac{d^2}{dt^2}(f \circ \gamma) = \zeta^2(f) = \kappa(r - 12)$ . Moreover,  $g(\dot{\gamma}, \dot{\gamma}) = -1$ , so  $t$  is the proper-time parameter along  $\gamma$ . Therefore, on any interval where  $\kappa(r - 12) > 0$  (resp.,  $\kappa(r - 12) < 0$ ), the function  $f \circ \gamma$  is strictly convex (resp., strictly concave). This proves the result.  $\square$

## 5. Example

The purpose of this example is to provide an explicit four-dimensional Lorentzian model satisfying the geometric assumptions used in the preceding sections and to illustrate the rigidity obtained for GRYSs when the scalar curvature is constant.

Let  $M^4 = \mathbb{R}^4$  be a four-dimensional smooth manifold with coordinates  $(x, y, z, t) \in \mathbb{R}^4$ , equipped with the Lorentzian metric

$$g = e^{-2t}(dx^2 + dy^2 + dz^2) - dt^2,$$

where the metric has the signature  $(-, +, +, +)$ .

Consider the following local frame adapted to the distinguished timelike direction:

$$l_1 = e^t \frac{\partial}{\partial x}, \quad l_2 = e^t \frac{\partial}{\partial y}, \quad l_3 = e^t \frac{\partial}{\partial z}, \quad l_4 = -\frac{\partial}{\partial t} = \zeta.$$

Then,  $\{l_1, l_2, l_3, l_4\}$  forms a local orthonormal frame field on  $M^4$ , with  $l_4 = \zeta$  as the unit timelike vector field, satisfying

$$[l_i, l_j] = \begin{cases} l_i, & 1 \leq i \leq 3, \quad j = 4, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $g$  be the Lorentzian metric defined by the following:

$$g(l_i, l_j) = \begin{cases} 1, & 1 \leq i = j \leq 3, \\ -1, & j = i = 4, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,  $(M^4, g)$  is a four-dimensional Lorentzian manifold. Let  $\omega$  be a 1-form on  $M^4$  defined by  $\omega(E) = g(E, l_4) = g(E, \zeta)$  for all  $E$  on  $M^4$ . By using the well-known Koszul's formula and the above facts, we obtain the following:

$$\nabla_{l_i} l_j = \begin{cases} l_4, & 1 \leq i = j \leq 3, \\ l_i, & 1 \leq i \leq 3, \quad j = 4, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently,

$$\nabla_E \zeta = \omega(E)\zeta + E.$$

Thus, the distinguished timelike vector field precisely satisfies the differential relation required in the geometric framework considered above. Hence,  $(M^4, g)$  provides an explicit four-dimensional model of the class of Lorentzian manifold under consideration. Now, the curvature tensor can be directly calculated from the above connection. Its non-zero components are as follows:

$$\begin{aligned} R(l_2, l_1)l_1 &= l_2, \quad R(l_1, l_2)l_2 = l_1, \quad R(l_3, l_1)l_1 = l_3, \quad R(l_1, l_3)l_3 = l_1, \\ R(l_4, l_1)l_1 &= l_4, \quad R(l_1, l_4)l_4 = -l_1, \quad R(l_3, l_2)l_2 = l_3, \quad R(l_2, l_3)l_3 = l_2, \\ R(l_4, l_2)l_2 &= l_4, \quad R(l_2, l_4)l_4 = -l_2, \quad R(l_3, l_4)l_3 = -l_4, \quad R(l_3, l_4)l_4 = -l_3. \end{aligned}$$

The remaining curvature components are obtained from the usual curvature symmetries  $R(X, Y) = -R(Y, X)$ ,  $g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$ , together with the Bianchi's first identity. We use the following convention for the Ricci tensor:

$$S(E, F) = \sum_{i=1}^4 \varepsilon_i g(R(l_i, E)F, l_i),$$

where  $\varepsilon_i = g(l_i, l_i) \in \{1, -1\}$  (i.e.,  $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1, \varepsilon_4 = -1$ ).

The components of  $S$  are computed as follows:

$$S(l_1, l_1) = \varepsilon_1 g(R(l_1, l_1)l_1, l_1) + \varepsilon_2 g(R(l_2, l_1)l_1, l_2) + \varepsilon_3 g(R(l_3, l_1)l_1, l_3) + \varepsilon_4 g(R(l_4, l_1)l_1, l_4) = 3$$

$$S(l_2, l_2) = \varepsilon_1 g(R(l_1, l_2)l_2, l_1) + \varepsilon_2 g(R(l_2, l_2)l_2, l_2) + \varepsilon_3 g(R(l_3, l_2)l_2, l_3) + \varepsilon_4 g(R(l_4, l_2)l_2, l_4) = 3$$

$$S(l_3, l_3) = \varepsilon_1 g(R(l_1, l_3)l_3, l_1) + \varepsilon_2 g(R(l_2, l_3)l_3, l_2) + \varepsilon_3 g(R(l_3, l_3)l_3, l_3) + \varepsilon_4 g(R(l_4, l_3)l_3, l_4) = 3$$

$$S(l_4, l_4) = \varepsilon_1 g(R(l_1, l_4)l_4, l_1) + \varepsilon_2 g(R(l_2, l_4)l_4, l_2) + \varepsilon_3 g(R(l_3, l_4)l_4, l_3) + \varepsilon_4 g(R(l_4, l_4)l_4, l_4) = -3$$

It is obvious that  $S = 3g$ . Thus, the manifold is Einstein. Hence, (2.9) is verified.

Next, we investigate the GRYS equation on this spacetime. With respect to the chosen orthonormal frame field  $\{l_1, l_2, l_3, l_4\}$ ,  $g(l_i, l_i) = 1; (i = 1, 2, 3)$ ,  $g(l_4, l_4) = -1$ , the gradient of  $f$  is given by the following:

$$Df = (l_1 f)l_1 + (l_2 f)l_2 + (l_3 f)l_3 - (l_4 f)l_4.$$

Using the product rule for the covariant derivative, for  $i = 1, 2, 3$ , we obtain the following:

$$\begin{aligned} \nabla_{l_i} Df &= \nabla_{l_i}((l_1 f)l_1) + \nabla_{l_i}((l_2 f)l_2) + \nabla_{l_i}((l_3 f)l_3) - \nabla_{l_i}((l_4 f)l_4) \\ &= l_i(l_1 f)l_1 + (l_1 f)\nabla_{l_i}l_1 + l_i(l_2 f)l_2 + (l_2 f)\nabla_{l_i}l_2 \\ &\quad + l_i(l_3 f)l_3 + (l_3 f)\nabla_{l_i}l_3 - l_i(l_4 f)l_4 - (l_4 f)\nabla_{l_i}l_4. \end{aligned}$$

Since  $\nabla_{l_i}l_j = \varepsilon_{ij}l_4$ , where  $\varepsilon_{ij}$  is the Kronecker delta defined by  $\varepsilon_{ij} = 1$ , if  $i = j$ , then  $\varepsilon_{ij} = 0$ , otherwise, and  $\nabla_{l_i}l_4 = l_i$ ,  $i, j = 1, 2, 3$ ; it follows that

$$\nabla_{l_i} Df = l_i(l_1 f)l_1 + l_i(l_2 f)l_2 + l_i(l_3 f)l_3 - l_i(l_4 f)l_4 + (l_i f)l_4 - (l_4 f)l_i, \quad i = 1, 2, 3.$$

Similarly, using  $\nabla_{l_4}l_j = 0$ ,  $j = 1, 2, 3, 4$ , we have

$$\begin{aligned} \nabla_{l_4} Df &= \nabla_{l_4}((l_1 f)l_1) + \nabla_{l_4}((l_2 f)l_2) + \nabla_{l_4}((l_3 f)l_3) - \nabla_{l_4}((l_4 f)l_4) \\ &= l_4(l_1 f)l_1 + l_4(l_2 f)l_2 + l_4(l_3 f)l_3 - l_4(l_4 f)l_4. \end{aligned}$$

Consequently, the Hessian of  $f$ ,  $\nabla^2 f(E, F) = g(\nabla_E Df, F)$ , has the following matrix representation with respect to the frame field  $\{l_1, l_2, l_3, l_4\}$ :

$$\text{Hess } f = \begin{pmatrix} l_1(l_1 f) - (l_4 f) & l_1(l_2 f) & l_1(l_3 f) & -l_1(l_4 f) + (l_1 f) \\ l_2(l_1 f) & l_2(l_2 f) - (l_4 f) & l_2(l_3 f) & -l_2(l_4 f) + (l_2 f) \\ l_3(l_1 f) & l_3(l_2 f) & l_3(l_3 f) - (l_4 f) & -l_3(l_4 f) + (l_3 f) \\ l_4(l_1 f) & l_4(l_2 f) & l_4(l_3 f) & -l_4(l_4 f) \end{pmatrix}.$$

Since  $S = 3g$  and  $r = 12$ , the GRYS equation (1.3) reduces to the following:

$$\nabla^2 f + (3\rho + \vartheta - 6\sigma)g = 0. \quad (5.1)$$

From (4.13), we obtain  $\zeta(f) = \vartheta + 3\rho - 6\sigma$ . Since  $Df = 0$  implies  $\zeta(f) = 0$ , it follows that  $\vartheta + 3\rho - 6\sigma = 0$ , and hence  $\vartheta = 6\sigma - 3\rho$ . Taking  $\vartheta = -3\rho + 6\sigma$  in (5.1), we obtain the following:

$$\nabla^2 f = 0.$$

This reduces the problem to the existence of a non-constant function with a vanishing Hessian.

Using the expressions of  $\nabla_{l_i} Df$  given above, the condition  $\nabla^2 f = 0$  yields the following system:

$$\begin{cases} l_1(l_1 f) - l_4 f = 0, \\ l_2(l_2 f) - l_4 f = 0, \\ l_3(l_3 f) - l_4 f = 0, \\ l_4(l_4 f) = 0, \\ l_i(l_j f) = 0, \quad j \neq i, \quad j, i \in 1, 2, 3, \\ l_i(l_4 f) - l_i f = 0, \quad i = 1, 2, 3, \\ l_4(l_i f) = 0, \quad i = 1, 2, 3. \end{cases}$$

Expressing the above system in local coordinates, we obtain the following:

$$e^{2t} \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial t} = 0, \quad (5.2)$$

$$e^{2t} \frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial t} = 0,$$

$$e^{2t} \frac{\partial^2 f}{\partial z^2} + \frac{\partial f}{\partial t} = 0,$$

$$\frac{\partial^2 f}{\partial t^2} = 0, \quad (5.3)$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial z} = 0, \quad (5.4)$$

$$\frac{\partial^2 f}{\partial t \partial x} + \frac{\partial f}{\partial x} = 0, \quad \frac{\partial^2 f}{\partial t \partial y} + \frac{\partial f}{\partial y} = 0, \quad \frac{\partial^2 f}{\partial t \partial z} + \frac{\partial f}{\partial z} = 0. \quad (5.5)$$

Since  $l_4 = -\partial_t$ , conditions  $l_4(l_i f) = 0$  and  $l_i(l_4 f) - l_i f = 0$  give the same mixed-derivative relations in (5.5), with the same sign. Therefore, these redundant relations are only listed once.

From (5.3), i.e.,  $\frac{\partial^2 f}{\partial t^2} = 0$ , it follows locally that

$$f(x, y, z, t) = A(x, y, z)t + B(x, y, z), \quad (5.6)$$

where  $A, B$  are smooth functions of  $(x, y, z)$ , independent of  $t$ .

Substituting (5.6) into (5.5), we obtain  $A_x = A_y = A_z = 0$ ,  $B_x = B_y = B_z = 0$ . Therefore,  $A = c_1$  and  $B = c_2$ , where  $c_1, c_2 \in \mathbb{R}$ . It follows that  $f = c_1 t + c_2$ , which using in (5.2), we deduce that  $c_1 = 0$ . Hence,  $f = c_2$ , and thus  $f \in \mathbb{R}$ . Hence,

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} = \frac{\partial f}{\partial t} = 0,$$

which implies that

$$Df = 0.$$

Therefore, for the given model and under the parameter condition  $\vartheta = -3\rho + 6\sigma$ , the potential function  $f$  is necessarily constant, and the corresponding GRYS is trivial. Thus, this example provides an explicit Einstein spacetime with a constant scalar curvature for which the GRYS has a constant potential function. Thus, the present example demonstrates the conclusion of Theorem 4.3 for the given model.

### Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

### Acknowledgments

The authors are thankful to the learned reviewers for their careful reading of our manuscript and their insightful comments and suggestions that have improved the quality of our manuscript.

### Conflict of interest

The authors declare there are no conflicts of interest.

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