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Research article

Finite-time synchronization and approximate controllability of neutral fractional stochastic systems with non-instantaneous impulses driven by Rosenblatt process

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Abstract: This paper investigates practical finite-time synchronization and approximate controllability of neutral fractional stochastic differential systems driven by the Rosenblatt process. The considered systems involve multiple time-varying delays, non-instantaneous impulses, and superlinear growth conditions. First, the existence and uniqueness of mild solutions are established by applying the Krasnoselskii fixed-point theorem and the Banach contraction principle. Second, a nonlinear finite-time feedback controller is designed for the drive-response systems, and sufficient conditions for practical finite-time synchronization under non-instantaneous impulses are derived. A key technical contribution is the development of a generalized Bihari-type inequality for fractional integral inequalities with weakly singular kernels and multiple time-varying delays, which provides explicit estimates for the synchronization settling time. Furthermore, based on the synchronization analysis, the approximate controllability of the response systems is investigated by employing the controllability Gramian and the resolvent operator approach. Finally, numerical simulations are presented to illustrate the effectiveness of the theoretical results.

Keywords: neutral fractional stochastic differential systems; Rosenblatt process; non-instantaneous impulses; practical finite-time synchronization; approximate controllability

1. Introduction

Fractional stochastic differential systems have emerged as powerful mathematical tools for modeling complex dynamical systems with memory effects and random perturbations. In contrast to integer-order differential frameworks, fractional-order operators accurately characterize long-range dependence and hereditary dynamics arising in a wide class of physical, economic, and financial evolutionary models [1, 2]. Among the various definitions of fractional derivatives, the Caputo derivative is widely adopted in theoretical analysis and possesses transparent physical interpretability, which greatly facilitates the formulation and qualitative investigation of evolutionary dynamical systems [3, 4].

In practical dynamical modeling, external random perturbations are unavoidable. Fractional Brownian motion (fBm), as a Gaussian self-similar process with long-range dependence, has been widely adopted to describe persistent stochastic fluctuations in fractional dynamical systems [5]. However, many natural and engineered systems exhibit non-Gaussian statistical behaviors together with long-range dependence, which cannot always be adequately characterized by Gaussian models. The Rosenblatt process, belonging to the second Wiener chaos, is a non-Gaussian self-similar process that preserves long-range dependence properties while providing a broader class of stochastic perturbations beyond fBm [6]. Therefore, introducing the Rosenblatt process in this work extends the existing fractional stochastic framework from Gaussian long-memory environments to non-Gaussian long-memory settings, leading to a more general synchronization and controllability analysis for fractional dynamical systems.

In addition, time-varying delays arise universally within dynamical systems, stemming from finite signal transmission velocities, signal processing lags, and network transmission latencies. Such state-dependent lag terms exert substantial impacts on the qualitative stability and control behavior of evolutionary systems [7–9]. Neutral differential equations are well-suited to describe these delay phenomena, as they encode dynamics where the instantaneous time derivative of the state depends on past derivatives. This property precisely captures hereditary lag and coupled delay structures inherent to many systems [10, 11]. Furthermore, non-instantaneous impulses represent another common feature of controlled evolution systems. These impulses occur when external control inputs or disturbances act continuously over finite time intervals, rather than taking effect instantaneously at discrete time points [12, 13].

Synchronization constitutes a core qualitative topic in dynamical systems theory, with applications spanning secure communication, multiagent coordination, and complex network dynamics. The fundamental goal of synchronization theory is to construct admissible feedback laws that force the trajectories of a response system to track those of a prescribed drive system [14, 15]. Finite-time synchronization further guarantees exact trajectory matching within a finite predetermined time horizon, making it theoretically and practically valuable for systems requiring rapid transient convergence, such as emergency regulation and real-time signal processing [16, 17].

Parallel to synchronization, controllability is another central qualitative property of dynamical systems analysis, which addresses whether a system trajectory can be transferred from an arbitrary initial state to a neighborhood of any preassigned target state via appropriate control inputs. Of particular interest is approximate controllability, which ensures that trajectories can reach an arbitrarily small neighborhood of the desired target. This property underpins theoretical investigations and practical applications including trajectory design and neural dynamical regulation [18–20].

The systems possess five characteristic dynamical features that inspire the construction of our mathematical model:

(1) Memory-dependent evolutionary behavior, naturally described by fractional derivatives;

(2) Non-Gaussian noise with long-range dependence, modeled via Rosenblatt process;

(3) State-coupled signal transmission that induces multiple time-varying delays;

(4) Intermittent control actions yielding non-instantaneous impulsive dynamics;

(5) Nonlinear functions with superlinear growth conditions.

Despite the widespread of the above dynamical features in real-world systems, several critical theoretical gaps remain unaddressed within the existing literature on fractional stochastic differential systems.

Existing studies on fixed-point methods have provided effective tools for the analysis of fractional differential systems. Ofem et al. [21] developed an implicit iterative scheme for delay Caputo fractional differential equations, and Ugboh et al. [22, 23] proposed efficient fixed-point iteration techniques for fractional delay and impulsive fractional differential equations. These results highlight the importance of fixed-point approaches in fractional systems with delays and impulses.

Further, Lyapunov-based methods have been widely applied to investigate the stability of delayed fractional systems [24]. However, existing studies mainly focus on deterministic systems or classical stability criteria, whereas the existence, synchronization, and controllability of neutral fractional stochastic systems with multiple time-varying delays, non-instantaneous impulses, and Rosenblatt processes remain challenging.

Recent studies on nonlinear dynamical systems have also investigated pattern formation, bifurcation phenomena, and finite-time stability in biological and epidemic models [25–27]. These works further demonstrate the importance of qualitative analysis tools for complex systems, but fractional stochastic systems with memory effects, non-Gaussian perturbations, and time-varying delays still require further investigation.

Although fractional Bihari-type inequalities have been widely studied, most existing results focus on nonlinear fractional integral inequalities without time-varying delay terms. In the present work, the main difficulty arises from the delayed nonlinear integral term $\int_0^t (t-s)^{2\alpha-2}\nu(s-\rho_i(s))ds$, where the delay function depends explicitly on time. By imposing suitable conditions on the delay functions and the delay-induced kernels, and introducing the inverse transformation of $s-\rho(s)$, we establish a suitable fractional Bihari-type estimate for variable-delay systems. This provides an effective tool for the finite-time synchronization analysis of fractional stochastic neural networks with time-varying delays.

Recent advances in nonlinear control have also explored neural-network- based and data-driven intermittent control strategies for uncertain nonlinear systems. For example, deep neural networks have been employed to improve function approximation capabilities in intermittent control designs, and data-driven learning-based control frameworks have further enhanced the adaptability of nonlinear systems under uncertainty [28, 29]. However, the finite-time synchronization and approximate controllability of neutral fractional stochastic systems driven by the Rosenblatt process remain largely unexplored.

To further illustrate the differences between the present work and existing results, a comparison with several related studies is provided in Tables 1 and 2.

It can be observed from Tables 1 and 2 that existing works mainly focus on partial aspects of these dynamical characteristics, whereas the present study simultaneously considers neutral terms, multiple time-varying delays, Rosenblatt stochastic perturbations, and non-instantaneous impulses. Moreover, this work establishes approximate controllability and finite-time synchronization results for fractional stochastic systems, where the generalized Bihari-type inequality serves as a key analytical tool rather than an independent dynamical property.

Table 1. Comparison of mathematical settings with related studies.

Reference	Neutral term	Time-varying delay	Noise type	Non-instantaneous impulse
[30]	×	×	Deterministic	×
[31]	×	×	fBm + Poisson jumps	×
[32]	×	×	fBm	✓
[33]	×	Single delay	Rosenblatt process	✓
This work	✓	✓	Rosenblatt process	✓

Table 2. Comparison of theoretical results with related studies.

Reference	Approximate controllability	Finite-time synchronization	Generalized Bihari inequality
[30]	✓	×	×
[31]	✓	×	×
[32]	✓	×	×
[33]	✓	×	×
This work	✓	✓	✓

To address these research gaps, the main contributions of this paper are summarized as follows:

(1) A generalized Bihari-type inequality is established to simultaneously accommodate weakly singular fractional kernels, multiple time-varying delays, and non-instantaneous impulses. The proposed inequality provides an effective analytical tool for the qualitative analysis of fractional stochastic systems with superlinear nonlinearities.

(2) A nonlinear finite-time feedback controller of the form $u(r) = -\eta e(r) - \xi \|e(r)\|^{q-1} e(r),\ 0 < q < 1$, is constructed. It is proved that practical finite-time synchronization is achieved for drive-response systems subject to the Rosenblatt process, multiple time-varying delays, and non-instantaneous impulses. Moreover, an explicit upper bound for the synchronization settling time is obtained.

(3) The approximate controllability of the response systems is established by using the controllability Gramian positivity and resolvent operator convergence. It is shown that the system can be steered from any initial state to an arbitrarily small neighborhood of any prescribed target state.

2. Preliminaries

We first introduce some notations and basic concepts that will be used throughout this paper. In this paper, the state space is specified as $Z = \mathbb{R}^n$, equipped with the standard Euclidean norm. The notation used in this paper is summarized in Table 3.

Table 3. Notation used throughout this paper.

Symbol	Description
$\mathbb{R}^n$	n-dimensional Euclidean space
α	fractional order, $0 < \alpha < 1$
$\mathcal{D}^\alpha$	Caputo fractional derivative of order α
$Z_H(t)$	Rosenblatt process with Hurst parameter H
$\rho_i(t)$	time-varying delay functions
$\varsigma(t)$	drive system state
$\varpi(t)$	response system state
$e(t)$	synchronization error, $e(t) = \varpi(t) - \varsigma(t)$
$u(t)$	feedback control input
$S_\alpha(t)$	fractional solution operator
$S_\alpha(t, \zeta_k)$	two-parameter fractional evolution operator
A, B, C, D	system coefficient matrices

Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete filtered probability space. Although the abstract formulation is presented in a Hilbert space, the considered fractional neural network system is studied in the finite dimensional state space $Z = \mathbb{R}^n$, equipped with the standard Euclidean inner product. Define

$$L_2(\Omega, Z) = \left\{X : \Omega \to Z : \mathbb{E}\|X\|^2 < \infty\right\}.$$

Set $J = [0, T]$. We define $C(J, L_2(\Omega, Z))$ as the Banach space of all continuous $\mathfrak{F}_t$-adapted stochastic processes satisfying the corresponding boundedness condition

$$\sup_{r\in J} \mathbb{E}\|\psi(r)\|^2 < \infty,$$

with the norm

$$\|\psi(\cdot)\|_{L_2} = \left(\sup_{r\in J} \mathbb{E}\|\psi(r)\|^2\right)^{\frac{1}{2}}.$$

Next, we introduce the relevant knowledge of the non-Gaussian Rosenblatt process.

Suppose $\{Z_H(r)\}$ is a one-dimensional Rosenblatt process and that its parameter obeys $H \in (1/2, 1)$. Referring to the literature [6] by Tudor, the corresponding formula of the process with $H > 1/2$ is shown as follows:

$$Z_H(r) = d(H) \int_0^r \int_0^r \left[\int_{u_1\vee u_2}^r \frac{\partial K^H}{\partial v}(v, u_1)\frac{\partial K^H}{\partial v}(v, u_2)dv\right] dB(u_1)dB(u_2),$$

where $K^H(v, u)$ is given by

$$K^H(v, u) = C_H u^{\frac{1}{2}-H} \int_u^v (z - u)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} dz,\ v > u,$$

with

$$C_H = \sqrt{\frac{H(2H - 1)}{\beta(2 - 2H, H - \frac{1}{2})}},$$

where $\beta(\cdot,\cdot)$ denotes the Beta function. For any $v \leq u$, the equality $K^H(v,u) = 0$ holds, and $B(r)$ is defined as Brownian motion. The covariance associated with $\{Z_H(r)\}$ is presented below:

$$\mathbb{E}(Z_H(r)Z_H(z)) = \frac{1}{2}\left(z^{2H} + r^{2H} - |z-r|^{2H}\right),$$

as well as the Rosenblatt process has the following property:

Lemma 1. [34] The function $\phi(r) \in C(J, L_2(\Omega, Z))$ is estimated by

$$\mathbb{E}\|\int_0^r \phi(z)dZ_H(z)\|^2 \leq K_H r^{2H-1}\int_0^r \mathbb{E}\|\phi(z)\|^2 dz,$$

where K_H is associated with the parameter H.

Because the Rosenblatt process is a non-Gaussian process belonging to the second Wiener chaos, the stochastic integrals with respect to $Z_H(t)$ cannot be defined through the classical Itô calculus. Throughout this paper, the integrals involving the Rosenblatt process are interpreted as Wiener integrals associated with the Hilbert space determined by the covariance structure of the Rosenblatt process. The integrands are assumed to belong to the Hilbert space $\mathcal{H}_{\mathcal{R}}$ induced by the covariance structure of the Rosenblatt process, namely $\|\psi\|^2_{\mathcal{H}_{\mathcal{R}}} < \infty$, which guarantees the existence of the Wiener integral with respect to Z_H. Here, $\mathcal{H}_{\mathcal{R}}$ denotes the deterministic Hilbert space associated with the covariance structure of the Rosenblatt process, which is different from the stochastic space $L_2(\Omega, Z)$.

In addition, relevant knowledge about fractional calculus can be found in [3].

Definition 1. The fractional integral of order $\alpha \in (0,1)$ of the integrable function $w(r) \in C(J,\mathbb{R}^n)$ is defined as

$$I^\alpha w(r) = \frac{1}{\Gamma(\alpha)}\int_0^r (r-z)^{\alpha-1}w(z)dz,$$

where the $\Gamma(\cdot)$ is the Gamma function.

Definition 2. The Caputo fractional derivative of order $\alpha \in (0,1)$ of the integrable function $w(r) \in C(J,\mathbb{R}^n)$ is defined as

$$\mathcal{D}^\alpha w(r) = \frac{1}{\Gamma(1-\alpha)}\int_0^r (r-z)^{-\alpha}w'(z)dz.$$

Our research targets the problems of fractional singular kernels and multiple time-varying delays. It is assumed that the drift term complies with the superlinear condition. For these reasons, the following generalized Bihari inequality will be employed.

Lemma 2. (Generalized Bihari inequality with time-varying delays) Assume that the delay functions $\rho_i : [0,T] \to [0,\rho]$, $i = 1,\ldots,m$ satisfy $\rho_i'(r) \leq \ell_i < 1, \quad r \in [0,T]$. Define $\eta_i(r) = r - \rho_i(r)$, and let $\phi_i = \eta_i^{-1}$ be the inverse function of η_i. Because $\eta_i'(r) = 1 - \rho_i'(r) \geq 1 - \ell_i > 0$, the mapping η_i is strictly increasing. For $-1 < \beta < 0$, define the variable-delay fractional kernel

$$\mathcal{K}_i(r,u) = \frac{(r-\phi_i(u))^\beta}{1-\rho_i'(\phi_i(u))}.$$

Furthermore, suppose that this delay-induced kernel satisfies the weakly singular estimate; that is, there exists a constant $C_i > 0$ depending on the delay functions such that

$$\mathcal{K}_i(r,u) \leq C_i(r-u)^\beta,\ 0 \leq u < r \leq T.$$

Let $v(r) \geq 0$ be continuous on $[-\rho, T]$, Φ be continuous and nondecreasing, $\Phi(s) > 0$ for $s > 0$, and $a(r)$ be non-negative and nondecreasing if

$$v(r) \leq a(r) + \sum_{i=1}^{m} \int_0^r \mathcal{K}_i(r,u)\Phi(v(u))du + \int_0^r (r-z)^\beta c(z)[v(z)]^p dz \tag{2.1}$$

holds, where $p > 0$, $p \neq 1$. Then,

$$v(r) \leq \bar{a}(r) + C \int_0^r (r-u)^\beta \Phi(v(u))du + \bar{c} \int_0^r (r-z)^\beta [v(z)]^p dz, \tag{2.2}$$

where $C = \sum\limits_{i=1}^{m} C_i$, $\bar{a}(r) = \max\left\{a(r), \sup\limits_{z\in[-\rho,0]} v(z)\right\}$, $\bar{c} = \sup\limits_{z\in[0,T]} c(z)$.

In particular, if $\Phi(s) = s^p$, then

$$v(r) \leq \left[\bar{a}(r)^{1-p} - (1-p)\bar{c}\frac{r^{\beta+1}}{\beta+1}\right]^{\frac{1}{1-p}} \exp\left(C\frac{r^{\beta+1}}{\beta+1}\right). \tag{2.3}$$

Proof. Because $\rho_i'(r) \leq \ell_i < 1$, we have $\eta_i'(r) = 1 - \rho_i'(r) \geq 1 - \ell_i > 0$. Hence, η_i is strictly increasing and admits an inverse function $\phi_i = \eta_i^{-1}$. Moreover, differentiating $\eta_i(\phi_i(u)) = u$, we obtain

$$\phi_i'(u) = \frac{1}{1 - \rho_i'(\phi_i(u))}.$$

Therefore, the delay-induced fractional kernel can be written as

$$\mathcal{K}_i(r,u) = (r - \phi_i(u))^\beta \phi_i'(u) = \frac{(r-\phi_i(u))^\beta}{1-\rho_i'(\phi_i(u))}.$$

By the weakly singular estimate of the delay-induced kernel,

$$\mathcal{K}_i(r,u) \leq C_i(r-u)^\beta, \ 0 \leq u < r \leq T.$$

Using this estimate, we obtain

$$v(r) \leq a(r) + \sum_{i=1}^{m} C_i \int_0^r (r-u)^\beta \Phi(v(u))du + \int_0^r (r-z)^\beta c(z)[v(z)]^p dz.$$

Because $a(r) \leq \bar{a}(r)$, $c(z) \leq \bar{c}$, and $C = \sum_{i=1}^m C_i$, we have

$$v(r) \leq \bar{a}(r) + C \int_0^r (r-u)^\beta \Phi(v(u))du + \bar{c} \int_0^r (r-z)^\beta [v(z)]^p dz.$$

Thus, (2.2) follows.

Finally, when $\Phi(s) = s^p$ with $p > 0$, $p \neq 1$, applying the generalized Bihari inequality with weakly singular kernels yields

$$v(r) \leq \left[\bar{a}(r)^{1-p} - (1-p)\bar{c}\frac{r^{\beta+1}}{\beta+1}\right]^{\frac{1}{1-p}} \exp\left(C\frac{r^{\beta+1}}{\beta+1}\right).$$

Therefore, (2.3) follows. □

Remark 1. It is worth noting that Lemma 2 contains the classical Bihari-type inequalities as a special case. When $\beta = 0$, the weakly singular kernel becomes $(r-u)^\beta = (r-u)^0 = 1$, and the integral inequality in Lemma 2 reduces to the standard nonlinear integral inequality

$$\nu(r) \le \bar{a}(r) + C\int_0^r \Phi(\nu(u))du + \bar{c}\int_0^r [\nu(z)]^p dz.$$

In particular, when $\Phi(s) = s^p$, the estimate in Lemma 2 becomes

$$\nu(r) \le \left[\bar{a}(r)^{1-p} - (p-1)\bar{c}r\right]^{\frac{1}{1-p}} \exp(Cr),$$

which reduces to a classical Bihari-type estimate. Therefore, Lemma 2 can be regarded as a weakly singular extension of the classical Bihari inequality.

We use Krasnoselskii and Banach fixed-point theorems to prove the existence and uniqueness of mild solutions.

Theorem 1. [35] (Krasnoselskii fixed-point theorem) Let Z be a nonempty, bounded, closed, and convex subset of a Banach space X. Suppose that two operators $A, B : Z \to X$ satisfy the following:
(I) $A\varsigma + B\varpi \in Z$ for all $\varsigma, \varpi \in Z$;
(II) A is a contraction mapping;
(III) B is completely continuous.
Then, the sum operator $A + B$ has at least one fixed point in Z; that is, there exists $\varsigma^* \in Z$ such that

$$\varsigma^* = A\varsigma^* + B\varsigma^*.$$

3. Existence and uniqueness

This section focuses on the following neutral fractional non-instantaneous impulsive stochastic functional differential equations with Rosenblatt process and multiple time-varying delays:

$$\begin{cases} \mathcal{D}^\alpha\left[\varsigma(r) - G(r, \varsigma(r-\rho(r))), \ldots, \varsigma(r-\rho(r))))\right] = \mathfrak{A}\varsigma(r) + w(r, \varsigma(r), \varsigma(r-\rho(r))), \ldots, \varsigma(r-\rho(r))))dr \\ \qquad\qquad + \sigma(r, \varsigma(r-\rho(r))), \ldots, \varsigma(r-\rho(r))))dZ_H(r),\ r \in \bigcup_k^n (\zeta_k, \theta_{k+1}], \\ \varsigma(r) = I_k(r, \varsigma(\theta_k^-)),\ r \in (\theta_k, \zeta_k],\ k = 0, 1, \ldots, n, \\ \varsigma(r) = \phi(r),\ r \in [-\rho, 0], \end{cases} \tag{3.1}$$

where $\alpha \in (1/2, 1)$, and the delay functions $\rho_i : \bigcup_k^n(\zeta_k, \theta_{k+1}] \to [0, \rho]$ satisfy $\rho_i'(r) \le \ell_i < 1$, $i = 1, \ldots, m$, for $r \in [0, T]$. Further, $\phi : [-\rho, 0] \to \mathbb{R}^n$, $\varsigma : [-\rho, T] \to \mathbb{R}^n$, $G, \sigma : \bigcup_k^n(\zeta_k, \theta_{k+1}] \times (\mathbb{R}^n)^m \to \mathbb{R}^n$, $w : \bigcup_k^n(\zeta_k, \theta_{k+1}] \times (\mathbb{R}^n)^{m+1} \to \mathbb{R}^n$, $I_k : (\theta_k, \zeta_k] \times \mathbb{R}^n \to \mathbb{R}^n$. $\{\theta_k\}$ and $\{\zeta_k\}$ are given as two increasing sequences of points satisfying $-\rho < 0 < \theta_1 \le \zeta_1 < \cdots < \theta_k \le \zeta_k < \theta_{k+1} \le \cdots \le \theta_n \le T$. $\{Z_H(r)\}$ is the Rosenblatt process, where $H \in (1/2, 1)$ is the Hurst parameter. $\mathfrak{A}$ constitutes the generator of a strongly continuous semigroup of operators $S_\alpha(r)$, and $S_\alpha(r)$ denote the solution operators generated by the linear fractional operator associated with the homogeneous system. It describes the fractional evolution from the initial time 0 to the time r. For the interval after the kth non-instantaneous impulse, the two-parameter operator $S_\alpha(r, \zeta_k)$ is introduced to represent the evolution from the impulse starting point ζ_k to the current time r, where $r \geq \zeta_k$. The operator satisfies $S_\alpha(\zeta_k, \zeta_k) = I$, and reduces to the single-parameter operator when $\zeta_k = 0$.

By performing α integration on both sides of Eq (3.1), we can obtain the following mild solution.

Lemma 3. The mapping $\varsigma : [-\rho, T] \to \mathbb{R}^n$ is the mild solution to Eq (3.1), if and only if $\varsigma(r)$ satisfies to the following integral equations:

$$\varsigma(r) = \begin{cases} S_\alpha(r)[\phi(0) - G(0, \phi(-\rho_1(0)), \ldots, \phi(-\rho_m(0)))] + G(r, \varsigma(r-\rho(r))), \ldots, \varsigma(r-\rho(r)))) \\ +\frac{1}{\Gamma(\alpha)}\Big(\int_0^r (r-z)^{\alpha-1} S_\alpha(r-z) w(z, \varsigma(z), \varsigma(z-\rho(z))), \ldots, \varsigma(z-\rho(z))))dz \\ + \int_0^r (r-z)^{\alpha-1} S_\alpha(r-z)\sigma(z, \varsigma(z-\rho(z))), \ldots, \varsigma(z-\rho(z))))dZ_H(z)\Big),\ r \in (0, \theta_1], \\ I_k(r, \varsigma(\theta_k^-)),\ r \in (\theta_k, \zeta_k],\ k = 0, 1, \ldots, n, \\ S_\alpha(r, \zeta_k)[\varsigma(\zeta_k^+) - G(\zeta_k^+, \varsigma(\zeta_k^+ - \rho_1(\zeta_k^+)), \ldots, \varsigma(\zeta_k^+ - \rho_m(\zeta_k^+)))] \\ +G(r, \varsigma(r-\rho(r))), \ldots, \varsigma(r-\rho(r)))) \\ +\frac{1}{\Gamma(\alpha)}\Big(\int_{\zeta_k}^r (r-z)^{\alpha-1} S_\alpha(r-z) w(z, \varsigma(z), \varsigma(z-\rho(z))), \ldots, \varsigma(z-\rho(z))))dz \\ + \int_{\zeta_k}^r (r-z)^{\alpha-1} S_\alpha(r-z)\sigma(z, \varsigma(z-\rho(z))), \ldots, \varsigma(z-\rho(z))))dZ_H(z)\Big),\ r \in \bigcup_k^n (\zeta_k, \theta_{k+1}], \\ \phi(r),\ r \in [-\rho, 0]. \end{cases}$$

Next, with the Krasnoselskii and Banach fixed-point theorems, we prove the existence and uniqueness of mild solutions for Eq (3.1).

Theorem 2. Assume that the following conditions hold:

(A_1) There exist constants $p > 1$ and $L_0, L_{i1} \geq 0$ such that

$$\|w(r, \varsigma, \varpi_1, \ldots, \varpi_m)\|^2 \leq L_0 \|\varsigma\|^{2p} + \sum_{i=1}^m L_{i1} \|\varpi_i\|^{2p}.$$

(A_2) For any bounded set $D \subset \mathbb{R}^n$, there exist constants $L_w(D), L_{i2}(D) > 0$ such that, for all $\varsigma_1, \varsigma_2, \varpi_i, \varrho_i \in D$,

$$\|w(r, \varsigma_1, \varpi_1, \ldots) - w(r, \varsigma_2, \varrho_1, \ldots)\| \leq L_w(D)\|\varsigma_1 - \varsigma_2\| + \sum_{i=1}^m L_{i2}(D)\|\varpi_i - \varrho_i\|.$$

Moreover, for any bounded set $D \subset \mathbb{R}^n$, $w(r, \cdot)$ is uniformly continuous with respect to r.

(A_3) There exist constants $M_{i1} \geq 0$, $C_G \geq 0$ such that

$$\|G(r, \varpi_1, \ldots, \varpi_m)\|^2 \leq \sum_{i=1}^m M_{i1} \|\varpi_i\|^2 + C_G.$$

(A_4) There exist constants $M_{i2} \geq 0$ such that

$$\|G(r, \varpi_1, \ldots, \varpi_m) - G(r, \varrho_1, \ldots, \varrho_m)\| \leq \sum_{i=1}^m M_{i2} \|\varpi_i - \varrho_i\|.$$

Moreover, for any bounded set $D \subset \mathbb{R}^n$, $G(r, \cdot)$ is uniformly continuous with respect to r.

(A_5) There exists $C_\sigma \geq 0$ such that

$$\|\sigma(r, \varpi_1, \ldots, \varpi_m)\| \leq C_\sigma \left(1 + \sum_{i=1}^{m} \|\varpi_i\|^2\right).$$

(A_6) There exists $L_\sigma \geq 0$ such that

$$\|\sigma(r, \varpi_1, \ldots, \varpi_m) - \sigma(r, \varrho_1, \ldots, \varrho_m)\| \leq L_\sigma \sum_{i=1}^{m} \|\varpi_i - \varrho_i\|.$$

Moreover, for any bounded set $D \subset \mathbb{R}^n$, $\sigma(r, \cdot)$ is uniformly continuous with respect to r.

(A_7) There exist $\mu_z \in (0, 1)$ such that

$$\|I_k(r, \varsigma) - I_k(r, \varpi)\| \leq \mu_k \|\varsigma - \varpi\|.$$

(A_8) The inequalities

$$4M^2(1 + (\sum_{i=1}^{m} M_{i2})^2) + \frac{2mM^2 K_H T^{2H-1} L_\sigma^2 L_k^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha - 1)} < 1, \ \sqrt{C_{A1} + C_{A2} R^2} + C_{B1} + C_{B2} R^p \leq R$$

hold, where

$$\begin{aligned}
C_{A1} &= 4M^2 C_G^2 + \frac{2M^2 K_H T^{2H-1} C_\sigma L_k^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha - 1)}, \\
C_{A2} &= 4M^2((1 + 2m(\sum_{i=1}^{m} M_{i1})^2) + \frac{2mM^2 K_H T^{2H-1} L_k^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha - 1)}, \\
C_{B1} &= 2C_G, C_{B2} = \sqrt{4m(\sum_{i=1}^{m} M_{i1})^2 + \frac{2M^2 L_k^{2\alpha}(L_0 \ + \sum_{i=1}^{m} L_{i1})}{\Gamma^2(\alpha)(2\alpha - 1)}}.
\end{aligned}$$

Then, the mild solution of Eq (3.1) exists.

Proof. Consider the space $X = \{\varsigma : (\zeta_k, \theta_{k+1}] \to \mathbb{R}^n \mid \sup_{r \in (\zeta_k, \theta_{k+1}]} \mathbb{E}\|\varsigma(r)\|^2 < \infty\}$ with the norm defined by $\|\varsigma\|_X = \Big(\sup_{r \in (\zeta_k, \theta_{k+1}]} \mathbb{E}\|\varsigma(r)\|^2\Big)^{\frac{1}{2}}$. Construct the closed ball $\mathcal{M} = \{\varsigma \in X : \|\varsigma\|_X \leq R\}$. Define the operator:

$$(\mathcal{F}\varsigma)(r) = \begin{cases}
S_\alpha(r)[\phi(0) - G(0, \phi(-\rho_1(0)), \ldots, \phi(-\rho_m(0)))] + G(r, \varsigma(r - \rho(r))), \ldots, \varsigma(r - \rho(r)))) \\
+\frac{1}{\Gamma(\alpha)}\Big(\int_0^r (r - z)^{\alpha-1} S_\alpha(r - z) w(z, \varsigma(z), \varsigma(z - \rho_1(z)), \ldots, \varsigma(z - \rho_m(z))) dz \\
+ \int_0^r (r - z)^{\alpha-1} S_\alpha(r - z) \sigma(z, \varsigma(z - \rho_1(z)), \ldots, \varsigma(z - \rho_m(z))) dZ_H(z)\Big), \ r \in (0, \theta_1], \\
I_k(r, \varsigma(\theta_k^-)), \ t \in (\theta_k, \zeta_k], \ k = 0, 1, \ldots, n, \\
S_\alpha(r, \zeta_k)[\varsigma(\zeta_k^+) - G(\zeta_k^+, \varsigma(\zeta_k^+ - \rho_1(\zeta_k^+)), \ldots, \varsigma(\zeta_k^+ - \rho_m(\zeta_k^+)))] \\
+G(r, \varsigma(r - \rho(r))), \ldots, \varsigma(r - \rho(r)))) \\
+\frac{1}{\Gamma(\alpha)}\Big(\int_{\zeta_k}^r (r - z)^{\alpha-1} S_\alpha(r - z) w(z, \varsigma(z), \varsigma(z - \rho_1(z)), \ldots, \varsigma(z - \rho_m(z))) dz \\
+ \int_{\zeta_k}^r (r - z)^{\alpha-1} S_\alpha(r - z) \sigma(z, \varsigma(z - \rho_1(z)), \ldots, \varsigma(z - \rho_m(z))) dZ_H(z)\Big), \ r \in \bigcup_k^n (\zeta_k, \theta_{k+1}], \\
\phi(r), \ r \in [-\rho, 0].
\end{cases} \tag{3.2}$$

To apply Theorem 1, we define the operators in $r \in (\zeta_k, \theta_{k+1}]$:

$$\begin{aligned}(\mathcal{A}\varsigma)(r) &= S_\alpha(r,\zeta_k)[\varsigma(\zeta_k^+) - G(\zeta_k^+, \varsigma(\zeta_k^+ - \rho_1(\zeta_k^+)), \ldots, \varsigma(\zeta_k^+ - \rho_m(\zeta_k^+)))] \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{\zeta_k}^r (r-z)^{\alpha-1} S_\alpha(r-z)\sigma(z, \varsigma(z-\rho_1(z)), \ldots, \varsigma(z-\rho_m(z)))dZ_H(z),\end{aligned}$$

and

$$\begin{aligned}(\mathcal{B}\varsigma)(r) &= G(r, \varsigma(r-\rho(r))), \ldots, \varsigma(r-\rho(r)))) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{\zeta_k}^r (r-z)^{\alpha-1} S_\alpha(r-z) w(z, \varsigma(z), \varsigma(z-\rho_1(z)), \ldots, \varsigma(z-\rho_m(z)))dz.\end{aligned}$$

Step 1. The operator $\mathcal{A}$ is a contraction mapping.

For any $\varsigma, \varpi \in \mathcal{M}$, because $\mathcal{M}$ is a bounded, closed subset of the phase space, the local Lipschitz conditions in (A_4) and (A_6) imply that there exist finite constants $M_{i2}(\mathcal{M})$ and $L_\sigma(\mathcal{M})$ such that the following estimates hold:

$$\begin{aligned}\mathbb{E}\|(\mathcal{A}\varsigma)(r) - (\mathcal{A}y)(r)\|^2 &\leq 2\mathbb{E}\|S_\alpha(r,\zeta_k)[\varsigma(\zeta_k^+) - \varpi(\zeta_k^+) + (G(\zeta_k^+, \varsigma(\zeta_k^+ - \rho_1(\zeta_k^+)), \ldots, \varsigma(\zeta_k^+ - \rho_m(\zeta_k^+)))) \\ &\quad - G(\zeta_k^+, \varpi(\zeta_k^+ - \rho_1(\zeta_k^+)), \ldots, \varpi(\zeta_k^+ - \rho_m(\zeta_k^+))))]\|^2 \\ &\quad + \frac{2}{\Gamma(\alpha)}\mathbb{E}\| \int_{\zeta_k}^r (r-z)^{\alpha-1} S_\alpha(r-z)[\sigma(z, \varsigma(z-\rho_1(z)), \ldots, \varsigma(z-\rho_m(z))) \\ &\quad - \sigma(z, \varpi(z-\rho_1(z)), \ldots, \varpi(z-\rho_m(z)))]dZ_H(z)\|^2 \\ &= I_1 + I_2.\end{aligned}$$

Because $\mathfrak{A}$ is a strongly continuous semigroup of operators $S_\alpha(r)$, there exsits a positive constant M such that $\|S_\alpha(r)\| \leq M$, and then, based on the condition (A_4), there is

$$\begin{aligned}I_1 &\leq 2M^2\mathbb{E}\|\varsigma(\zeta_k^+) - \varpi(\zeta_k^+) + (G(\zeta_k^+, \varsigma(\zeta_k^+ - \rho_1(\zeta_k^+)), \ldots, \varsigma(\zeta_k^+ - \rho_m(\zeta_k^+))) \\ &\quad - G(\zeta_k^+, \varpi(\zeta_k^+ - \rho_1(\zeta_k^+)), \ldots, \varpi(\zeta_k^+ - \rho_m(\zeta_k^+))))\|^2 \\ &\leq 4M^2\mathbb{E}\|\varsigma(\zeta_k^+) - \varpi(\zeta_k^+)\|^2 + 4M^2\mathbb{E}\|G(\zeta_k^+, \varsigma(\zeta_k^+ - \rho_1(\zeta_k^+)), \ldots, \varsigma(\zeta_k^+ - \rho_m(\zeta_k^+))) \\ &\quad - G(\zeta_k^+, \varpi(\zeta_k^+ - \rho_1(\zeta_k^+)), \ldots, \varpi(\zeta_k^+ - \rho_m(\zeta_k^+))))\|^2 \\ &\leq 4M^2\mathbb{E}\|\varsigma(\zeta_k^+) - \varpi(\zeta_k^+)\|^2 + 4M^2(\sum_{i=1}^m M_{i2})^2 \sum_{i=1}^m \mathbb{E}\|\varsigma(\zeta_k^+ - \rho_i(\zeta_k^+)) - \varpi(\zeta_k^+ - \rho_i(\zeta_k^+))\|^2.\end{aligned}$$

Let $C_1 = 4M^2(1 + (\sum_{i=1}^m M_{i2}(M))^2)$, as $\mathbb{E}\|\varsigma(\zeta_k^+ - \rho_i(\zeta_k^+)) - \varpi(\zeta_k^+ - \rho_i(\zeta_k^+))\|^2 \leq \|\varsigma - \varpi\|_X^2$. Then, $I_1 \leq C_1\|\varsigma - \varpi\|_X^2$.

Define $L_k = \theta_{k+1} - \zeta_k$. Based on the condition (A_6) and Lemma 1, there is

$$\begin{aligned}I_2 &\leq \frac{2M^2}{\Gamma^2(\alpha)}\mathbb{E}\| \int_{\zeta_k}^r (r-z)^{\alpha-1}[\sigma(z, \varsigma(z-\rho_1(z)), \ldots, \varsigma(z-\rho_m(z))) \\ &\quad - \sigma(z, \varpi(z-\rho_1(z)), \ldots, \varpi(z-\rho_m(z)))]dZ_H(z)\|^2 \\ &\leq \frac{2M^2 K_H T^{2H-1} L_\sigma^2}{\Gamma^2(\alpha)} \int_{\zeta_k}^r (r-z)^{2\alpha-2} \sum_{i=1}^m \mathbb{E}\|\varsigma(z-\rho_i(z)) - \varpi(z-\rho_i(z))\|^2 dz \\ &\leq \frac{2mM^2 K_H T^{2H-1} L_\sigma^2 L_k^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}\|\varsigma - \varpi\|_X^2.\end{aligned}$$

Letting

$$C_2 = \frac{2mM^2 K_H T^{2H-1} L_\sigma^2(M) L_k^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)},$$

it follows that

$$\mathbb{E}\|(\mathcal{A}\varsigma)(r) - (\mathcal{A}\varpi)(r)\|^2 \leq (C_1 + C_2)\|\varsigma - \varpi\|_X^2.$$

Let $q = C_1 + C_2$. According to the condition (A_8), the constants involved in the above estimates satisfy $q < 1$. Therefore,

$$\mathbb{E}\|(\mathcal{A}\varsigma)(r) - (\mathcal{A}\varpi)(r)\|^2 \leq q\|\varsigma - \varpi\|_X^2,$$

which implies that $\mathcal{A}$ is a contraction mapping.

Step 2. The operator $\mathcal{B}$ is a compact continuous mapping.

For any $\varsigma,\ \varpi \in \mathcal{M}$, based on Minkowski inequality, we have

$$\begin{aligned}
&\mathbb{E}\|(\mathcal{B}\varsigma)(r) - (\mathcal{B}\varpi)(r)\|^2 \\
\leq\ & 2\mathbb{E}\|G(r, \varsigma(r-\rho_1(r)), \ldots, \varsigma(r-\rho_m(r)))) - G(r, \varpi(r-\rho_1(r)), \ldots, \varpi(r-\rho_m(r)))\|^2 \\
&+ \frac{2}{\Gamma^2(\alpha)}\mathbb{E}\|\int_{\zeta_k}^r (r-z)^{\alpha-1} S_\alpha(r-z)[w(z, \varsigma(z-\rho_1(z)), \ldots, \varsigma(z-\rho_m(z))) \\
&- w(z, \varpi(z-\rho_1(z)), \ldots, \varpi(z-\rho_m(z)))]dz\|^2 \\
=\ & J_1 + J_2.
\end{aligned}$$

According to the condition (A_4), there is

$$J_1 \leq 2(\sum_{i=1}^m M_{i2})^2 \sum_{i=1}^m \mathbb{E}\|\varsigma(r-\rho_i(r)) - \varpi(r-\rho_i(r))\|^2 \leq 2m(\sum_{i=1}^m M_{i2})^2\|\varsigma - \varpi\|_X^2.$$

Based on Hölder's inequatlity and the condition (A_2), we have

$$\begin{aligned}
J_2 \leq\ & \frac{2M^2}{\Gamma^2(\alpha)}(\int_{\zeta_k}^r (r-z)^{2\alpha-2}dz)(\int_{\zeta_k}^r \mathbb{E}\|w(z, \varsigma(z-\rho_1(z)), \ldots, \varsigma(z-\rho_m(z))) \\
&- w(z, \varpi(z-\rho_1(z)), \ldots, \varpi(z-\rho_m(z)))\|^2 dz \\
\leq\ & \frac{2M^2 L_k^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \int_{\zeta_k}^r \sum_{i=1}^m (L_w \mathbb{E}\|\varsigma(z) - \varpi(z)\|dz + \sum_{i=1}^m L_{i2}\mathbb{E}\|\varsigma(z-\rho_i(z)) - \varpi(z-\rho_i(z))\|)^2 dz \\
\leq\ & \frac{2M^2 L_k^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}\Big(2L_w^2 \int_{\zeta_k}^r \sum_{i=1}^m \mathbb{E}\|\varsigma(z) - \varpi(z)\|^2 dz + 2(\sum_{i=1}^m L_{i2})^2 \int_{\zeta_k}^r \sum_{i=1}^m \mathbb{E}\|\varsigma(z-\rho_i(z)) - \varpi(z-\rho_i(z))\|^2 dz\Big) \\
\leq\ & \frac{4mM^2 L_k^{2\alpha}(L_w^2 + (\sum_{i=1}^m L_{i2})^2)}{\Gamma^2(\alpha)(2\alpha-1)}\|\varsigma - \varpi\|_X^2.
\end{aligned}$$

Defining

$$C_3 = 2m(\sum_{i=1}^m M_{i2})^2,\ C_4 = \frac{4mM^2 L_k^{2\alpha}(L_w^2 + (\sum_{i=1}^m L_{i2})^2)}{\Gamma^2(\alpha)(2\alpha-1)},$$

it follows that

$$\mathbb{E}\|(\mathcal{B}\varsigma)(r) - (\mathcal{B}\varpi)(r)\|^2 \leq (C_3 + C_4)\|\varsigma - \varpi\|_X^2,$$

and thus, the operator $\mathcal{B}$ is continuous.

In addition, based on the conditions (A_1) and (A_3), there is

$$
\begin{aligned}
\mathbb{E}\|(\mathcal{B}\varsigma)(r)\|^2 \le{}& 2\mathbb{E}\|G(r,\varsigma(r-\rho(r))),\dots,\varsigma(r-\rho(r))))\|^2 \\
&+\frac{2M^2}{\Gamma^2(\alpha)}\int_{\zeta_k}^{r}(r-z)^{\alpha-1}w(z,\varsigma(z),\varsigma(z-\rho_1(z)),\dots,\varsigma(z-\rho_m(z)))dz \\
\le{}& 2(\sum_{i=1}^{m}M_{i1})^2\|\varsigma\|_X^2+\frac{4M^2L_k^{2\alpha}}{\Gamma^2(\alpha)(2\alpha-1)}(L_0\|\varsigma\|_X^{2p}+\sum_{i=1}^{m}L_{i1}\|\varsigma\|_X^{2p})+2C_G^2.
\end{aligned}
$$

Because $\varsigma\in\mathcal{M}$, there is $\|\varsigma\|_X\le R$; thus, the operator $\mathcal{B}$ is uniformly bounded.

Furthermore, for any $r_1<r_2\in(\zeta_k,\theta_{k+1}]$, we have

$$
\begin{aligned}
&\mathbb{E}\|(\mathcal{B}\varsigma)(r_2)-(\mathcal{B}\varsigma)(r_1)\|^2 \\
\le{}& 2\mathbb{E}\|G(r_2,\varsigma(r_2-\rho_1(r_2)),\dots,\varsigma(r_2-\rho_m(r_2)))-G(r_1,\varsigma(r_1-\rho_1(r_1)),\dots,\varsigma(r_1-\rho_m(r_1)))\|^2 \\
&+\frac{4}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_{r_1}^{r_2}(r_2-s)^{\alpha-1}S_\alpha(r_2-sz)w(s,\varsigma(z-\rho_1(z)),\dots,\varsigma(z-\rho_m(z)))dz\right\|^2 \\
&+\frac{4}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_{\zeta_k}^{r_1}(r_2-z)^{\alpha-1}S_\alpha(r_2-z)-(r_1-z)^{\alpha-1}S_\alpha(r_1-z)w(z,\varsigma(z-\rho_1(z)),\dots,\varsigma(z-\rho_m(z)))dz\right\|^2.
\end{aligned}
$$

By the continuity of G with respect to the time variable, we have

$$
\lim_{r_2\to r_1}\mathbb{E}\|G(r_2,\varsigma(r_2-\rho_1(r_2)),\dots,\varsigma(r_2-\rho_m(r_2)))-G(r_1,\varsigma(r_1-\rho_1(r_1)),\dots,\varsigma(r_1-\rho_m(r_1)))\|^2=0.
$$

For the second term, by the condition (A_1), the boundedness of $\varsigma\in M$, and the boundedness of $S_\alpha(r)$, we obtain

$$
\begin{aligned}
&\frac{4}{\Gamma^2(\alpha)}\mathbb{E}\left\|\int_{r_1}^{r_2}(r_2-z)^{\alpha-1}S_\alpha(r_2-z)w(z,\varsigma(z-\rho_1(z)),\dots,\varsigma(z-\rho_m(z)))dz\right\|^2 \\
\le{}& \frac{4M^2}{\Gamma^2(\alpha)}\left(L_0+\sum_{i=1}^{m}L_{i1}\right)R^{2p}\frac{(r_2-r_1)^{2\alpha-1}}{2\alpha-1}.
\end{aligned}
$$

Hence, this term tends to zero as $r_2\to r_1$.

For the third term, because the operator family $S_\alpha(r)$ is strongly continuous, and w is continuous with respect to the time variable, we have $(r_2-z)^{\alpha-1}S_\alpha(r_2-z)\to(r_1-z)^{\alpha-1}S_\alpha(r_1-z)$ uniformly for $z\in[\zeta_k,r_1]$. Therefore, by the dominated convergence theorem, there is

$$
\lim_{r_2\to r_1}\mathbb{E}\left\|\int_{\zeta_k}^{r_1}\left[(r_2-z)^{\alpha-1}S_\alpha(r_2-z)\ \ -(r_1-z)^{\alpha-1}S_\alpha(r_1-z)\right]w(z,\varsigma(z-\rho_1(z)),\dots,\varsigma(z-\rho_m(z)))dz\right\|^2=0.
$$

Consequently,

$$
\lim_{r_2\to r_1}\mathbb{E}\|(\mathcal{B}\varsigma)(r_2)-(\mathcal{B}\varsigma)(r_1)\|^2=0.
$$

Thus, $\mathcal{B}(M)$ is equicontinuous. Because $\mathcal{B}(M)$ is also uniformly bounded, the Arzelà–Ascoli theorem implies that $\mathcal{B}(M)$ is relatively compact. Combining this with the continuity of $\mathcal{B}$, we conclude that $\mathcal{B}$ is a completely continuous operator. The equicontinuity here is understood in the mean-square sense. For the stochastic integral term driven by the Rosenblatt process, the continuity follows from Lemma 1 and the condition (A_6).

Step 3. $\mathcal{A}(\mathcal{M}) + \mathcal{B}(\mathcal{M}) \subset \mathcal{M}$.

For any $\varsigma,\ \varpi \in \mathcal{M}$, we have $\|\mathcal{A}\varsigma + \mathcal{B}\varpi\|_X \leq \|\mathcal{A}\varsigma\|_X + \|\mathcal{B}\varpi\|_X$, and

$$\begin{aligned}\mathbb{E}\|(\mathcal{A}\varsigma)(r)\|^2 &\leq 2\mathbb{E}\|S_\alpha(r,\zeta_k)[\varsigma(\zeta_k^+) - G(\zeta_k^+, \varsigma(\zeta_k^+ - \rho_1(\zeta_k^+)), \ldots, \varsigma(\zeta_k^+ - \rho_m(\zeta_k^+)))]\|^2 \\ &+ \frac{2}{\Gamma^2(\alpha)}\mathbb{E}\|\int_{\zeta_k}^{r}(r-z)^{\alpha-1}S_\alpha(r-z)\sigma(z, \varsigma(z-\rho_1(z)), \ldots, \varsigma(z-\rho_m(z)))dZ_H(z)\|^2 \\ &= K_1 + K_2.\end{aligned}$$

Because $\|\varsigma\|_X \leq R$, we have $\mathbb{E}\|\varsigma(\zeta_k^+ - \rho_i(\zeta_k^+)\|^2 \leq R^2$; thus, based on the conditions (A_3) and (A_5) and Lemma 1, there are

$$K_1 \leq 4M^2((1+2m(\sum_{i=1}^{m} M_{i1})^2)R^2 + C_G^2),$$

and

$$\begin{aligned}K_2 &\leq \frac{2M^2K_HT^{2H-1}}{\Gamma^2(\alpha)}\int_{\zeta_k}^{r}(r-z)^{2\alpha-2}\mathbb{E}\|\sigma(z, \varsigma(z-\rho_1(z)), \ldots, \varsigma(z-\rho_m(z)))\|^2 dz \\ &\leq \frac{2M^2K_HT^{2H-1}L_k^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}(C_\sigma(1+mR^2)).\end{aligned}$$

Then, there is

$$\|\mathcal{A}\varsigma\|_X^2 \leq 4M^2((1+2m(\sum_{i=1}^{m} M_{i1})^2)R^2 + C_G^2) + \frac{2M^2K_HT^{2H-1}L_k^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}(C_\sigma(1+mR^2)) = C_{A1} + C_{A2}R^2.$$

In addition, it follows from **Step 2** that

$$\begin{aligned}\|\mathcal{B}\varsigma\|_X^2 &\leq 2(\sum_{i=1}^{m} M_{i1})^2\|\varsigma\|_X^2 + 2C_G^2 + \frac{2M^2L_k^{2\alpha}}{\Gamma^2(\alpha)(2\alpha-1)}(L_0\|\varsigma\|_X^{2p} + \sum_{i=1}^{m} L_{i1}\|\varsigma\|_X^{2p}) \\ &\leq 2m(\sum_{i=1}^{m} M_{i1})^2R^2 + \frac{2M^2L_k^{2\alpha}}{\Gamma^2(\alpha)(2\alpha-1)}\Big((L_0 + \sum_{i=1}^{m} L_{i1})R^{2p}\Big) + 2C_G^2.\end{aligned}$$

Because $p > 1$, the operator $\mathcal{B}$ satisfies the estimate $\|\mathcal{B}\varsigma\|_X \leq C_{B1} + C_{B2}R^p$. Although the growth term is superlinear, this does not affect the application of Krasnoselskii's fixed-point theorem, as $\mathcal{B}$ is only required to be completely continuous rather than contractive. The superlinear term is controlled by the ball invariance condition (A_8), which ensures that the operator mapping remains in the closed ball $\mathcal{M}$.

For $r \in (\theta_k, \zeta_k]$, by the condition (A_7), $I_k(r,\cdot)$ is Lipschitz continuous, and $I_k(\cdot,\varsigma)$ is continuous. Consequently, for any given $\varsigma(\theta_k^-) \in \mathbb{R}^n$, the function $\varsigma(r)$ defined by (3.2) is uniquely determined and continuous on $(\theta_k, \zeta_k]$. Thus, the existence and uniqueness on the non-instantaneous impulse interval follow directly from the definition of I_k.

It should be noted that the superlinear growth condition in (A_1) only affects the boundedness estimate of $\mathcal{B}$, whereas the contraction property of $\mathcal{A}$ is determined by the Lipschitz conditions in (A_4) and (A_6). Because $\mathcal{B}$ is required to be completely continuous rather than contractive in Krasnoselskii's fixed-point theorem, the superlinear term is controlled by the ball invariance condition (A_8).

In conclusion, based on the Krasnoselskii fixed-point theorem, Eq (3.1) admits the mild solution. □

Theorem 3. If assumptions $(A_2), (A_4), (A_6)$, and (A_7) hold on the bounded closed ball $\mathcal{M}_R$, and

$$3m(\sum_{i=1}^{m} M_{i2})^2 + \frac{3M^2 L_k^{2\alpha-1}(2L_k(L_w^2 + m(\sum_{i=1}^{m} L_{i2})^2) + m^2 K_H T^{2H-1} L_\sigma^2)}{\Gamma^2(\alpha)(2\alpha - 1)} < 1, \tag{3.3}$$

holds, then Eq (3.1) has a unique solution.

Proof. Because the operator is considered on the bounded closed ball $\mathcal{M}_R$, the local Lipschitz conditions in $(A_2), (A_4)$, and (A_6) imply that the corresponding Lipschitz constants are finite on $\mathcal{M}_R$. Let $\varsigma(r)$, $\varpi(r)$ be mild solutions of Eq (3.1), and let $e(r) = \varsigma(r) - \varpi(r)$. Then, for $r \in (\zeta_k, \theta_{k+1}]$, we have

$$\begin{aligned}
\mathbb{E}\|e(r)\|^2 \leq{}& 3\mathbb{E}\|G(r, \varsigma(r - \rho(r))), \ldots, \varsigma(r - \rho(r)))) - G(r, \varpi(r - \rho_1(r)), \ldots, \varpi(r - \rho_m(r)))\|^2 \\
&+ \frac{3}{\Gamma^2(\alpha)}\mathbb{E}\|\int_{\zeta_k}^{r} (r - z)^{\alpha-1} S_\alpha(r - z)(w(z, \varsigma(z), \varsigma(z - \rho_1(z)), \ldots, \varsigma(z - \rho_m(z)) \\
&- w(z, \varpi(z), \varpi(z - \rho_1(z)), \ldots, \varpi(z - \rho_m(z)))))dz\|^2 \\
&+ \frac{3}{\Gamma^2(\alpha)}\mathbb{E}\|\int_{\zeta_k}^{r} (r - z)^{\alpha-1} S_\alpha(r - z)(\sigma(z, \varsigma(z - \rho_1(z)), \ldots, \varsigma(z - \rho_m(z))) \\
&- \sigma(z, \varpi(z - \rho_1(z)), \ldots, \varpi(z - \rho_m(z))))dZ_H(z)\|^2 \\
={}& Q_1 + Q_2 + Q_3.
\end{aligned}$$

Based on the conditions $(A_2), (A_4)$, and (A_6), there is

$$Q_1 \leq 3(\sum_{i=1}^{m} M_{i2})^2 \sum_{i=1}^{m} \mathbb{E}\|z(r - \rho_i(r))\|^2 \leq 3m(\sum_{i=1}^{m} M_{i2})^2 \sup_{r\in(\zeta_k,\theta_{k+1}]} \mathbb{E}\|e(r)\|^2,$$

$$Q_2 \leq \frac{6M^2 L_k^{2\alpha}(L_w^2 + m(\sum_{i=1}^{m} L_{i2})^2)}{\Gamma^2(\alpha)(2\alpha - 1)} \sup_{r\in(\zeta_k,\theta_{k+1}]} \mathbb{E}\|e(r)\|^2,$$

$$Q_3 \leq \frac{3m^2 M^2 K_H T^{2H-1} L_\sigma^2 L_k^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha - 1)} \sup_{r\in(\zeta_k,\theta_{k+1}]} \mathbb{E}\|e(r)\|^2,$$

and then, we have

$$\mathbb{E}\|e(r)\|^2 \leq 3m(\sum_{i=1}^{m} M_{i2})^2 + \frac{3M^2 L_k^{2\alpha-1}(2L_k(L_w^2 + m(\sum_{i=1}^{m} L_{i2})^2) + m^2 K_H T^{2H-1} L_\sigma^2)}{\Gamma^2(\alpha)(2\alpha - 1)} \sup_{r\in(\zeta_k,\theta_{k+1}]} \mathbb{E}\|e(r)\|^2.$$

According to (3.3) and the Banach fixed point theorem, we obtain $\mathbb{E}\|e(r)\|^2 = 0$; that is, Eq (3.1) has a unique mild solution on $r \in (\zeta_k, \theta_{k+1}]$.

On each impulse interval $(\theta_k, \zeta_k]$, the solution is given explicitly by

$$\varsigma(r) = I_k(r, \varsigma(\theta_k^-)),\ r \in (\theta_k, \zeta_k].$$

By Assumption (A_7), $I_k(r, \cdot)$ is Lipschitz continuous with constant $\mu_k < 1$, and hence, for any given $\varsigma(\theta_k^-)$, the right-hand side is uniquely determined. Consequently, existence and uniqueness on the impulse interval are immediate.

By induction, the mild solution exists uniquely on the whole interval $[0, T]$. □

4. Finite-time synchronization

The well-posedness is verified for neutral fractional stochastic systems with non-instantaneous impulses. Beyond single-system qualitative analysis, synchronization characterizes coordinated dynamics of coupled systems and has garnered widespread research interest, particularly for neural network configurations. This section focuses on the practical finite-time synchronization of neutral fractional neural networks driven by the Rosenblatt process. We write the drive systems as:

$$\begin{cases} \mathcal{D}^{\alpha}\Big[\varsigma_i(r) - D\varsigma_i(r-\rho_1(r))\Big] = -C\varsigma_i(r) + Aw(\varsigma_i(r)) + Bh(\varsigma_i(r-\rho_2(r))) + \sum_{j=1}^{N} v_{ij}\Gamma\varsigma_j(r) \\ \qquad\qquad + \sigma(r, \varsigma_i(r-\rho_1(r)), \dots, \varsigma_i(r-\rho_m(r)))dZ_H(r), \ r \in (0, T], \\ \varsigma(r) = \phi(r), \ r \in [-\rho, 0], \end{cases} \tag{4.1}$$

and the response systems with non-instantaneous impulses are given as follows:

$$\begin{cases} \mathcal{D}^{\alpha}\Big[\varpi_i(r) - D\varpi_i(r-\rho_1(r))\Big] = -C\varpi_i(r) + Aw(\varpi_i(r)) + Bh(\varpi_i(r-\rho_2(r))) + \sum_{j=1}^{N} v_{ij}\Gamma\varpi_j(r) \\ \qquad\qquad + \sigma(r, \varpi_i(r-\rho_1(r)), \dots, \varpi_i(r-\rho_m(r)))dZ_H(r) \\ \qquad\qquad + u_i(r), \ r \in \bigcup_{k}^{n}(\zeta_k, \theta_{k+1}], \\ \varpi(r) = I_k(r, \varpi(\theta_k^-)), \ t \in (\theta_k, \zeta_k], \ k = 0, 1, \dots, n, \\ \varpi(r) = \phi(r), \ r \in [-\rho, 0], \end{cases} \tag{4.2}$$

where $\varsigma_i(r)$ denotes the state variable of the ith neuron in the drive system; $\varpi_i(r)$ denotes the state variable of the ith neuron; $\rho_i(r), \ i = 1, 2$ denote the time-varying delays satisfying $\rho_i'(r) \leq \ell_i < 1$; D denotes the coefficient matrix of the neutral term; $C = \operatorname{diag}(c_1, c_2, \dots, c_n) > 0$ denotes the self-inhibition matrix; A, B denote the connection weight matrices; $w(\cdot), h(\cdot)$ denote the neuron activation functions; N denotes the number of neurons; v_{ij} denotes the coupling weight from the jth neuron to the ith neuron; Γ denotes the internal coupling matrix; $u_i(r)$ denotes the external control input. Define $e(r) = \varpi(r) - \varsigma(r)$; then, the error systems are given by

$$\begin{cases} \mathcal{D}^{\alpha}\Big[e(r) - De(r-\rho_1(r))\Big] = -Ce(r) + A[w(\varpi(r)) - w(\varsigma(r))] + \sum_{j=1}^{N} v_{ij}\Gamma e_j(r) \\ \qquad\qquad + B[h(\varpi(r-\rho_2(r))) - h(\varsigma(r-\rho_2(r)))] + u(r) \\ \qquad\qquad + [\sigma(r, \varpi_i(r-\rho_1(r)), \dots, \varpi_i(r-\rho_m(r))) \\ \qquad\qquad - \sigma(r, \varsigma_i(r-\rho_1(r)), \dots, \varsigma_i(r-\rho_m(r)))]dZ_H(r), \ r \in \bigcup_{k}^{n}(\zeta_k, \theta_{k+1}], \\ e(r) = I_k(r, \varpi(\theta_k^-)) - \varsigma(r), \ r \in (\theta_k, \zeta_k], \ k = 0, 1, \dots, n, \\ e(r) = 0, \ r \in [-\rho, 0]. \end{cases} \tag{4.3}$$

Definition 3. [17] The systems (4.1) and (4.2) are said to achieve practical finite-time synchronization if for any prescribed constant $\varepsilon > 0$, there exists a finite time $T_\varepsilon \leq T$ such that

$$\mathbb{E}\|e(r)\|^2 \leq \varepsilon, \ r \geq T_\varepsilon.$$

Here, T_ε denotes the practical settling time, and ε represents the prescribed synchronization error bound.

Based on the above definition, the main synchronization result is presented as follows.

Theorem 4. Under the conditions of Theorem 3, the following conditions also hold:

(B_1) There exist constants $L_w, L_h \geq 0$ such that

$$\|w(\varpi) - w(\varsigma)\| \leq L_w\|\varpi - \varsigma\|, \|h(\varpi) - h(\varsigma)\| \leq L_h\|\varpi - \varsigma\|.$$

(B_2) $D = \mathrm{diag}(d_1, \ldots, d_n)$ is a constant diagonal matrix with $\|D\| = \max\limits_{1\leq i\leq n} |d_i|$. Assume $d = \|D\| < 1/2$. Then,

$$\|De(r - \rho_1(r))\|^2 \leq d^2\|e(r - \rho_1(r))\|^2.$$

(B_3) The controller is designed as $u(r) = -\eta e(r) - \xi\|e(r)\|^{q-1}e(r),\ 0 < q < 1$, where $\eta > 0$, and $\xi > 0$. Moreover, the controller satisfies

$$\langle e(r), u(r)\rangle = -\eta\|e(r)\|^2 - \xi\|e(r)\|^{q+1}.$$

Then, the drive-response systems achieve practical finite-time synchronization, and the synchronization time can be estimated by

$$T_\varepsilon \leq \left(\frac{\varepsilon^{1-\gamma}(2\alpha - 1)}{(1 - \gamma)\bar{B}}\right)^{\frac{1}{2\alpha-1}},$$

where C_γ is a constant, and

$$\gamma = \frac{q + 1}{2},\ \ \bar{B} = \frac{4M^2\xi C_\gamma}{\Gamma^2(\alpha)(1 - 4d^2)}.$$

Proof. Based on Lemma 3, the synchronization error system can be expressed as

$$e(r) = \begin{cases} S_\alpha(r, \zeta_k)[e(\zeta_k^+) - De(\zeta_k^+ - \rho_1(\zeta_k^+))] + De(r - \rho_1(r)) + \frac{1}{\Gamma(\alpha)}\Big(\int_{\zeta_k}^r (r - z)^{\alpha-1}S_\alpha(r - z) \\ \times\Big[- Ce(z) + A(w(\varpi(z)) - w(\varsigma(z))) + B(h(\varpi(z - \rho_2(z))) - h(\varsigma(z - \rho_2(z)))) \\ + \sum_{j=1}^N v_{ij}\Gamma e_j(r) + u(r)\Big]dz + \int_{\zeta_k}^r (r - z)^{\alpha-1}S_\alpha(r - z)(\sigma(z, \varpi(z - \rho_1(z)), \ldots, \varpi(z - \rho_m(z))) \\ - \sigma(z, \varsigma(z - \rho_1(z)), \ldots, \varsigma(z - \rho_m(z)))dZ_H(z)\Big),\ r \in \bigcup_k^n(\zeta_k, \theta_{k+1}], \\ \\ I_k(r, \varpi(\theta_k^-)) - \varsigma(r),\ r \in (\theta_k, \zeta_k]. \end{cases}$$

For the first nonimpulse interval $r \in (0, \theta_1]$, because $e(0) = 0$, and $e(-\rho_1(0)) = 0$, we have

$$\begin{aligned} e(r) = De(r - \rho_1(r)) &+ \frac{1}{\Gamma(\alpha)}\int_0^r (r - z)^{\alpha-1}S_\alpha(r - z)\Big[- Ce(z) + A(w(\varpi(z)) - w(\varsigma(z))) \\ &+ B(h(\varpi(z - \rho_2(z))) - h(\varsigma(z - \rho_2(z)))) + \sum_{j=1}^N v_{ij}\Gamma e_j(z) + u(z)\Big]dz \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^r (r - z)^{\alpha-1}S_\alpha(r - z)(\sigma(z, \varpi(z - \rho_1(z)), \ldots, \varpi(z - \rho_m(z))) \\ &- \sigma(z, \varsigma(z - \rho_1(z)), \ldots, \varsigma(z - \rho_m(z))))dZ_H(z). \end{aligned}$$

Let $v(r) = \mathbb{E}\|e(r)\|^2$. Using the Minkowski inequality, we obtain

$$v(r) \leq P_1 + P_2 + P_3,$$

where $P_1 = 4\mathbb{E}\|De(r-\rho_1(r))\|^2$, and P_2, P_3 respectively denote the deterministic integral term and the stochastic integral term.

By the condition (B_2), we have

$$P_1 \leq 4d^2 v(r-\rho_1(r)).$$

For the deterministic integral term, according to (B_1) and the boundedness of $S_\alpha(r)$, there exists a constant $C_e > 0$ such that

$$P_2 \leq \frac{4M^2 C_e}{\Gamma^2(\alpha)} \int_0^r (r-z)^{2\alpha-2}(v(z) + v(z-\rho_2(z)))dz.$$

For the stochastic term, according to (A_6) and Lemma 1,

$$P_3 \leq \frac{4M^2 K_H T^{2H-1} L_\sigma}{\Gamma^2(\alpha)} \sum_{i=1}^m \int_0^r (r-z)^{2\alpha-2} v(z-\rho_i(z))dz.$$

Next, we consider the contribution of the controller. Different from the estimate of the controller norm, the dissipative effect of the controller is characterized by its inner product with the error state. By (B_3), $u(r) = -\eta e(r) - \xi\|e(r)\|^{q-1}e(r),\ 0 < q < 1$. Therefore, there is

$$\mathbb{E}\langle e(r), u(r)\rangle = -\eta\mathbb{E}\|e(r)\|^2 - \xi\mathbb{E}\|e(r)\|^{q+1}.$$

Because $e(r) \in \mathbb{R}^n$, and all norms in finite-dimensional spaces are equivalent, there exists a positive constant C_γ such that

$$\mathbb{E}\|e(r)\|^{q+1} \geq C_\gamma \left(\mathbb{E}\|e(r)\|^2\right)^{\frac{q+1}{2}}.$$

Therefore, we have

$$\mathbb{E}\|e(r)\|^{q+1} \geq C_\gamma [v(r)]^{\frac{q+1}{2}}.$$

Consequently, there is

$$\mathbb{E}\langle e(r), u(r)\rangle = -\eta\mathbb{E}\|e(r)\|^2 - \xi\mathbb{E}\|e(r)\|^{q+1} \leq -\eta v(r) - \xi C_\gamma [v(r)]^{\frac{q+1}{2}}.$$

Combining the above estimates, we obtain

$$\begin{aligned} v(r) \leq{}& 4d^2 v(r-\rho_1(r)) + \frac{4M^2(C_e+\eta)}{\Gamma^2(\alpha)} \int_0^r (r-z)^{2\alpha-2} v(z)dz + \frac{4M^2 C_e}{\Gamma^2(\alpha)} \int_0^r (r-z)^{2\alpha-2} v(z-\rho_2(z))dz \\ &+ \frac{4M^2 K_H T^{2H-1} L_\sigma}{\Gamma^2(\alpha)} \sum_{i=1}^m \int_0^r (r-z)^{2\alpha-2} v(z-\rho_i(z))dz - \frac{4M^2 \xi C_\gamma}{\Gamma^2(\alpha)} \int_0^r (r-z)^{2\alpha-2} [v(z)]^{\frac{q+1}{2}} dz. \end{aligned} \tag{4.4}$$

Because the delay functions satisfy $\rho_i'(r) \leq \ell_i < 1,\ r \in [0, T]$, we introduce the transformation $u = z - \rho_i(z)$. Let $\phi_i = \eta_i^{-1}$ be the inverse function of $\eta_i(z) = z - \rho_i(z)$. Then,

$$dz = \frac{du}{1-\rho_i'(\phi_i(u))}.$$

Therefore,

$$\int_0^r (r-z)^{2\alpha-2}\nu(z-\rho_i(z))dz = \int_{-\rho_i(0)}^{r-\rho_i(r)} \frac{(r-\phi_i(u))^{2\alpha-2}}{1-\rho_i'(\phi_i(u))}\nu(u)du.$$

Because $d < 1/2$, we have $1-4d^2 > 0$. Moving the delayed term to the left-hand side and dividing by $1-4d^2$, we obtain the following inequality. Define

$$\mathcal{K}_i(r,u) = \frac{(r-\phi_i(u))^{2\alpha-2}}{1-\rho_i'(\phi_i(u))}.$$

Then, from (4.4), we have

$$\nu(r) \le 4d^2\nu(r-\rho_1(r)) + A\int_0^r (r-z)^{2\alpha-2}\nu(z)dz + \sum_{i=1}^m A_i \int \mathcal{K}_i(r,u)\nu(u)du - B\int_0^r (r-z)^{2\alpha-2}[\nu(z)]^{\frac{q+1}{2}}dz,$$

where

$$A = \frac{4M^2(C_e+\eta)}{\Gamma^2(\alpha)},\ B = \frac{4M^2\xi C_\gamma}{\Gamma^2(\alpha)}.$$

Let $\gamma = (q+1)/2$. Because $0 < q < 1$, we have $0 < \gamma < 1$. Applying Lemma 2 with $\beta = 2\alpha-2$, we obtain

$$\nu(r) \le \left[\varepsilon^{1-\gamma} - (1-\gamma)\bar{B}\frac{r^{2\alpha-1}}{2\alpha-1}\right]^{\frac{1}{1-\gamma}} \exp\left(\bar{A}\frac{r^{2\alpha-1}}{2\alpha-1}\right),$$

where

$$\bar{A} = \frac{4M^2}{\Gamma^2(\alpha)(1-4d^2)}\left(C_e+\eta+mC_K K_H T^{2H-1}L_\sigma\right),\ \bar{B} = \frac{4M^2\xi C_\gamma}{\Gamma^2(\alpha)(1-4d^2)}.$$

On the impulse interval $(\theta_k, \zeta_k]$, the error satisfies

$$\|e(r)\| = \|I_k(r, \varpi(\theta_k^-)) - \varsigma(r)\|.$$

By adding and subtracting $I_k(r, \varsigma(\theta_k^-))$, we obtain

$$\|e(r)\| \le \|I_k(r, \varpi(\theta_k^-)) - I_k(r, \varsigma(\theta_k^-))\| + \|I_k(r, \varsigma(\theta_k^-)) - \varsigma(r)\|.$$

According to (A_7) and the continuity of the impulse operator, we have

$$\|e(r)\| \le \mu_k\|e(\theta_k^-)\| + C_k,$$

where $\mu_k < 1$, and C_k is a bounded constant determined by the continuity of the non-instantaneous impulse operator on $(\theta_k, \zeta_k]$. Hence, the non-instantaneous impulse interval provides an additional contraction effect. Therefore, after each impulse interval, the error bound obtained from the preceding nonimpulse interval is preserved and further reduced.

By repeating the above procedure over all nonimpulse and non-instantaneous impulse intervals, the synchronization error satisfies

$$\mathbb{E}\|e(r)\|^2 \le \varepsilon$$

within a finite time T_ε. Thus, the drive-response systems achieve practical finite-time synchronization. □

Example 1. We consider the following neural network systems. The drive systems are given by:

$$\begin{cases} {}^{C}D_t^{0.85}[\varsigma_1(r) - 0.25\varsigma_1(r-\rho_1(r))] = & -0.3\varsigma_1(r) + 2.5\tanh(\varsigma_1(r)) - 1.5\tanh(\varsigma_2(r)) \\ & + 1.2\tanh(\varsigma_1(r-\rho_2(r))) - 0.8\tanh(\varsigma_2(r-\rho_2(r))) \\ & + \dfrac{0.01}{1+\|\varsigma(r-\rho_1(r))\|}dZ_{0.7}(r), \\ {}^{C}D_t^{0.85}[\varsigma_2(r) - 0.25\varsigma_2(r-\rho_1(r))] = & -0.3\varsigma_2(r) + 1.0\tanh(\varsigma_1(r)) + 2.5\tanh(\varsigma_2(r)) \\ & - 0.6\tanh(\varsigma_1(r-\rho_2(r))) + 1.0\tanh(\varsigma_2(r-\rho_2(r))) \\ & + \dfrac{0.01}{1+\|\varsigma(r-\rho_1(r))\|}dZ_{0.7}(r). \end{cases}$$

The response systems are given by:

$$\begin{cases} {}^{C}D_t^{0.85}[\varpi_1(r) - 0.25\varpi_1(r-\rho_1(r))] = & -0.3\varpi_1(r) + 2.5\tanh(\varpi_1(r)) - 1.5\tanh(\varpi_2(r)) \\ & + 1.2\tanh(\varpi_1(r-\rho_2(r))) - 0.8\tanh(\varpi_2(r-\rho_2(r))) \\ & + u_1(r) + \dfrac{0.01}{1+\|\varpi(r-\rho_1(r))\|}dZ_{0.7}(r), \\ {}^{C}D_t^{0.85}[\varpi_2(r) - 0.25\varpi_2(r-\rho_1(r))] = & -0.3\varpi_2(r) + 1.0\tanh(\varpi_1(r)) + 2.5\tanh(\varpi_2(r)) \\ & - 0.6\tanh(\varpi_1(r-\rho_2(r))) + 1.0\tanh(\varpi_2(r-\rho_2(r))) \\ & + u_2(r) + \dfrac{0.01}{1+\|\varpi(r-\rho_1(r))\|}dZ_{0.7}(r), \end{cases}$$

where $\alpha = 0.85$, and $H = 0.7$ denotes the Hurst parameter of the Rosenblatt-driven stochastic perturbation. The time-varying delays are chosen as

$$\rho_1(r) = 0.2 + 0.05\sin(r), \qquad \rho_2(r) = 0.3 + 0.05\cos(r).$$

The initial states are selected as

$$\varsigma(0) = \begin{bmatrix} 3.0 \\ -2.5 \end{bmatrix}, \qquad \varpi(0) = \begin{bmatrix} -3.5 \\ 2.8 \end{bmatrix}.$$

Therefore, the initial synchronization error is

$$e(0) = \varpi(0) - \varsigma(0) = \begin{bmatrix} -6.5 \\ 5.3 \end{bmatrix}, \qquad \|e(0)\| \approx 8.38,$$

where $e(r) = \varpi(r) - \varsigma(r)$ denotes the synchronization error, and $\|e(r)\| = \sqrt{e_1^2(r) + e_2^2(r)}$. The system parameters are selected as

$$A = \begin{bmatrix} 2.5 & -1.5 \\ 1.0 & 2.5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1.2 & -0.8 \\ -0.6 & 1.0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \qquad D = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}.$$

According to the Lipschitz properties of the nonlinear activation functions, the corresponding constants are chosen as

$$||A|| \approx 3.5, \qquad ||B|| \approx 2.0, \qquad ||C|| = 0.3,$$

and

$$L_w = L_h = 1, \qquad ||\Gamma|| = ||W|| = 1.$$

The nonlinear finite-time controller is designed as

$$\begin{aligned} u_1(r) &= -\,4e_1(r) - 30.5||e(r)||^{-0.1}e_1(r), \\ u_2(r) &= -\,4e_2(r) - 30.5||e(r)||^{-0.1}e_2(r), \end{aligned}$$

which corresponds to $\eta = 4,\ \xi = 30.5,\ p = 0.9$.

For comparison, the linear controller is selected as

$$u_l(r) = -4e(r),$$

which corresponds to $\xi = 0$.

For the numerical implementation, the interval $[0, T]$ is uniformly divided into $N = 2000$ grid points with step size

$$\Delta t = \frac{T}{N} = 0.01.$$

The fractional integral term is approximated by a fractional convolution memory scheme. The stochastic perturbation is generated using a covariance-consistent approximation associated with the Rosenblatt-driven system, preserving the required second-order statistical properties. To reduce the influence of randomness, $M = 10$ independent realizations are considered. The delayed states are obtained according to the time-varying delay functions on the discrete grid. The selected parameters satisfy the boundedness and Lipschitz assumptions required in the theoretical analysis.

The numerical simulations are performed to illustrate the synchronization and stabilization performance of the proposed controller. The synchronization error under the nonlinear controller is shown in Figure 1.

As shown in Figure 2, under the nonlinear controller, the synchronization error converges from its initial value of 10 to below the threshold of 10^{-6} at approximately 2.56 s, which is consistent with the theoretical upper bound. In contrast, the linear controller requires approximately 5.57 s to reach the same threshold, and the uncontrolled case fails to synchronize.

To further investigate the influence of the fractional order α and the Hurst parameter H on the synchronization performance, additional sensitivity analyses are carried out. Because the proposed controller guarantees practical finite-time synchronization, the influence of these parameters is evaluated by two quantitative indicators: the integral synchronization error

$$J_e = \int_0^T ||e(r)||dr,$$

and the maximum steady-state synchronization error

$$E_s = \max_{r \geq T/2} ||e(r)||.$$

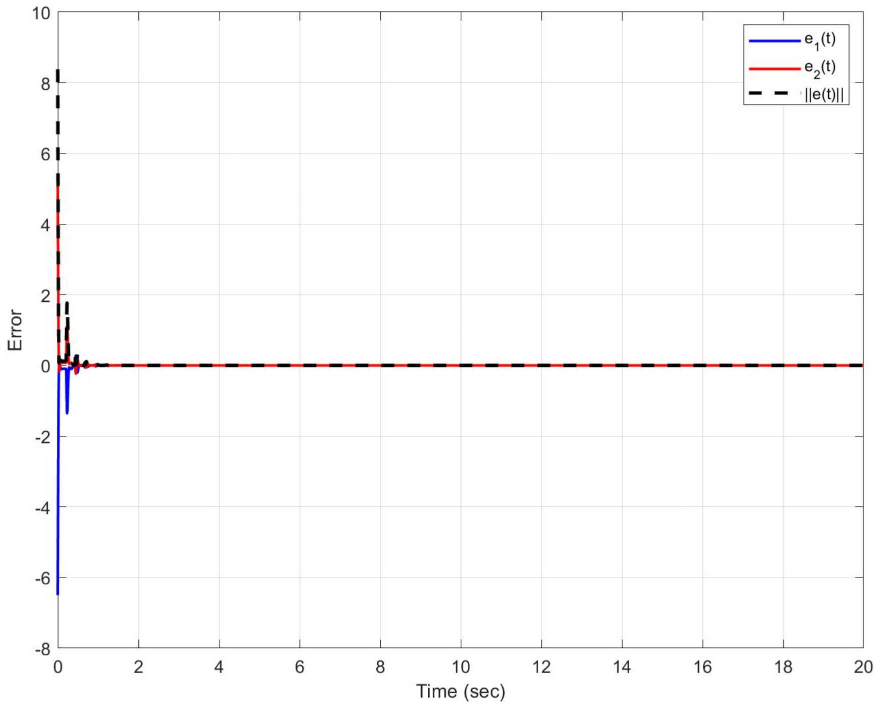

Figure 1. Synchronization error under nolinear control.

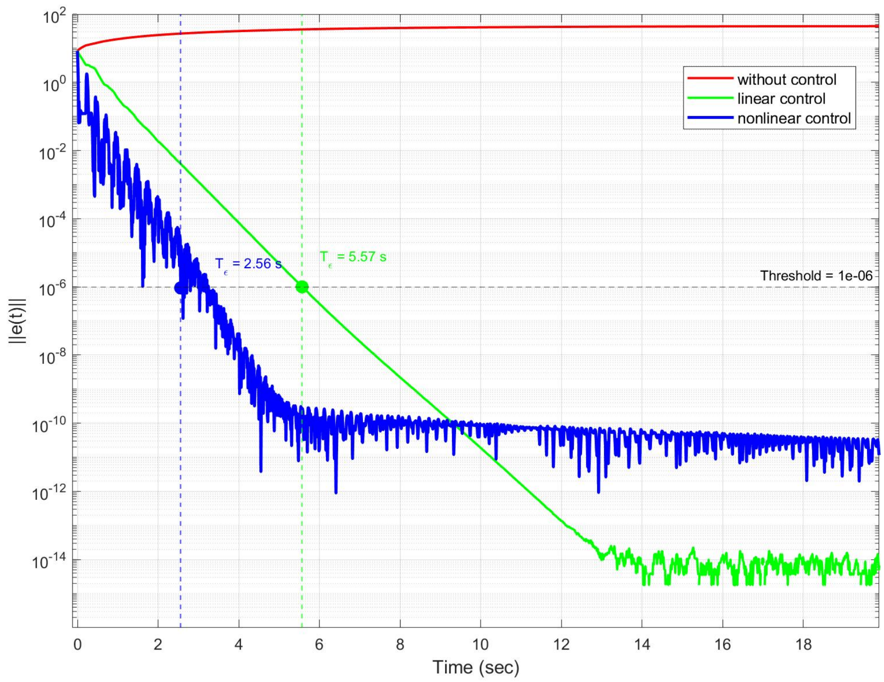

Figure 2. Synchronization time comparison under different control strategies.

The obtained results are summarized in Tables 4 and 5.

Table 4. Sensitivity analysis with respect to the Hurst parameter H.

H	J_e	E_s
0.51	11.667	1.8414×10^{-6}
0.70	11.680	1.8474×10^{-6}
0.95	11.683	1.8528×10^{-6}

Table 5. Sensitivity analysis with respect to the fractional order α.

α	J_e	E_s
0.65	12.846	3.0806×10^{-3}
0.80	11.829	2.0934×10^{-5}
0.95	11.525	1.5891×10^{-9}

It can be observed from Table 4 that different values of the Hurst parameter result in only slight variations of the synchronization indices. This indicates that the proposed controller is robust against different long-range dependence intensities of the Rosenblatt process.

Moreover, Table 5 shows that the fractional order has a more significant influence on synchronization performance. A larger fractional order leads to smaller synchronization errors in the considered example, which is attributed to the direct effect of α on the memory kernel of the fractional system.

Figures 3 and 4 illustrate the sensitivity of the synchronization time to the parameters p and ξ. For the linear control scheme, the synchronization time remains nearly constant over the entire range of p, confirming that linear control is independent of this parameter. In contrast, under the proposed nonlinear control, the synchronization time decreases monotonically as p increases, which is consistent with the theoretical prediction that a larger superlinear exponent accelerates finite-time convergence. Similarly, as ξ increases, the synchronization time exhibits a clear decreasing trend, verifying that a larger nonlinear gain contributes to faster synchronization. These numerical observations are in full agreement with the theoretical results established in Theorem 4.

It should be pointed out that the minor fluctuations observed in the two figures are reasonable. They mainly stem from the randomness of the Rosenblatt process and numerical discretization errors. Despite such fluctuations, the overall monotonically decreasing trend remains distinct, which fully verifies the correctness of the theoretical analysis.

In addition, while keeping the above nonlinear controller unchanged, the non-instantaneous impulses are implemented by modifying the response system state within each impulse interval $(\theta_z, \zeta_z]$. The pulse function is defined as

$$\varpi(r) = \varsigma(r) + \mu(\varpi(r^-) - \varsigma(r^-)), t \in (\theta_z, \zeta_z],$$

where $\mu = 0.005$ is the contraction coefficient. The non-instantaneous impulse intervals are set as $(\vartheta_z, \zeta_z] \in \{(1, 1.6], (3, 3.6], (5, 5.6], (7, 7.6], (9, 9.6], (11, 11.6], (13, 13.6], (15, 15.6]\}$.

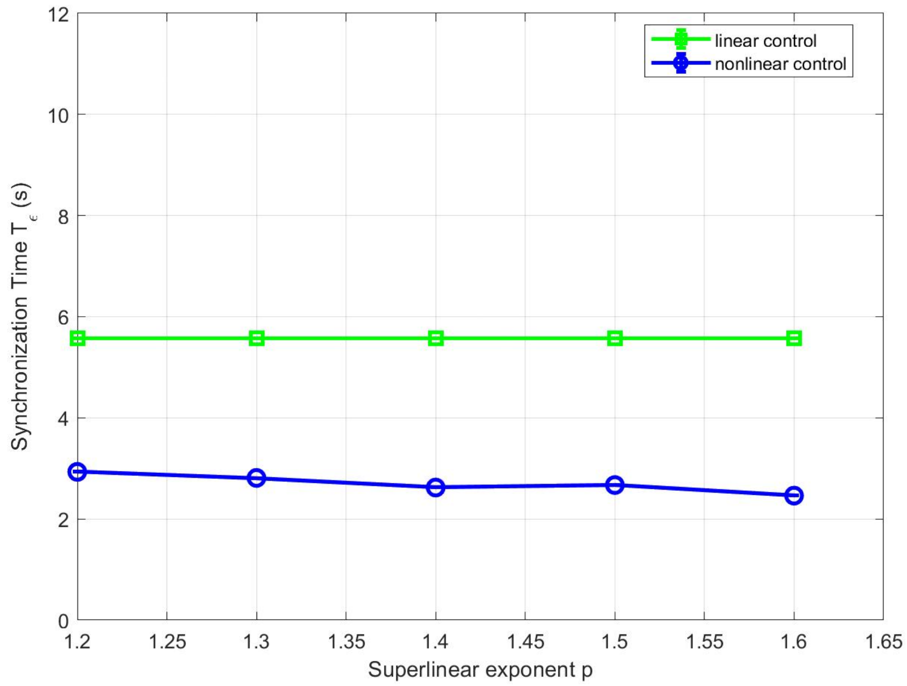

Figure 3. Effect of p on synchronization time.

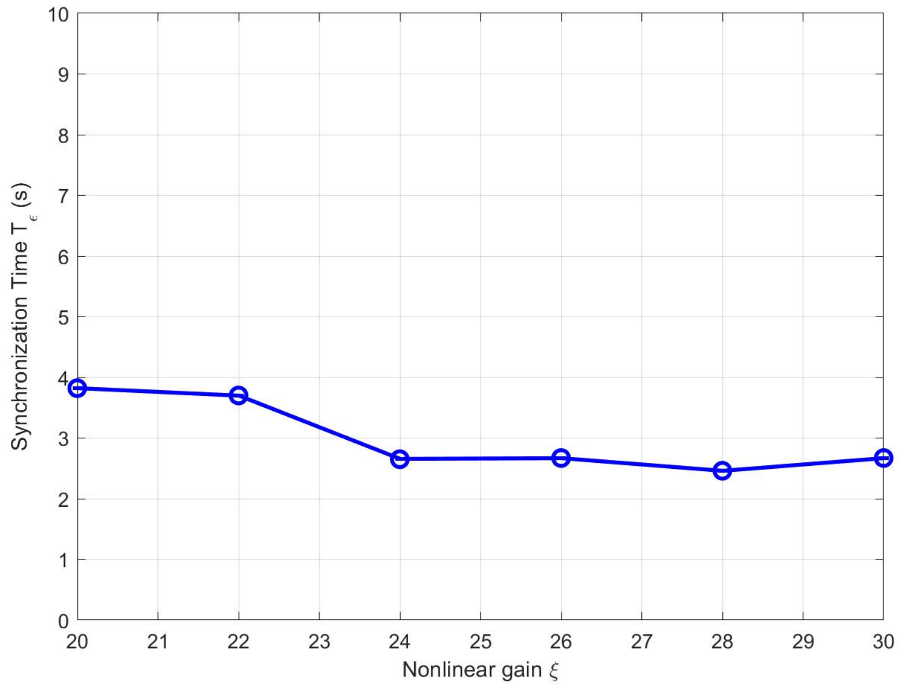

Figure 4. Effect of ξ on synchronization time.

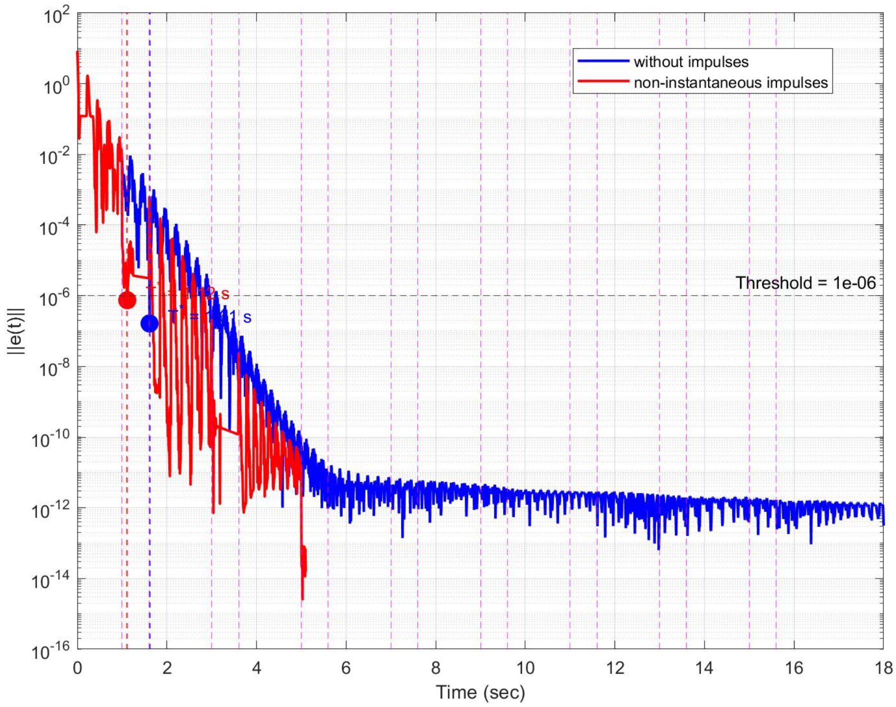

Figure 5. Comparison of synchronization error with and without non-instantaneous impulses under nonlinear control.

Figure 5 demonstrates that, with the nonlinear controller applied, the proposed non-instantaneous impulse control achieves faster practical finite-time synchronization than the system without impulses, which is consistent with the theoretical analysis that non-instantaneous impulses can further accelerate the synchronization process.

The numerical results are consistent with the theoretical conclusions. Compared with linear control, the proposed nonlinear controller, combined with non-instantaneous impulse control, achieves practical finite-time synchronization with faster convergence and higher precision, as predicted by the theoretical analysis. The parameter sensitivity analysis further illustrates the qualitative dependence of the synchronization time on the controller parameters.

5. Approximate controllability

After synchronization is achieved, it becomes a natural question whether the response system can be driven to an arbitrary target state rather than merely following the drive system. This is the controllability problem. Therefore, we analyze the approximate controllability of the above fractional stochastic neural networks with non-instantaneous impulses, multiple time-varying delays, and the Rosenblatt process:

$$
\begin{cases}
\mathcal{D}^{\alpha}\Big[\varpi(r) - D\varpi(r-\rho_1(r))\Big] = -C\varpi(r) + Aw(\varpi(r)) + Bh(\varpi(r-\rho_2(r))) + \sum_{j=1}^{N} v_{ij}\Gamma\varpi_j(r) \\
\qquad\qquad + \sigma(r, \varpi(r-\rho_1(r)), \dots, \varpi(r-\rho_m(r)))dZ_H(r) \\
\qquad\qquad + u(r),\ r \in \bigcup_{k}^{n}(\zeta_k, \theta_{k+1}], \\
\varpi(r) = I_k(r, \varpi(\theta_k^-)),\ r \in (\theta_k, \zeta_k],\ k = 0, 1, \dots, n, \\
\varpi(r) = \phi(r),\ r \in [-\rho, 0].
\end{cases} \tag{5.1}
$$

Definition 4. [20] System (5.1) is said to be approximately controllable on $[0, T]$, if for any initial state ϖ_0, any target state ϖ_T, and any $\varepsilon > 0$, there exists a control $u(r)$ such that the corresponding mild solution satisfies

$$\mathbb{E}\|\varpi(T) - \varpi_T\|^2 < \varepsilon.$$

First, we introduce the controllability operator associated with the considered system.

Let U be the control space and $C : U \to X$ be the bounded control operator. The controllability operator is defined by

$$\mathcal{L}_T u = \frac{1}{\Gamma(\alpha)} \int_0^T (T-z)^{\alpha-1} S_\alpha(T-z) C u(z) dz.$$

The adjoint operator is

$$\mathcal{L}_T^* x = \frac{1}{\Gamma(\alpha)} C^* S_\alpha^*(T-z)(T-z)^{\alpha-1} x.$$

Hence, the controllability Gramian is

$$\mathcal{W}_T = \mathcal{L}_T \mathcal{L}_T^*.$$

Theorem 5. Under the conditions of Theorem 2, the following conditions also hold:

(C_1) the controllability Gramian

$$\mathcal{W}_T = \frac{1}{\Gamma^2(\alpha)} \int_0^T (T-z)^{2\alpha-2} S_\alpha(T-z) C C^* S_\alpha^*(T-z) dz \tag{5.2}$$

satisfies

$$\lambda(\lambda I + \mathcal{W}_T)^{-1} \to 0,\ \lambda \to 0^+.$$

Equivalently, the resolvent operator $\mathcal{R}(\lambda, \mathcal{W}_T) = (\lambda I + \mathcal{W}_T)^{-1}$ satisfies

$$\lambda \mathcal{R}(\lambda, \mathcal{W}_T) \to 0,\ \lambda \to 0^+. \tag{5.3}$$

(C_2) There exist the constants $L_N > 0$ such that

$$\|N(r, \varpi, \varpi(r-\rho_1(r)), \dots)\| \le L_N \left(1 + \|\varpi\| + \sum_{i=1}^{m} \|\varpi(r-\rho_i(r))\|\right),$$

where $N(r, \cdot) = -C\varpi(r) + Aw(\varpi(r)) + Bh(\varpi(r-\rho_2(r))) + \sum v_{ij}\Gamma\varpi_j(r)$.

(C_3) For each $k = 0, 1, \dots,$ the impulse function $I_k(r, \cdot)$ is bijective, and its inverse satisfies the Lipschitz condition.

Then, the system (5.1) is approximately controllable on $[0, T]$.

Proof. Recalling the mild solution of system (5.1), and letting $\zeta_0 = 0$, $\varpi_0 = \varpi(0)$, there is

$$
\begin{aligned}
\varpi(r) = {} & S_\alpha(r)\varpi(0) - S_\alpha(r)D\varpi(-\rho_1(0)) + D\varpi(r-\rho_1(r)) \\
& + \frac{1}{\Gamma(\alpha)}\int_0^r (r-z)^{\alpha-1}S_\alpha(r-z)N(z,\varpi(z),\dots)dz \\
& + \frac{1}{\Gamma(\alpha)}\int_0^r (r-z)^{\alpha-1}S_\alpha(r-z)u(z)dz \\
& + \frac{1}{\Gamma(\alpha)}\int_0^r (r-z)^{\alpha-1}S_\alpha(r-z)\sigma(z,\varpi(z-\rho_1),\dots)dZ_H(z).
\end{aligned}
$$

Define the noise term

$$
R(r) = \frac{1}{\Gamma(\alpha)}\int_0^r (r-z)^{\alpha-1}S_\alpha(r-z)\sigma(z,\varpi(z-\rho_1),\dots)dZ_H(z).
$$

For any $\lambda > 0$, define the control

$$
u_\lambda(r) = C^* S_\alpha^*(T-r)^T \mathcal{R}(\lambda, \mathcal{W}_T)\mathcal{P}(\varpi),
$$

where

$$
\begin{aligned}
\mathcal{P}(\varpi) = {} & y_\varpi - S_\alpha(T)\varpi(0) + S_\alpha(T)D\varpi(-\rho_1(0)) - D\varpi(T-\rho_1(T)) \\
& - \frac{1}{\Gamma(\alpha)}\int_0^T (T-z)^{\alpha-1}S_\alpha(T-z)N(z,\varpi(z),\dots)dz - R(T).
\end{aligned}
$$

For any $\lambda > 0$, because C and $S_\alpha(t)$ are bounded operators, and $\mathcal{R}(\lambda, \mathcal{W}_T)$ is bounded, we have

$$
\mathbb{E}\int_0^T \|u_\lambda(r)\|^2 dr < \infty.
$$

Hence, $u_\lambda \in L_2([0,T],U)$, and the control is admissible. Moreover, u_λ is adapted with respect to the filtration generated by the stochastic process.

Define the operator $\mathcal{F}_\lambda = \mathcal{A}_\lambda + \mathcal{B}_\lambda$ on the Banach space X by

$$
\begin{aligned}
(\mathcal{A}_\lambda\varpi)(r) &= S_\alpha(r)\varpi(0) - S_\alpha(r)D\varpi(-\rho_1(0)) + D\varpi(r-\rho_1(r)) + R(r), \\
(\mathcal{B}_\lambda\varpi)(r) &= \frac{1}{\Gamma(\alpha)}\int_0^r (r-z)^{\alpha-1}S_\alpha(r-z)N(z,\varpi(z),\dots)dz + \frac{1}{\Gamma(\alpha)}\int_0^r (r-z)^{\alpha-1}S_\alpha(r-z)u_\lambda(z)dz.
\end{aligned}
$$

Based on the condition (C_2) and following the reasoning of Theorem 2, we apply the Krasnoselskii fixed-point theorem to obtain that $\mathcal{F}_\lambda$ has a fixed point $\varpi_\lambda \in X$. Therefore, the controlled mild solution exists. Substituting the fixed point ϖ_λ into the mild solution and using the identity $I - \mathcal{W}_T\mathcal{R}(\lambda, \mathcal{W}_T) = \lambda\mathcal{R}(\lambda, \mathcal{W}_T)$, we obtain

$$
\varpi_\lambda(T) = \varpi_T - \lambda\mathcal{R}(\lambda, \mathcal{W}_T)\mathcal{P}(\varpi_\lambda).
$$

Taking expectation and using the boundedness of $\lambda\mathcal{R}(\lambda, \mathcal{W}_T)$,

$$
\mathbb{E}\|\varpi_\lambda(T) - \varpi_T\|^2 = \mathbb{E}\|\lambda\mathcal{R}(\lambda, \mathcal{W}_T)\mathbb{P}(\varpi_\lambda)\|^2 \le \|\lambda\mathcal{R}(\lambda, \mathcal{W}_T)\|^2\mathbb{E}\|\mathcal{P}(\varpi_\lambda)\|^2.
$$

By the condition (C_1), $\lambda\mathcal{R}(\lambda, \mathcal{W}_T) \to 0$ strongly as $\lambda \to 0^+$. Moreover, $\mathbb{E}\|\mathcal{P}(\varpi_\lambda)\|^2$ is uniformly bounded. Therefore,

$$
\lim_{\lambda\to 0^+}\mathbb{E}\|\varpi_\lambda(T) - \varpi_T\|^2 = 0.
$$

For any given $\varepsilon > 0$, choose $\lambda > 0$ sufficiently small such that $\mathbb{E}\|\varpi_\lambda(T) - \varpi_T\|^2 < \varepsilon$. This proves that the terminal state can be driven arbitrarily close to the target state, which is exactly the definition of approximate controllability. Then, $u = u_\lambda$ is the desired control.

For systems with non-instantaneous impulses, we use backward induction. Let $\varepsilon > 0$ be given, and then the following:

(I) On the last non-impulse interval $(\zeta_k, \theta_{k+1}]$, by the controllability property established above, there exists a control $u_{\lambda k}(r)$ such that the state reaches the prescribed neighborhood of the target state.

(II) On the impulse interval $(\theta_k, \zeta_k]$, by the condition (C_3), we have

$$\varpi_\lambda(\theta_k^-) = I_k^{-1}(\zeta_k, \varpi_\lambda(\zeta_k^+)).$$

(III) On the preceding nonimpulse interval $(\zeta_{k-1}, \theta_k]$, we repeat the construction with target state $\varpi_\lambda(\theta_k^-)$ and accuracy $\varepsilon/2^{k+1}$.

By induction, we obtain controls on all intervals. The total error satisfies

$$\mathbb{E}\|\varpi_\lambda(T) - \varpi_T\|^2 \leq \sum_{j=0}^{k} \frac{\varepsilon}{2^{j+1}} < \varepsilon.$$

□

Example 2. We consider the following fractional neural network systems:

$$\begin{cases} {}^C D_t^{0.85}[\varpi_1(r) - 0.2\varpi_1(r - \rho_1(r))] = -0.3\varpi_1(r) + 2.0\tanh(\varpi_1(r)) - 1.0\tanh(\varpi_2(r)) \\ \qquad + 1.0\tanh(\varpi_1(r - \rho_2(r))) - 0.6\tanh(\varpi_2(r - \rho_2(r))) \\ \qquad + u_1(r) + \dfrac{0.005}{1 + \|\varpi(r - \rho_1(r))\|} dZ_{0.7}(r), \\ {}^C D_t^{0.85}[\varpi_2(r) - 0.2\varpi_2(r - \rho_1(r))] = -0.3\varpi_2(r) + 0.8\tanh(\varpi_1(r)) + 2.0\tanh(\varpi_2(r)) \\ \qquad - 0.5\tanh(\varpi_1(r - \rho_2(r))) + 0.8\tanh(\varpi_2(r - \rho_2(r))) \\ \qquad + u_2(r) + \dfrac{0.005}{1 + \|\varpi(r - \rho_1(r))\|} dZ_{0.7}(r), \end{cases}$$

where $\alpha = 0.85$, and $H = 0.7$ denotes the Hurst parameter of the Rosenblatt-driven stochastic perturbation. The noise intensity is chosen as $\sigma_{\max} = 0.005$, and the terminal time is $T = 30$. The time-varying delays are selected as

$$\rho_1(r) = 0.25 + 0.05\sin(0.5r),$$
$$\rho_2(r) = 0.35 + 0.05\cos(0.5r).$$

The non-instantaneous impulse operator is defined by

$$\varpi(r) = 0.3\varpi(\theta_k^-) + 0.7\varpi_T, \ r \in (\theta_k, \zeta_k],$$

where the contraction coefficient is $\mu = 0.3$. The impulse intervals are chosen as

$$(\theta_k, \zeta_k], \ \theta_k = 1.5k, \ \zeta_k = \theta_k + 0.3,$$

for $k = 0, 1, \ldots, 19$. Choose

$$A = \begin{bmatrix} 2.0 & -1.0 \\ 0.8 & 2.0 \end{bmatrix}, \quad B = \begin{bmatrix} 1.0 & -0.6 \\ -0.5 & 0.8 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Then, there is $\|C\| = 0.3$, $\|A\| \approx 2.5$, $\|B\| \approx 1.3$, and the activation functions satisfy $L_w = L_h = 1$; we assume $\|W\| = \|\Gamma\| = 1$. By the triangle inequality, we have

$$\|N\| \leq (0.3 + 2.5 + 1)\|\varpi\| + 1.3\|\varpi(r - \rho_2)\| = 3.8\|\varpi\| + 1.3\|\varpi(r - \rho_2)\|.$$

Thus, $L_N = 3.8$.

Let the target state be

$$\varpi_T = \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix}.$$

The nonlinear feedback controller is designed as

$$\begin{aligned} u_1(r) &= -\,20(\varpi_1(r) - 1.0) - 50\|e(r)\|^{-0.1}(\varpi_1(r) - 1.0), \\ u_2(r) &= -\,20(\varpi_2(r) - 0.5) - 50\|e(r)\|^{-0.1}(\varpi_2(r) - 0.5), \end{aligned}$$

where $\eta = 20$, $\xi = 50$, $q = 0.9$, and

$$e(r) = \varpi(r) - \varpi_T = \begin{bmatrix} \varpi_1(r) - 1.0 \\ \varpi_2(r) - 0.5 \end{bmatrix},$$

and $\|e(r)\| = \sqrt{(\varpi_1(r) - 1.0)^2 + (\varpi_2(r) - 0.5)^2}$.

For the numerical implementation, the time interval $[0, T]$ is divided into uniform subintervals with step size Δt. The Caputo fractional operator is approximated by the fractional convolution memory scheme, where the historical states are accumulated according to the fractional kernel.

The stochastic perturbation driven by the Rosenblatt process is approximated by a covariance-consistent numerical scheme preserving the required second-order statistical properties. The Hurst parameter is chosen as $H = 0.7$. In the simulation, M independent realizations are used to reduce the influence of stochastic fluctuations. The time-varying delays are implemented on the discrete time grid according to the functions $\rho_1(r)$ and $\rho_2(r)$. The non-instantaneous impulse operator is applied on the intervals $(\theta_k, \zeta_k]$ according to the prescribed impulse mapping. The selected parameters satisfy the boundedness and Lipschitz conditions required in the theoretical results.

The numerical parameters are selected as $T = 30$, $N = 3000$, and $\Delta t = T/N$. We simulate the phase portraits of the solution trajectories under six different initial values using MATLAB.

$$\varpi_0^{(1)} = \begin{bmatrix} 4.0 \\ -3.0 \end{bmatrix}, \varpi_0^{(2)} = \begin{bmatrix} -3.0 \\ 3.5 \end{bmatrix}, \varpi_0^{(3)} = \begin{bmatrix} 3.0 \\ 2.5 \end{bmatrix}, \varpi_0^{(4)} = \begin{bmatrix} -3.5 \\ -2.5 \end{bmatrix}, \varpi_0^{(5)} = \begin{bmatrix} 1.0 \\ -3.5 \end{bmatrix}, \varpi_0^{(6)} = \begin{bmatrix} -2.0 \\ -2.0 \end{bmatrix}.$$

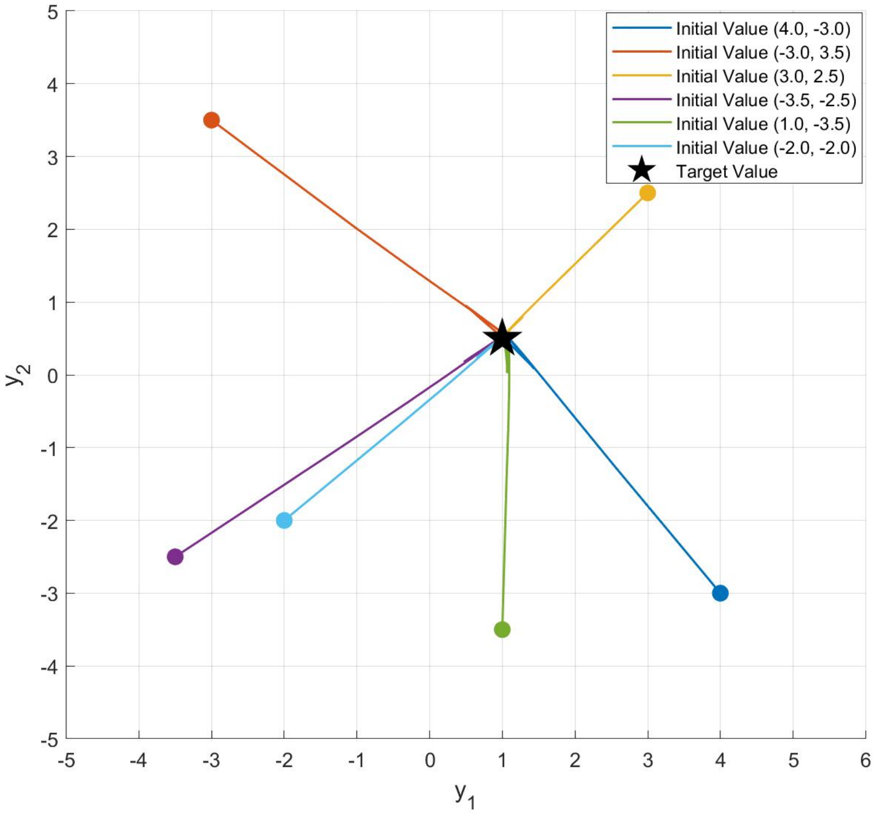

Figure 6. Phase portrait of target stabilization under non-instantaneous impulsive feedback control.

Figure 6 illustrates the evolution of six trajectories with different initial states under the designed feedback controller and non-instantaneous impulse mechanism. It can be observed that all trajectories gradually approach the prescribed target state $(1.0, 0.5)$.

This numerical example demonstrates the target-reaching and stabilization performance of the proposed controller. It should be emphasized that the approximate controllability property is established theoretically in Theorem 5 through the controllability operator and controllability Gramian analysis, whereas the present simulation only provides a numerical illustration of the effectiveness of the control strategy.

6. Conclusions

This paper investigates practical finite-time synchronization and approximate controllability of neutral fractional stochastic systems driven by the Rosenblatt process and subjected to multiple time-varying delays, non-instantaneous impulses, and superlinear nonlinearities. First, a fractional Bihari-type estimate is developed to handle weakly singular kernels and nonlinear integral inequalities involving multiple time-varying delay terms. By exploiting the condition on the delay functions and the inverse transformation of the delay mapping, the proposed inequality provides an effective tool for the qualitative analysis of fractional stochastic systems with variable delays.

Second, the Rosenblatt process is incorporated into the synchronization and controllability analysis of fractional neural network models, which extends existing results from Gaussian long-memory stochastic systems to non-Gaussian long-memory environments. Third, the existence and uniqueness of mild solutions are established by applying the Krasnoselskii fixed-point theorem and the Banach contraction principle. Based on the proposed nonlinear feedback controller, finite-time synchronization is achieved, and approximate controllability is proved through the controllability Gramian and resolvent operator approach.

Future research may focus on several extensions of the present work. For example, it is of interest to develop new fractional integral inequalities for systems with distributed delays or more general delay kernels. Moreover, extending the current results to generalized impulse operators, adaptive or event-triggered control strategies, and other types of non-Gaussian stochastic processes provides promising directions for further investigation. It is also of interest to investigate data-driven synchronization strategies and infinite-dimensional fractional stochastic systems. These developments may offer a broader theoretical framework for synchronization and controllability of complex fractional stochastic systems.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions disclosure statement

Bing Han: Writing-Original draft, Methodology, Investigation; Zhaoyan Zhang: Methodology, Supervision; Peiguang Wang: Conceptualization, Methodology, Writing-review and editing.

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Declaration of competing interest

The authors declare no competing interests.

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