



Research article

Global exponential stability of a generalized non-autonomous Nicholson's blowflies model with three time-varying delays

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Abstract: This paper investigated the asymptotic behavior of a generalized Nicholson's blowflies model characterized by nonautonomous coefficients and three distinct time-varying delays. The dynamics of the population were governed by the following delay differential equation:

$$x'(t) = -p_1(t)x(t - h_1(t)) + p_2(t)x(t - h_2(t))e^{-p_3(t)x(t - h_3(t))}.$$

Using a rigorous analytical approach, we established new sufficient conditions that guaranteed the global exponential stability of the zero equilibrium. These conditions provided valuable insights into scenarios of population extinction or suppression under fluctuating environmental influences. Our results extended and improved upon existing literature by incorporating three independent delays $h_1(t)$, $h_2(t)$, and $h_3(t)$, offering a more comprehensive framework for understanding the complex feedback mechanisms in biological systems. To illustrate the practical applicability of our theoretical findings, a numerical example demonstrated the convergence of solutions toward the equilibrium, confirming the robustness of the stability criteria under various conditions.

Keywords: Nicholson's blowflies model; time-varying delays; non-autonomous coefficients; global exponential stability; asymptotic behavior; delay differential equations

1. Introduction

Delay differential equations occupy a distinctive position in mathematical biology: a single lagged term, seemingly innocuous, can transform a stable population into one exhibiting oscillations, collapse, or intricate persistence patterns. Nicholson's blowflies model exemplifies this phenomenon. Introduced by Nicholson (1954) [1] to account for the erratic cycles observed in laboratory cultures of *Lucilia cuprina*, its classical form reads

$$x'(t) = -k_1x(t) + k_2x(t - \tau)e^{-k_3x(t - \tau)}, \quad (1.1)$$

where $x(t)$ is the adult population, τ the maturation delay, and k_1, k_2, k_3 are positive parameters representing mortality, reproductive capacity, and density regulation, respectively. Equation (1.1) and its variants have become central in delay differential equation (DDE) theory, chiefly because the delayed reproductive term alone can induce sustained oscillations, multistability, and bifurcations that are not achievable with instantaneous feedback.

Classical models assume constant delays and time-invariant vital rates, effectively presuming stationary environments. This variability has prompted an extensive body of research replacing fixed parameters with time-dependent functions, leading to nonautonomous Nicholson-type equations. Stability under distributed and delay-dependent settings was studied by Braverman and Zhukovskiy [2], the foundational framework for such models was surveyed by Berezensky et al. [3], and the influence of delays in the mortality term was examined by Berezensky and Braverman [4]. Global exponential stability for periodic nonautonomous models was established by Cao et al. [5], while attractivity, mixed monotonicities, and stability were further investigated by El-Morshedy and Ruiz-Herrera [6], Faria and Prates [7–9]. The classical formulation of Nicholson’s blowflies model was revisited by Gurney et al. [10], exponential stability for the diffusive case with multiple time-varying delays was treated by Huang and Liu [11], and global attractivity under constant delays was addressed by So and Yu [12]. The stability and persistence of these systems hinge on the boundedness, variation rates, and mutual interactions of the time-dependent functions involved. These generalizations present significant analytical hurdles, especially in characterizing long-term behavior under nonlinear couplings of time-dependent parameters [13, 14]. Results for systems with multiple time-varying delays have refined stability thresholds [6, 15], and periodic-environment analogues have been established under time-varying delays [5]. Collectively, these studies demonstrate that, under mild regularity assumptions on $h(t)$ and $p_i(t)$, properties such as persistence, positive equilibrium existence, and sometimes global attractivity can be established, often at the expense of more complex, delay-dependent criteria compared to autonomous models.

A second line of inquiry examines the biological realism of the reproduction term itself. Following Huang et al. [16], a structured-population derivation tracking immature cohorts yields a mixed-monotone birth term $p(t)x(t - h(t))e^{-ax(t - g(t))}$, where $h(t) \neq g(t)$, with $g(t) < h(t)$. This models stage-specific effects: for example, nymphal crowding regulating adult reproduction acts at a different time than the recruitment process. The term is monotone increasing in $x(t - h(t))$ but decreasing in $x(t - g(t))$, complicating the use of standard monotone iteration or fixed-point methods, which rely on cooperative or competitive comparisons. Faria [7] embedded this structure within the broader class of canonical mixed-monotone DDEs, establishing permanence and attractivity of the positive equilibrium through auxiliary maps with negative Schwarzian derivatives techniques now standard in the analysis of such equations. Related persistence results for Nicholson systems were developed by Amster and Bondorevsky [17]. However, these methods typically presuppose the positivity of solutions, which must be established separately.

A parallel, largely independent thread concerns neutral Nicholson equations under periodic forcing. Huang et al. [18] used the fluctuation lemma to prove the global exponential stability of positive periodic solutions, revealing that neutral history data require a strict initial condition $\varphi(0) - \varphi(-\mu_0(t_0)) > 0$, not merely nonnegativity. This early indication suggests that, for Nicholson-type DDEs, solution positivity from nonnegative initial data alone is not guaranteed; rather, it depends critically on the relationship between the history function and the delays.

Berezansky et al. [3] first raised the more challenging question: how does the delay in mortality influence the interaction with the delayed, nonlinear reproductive term, and whether solutions remain positive and bounded. They recently addressed this for a special case:

$$N'(t) = m(t) \left(-\delta N(t - h_1(t)) + PN(t - h_2(t))e^{-\gamma N(t - h_2(t))} \right) \quad (1.2)$$

with $P > \delta$, establishing positivity, persistence, boundedness, and two global stability criteria for the positive equilibrium via a linearized comparison approach [4]. This marked a significant advance but is limited to a single reproductive delay with monotone dependence.

While this progress is notable, the case where crowding and recruitment effects act at two genuinely distinct, independently varying delays corresponding to the mixed-monotone structure introduced by [16] combined with delayed mortality remains unresolved. The obstacles are twofold: first, the positivity of solutions, already subtle under a single delay, becomes more delicate when reproduction is mixed-monotone because nonnegative initial histories can, as shown in Section 2, lead to solutions becoming negative in finite time. This reflects, in a non-neutral setting, the phenomenon observed by Huang et al. [18], emphasizing that positivity depends not just on initial nonnegativity but on structural conditions involving delays and their interaction with initial histories. Second, once positivity and boundedness are secured, the bifurcation analysis between the zero and positive equilibria is significantly more complex: extinction (the zero equilibrium) can be shown via comparison principles once positivity is established, but excluding sustained oscillations around a positive equilibrium, particularly with lagged mortality and mixed-monotone reproduction requires new analytical tools beyond the linearization and Schwarzian derivative techniques previously employed.

Motivated by these challenges, we analyze a fully generalized Nicholson-type model with delayed mortality and a two-delay, mixed-monotone reproduction term:

$$x'(t) = -p_1(t)x(t - h_1(t)) + p_2(t)x(t - h_2(t))e^{-p_3(t)x(t - h_3(t))}, \quad (1.3)$$

where $h_i(t)$ and $p_i(t)$ are time-dependent delays and coefficients, with $h_3(t) \neq h_2(t)$, decoupling recruitment and crowding timing. Replacing the instantaneous mortality term $-p_1(t)x(t)$ with a delayed version shifts the dominant diagonal component to a delayed state, invalidating the absolute-stability framework used in prior analyses and extending the two-delay results of [4] to the fully general, three-delay, mixed-monotone setting.

This paper makes three main contributions. First, we identify the necessary and sufficient conditions on the initial history that guarantee positivity of solutions in this complex setting; this extends the positivity and boundedness results from the single-delay case in the reproduction term ($h_2 = h_3$ in our notation) [4] to the mixed-monotone framework of [7, 16], supported by explicit counterexamples in Section 2. Second, leveraging this positivity and uniform boundedness, we derive explicit, delay-dependent criteria for the global exponential stability of the zero solution, employing a generalized Halanay-type inequality accommodating three independent, time-varying delays. This complements the linearized stability approach and provides practical thresholds for extinction. Third, we validate these theoretical thresholds through numerical simulations illustrating the boundary between extinction and persistence. Extending these results to the stability of positive equilibria or periodic solutions remains an open and challenging problem, which we identify as a core direction for future research.

The remainder of this paper is organized as follows. Section 2 introduces the essential preliminaries, notations, and auxiliary lemmas underpinning our analysis; in particular, we present the motivating counterexample that justifies the necessity of our underlying assumptions, followed by a rigorous proof of the positivity and uniform boundedness of solutions. Section 3 develops explicit, readily verifiable criteria for the global exponential stability of the zero equilibrium. Section 4 provides comprehensive numerical simulations corroborating the theoretical results. Section 5 concludes the paper with a discussion of the main findings and an outline of promising directions for future research.

2. Positivity and boundedness of solutions

We first recall some notions. For a bounded continuous function f defined on $\mathbb{R}$, we denote by $f^- := \inf_{t \in [t_0, +\infty)} f(t)$ and $f^+ := \sup_{t \in [t_0, +\infty)} f(t)$. We define $\bar{h} = \max(h_1^+, h_2^+, h_3^+)$.

Let $C = C([- \bar{h}, 0], \mathbb{R})$ represent the set of all continuous functions from $[- \bar{h}, 0]$ to $\mathbb{R}$. We denote $C_+ = \{\varphi \in C : \varphi(\theta) \geq 0 \text{ on } [- \bar{h}, 0]\}$ as a subset of C . If $x(\cdot)$ is defined on $[- \bar{h} + t_0, \xi)$ where $t_0, \xi \in \mathbb{R}$, then we define $x_t \in C$ as $x_t(\theta) = x(t + \theta)$ for all $\theta \in [- \bar{h}, 0]$. As the present equations are inspired in mathematical biology models, only positive or nonnegative solutions are to be considered. In this way, we consider the following initial condition

$$x_{t_0} = \varphi \text{ where } \varphi \in C_+ \text{ and } \varphi(0) > 0. \quad (2.1)$$

Furthermore, the solution $x(t)$ verifies

$$x(t) = \varphi(t), \quad t < t_0 \text{ and } x(t_0) = x_0.$$

We introduce the following hypothesis regarding the initial data:

(A₀). The function φ satisfies

$$0 \leq \varphi(t) < x(t_0). \quad (2.2)$$

Under these assumptions, we denote by $x(t; t_0, \varphi)$ the solution of Eq (1.3) with the initial condition (2.1) that complies with assumption (A₀).

In this section, we investigate the positivity and the boundedness of solutions to the system (1.3)–(2.1), which are vital for understanding their long-term qualitative behavior and stability. To this end, we analyze the general nonautonomous model (1.3)–(2.1) under the following fundamental assumptions.

(A₁). For all $i = 1, 2, 3$, the functions p_i and $h_i : [t_0, +\infty) \rightarrow (0, +\infty)$ are continuous and there exist constants $h_i^+ > 0$ such that $h_i(t) \leq h_i^+$ for all $t \geq t_0$. Each $p_{i=1,2,3}$ is bounded between positive constants: $p_i^- \leq p_i(s) \leq p_i^+$, where $p_i^- > 0$ and $p_i^+ > 0$.

The functions p_i satisfy:

$$(A_2). \quad \int_{\max(t-h_1(t), t_0)}^t p_1(s) ds \leq \frac{1}{e}.$$

$$(A_3). \int_{t_0}^{\infty} p_i(s) ds = \infty \text{ for } i = 1, 2, 3.$$

$$(A_4). -p_1^- + p_1^- p_2^+ h_1^+ + p_2^+ < 0.$$

Hypotheses (A_0) and (A_2) are imposed to ensure the positivity of solutions to the system (1.3)–(2.1). In particular, the boundedness of the coefficients and of the delays in (1.3), as prescribed in (A_1) and (A_3) , are fundamental to the analytical techniques employed herein. Furthermore, hypothesis (A_4) is introduced to guarantee the boundedness of the solutions. A detailed discussion and justification of these assumptions are available in [4, 7].

The positivity of solutions in the Nicholson model differential equations presents significant challenges. When $h_1 = 0$, the comparison differential theorem can be effectively employed to analyze the system, as demonstrated in references such as Faria [7] and Long and Gong [15]. However, when $h_1 \neq 0$ and $p_2 = 0$, the differential equation simplifies to:

$$x'(t) = -p_1(t)x(t - h_1(t)). \quad (2.3)$$

In this case, analyzing the positivity of solutions becomes more intricate; see [19–22] and references therein. The direct application of the comparison theorem becomes less straightforward when $h_1 \neq 0$ and $p_2 = 0$, making the analysis of non-oscillation a particularly challenging problem, as discussed by Agarwal et al. [19] and references therein. In cases where $h_1 \neq 0$ and $h_2 = h_3$, the positivity was studied by Berezansky and Braverman [4].

Regarding the model (1.3) with three varying time delays, proving the positivity of solutions is among the most complex aspects of this study. Certain technical conditions on the initial data are essential to guarantee positivity. To illustrate this, the paper first provides an example showing that, under nonnegative initial functions, the initial value problem may not admit a positive solution. Subsequently, it introduces the necessary technical conditions on the initial functions to ensure positivity, along with a detailed and comprehensive proof of Theorem 2.4.

To illustrate the significance of the conditions (A_0) and (A_2) , we can provide counterexamples where violating these conditions causes solutions starting from nonnegative initial data to reach zero in finite time. This highlights the importance of these assumptions for maintaining positivity.

Example 2.1. (A_2) : Consider Eq (1.3) with constant coefficients and constant delays

$$x'(t) = -2x(t - 1) + x(t - 1)e^{-x(t-1)}, \quad t \geq 0, \quad (2.4)$$

that is,

$$p_1(t) = 2, \quad p_2(t) = 1, \quad p_3(t) = 1, \quad h_1(t) = h_2(t) = h_3(t) = 1. \quad (2.5)$$

Let

$$\varphi(t) = 1 \quad (t < 0), \quad x(0) = 1.5.$$

This satisfies (A_0) strictly: $0 \leq \varphi(t) = 1 < x(0) = 1.5$. Condition (A_2) remains violated regardless of the initial data, since $p_1 \equiv 2$ is constant and

$$\int_{t-1}^t p_1(s) ds = 2 > \frac{1}{e} \quad \text{for all } t,$$

which is exactly the quantity (A_2) constrains and depends only on p_1, h_1 , not on φ . For $t \in [0, 1)$, $t - 1 < 0$, so $x(t - 1) = \varphi(t - 1) = 1$ and

$$x'(-1) = e^{-1} - 2 \approx -1.6321,$$

giving $x(t) = 1.5 - 1.6321 t$ on $[0, 1)$, which reaches zero at

$$t^* = \frac{1.5}{2 - e^{-1}} \approx 0.919 < 1,$$

with $x(t) < 0$ for $t > t^*$. Thus the solution loses positivity while (A_0) holds strictly, correctly isolating the necessity of (A_2) . So, assumption (A_2) is violated, indicating that (A_2) is essential for the positivity of solutions.

Example 2.2. (A_0) : Consider

$$x'(t) = -x(t - 1) + \frac{1}{2}x(t - 1)e^{-x(t-1)}, \quad t \geq 0, \quad (2.6)$$

that is,

$$p_1(t) = 1, \quad p_2(t) = \frac{1}{2}, \quad p_3(t) = 1, \quad h_1 = h_2 = h_3 = 1. \quad (2.7)$$

Take the nonnegative initial data

$$x(t) = 1, \quad t < 0, \quad x(0) = 0.$$

Notice that this does not satisfy (A_0) , since

$$\varphi(t) = 1 \not\leq x(0) = 0.$$

For $t \in [0, 1]$,

$$x'(t) = -1 + \frac{1}{2}e^{-1} < 0.$$

Since $x(0) = 0$, it follows that

$$x(t) < 0,$$

for sufficiently small $t > 0$. Hence, no positive solution exists, although the initial function is nonnegative. Therefore, the strict condition

$$0 \leq \varphi(t) < x(t_0)$$

in (A_0) is also important.

These two examples demonstrate that assumptions (A_0) and (A_2) cannot be omitted in order to guarantee the positivity of solutions to problems (1.3)–(2.1).

Because the birth and mortality terms in Eq (1.3) depend on the population state at three past times, $t - h_1(t)$, $t - h_2(t)$, $t - h_3(t)$, reflecting maturation, incubation, and related developmental delays, the present state $x(t_0)$ alone cannot determine the future evolution; the model requires the population's trajectory over the entire preceding delay window. This is why the initial condition is prescribed as a continuous history function $x_{t_0} = \varphi$, $\varphi \in C_+$ with $\varphi(0) > 0$, rather than a single value: φ is the colony's recorded population history over the most recent maturation-incubation cycle, and $\varphi(0) = x(t_0)$ is the

population level at the onset of observation. Nonnegativity of φ reflects that population densities cannot be negative, while $\varphi(0) > 0$ excludes an already-extinct colony.

Hypothesis (A_0) , $0 \leq \varphi(t) < x(t_0)$, imposes a stronger requirement: every past value in the delay window must lie strictly below the level at t_0 , so that $x(t_0)$ is the largest value the population has attained to date. This is essential rather than merely technical. When $h_1 = 0$, or $h_1 \neq 0$ with $p_2 = 0$, the loss term is undelayed or acts alone, and classical comparison/non-oscillation arguments [7, 15, 19] yield positivity from nonnegativity of φ alone, since delayed terms cannot outpace the present state. In the general model, with three independently varying delays and $p_2 \neq 0$, this fails: loss and gain terms act at unsynchronized delayed times, so a nonnegative but poorly shaped history can let mortality dominate the saturating gain term immediately after t_0 , driving the solution negative in finite time, as our counterexample shows. Condition (A_0) closes this gap by excluding histories that allow mortality to overwhelm reproduction before the population can respond, substituting a direct constraint on the initial data for the comparison argument unavailable in this setting. Together with $\varphi(0) > 0$, it guarantees (Theorem 2.4) that the solution remains strictly positive on its maximal interval of existence and satisfies $x(t; t_0, \varphi) \leq x(t_0)$ for all $t \geq t_0$: biologically, the colony neither collapses nor grows without bound, but persists within the range set by its own recent history.

We will now establish the positivity of solutions of the system (1.3)–(2.1). We start the proof by recalling the following comparison lemma.

Lemma 2.3. (Gyori–Ladas [20]; see also Agarwal et al. [19].) Suppose $p_1 \geq 0$ and the Cauchy function of the linear delay differential equation

$$z'(t) + p_1(t)z(t - h_1(t)) = 0$$

is positive on $[t_0, t_1]$. If u, v agree on the initial interval $[t_0 - h, t_0]$ and satisfy

$$u'(t) + p_1(t)u(t - h_1(t)) \geq 0 \geq v'(t) + p_1(t)v(t - h_1(t)), \quad t \in [t_0, t_1],$$

then

$$u(t) \geq v(t), \quad t \in [t_0, t_1].$$

Theorem 2.4. Suppose assumptions (A_0) – (A_3) hold, and let $[t_0, \eta(\varphi))$ be the maximal interval of existence of the solution $x(t; t_0, \varphi)$ of the initial value problem (1.3)–(2.1). Then, for all $t \in [t_0, \eta(\varphi))$, we have

$$x(t; t_0, \varphi) > 0. \quad (2.8)$$

Proof. Suppose, for contradiction, that the claim fails. Since x is continuous, the zero set

$$Z := \{t \in [t_0, \eta(\varphi)) : x(t) \leq 0\}$$

is nonempty and closed in $[t_0, \eta(\varphi))$. Let $\zeta := \inf Z$. Since $x(t_0) = \varphi(t_0) > 0$ and x is continuous, $t_0 \notin Z$, which implies $\zeta > t_0$. Furthermore, since $Z \neq \emptyset$, we have $\zeta < \eta(\varphi)$. Because Z is closed in $[t_0, \eta(\varphi))$, it contains its infimum, so $\zeta \in Z$. By definition of the infimum, $x > 0$ on $[t_0, \zeta)$, and continuity together with $x(\zeta) \leq 0$ forces $x(\zeta) = 0$. On the history piece $[t_0 - \bar{h}, t_0)$, assumption (A_0) gives $x = \varphi \geq 0$. Combined with $x > 0$ on $[t_0, \zeta)$, we have:

$$x \geq 0 \text{ for } t \in [t_0 - \bar{h}, \zeta), \quad x > 0 \text{ for } t \in [t_0, \zeta).$$

Since $\zeta < \eta(\varphi)$, the interval $[t_0, \zeta]$ is compact and x is bounded there.

For $t \in [t_0, \zeta]$ we have $t - h_2(t) \in [t_0 - \bar{h}, \zeta]$, so $x(t - h_2(t)) \geq 0$. In addition, $e^{-p_3(t)x(t-h_3(t))} > 0$, and $p_2(t) \geq 0$ by (A_1) . Hence the birth term in (1.3), being a product of nonnegative and positive factors, is nonnegative on $[t_0, \zeta]$. Discarding this nonnegative term gives the one-sided inequality

$$x'(t) + p_1(t)x(t - h_1(t)) \geq 0, \quad t \in [t_0, \zeta], \quad (2.9)$$

with $x(t) = \varphi(t) \geq 0$ on $[t_0 - \bar{h}, t_0]$ and $x(t_0) = \varphi(t_0) > 0$. Now, let y solve the linear comparison equation

$$y'(t) + p_1(t)y(t - h_1(t)) = 0, \quad t \geq t_0, \quad y(t) = \varphi(t) \text{ on } [t_0 - \bar{h}, t_0]. \quad (2.10)$$

By the classical positivity and nonoscillation criterion for linear delay equations with a delay-dependent coefficient, see Koplatadze and Chanturiya [21]; Ladas et al. [22]; presented in textbook form in Gyori and Ladas [20], assumption (A_2) guarantees that the solution $y(t)$ of (2.10) with positive initial history remains strictly positive for all $t \geq t_0$. In particular,

$$y(\zeta) > 0. \quad (2.11)$$

Applying Lemma 2.3 with $u = x$ satisfying (2.9) and $v = y$ satisfying the Eq (2.10), both sharing with the same initial condition φ , we obtain

$$x(t) \geq y(t), \quad t \in [t_0, \zeta]. \quad (2.12)$$

Evaluating at $t = \zeta$ yields

$$x(\zeta) \geq y(\zeta) > 0,$$

which directly contradicts $x(\zeta) = 0$. Hence, no such $\zeta < \eta(\varphi)$ exists, proving that $x(t) > 0$ for all $t \in [t_0, \eta(\varphi))$. $\square$

Remark 2.5. Theorem 2.4 extends previous positivity results to a broader setting. Past studies (such as Berezhansky and Braverman [4]) required matching delays ($h_2(t) = h_3(t)$) and a constant parameter p_3 to keep solutions positive. With three separate, time-varying delays, those standard math bounds no longer work. Instead, we show that solutions stay positive because the system's direction always pushes back inside the positive region whenever it gets close to zero. This broadens earlier results by Faria and Prates [8] and Berezhansky and Braverman [4] to time-dependent delays $h_1(t)$, $h_2(t)$, and $h_3(t)$.

The approach for establishing the boundedness of positive solutions to the initial value problems (1.2) and (1.3) is based on several novel concepts.

Theorem 2.6. Suppose assumptions (A_0) – (A_4) hold. Then, every positive solution $x(t) = x(t; t_0, \varphi)$ of the delay differential equations (1.3)–(2.1) satisfies the uniform bound

$$x(t; t_0, \varphi) \leq x(t_0), \text{ for all } t \geq t_0. \quad (2.13)$$

Proof. Set $x_0 := x(t_0)$. Suppose, for contradiction, that $x(t) > x_0$ at some point in the maximal interval $[t_0, \eta(\varphi))$. Let t_1 be the first such time: by continuity, at t_1 , we have $x(t_1) = x_0$, and for $t \in [t_0, t_1)$, $x(t) < x_0$.

Since at t_1 , the right derivative satisfies $x'(t_1) \geq 0$, and from the differential equation (1.3), dropping the nonpositive term $-p_1(t)x(t-h_1(t))$ yields an upper bound:

$$x'(s) \leq p_2(s)x(s-h_2(s)), \quad s \in [t_0, t_1]. \quad (2.14)$$

Integrating (2.14) over $[t_1-h_1(t_1), t_1]$, and using the bounds $h_1(t) \leq h_1^+$, along with $p_2(t) \leq p_2^+$, we get

$$x(t_1-h_1(t_1)) \geq x(t_1) - h_1^+ p_2^+ x_0. \quad (2.15)$$

Next, evaluating the differential inequality (1.3) at t_1 , and bounding $e^{-p_3 x(\cdot)} \leq 1$, as well as $x(t_1-h_2(t_1)) \leq x_0$, we obtain:

$$x'(t_1) \leq -p_1(t_1)x(t_1-h_1(t_1)) + p_2(t_1)x_0 \leq x_0(p_1^- p_2^+ h_1^+ - p_1^- + p_2^+).$$

The condition (A₄) ensures that this quantity is negative, i.e.,

$$p_1^- p_2^+ h_1^+ - p_1^- + p_2^+ < 0,$$

which implies

$$x'(t_1) < 0, \quad (2.16)$$

contradicting the assumption that $x'(t_1) \geq 0$. Therefore, no such t_1 exists, and the bound (2.13) holds for all $t \geq t_0$. Since the solution remains bounded, it follows from Theorem 2.3.1 [23] that the maximal interval of existence is infinite. The proof is complete. $\square$

3. Global exponential stability for the zero equilibrium point

This section presents sufficient conditions ensuring the global exponential stability of the zero equilibrium point for the following nonautonomous form of Nicholson's blowflies equation

$$x'(t) = \xi(t) \left(-q_1 x(t-h_1(t)) + q_2 x(t-h_2(t)) e^{-q_3 x(t-h_3(t))} \right), \quad (3.1)$$

where q_i for $i = 1, 2, 3$ are positive constants, and the nonautonomous function $\xi(t)$ satisfies the specified assumptions

(B₁). The positive measurable function $\xi^+ \geq \xi(t) \geq \xi^- > 0$ satisfies $\int_{t_0}^{\infty} \xi(s) ds = \infty$.

(B₂). $q_1 \int_{\max(t-h_1(t), t_0)}^t \xi(s) ds \leq \frac{1}{e}$.

(B₃). $-q_1 + q_1 q_2 \xi^+ h_1^+ + q_2 < 0$.

(B₄). $-q_1 + q_2 \leq 0$.

(B₅). $-q_1 \xi^- + q_1 q_2 h_1^+ (\xi^+)^2 + q_2 \xi^+ < 0$.

In (B₄), the condition $q_2 \leq q_1$ implies that Eq (3.1) has a zero equilibrium.

Since the argument used in the proof of Theorem 3.3 below reapplies the delay h_2 inside the estimate for the delay h_1 , we define the extended delay bound

$$\widehat{h} := \max(h_1^+ + h_2^+, \bar{h}), \quad \bar{h} = \max(h_1^+, h_2^+, h_3^+),$$

and let $\widehat{C} := C([- \widehat{h}, 0], \mathbb{R})$ denote the corresponding extended history space, with $\widehat{C}^+ := \{\varphi \in \widehat{C} : \varphi(\theta) \geq 0 \text{ on } [- \widehat{h}, 0]\}$. Theorem 3.3 and Corollary 3.4 below require the initial history $\varphi \in \widehat{C}^+$, satisfying (A₀) on $[- \widehat{h}, 0]$, rather than the shorter window $[- \bar{h}, 0]$ used in Theorem 3.1.

We aim to establish the boundedness of the solutions of the model (2.1)–(3.1).

Theorem 3.1. Suppose (B_1) – (B_3) and (A_0) – (A_1) hold. Then, every positive solution $x(t) = x(t; t_0, \varphi)$ of (2.1)–(3.1) is bounded, with

$$x(t; t_0, \varphi) \leq x_0 := \max_{\theta \in [-\bar{h}, 0]} \varphi(\theta), \quad t \geq t_0. \quad (3.2)$$

Proof. By the initial condition, $x(t) \leq x_0$ for $t \in [t_0 - \bar{h}, t_0]$. Suppose, for contradiction, that $x(t) > x_0$ for some t in the maximal interval $[t_0, \eta(\varphi))$, and define

$$t_1 = \inf\{t > t_0 : x(t) > x_0\}.$$

By continuity, $t_1 > t_0$, $x(t_1) = x_0$, and $x(t) \leq x_0$ for all $t \in [t_0 - \bar{h}, t_1]$. Since $x(t) > x_0$ for points arbitrarily close to t_1 from the right, the righthand derivative satisfies $x'(t_1) \geq 0$. Now, instead, we will show that $x'(t_1) < 0$, which yields a contradiction. On $[t_1 - h_1(t_1), t_1]$, dropping the (nonpositive) term $-q_1 \xi(t)x(t - h_1(t))$ in (3.1) and using $x(s - h_2(s)) \leq x_0$, we get $x'(s) \leq q_2 \xi^+ x_0$. Integrating over this interval, and using $h_1(t_1) \leq h_1^+$,

$$x(t_1 - h_1(t_1)) \geq x(t_1) - h_1^+ q_2 \xi^+ x_0 = x_0(1 - h_1^+ q_2 \xi^+).$$

Evaluating (3.1) at $t = t_1$, and bounding $x(t_1 - h_2(t_1)) \leq x_0$ and noting $e^{-q_3 x(t_1 - h_3(t_1))} \leq 1$ (since $x \geq 0$, $q_3 > 0$),

$$x'(t_1) \leq -q_1 \xi(t_1)x(t_1 - h_1(t_1)) + q_2 \xi(t_1)x_0 \leq \xi(t_1)x_0(q_1 h_1^+ q_2 \xi^+ - q_1 + q_2).$$

By (B_3) , the bracketed quantity is negative, so $x'(t_1) < 0$. This contradicts the earlier conclusion that $x'(t_1) \geq 0$. Hence, no such t_1 exists, and the bound (3.2) holds for all $t \in [t_0, \eta(\varphi))$. The boundedness on the maximal interval implies $\eta(\varphi) = +\infty$ by Theorem 2.3.1 from [23], so the solution exists globally and satisfies (3.2) for all $t \geq t_0$. Thus, the proof is concluded. $\square$

In model (3.1), the conditions for population persistence are defined by the ratio $\frac{q_2}{q_1}$. The system undergoes a stability shift across two distinct phases:

Extinction Phase: For $0 < \frac{q_2}{q_1} \leq 1$, the zero equilibrium ($\kappa = 0$) is globally stable, representing the only biologically relevant state.

Persistence Phase: When $\frac{q_2}{q_1} > 1$, the zero equilibrium loses stability. At this threshold, a unique positive equilibrium emerges, defined by: $\kappa^* = \frac{1}{q_3} \ln \frac{q_2}{q_1}$.

To establish the stability of this zero equilibrium, we apply Halanay's theorem. In 1966, Halanay [24] first proved the following theorem:

Theorem 3.2. Suppose that $\tau \geq 0$ and $y(t)$ is a positive function defined on $[t_0 - \tau, \infty)$. If y satisfies the differential inequality

$$y'(t) \leq -ay(t) + b \sup_{s \in [t-\tau, t]} y(s), \quad t \geq t_0, \quad (3.3)$$

where: $a > b > 0$. Then, there exist positive constants α and β such that

$$y(t) \leq \alpha e^{-\beta(t-t_0)} \text{ for } t \geq t_0. \quad (3.4)$$

In 2000, Mohamad and Gopalsamy gave a new version for nonautonomous coefficients and time-varying delay [25]. Many results have been obtained for Halanay's inequality and its generalizations; see [26] and related references. The main result of this paper is stated below.

Theorem 3.3. Suppose that assumptions (B_1) – (B_5) and (A_1) are satisfied, and assume that (A_0) holds on the extended interval $[-\widehat{h}, 0]$. Let the initial function $\varphi \in \widehat{C} = C([-\widehat{h}, 0], \mathbb{R})$ satisfy $0 \leq \varphi(\theta) < x(t_0)$, for all $\theta \in [-\widehat{h}, 0]$. Then, the zero equilibrium of the nonautonomous delay Nicholson's blowflies model (2.1)–(3.1) is globally exponentially stable.

Proof. Since $e^{-q_3 x(t-h_3(t))} \leq 1$, every positive solution $x(t)$ of (3.1) satisfies

$$x'(t) \leq -q_1 \xi(t)x(t-h_1(t)) + q_2 \xi(t)x(t-h_2(t)). \quad (3.5)$$

Integrating the inequality

$$x'(s) \leq q_2 \xi(s)x(s-h_2(s)) \quad (3.6)$$

over $[t-h_1(t), t]$, and using the bound $h_1(t) \leq h_1^+$, as well as $\xi(t)$ bounded between ξ^- and ξ^+ . Since the estimate further applies the delay h_2 inside the bound for $x(t-h_1(t))$, for $r \in [t-h_1^+, t]$,

$$r-h_2(r) \in [t-h_1^+-h_2^+, t],$$

which is contained within $[t-\widehat{h}, t]$. We obtain

$$x(t-h_1(t)) \geq x(t) - h_1^+ q_2 \xi^+ \sup_{s \in [t-\widehat{h}, t]} x(s). \quad (3.7)$$

Substituting this lower bound into the inequality (3.5), and bounding both $x(s-h_i(s))$ by $\sup_{r \in [t-\widehat{h}, t]} x(r)$, we arrive at the estimate

$$x'(t) \leq -q_1 \xi^- x(t) + \xi^+ (q_1 h_1^+ q_2 \xi^+ + q_2) \sup_{s \in [t-\widehat{h}, t]} x(s), \quad t \geq t_0. \quad (3.8)$$

Condition (B_5) states exactly that the coefficient of the delayed term in (3.8) is less than $q_1 \xi^-$. Thus, inequality (3.8) is of the form required by Halanay's inequality (Theorem 3.2) with delay parameter $\tau = \widehat{h}$, which guarantees the existence of constants $K, \eta > 0$ such that

$$x(t) \leq K e^{-\eta(t-t_0)}, \quad t \geq t_0, \quad (3.9)$$

and consequently $x(t) \rightarrow 0$ exponentially as $t \rightarrow \infty$. This completes the proof. $\square$

As a consequence of Theorem 3.3, we obtain the following corollary.

Corollary 3.4. Assume that conditions (B_1) – (B_5) , (A_0) on $[-\widehat{h}, 0]$ (i.e., $\varphi \in \widehat{C}^+$), and (A_1) are satisfied. Then, the equilibrium $x = 0$ of the nonautonomous delay Nicholson's blowflies model (2.1)–(3.1) is globally exponentially stable. Moreover, there exist constants $M > 0$ and $\lambda > 0$ such that every solution $x(t; t_0, \varphi)$ of the system satisfies

$$x(t; t_0, \varphi) \leq M e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0. \quad (3.10)$$

Remark 3.5. Theorem 3.3 generalizes previous stability results for the zero equilibrium of the Nicholson's blowflies model by incorporating nonautonomous coefficients and extending the framework from two to three time-varying delays. This transition from autonomous to dynamic parameters allows for a more robust characterization of global exponential stability under fluctuating environmental conditions.

4. A numerical example

In this section, we present a numerical example to illustrate the effectiveness of the theoretical results established in the previous sections.

Example 4.1. Consider the generalized Nicholson's blowflies model with three time-varying delays:

$$x'(t) = -q_1\xi(t)x(t-h_1(t)) + q_2\xi(t)x(t-h_2(t))e^{-q_3x(t-h_3(t))}, \quad (4.1)$$

where the functions and parameters are defined as follows:

$$\xi(t) = 1 + \frac{1}{2}\sin^2(t), \quad h_1(t) = \frac{1}{11 + \sin(t)}, \quad h_2(t) = \frac{1}{12 + \cos^2(t)}, \quad h_3(t) = \frac{1}{15 + \sin^2(t)},$$

and $q_1 = 2$, $q_2 = 1$, $q_3 = 1$. We verify that the conditions of Theorem 3.3 are satisfied. First, note that:

$$1 \leq \xi(t) \leq \frac{3}{2}.$$

Since $\int_{t_0}^t \xi(s)ds = +\infty$, the required integral condition B_1 holds. Next, the delay functions are bounded: because $-1 \leq \sin t \leq 1$, we have $10 \leq 11 + \sin t \leq 12$, which implies

$$h_1(t) = \frac{1}{11 + \sin t} \leq \frac{1}{10}.$$

Similarly,

$$h_2(t) \leq \frac{1}{12}, \quad h_3(t) \leq \frac{1}{15}.$$

Hence, we can take

$$h_1^+ = \frac{1}{10}, \quad h_2^+ = \frac{1}{12}, \quad h_3^+ = \frac{1}{15}.$$

Correspondingly, the extended delay bound is

$$\widehat{h} = \max(h_1^+ + h_2^+, \bar{h}) = \max(0.1 + 0.0833, 0.1) \approx 0.183.$$

The history space for the initial data in this example is therefore taken as $\hat{C}^+ = C([-0.183, 0], \mathbb{R})$ rather than $[-0.1, 0]$. The assumption A_2 is satisfied. For condition B_2 , we estimate:

$$q_1 \int_{\max(t-h_1(t), t_0)}^t \xi(s)ds \leq q_1 \xi^+ h_1^+ = 0.3,$$

which is less than approximately $\frac{1}{e} = 0.3679$, so condition B_2 holds. Condition B_4 is straightforward:

$$-q_1 + q_2 = -1 < 0,$$

thus satisfied. For condition B_3 , we verify:

$$-q_1 + q_1 q_2 \xi^+ h_1^+ + q_2 = -0.7 < 0.$$

and for B_5 , we have

$$-q_1\xi^- + q_1\xi^+ h_1^+ q_2\xi^+ + q_2\xi^+ = -0.05 < 0.$$

For each of the trajectories shown in Figure 1, corresponding to initial levels $x_0 \in \{2, 4, 6, \dots, 14\}$, we will specify an explicit constant history

$$\varphi(t) = c, \quad t \in [t_0 - \widehat{h}, t_0),$$

where $0 \leq c < x_0$. For example, choosing $c = 0.9x_0$. The effective window is $\widehat{h} \approx 0.183$ for this setup, along with the condition $x(t_0) = x_0$. This construction ensures that the initial history strictly satisfies assumption (A_0) for each curve depicted in the figure. Thus, all the conditions in Theorem 3.3 are satisfied for the model in Eq (4.1). It follows that all solutions with positive initial conditions converge to the zero equilibrium point as $t \rightarrow \infty$. This fact is verified by the numerical simulations shown in Figure 1.

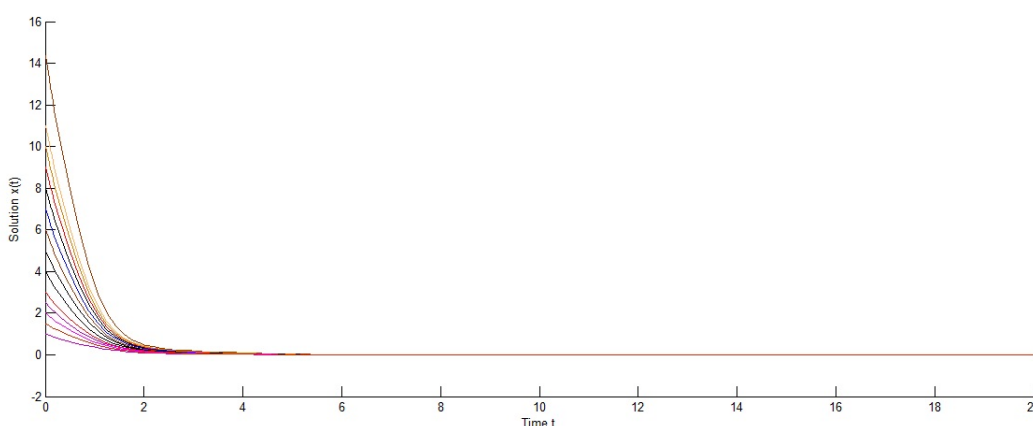


Figure 1. Numerical solutions $x(t)$ of Eq (4.1) with positive initial values x_0 .

The numerical simulations demonstrate that all solution trajectories converge toward the zero equilibrium, thereby confirming the global exponential stability of the system. Notwithstanding the complexities introduced by time-varying delays and nonautonomous coefficients, the solutions display a pronounced damping effect. These observations align with the theoretical conditions (B_1) – (B_5) and (A_0) – (A_1) , providing empirical validation for our analytical results.

Remark 4.2. To the best of the author's knowledge, the literature contains no results regarding the dynamics of Eq (4.1) under the influence of three time-varying delays. Moreover, the frameworks developed in [4–15] and [27–29] fail to apply to the present model. Therefore, the results established herein are novel and substantially generalize existing work in this direction.

5. Discussion and conclusions

It is important to highlight that replacing the classical instantaneous linear mortality term $-d(t)x(t)$ with the time-varying delayed death term $-p_1(t)x(t - h_1(t))$ introduces formidable mathematical challenges. Chief among these are preserving the strict positivity of solutions and constructing

appropriate, nontrivial Lyapunov functionals capable of dominating the delayed negative feedback. Despite these analytical complexities, this formulation yields profound advantages in both modeling and practical perspectives.

Biologically, this framework captures more realistic population mechanisms where the mortality rate is inherently dependent on past history, effectively reflecting delayed physiological effects such as incubation periods, metabolic lag, or prolonged resource depletion. Furthermore, by simultaneously integrating three distinct time-varying delays $h_1(t)$ for delayed mortality, $h_2(t)$ for maturity and reproductive processes, and $h_3(t)$ for density-dependent crowding effects, the proposed model establishes a highly nuanced and comprehensive representation of complex biological interactions.

From an analytical standpoint, the presence of the delay $h_1(t)$ within the governing linear operator significantly complicates the verification of solution boundedness and positivity. Overcoming these challenges is a critical prerequisite for establishing meaningful, verifiable criteria for global attractivity and exponential stability. Ultimately, managing this structural complexity enhances the model's overall robustness and its fidelity in describing real-world ecological systems.

In summary, this work provides a rigorous qualitative analysis of a nonautonomous, nonperiodic Nicholson's blowflies model governed by three distinct time-varying delays. By introducing novel analytical techniques and deriving explicit algebraic conditions, we have established verifiable criteria that guarantee the global exponential stability of the system, successfully addressing a prominent gap in the existing literature. These findings offer deeper insights into the intricate role that multi-delay feedback loops play in stabilizing or destabilizing long-term population dynamics.

Furthermore, the mathematical framework and analytical machinery developed in this paper lay a solid foundation for investigating a new class of generalized Mackey-Glass systems characterized by multiple delays, formulated as:

$$x'(t) = -p_1(t)x(t - h_1(t)) + \frac{p_2(t)x(t - h_2(t))}{1 + p_3(t)x^\sigma(t - h_3(t))}, \quad \sigma > 0.$$

Prospective extensions of this research will focus on expanding these techniques to higher-dimensional systems, incorporating spatial heterogeneity via reaction-diffusion equations, and exploring the intricate bifurcations induced by state-dependent nonlinearities.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there are no conflicts of interest.

References

1. A. Nicholson, An outline of the dynamics of animal populations, *Aust. J. Zool.*, **2** (1954), 19-65. <https://doi.org/10.1071/ZO9540009>

2. E. Braverman, S. Zhukovskiy, Absolute and delay-dependent stability of equations with a distributed delay, *Discrete Contin. Dyn. Syst. Ser. A*, **32** (2012), 2041–2061. <https://doi.org/10.3934/dcds.2012.32.2041>
3. L. Berezansky, E. Braverman, L. Idels, Nicholson's blowflies differential equations revisited: Main results and open problems, *Appl. Math. Model.*, **34** (2010) 1405–1417. <https://doi.org/10.1016/j.apm.2009.08.027>
4. L. Berezansky, E. Braverman, Nicholson's blowflies differential equations with a small delay in the mortality term, *Nonlinear Anal. Real World Appl.*, **81** (2025), 104193. <https://doi.org/10.1016/j.nonrwa.2024.104193>
5. Q. Cao, G. Wang, C. Qian, New results on global exponential stability for a periodic Nicholson's blowflies model involving time-varying delays, *Adv. Differ. Equ.*, **2020** (2020), 43. <https://doi.org/10.1186/s13662-020-2495-4>
6. H. El-Morshedy, A. Ruiz-Herrera, Global attractivity for nonautonomous delay-differential equations with mixed monotonicity and two delays, *J. Dyn. Differ. Equ.*, **36** (2024), 757–774. <https://doi.org/10.1007/s10884-022-10163-0>
7. T. Faria, Asymptotic behaviour of general nonautonomous Nicholson equations with mixed monotonicities, *Nonlinear Anal. Real World Appl.*, **77** (2024), 104044. <https://doi.org/10.1016/j.nonrwa.2023.104044>
8. T. Faria, H. C. Prates, Global attractivity for a nonautonomous Nicholson's equation with mixed monotonicities, *Nonlinearity*, **35** (2022), 589–607. <https://doi.org/10.1088/1361-6544/ac3c2a>
9. T. Faria, Stability and attractivity for Nicholson systems with time-dependent delay, *Electron. J. Qual. Theory Differ. Equ.*, **63** (2017), 1–19. <https://doi.org/10.14232/ejqtde.2017.1.63>
10. W. S. C. Gurney, S. P. Blythe, R. M. Nisbet, Nicholson's blowflies revisited, *Nature*, **287** (1980), 17–21. <https://doi.org/10.1038/287017a0>
11. C. Huang, B. Liu, Exponential stability of a diffusive Nicholson's blowflies equation accompanying multiple time-varying delays, *Appl. Math. Lett.*, **163** (2025), 109451. <https://doi.org/10.1016/j.aml.2024.109451>
12. J. W. H. So, J. S. Yu, Global attractivity and uniform persistence in Nicholson's blowflies, *Differ. Equ. Dyn. Syst.*, **2** (1994), 11–18.
13. C. K. Lin, M. Mei, On travelling wavefronts of Nicholson's blow flies equation with diffusion, in *Proc. R. Soc. Edinb. Sect. A Math.*, **140** (2010), 135–152. <https://doi.org/10.1017/S0308210508000784>
14. T. Yi, X. Zou, Global attractivity of the diffusive Nicholson's blowflies equation with Neumann boundary condition: A non-monotone case, *J. Differ. Equ.*, **245** (2008), 3376–3388. <https://doi.org/10.1016/j.jde.2008.03.007>
15. X. Long, S. Gong, New results on stability of Nicholson's blowflies equation with multiple pairs of time-varying delays, *Appl. Math. Lett.*, **100** (2020), 106027. <https://doi.org/10.1016/j.aml.2019.106027>

16. C. Huang, L. Huang, J. Wu, Global population dynamics of a single species structured with distinctive time-varying maturation and self-limitation delays, *Discrete Contin. Dyn. Syst. Ser. B*, **27** (2022), 2427–2440. <https://doi.org/10.3934/dcdsb.2021138>
17. P. Amster, M. Bondorevsky, Persistence and periodic solutions in systems of delay differential equations, *Appl. Math. Comput.*, **403** (2021), 126193. <https://doi.org/10.1016/j.amc.2021.126193>
18. C. Huang, X. Ding, Q. Wang, Positive periodic stability to a neutral Nicholson's blowflies model, *Qual. Theory Dyn. Syst.*, **24** (2025), 140. <https://doi.org/10.1007/s12346-025-01295-x>
19. R. P. Agarwal, L. Berezansky, E. Braverman, A. Domoshnitsky, *Nonoscillation Theory of Functional Differential Equations with Applications*, Springer, New York, 2012. <https://doi.org/10.1007/978-1-4614-3455-9>
20. I. Gyori, G. Ladas, *Oscillation Theory of Delay Differential Equations: With Applications*, Oxford University Press, 1991. <https://doi.org/10.1093/oso/9780198535829.001.0001>
21. R. G. Koplatadze, T. A. E. Chanturiya, Oscillating and monotone solutions of first-order differential equations with deviating argument, *Differ. Uravn.*, **18** (1982), 1463–1465.
22. G. Ladas, Y. G. Sficas, I. P. Stavroulakis, Functional differential inequalities and equations with oscillating coefficients, in *Trends in Theory and Practice of Nonlinear Differential Equations*, CRC Press, 1984.
23. J. K. Hale, S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1993. <https://doi.org/10.1007/978-1-4612-4342-7>
24. A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, New York, 1966.
25. S. Mohamad, K. Gopalsamy, Continuous and discrete Halanay-type inequalities, *Bull. Aust. Math. Soc.*, **61** (2000), 371–385. <https://doi.org/10.1017/S0004972700022413>
26. B. Liu, W. Lu, T. Chen, Generalized Halanay inequalities and their applications to neural networks with unbounded time-varying delays, *IEEE Trans. Neural Netw.*, **22** (2011), 1508–1513. <https://doi.org/10.1109/TNN.2011.2160987>
27. C. Huang, B. Liu, Traveling wave fronts for a diffusive Nicholson's Blowflies equation accompanying mature delay and feedback delay, *Appl. Math. Lett.*, **134** (2022), 108321. <https://doi.org/10.1016/j.aml.2022.108321>
28. M. Mei, J. W. H. So, M. Y. Li, S. S. P. Shen, Asymptotic stability of travelling waves for Nicholson's blow flies equation with diffusion, *Proc. R. Soc. Edinb. A*, **134** (2004), 579–594. <https://doi.org/10.1017/S0308210500003358>
29. X. Wei, J. Wei, Neimark-Sacker bifurcation analysis in a discrete neutral Nicholson's blowflies system with delay, *J. Differ. Equ. Appl.*, **22** (2016), 865–877. <https://doi.org/10.1080/10236198.2016.1154953>