



Research article

Barycentric rational collocation-based IMEX-BDF2 scheme for option pricing under jump-diffusion models

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Abstract: Jump-diffusion models, such as Merton's and Kou's models, are essential in option pricing for capturing the discontinuous jump characteristics of asset prices. However, the nonlocal integral term in the resulting partial integro-differential equation (PIDE) presents substantial numerical challenges. This paper proposes an efficient and high-order numerical framework for pricing European and American options under these models. We employ the barycentric rational interpolation (BRI) collocation method for spatial discretization to achieve high-order accuracy and ensure numerical stability. For temporal discretization, the second-order implicit–explicit backward differentiation formula (IMEX-BDF2) is utilized, and the nonlocal integral term is evaluated via the Gauss–Legendre quadrature. To handle the early exercise constraint of American options, the operator splitting method is integrated into the framework. A rigorous error estimation and convergence analysis are provided to establish the mathematical reliability of the scheme. Numerical experiments demonstrate that the proposed method is highly efficient, providing accurate estimates for option values and stable numerical approximations for their Greeks (delta and gamma), validating its effectiveness for solving PIDEs in financial engineering.

Keywords: jump-diffusion models; barycentric rational interpolation; IMEX-BDF2 scheme; operator splitting method; Gauss–Legendre quadrature; option pricing

1. Introduction

Option pricing is one of the core topics in modern financial mathematics. Since the proposal of the Black–Scholes model, option pricing theory has advanced significantly. However, extensive empirical evidence suggests that financial asset returns frequently exhibit volatility smiles and fat-tail distributions, which deviate from the classical assumption of continuous geometric Brownian motion.

To overcome these limitations, various enhanced models have been developed. Notably, the jump-diffusion models introduced by Merton [1] and further refined by Kou [2] have been widely adopted. By incorporating a Poisson jump process, these models effectively capture market discontinuities and sudden price shocks. Specifically, Merton employed a log-normal distribution to characterize jump amplitudes, and Kou proposed a double-exponential distribution to better account for the observed asymmetry and leptokurtic nature of asset returns. Jump-diffusion models are much more realistic than the standard Black–Scholes framework because they can capture asset price discontinuities. Accurate and rapid pricing of both European and American options, as well as their sensitivities (Greeks such as delta and gamma), is essential for real-time hedging and risk management in volatile financial markets.

Under jump-diffusion models, the evolution of option prices is typically governed by partial integro-differential equations (PIDEs). Compared with standard partial differential equations, the presence of nonlocal integral terms in PIDEs introduces significant computational complexity. Moreover, for American options, because the holder has the right to exercise at any time, the pricing problem is formulated as a linear complementarity problem (LCP) with a free boundary. How to efficiently and accurately solve such mathematical models involving both nonlocal operators and inequality constraints has been a focal point and a significant challenge in computational finance. Consequently, various numerical methods have been proposed in the literature to address these PIDEs, including the finite volume method [3], high order semi-imex backward differentiation formula (BDF) schemes [4], the front-fixing exponential time differencing (ETD) method [5], the variable step-size method [6, 7], and the collocation method [8, 9]. The finite difference method [10, 11] is widely adopted in numerical simulations owing to its straightforward implementation; however, it is typically constrained by relatively low spatial algebraic accuracy. To improve precision, [12, 13] employed radial basis function methods for option pricing under jump-diffusion models. Although the radial basis function methods offers the advantages of mesh free formulation and implementation simplicity, their numerical stability and accuracy are often highly sensitive to the choice of shape parameters. Regarding temporal discretization, Kadalbajoo et al. [14] proposed a class of implicit–explicit schemes for European options and established the unconditional stability of their fully implicit variants. Compared with European options, the mathematical modeling of American options is significantly more complex, requiring the solution of an LCP involving a nonlocal integral operator. To address the discrete LCP arising in American option pricing, many methods have been proposed in the literature, such as the iterative method [11, 15, 16], operator splitting method [17–19], and penalty method [20, 21]. Although the commonly used penalty methods are effective, they often result in huge computational costs when dealing with complex integral terms. Currently, there are few studies that combine the highly stable rational interpolation method with the efficient operator splitting method and conduct a systematic analysis of their theoretical convergence.

The barycentric interpolation collocation method, originally introduced by Baltensperger and Berrut [22, 23], is a numerical collocation technique that employs barycentric interpolation to approximate unknown functions. This interpolation method exhibits exceptional numerical stability, high approximation accuracy, and straightforward implementation in computational algorithms. This method has been extensively adopted to solve various complex integro-differential equations, such as the Sine–Gordon equation [24, 25], the Allen–Cahn equation [26], the viscoelastic wave equation [27], and time-fractional evolution equations [28]. In this paper, we develop an efficient and high-precision numerical framework for pricing European and American options under Merton’s and Kou’s jump-diffusion models. For spatial discretization, we employ the barycentric rational interpolation (BRI)

collocation method. By strategically designing the rational weights, this approach inherits the high-precision characteristics of the collocation methods, thereby improving spatial numerical robustness. The temporal evolution is addressed using the second-order implicit—explicit backward differentiation formula (IMEX-BDF2) scheme, and the non-local integral term is evaluated via the Gauss–Legendre quadrature formula. Under an explicit time-step restriction, the proposed scheme is shown to be conditionally stable for European options. For American options, the operator splitting method is integrated to decouple the LCP. This strategy bypasses the need for large-scale nonlinear iterations, improving computational efficiency without compromising accuracy. Furthermore, a rigorous theoretical analysis is provided to establish the stability and convergence of the proposed numerical framework.

The structure of this paper is organized as follows. Section 2 introduces the mathematical models for option pricing under jump-diffusion processes. Section 3 presents the fundamentals of the BRI method. Building upon these foundations, Section 4 provides a rigorous derivation of the fully discrete numerical scheme for European options, integrating the IMEX-BDF2 temporal discretization with the BRI spatial collocation method; furthermore, the convergence of the scheme is theoretically established through detailed error estimations. Section 5 extends the framework to American option pricing via the operator splitting method and provides a detailed stability analysis and error estimation. Section 6 conducts four numerical experiments to validate the accuracy of the proposed schemes. Notably, the efficiency of the method in approximating both option values and their Greeks (delta and gamma) is also evaluated. Finally, Section 7 serves as the conclusion of this study.

2. Option pricing under the jump-diffusion model

The following section elaborates on the mathematical foundations of the jump-diffusion process as applied to option pricing models. Consider an asset with price denoted by S . Its price evolution can be described by the following stochastic differential equation:

$$\frac{dS}{S} = (\mu - \xi\lambda)d\tau + \sigma dW + (\eta - 1)dq, \quad (2.1)$$

where μ , σ , τ represent the drift rate, constant volatility, and time to maturity, respectively. dW is an increment term of the standard Gauss–Wiener process. ξ is related to the expected relative jump size $\mathbb{E}(\eta - 1)$, where $\mathbb{E}[\cdot]$ is the expectation operator, and $\eta - 1$ is a impulse function producing jump from S to $S\eta$. λ denotes the intensity parameter of the independent Poisson process dq with

$$dq = \begin{cases} 0, & \text{with probability } 1 - \lambda d\tau, \\ 1, & \text{with probability } \lambda d\tau. \end{cases}$$

Let $V(S, \tau)$ denote the value of a variable equity linked to the underlying asset, with its valuation determined by the underlying asset price S with current time τ . The function $V(S, \tau)$ satisfies the following backward PIDE [29]:

$$\frac{\partial V}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \lambda\xi)S \frac{\partial V}{\partial S} - (r + \lambda)V + \lambda \int_0^\infty V(S\eta)g(\eta)d\eta = 0, \quad (2.2)$$

for $(S, \tau) \in (0, \infty) \times (0, T]$, where r denotes the risk free interest rate, and $g(\eta)$ represents the probability density function of the jump amplitude η , satisfying the non-negativity condition $g(\eta) \geq 0$ for $\forall \eta$ and

$\int_0^\infty g(\eta) d\eta = 1$. The value of V at maturity is defined as follows:

$$V(S, T) = \mathcal{P}(S), \quad S \in (0, \infty), \quad (2.3)$$

where $\mathcal{P}(S)$ is the payoff function. For a put option, the payoff function is defined as $\mathcal{P}(S) = \max(K - S, 0)$, where K denotes the strike price; for a call option, the payoff function is defined as $\mathcal{P}(S) = \max(S - K, 0)$. The symbol $g(\eta)$ denotes the probability density function of the log-normal distribution in the Merton jump-diffusion model

$$g(\eta) := \frac{1}{\sqrt{2\pi}\sigma_j\eta} e^{-\frac{(\ln(\eta)-\mu_j)^2}{2\sigma_j^2}},$$

where $\xi := \mathbb{E}(\eta - 1) = \exp(\mu_j + \frac{\sigma_j^2}{2}) - 1$, μ_j , and σ_j^2 denote the mean and the variance of jump in return, respectively. According to the Kou jump-diffusion model, $g(\eta)$ follows a log-double-exponential probability density function

$$g(\eta) := \frac{1}{\eta} (p\eta_1 e^{-\eta_1 \ln(\eta)} \mathcal{H}(\ln(\eta)) + (1-p)\eta_2 e^{-\eta_2 \ln(\eta)} \mathcal{H}(-\ln(\eta))),$$

where $\mathcal{H}(\cdot)$ is the Heaviside function with $p > 0$, $\eta_1 > 1$, $\eta_2 > 0$, and ξ is given by $\xi := \frac{p\eta_1}{\eta_1-1} + \frac{(1-p)\eta_2}{\eta_2+1} - 1$.

Let $t = T - \tau$ and denote the new variables $x = \ln(\frac{S}{K})$, $y = \ln(\eta)$. The model (2.2) can be expressed as the following PIDE:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + (r - \frac{\sigma^2}{2} - \lambda\xi) \frac{\partial u}{\partial x} - (r + \lambda)u + \lambda \int_{-\infty}^{\infty} u(y, t) f(y - x) dy, \quad (2.4)$$

where $u(x, t) := V(Ke^x, T - t)$, and $f(y) = g(e^y)e^y$. Due to the above transformation, $f(y)$ can be expressed as

$$f(y) := \begin{cases} \frac{1}{\sqrt{2\pi}\sigma_j} e^{-\frac{(y-\mu_j)^2}{2\sigma_j^2}}, & \text{Merton's model,} \\ (p\eta_1 e^{-\eta_1 y} \mathcal{H}(y) + (1-p)\eta_2 e^{-\eta_2 y} \mathcal{H}(-y)), & \text{Kou's model.} \end{cases}$$

For the price of a European put option, the initial condition and asymptotic behavior are given by

$$u(x, 0) = \max\{K - Ke^x, 0\},$$

$$u(x, t) = \begin{cases} Ke^{-rt} - Ke^x, & x \rightarrow -\infty, \\ 0, & x \rightarrow +\infty. \end{cases}$$

Moreover, the American option may be modeled as the following LCP:

$$\begin{cases} \frac{\partial u}{\partial t} - \mathcal{L}u \geq 0, & (x, t) \in \mathbb{R} \times J, \\ u(x, t) - \mathcal{P}(Ke^x) \geq 0, & (x, t) \in \mathbb{R} \times J, \\ \left(\frac{\partial u}{\partial t} - \mathcal{L}u\right)(u(x, t) - \mathcal{P}(Ke^x)) = 0, & (x, t) \in \mathbb{R} \times J, \\ u(x, 0) = \mathcal{P}(Ke^x), & x \in \mathbb{R}, \end{cases} \quad (2.5)$$

where the operator $\mathcal{L}u = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + (r - \frac{\sigma^2}{2} - \lambda\xi) \frac{\partial u}{\partial x} - (r + \lambda)u + \lambda \int_{-\infty}^{\infty} u(y, t) f(y - x) dy$, and $J = (0, T]$. For American options, we restrict our attention to put options. In this case, we additionally impose the following boundary conditions:

$$u(x, t) = \begin{cases} K - Ke^x, & x \rightarrow -\infty, \\ 0, & x \rightarrow +\infty. \end{cases}$$

3. Barycentric form of the rational interpolation

As a form of rational interpolation, BRI features outstanding numerical stability, a high convergence order, and strong nodal adaptability, even when the nodes are equidistant. Moreover, the differential matrix of BRI can be derived easily. In this subsection, we mainly review the barycentric form of the rational interpolation and its associated differential matrix.

Let $u : \Omega_1 \mapsto \mathbb{R}$ be a univariate function with $\Omega_1 = [a, b]$, $\{x_i\}_{i=0}^m$ be $m + 1$ distinct nodes on Ω_1 , and $u_i := u(x_i)$ be the interpolated value at the interpolation nodes x_i . For an integer $d \in \{0, 1, \dots, m\}$, let $p_i(x)$ be the polynomials of degree at most d which interpolate the $d + 1$ data pairs $\{(x_i, u_i), (x_{i+1}, u_{i+1}), \dots, (x_{i+d}, u_{i+d})\}$ of $u(x)$. Then, a family of rational interpolating functions can be obtained as

$$r_m[u](x) = \sum_{i=0}^{m-d} \lambda_i(x) p_i(x) \Bigg/ \sum_{i=0}^{m-d} \lambda_i(x), \quad \lambda_i(x) = \frac{(-1)^i}{(x - x_i)(x - x_{i+1}) \cdots (x - x_{i+d})}, \quad (3.1)$$

where $\lambda_i(x)$ are the rational interpolation weight functions, and d is the rational interpolation parameter. In [30], Floater and Hormann established that Eq (3.1) achieves a convergence order of $O(h^{d+1})$ provided the interpolated function $u(x)$ is sufficiently smooth, where $h = \max_{0 \leq i \leq m-1} (x_{i+1} - x_i)$. Furthermore, they introduced the following barycentric weights:

$$\varpi_i = \sum_{\kappa \in J_i} (-1)^\kappa \prod_{j=\kappa, j \neq i}^{\kappa+d} \frac{1}{x_i - x_j},$$

where $J_i = \{\kappa \in I : i - d \leq \kappa \leq i, I = \{0, 1, \dots, m\}\}$ is an index set. In accordance with, the standard barycentric form of the rational interpolation, Eq (3.1) can be rewritten as

$$u_m(x) = \sum_{i=0}^m \varphi_i(x) u_i, \quad \varphi_i(x) = \frac{\varpi_i}{x - x_i} \Bigg/ \sum_{k=0}^m \frac{\varpi_k}{x - x_k}. \quad (3.2)$$

To enhance clarity and consistency, we substitute $r_m[u](x)$ with $u_m(x)$. Here, $u_m(x)$ is referred to as the BRI of $u(x)$ on the nodes $\{x_i\}_{i=0}^m$, and $\varphi_i(x)$ are called the BRI basis functions, which satisfy the character $\varphi_i(x_j) = \delta_{ij}$, $j = 0, 1, 2, \dots, m$, with δ_{ij} being the Kronecker delta.

Equation (3.2) shows that $u_m(x)$ is linear in $\{u_i\}_{i=0}^m$. Therefore, a straightforward way to compute the derivatives of $u_m(x)$ is to directly differentiate $\varphi_i(x)$. Then, the μ -th order derivatives of $u_m(x)$ evaluated at the nodes $\{x_j\}_{j=0}^m$ are

$$u_{m,j}^{(\mu)} := u_m^{(\mu)}(x_j) := \frac{d^\mu u_m(x_j)}{dx^\mu} = \sum_{i=0}^m \varphi_i^{(\mu)}(x_j) u_i := \sum_{i=0}^m D_{ij}^{(\mu)} u_i,$$

where $D_{ij}^{(\mu)} = \varphi_i^{(\mu)}(x_j)$, $j = 0, 1, \dots, m$. Moreover, the above equality can be equivalently expressed in matrix-vector form as follows:

$$\mathbf{u}_m^{(\mu)} = \mathbf{D}^{(\mu)} \mathbf{u},$$

where $\mathbf{u}_m^{(\mu)} = [u_{m,0}^{(\mu)}, u_{m,1}^{(\mu)}, \dots, u_{m,m}^{(\mu)}]^T$ represents the column vector of approximated μ -order derivative values, $\mathbf{u} = [u_0, u_1, \dots, u_m]^T$ represents the function value column vector, and the matrix $\mathbf{D}^{(\mu)}$ denotes a μ -order BRI differential matrix whose elements are $[D_{ij}^{(\mu)}]_{i,j=0}^m$. According to [31], the elements of $D_{ij}^{(1)}$ are formulated as

$$D_{ij}^{(1)} = \begin{cases} \frac{\varpi_i / \varpi_j}{x_j - x_i}, & i \neq j, \\ -\sum_{j \neq i} \frac{\varpi_i / \varpi_j}{x_j - x_i}, & i = j. \end{cases} \quad (3.3)$$

The elements of $D_{ij}^{(\mu)}$ can be computed using the following expressions:

$$D_{ij}^{(\mu)} = \begin{cases} \mu \left(D_{jj}^{(\mu-1)} D_{ij}^{(1)} - \frac{D_{ij}^{(\mu-1)}}{x_j - x_i} \right), & i \neq j, \\ -\sum_{i \neq j} D_{ij}^{(\mu)}, & i = j. \end{cases} \quad (3.4)$$

The following lemma in [32] provides an estimation of $e^{(\mu)}(x) = u^{(\mu)}(x) - u_m^{(\mu)}(x)$, $\mu \in \{0, 1, \dots, m\}$, where $u^{(\mu)}(x)$ denotes the μ -th derivative of $u(x)$, and $u_m^{(\mu)}(x)$ denotes the μ -th derivative of the BRI approximation $u_m(x)$.

Lemma 3.1. [32] *Given a function $u(x) \in C^{d+3}[a, b]$ with $d \in \{1, \dots, m\}$ and letting $u_m(x)$ denote the BRI defined in (3.2), there exists a non-negative constant c for sufficiently large m such that*

$$|u^{(\mu)}(x) - u_m^{(\mu)}(x)| \leq ch^{d-\mu+1}, \quad \mu \in \{0, 1, \dots, m\},$$

where $h = \max_{0 \leq i \leq m-1} (x_{i+1} - x_i)$, $\{x_i\}_{i=0}^m$ is $m+1$ different interpolation nodes on $[a, b]$.

4. Numerical method for solving European options

In this section, we will combine the IMEX-BDF2 time discretization techniques and the BRI space discretization techniques to discuss the fully discrete scheme for the following PIDE on the reduced domain $\Omega := (x_L, x_R)$:

$$\frac{\partial u}{\partial t} = \mathcal{D}u(x, t) + \lambda \mathcal{I}u(x, t), \quad (x, t) \in \Omega \times [0, T], \quad (4.1)$$

$$u(x, 0) = \mathcal{P}(x) = \max\{K - Ke^x, 0\}, \quad x \in \bar{\Omega}, \quad (4.2)$$

where $\mathcal{D}u(x, t) = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + (r - \frac{\sigma^2}{2} - \lambda\xi) \frac{\partial u}{\partial x} - (r + \lambda)u$ is differential operator, and

$$\mathcal{I}u(x, t) = \int_{-\infty}^{\infty} u(y, t) f(y - x) dy$$

is the integral operator.

To approximate the integral operator $\mathcal{I}u$ in (4.1), the integration domain is partitioned into two disjoint subdomains: a bounded interval Ω and an unbounded interval $\Omega^c := \mathbb{R} \setminus \Omega$. Therefore, the aforementioned integral $\mathcal{I}u(x, t)$ can be decomposed as shown in reference [33]:

$$\mathcal{I}u(x, t) = \int_{\Omega} u(y, t) f(y - x) dy + \mathcal{R}(x, t). \quad (4.3)$$

The integral $\mathcal{R}(x, t)$ in the infinite domain Ω^c can be calculated directly. In the case of the Merton model, the residual term $\mathcal{R}(x, t)$ is expressed as follows:

$$\mathcal{R}(x, t) = \begin{cases} Ke^{-rt} \mathcal{N}\left(\frac{x_L - x - \mu_j}{\sigma_j}\right) - Ke^{x + \mu_j + \frac{\sigma_j^2}{2}} \mathcal{N}\left(\frac{x_L - x - \mu_j - \sigma_j^2}{\sigma_j}\right), & \text{European put option,} \\ K \mathcal{N}\left(\frac{x_L - x - \mu_j}{\sigma_j}\right) - Ke^{x + \mu_j + \frac{\sigma_j^2}{2}} \mathcal{N}\left(\frac{x_L - x - \mu_j - \sigma_j^2}{\sigma_j}\right), & \text{American put option,} \end{cases}$$

where $\mathcal{N}(x)$ denotes the cumulative distribution function of the standard normal distribution, with $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{\xi^2}{2}} d\xi$; thus, its value can be computed directly. In the case of the Kou model, the residual term $\mathcal{R}(x, t)$ is given as

$$\mathcal{R}(x, t) = \begin{cases} K(1 - p)e^{-rt + \eta_2(x_L - x)} - K(1 - p)\frac{\eta_2}{\eta_2 + 1}e^{-\eta_2 x + (\eta_2 + 1)x_L}, & \text{European put option,} \\ K(1 - p)e^{\eta_2(x_L - x)} - K(1 - p)\frac{\eta_2}{\eta_2 + 1}e^{-\eta_2 x + (\eta_2 + 1)x_L}, & \text{American put option.} \end{cases}$$

4.1. Temporal discretization

We present the IMEX-BDF2 time discretization scheme applied to the PIDE in (4.1) on the truncated domain Ω . To discretize the time variable, we introduce a uniform temporal grid by defining $t_n = n\Delta t$, where $n = 1, \dots, N$, and Δt denotes the time step size. Let u^n denote the numerical approximation of $u(x, t_n)$. Then, Eq (4.1) will be discretized by following implicit–explicit time-stepping scheme:

$$\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} = \mathcal{D}u^{n+1} + \lambda \mathcal{I}(\mathcal{E}u^n) + R_{time}, \quad n \geq 1, \quad (4.4)$$

with the boundary conditions

$$u^{n+1}(x) = \begin{cases} Ke^{-rt_{n+1}} - Ke^{x_L}, & x = x_L, \\ 0, & x = x_R, \end{cases} \quad \text{ffl}$$

where $\mathcal{E}u^n = 2u^n - u^{n-1}$, and R_{time} denotes the temporal discretization error including the extrapolation of the integral term.

The aforementioned numerical method is referred to as the IMEX-BDF2 scheme involving three time levels. This scheme requires two initial values: the solution at the zeroth time level, denoted by u^0 ,

and the solution at the first time level, denoted by u^1 . The value u^0 is directly determined from the given initial conditions of the problem, whereas u^1 can be computed using the implicit-explicit backward difference method of order one. Specifically, by setting $n = 0$ in the first-order starting scheme, u^1 is obtained by solving the following discrete system:

$$\frac{u^1 - u^0}{\Delta t} = \mathcal{D}u^1 + \lambda I(u^0).$$

It is worth noting that although this localized starting step possesses first-order temporal accuracy, it is applied only once at the very beginning. Therefore, according to standard numerical analysis for multistep methods, it does not degrade the global second-order convergence rate of the overall IMEX-BDF2 scheme.

As demonstrated in [34], the stability of the aforementioned time discrete scheme (4.4) is rigorously established.

Theorem 4.1. [34] *If Δt is sufficiently small, specifically $\Delta t < \frac{1}{2(2\alpha+1+\lambda C_I)}$, the IMEX-BDF2 scheme is stable in the following sense:*

$$\|\mathcal{B}^k\|^2 \leq C[\|\mathcal{B}^0\|^2 + \|\mathcal{B}^1\|^2 + \max_{2 \leq n \leq k} \sum_{n=2}^k \|\mathcal{F}^n\|^2],$$

where C is a constant that depends on the parameters C_I, r, σ, λ and T , and $\mathcal{B}^n$ denotes the error at time t_n .

Proof. The stability proof is analogous to that in [34]. □

4.2. Spatial approximation

Subsequently, we discretize Eq (4.4) in the spatial domain using the BRI scheme. First, we consider the discretization of the spatial integral operator. Let $\mathcal{I}_1 u(x, t) := \int_{\Omega} u(y, t) f(y - x) dy$. For the discretization of the integral operator $\mathcal{I}_1 u$ on Ω , we employ the Gauss–Legendre quadrature formula, which enables us to obtain an approximation of the integral. We propose the following linear transformation:

$$y(\sigma) = \frac{x_R + x_L}{2} + \frac{x_R - x_L}{2} \sigma, \quad \sigma \in [-1, 1]. \quad (4.5)$$

By using transformations (4.5) and the Gauss–Legendre quadrature formula, the approximate integration formula can be expressed as follows:

$$\begin{aligned} \mathcal{I}_1 u(x, t) &= \epsilon \int_{-1}^1 u(y(\sigma), t) f(y(\sigma) - x) d\sigma \\ &\approx \epsilon \sum_{q=1}^Q \omega_q u(y(\sigma_q), t) f(y(\sigma_q) - x) \\ &= Qu(x, t), \end{aligned}$$

where $\epsilon = \frac{x_R - x_L}{2}$, σ_q , and ω_q are Gauss–Legendre nodes and weights, respectively.

Then we derive the fully discrete formulation of the model. Equation (4.4) can be equally rewritten as

$$\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} = \mathcal{D}u^{n+1} + \lambda Q(2u^n - u^{n-1}) + \lambda \mathcal{R}^n + R_{time} + R_{quad}, \quad (4.6)$$

where R_{quad} is integral error. Then, substituting Eq (3.2) into Eq (4.6), we can obtain

$$\begin{aligned} \frac{1}{2\Delta t} \left[3 \sum_{i=0}^M \varphi_i(x) u_i^{n+1} - 4 \sum_{i=0}^M \varphi_i(x) u_i^n + \sum_{i=0}^M \varphi_i(x) u_i^{n-1} \right] = \\ \frac{\sigma^2}{2} \sum_{i=0}^M \varphi_i''(x) u_i^{n+1} + \left(r - \frac{\sigma^2}{2} - \lambda \xi \right) \sum_{i=0}^M \varphi_i'(x) u_i^{n+1} - (r + \lambda) \sum_{i=0}^M \varphi_i(x) u_i^{n+1} \\ + \lambda \epsilon \sum_{q=1}^Q \omega_q \left(\sum_{i=0}^M \varphi_i(y(\sigma_q)) (2u_i^n - u_i^{n-1}) \right) f(y(\sigma_q) - x) + \lambda \mathcal{R}^n + R_{time} + R_{quad} + R_{space}, \end{aligned} \quad (4.7)$$

where $u_i^n = u^n(x_i) = u(x_i, t_n)$, $\mathcal{R}^n = \mathcal{R}(x, t_n)$, and R_{space} is the space interpolation error. Let Eq (4.7) be exact at the nodes $\{x_j\}_{j=0}^M$, and then,

$$\begin{aligned} \frac{3\delta_{ij}u_i^{n+1} - 4\delta_{ij}u_i^n + \delta_{ij}u_i^{n-1}}{2\Delta t} = \frac{\sigma^2}{2} D_{ij}^{(2)} u_i^{n+1} + \left(r - \frac{\sigma^2}{2} - \lambda \xi \right) D_{ij}^{(1)} u_i^{n+1} - (r + \lambda) \delta_{ij} u_i^{n+1} \\ + \lambda Q_h(2u_i^n - u_i^{n-1}) + \lambda \mathcal{R}_j^n + R_{time} + R_{quad} + R_{space}, \end{aligned} \quad (4.8)$$

where

$$Q_h(2u_i^n - u_i^{n-1}) = \epsilon \sum_{q=1}^Q \omega_q \left(\sum_{i=0}^M \varphi_i(y(\sigma_q)) (2u_i^n - u_i^{n-1}) \right) f(y(\sigma_q) - x_j),$$

and the above expression uses properties $\varphi_i(x_j) = \delta_{ij}$, $\varphi_i''(x_j) = D_{ij}^{(2)}$ and $\varphi_i'(x_j) = D_{ij}^{(1)}$.

Note $\mathcal{D}_h u_i^{n+1} := \frac{\sigma^2}{2} D_{ij}^{(2)} u_i^{n+1} + \left(r - \frac{\sigma^2}{2} - \lambda \xi \right) D_{ij}^{(1)} u_i^{n+1} - (r + \lambda) \delta_{ij} u_i^{n+1}$. Ignoring the remainders $R_j^{n+1} := R_{time} + R_{quad} + R_{space}$, we use U_i^n instead of u_i^n and then obtain the following fully discrete equation:

$$\frac{3\delta_{ij}U_i^{n+1} - 4\delta_{ij}U_i^n + \delta_{ij}U_i^{n-1}}{2\Delta t} = \mathcal{D}_h U_i^{n+1} + \lambda Q_h(2U_i^n - U_i^{n-1}) + \lambda \mathcal{R}_j^n, \quad 1 \leq i \leq M, 1 \leq n \leq N. \quad (4.9)$$

4.3. Error estimation and convergence analysis

In this subsection, we rigorously analyze the local truncation error estimation and convergence of the proposed numerical scheme. First, we present several fundamental inequalities and identities that are frequently employed in the analysis of the PIDE (see References [34, 35]). In particular, the following relationships among differential operators hold for such inequations:

$$(\mathcal{D}u, u) \leq \alpha \|u\|^2, \quad (4.10)$$

where

$$\begin{aligned} (\mathcal{D}u, u) &= \frac{1}{2}\sigma^2 \|u_x\|^2 + \left(r - \frac{1}{2}\sigma^2 - \lambda\xi\right)(u_x, u) - (r + \lambda)\|u\|^2 \\ &= -\frac{1}{2}\sigma^2 \left\|u_x - \frac{(r - \frac{1}{2}\sigma^2 - \lambda\xi)}{\sigma^2}u\right\|^2 + \frac{(r - \frac{1}{2}\sigma^2 - \lambda\xi)^2 - 2(r + \lambda)\sigma^2}{2\sigma^2} \|u\|^2 \\ &\leq \frac{(r - \frac{1}{2}\sigma^2 - \lambda\xi)^2 - 2(r + \lambda)\sigma^2}{2\sigma^2} \|u\|^2 \leq \alpha \|u\|^2, \end{aligned}$$

and $\alpha = \left| \frac{(r - \frac{1}{2}\sigma^2 - \lambda\xi)^2 - 2(r + \lambda)\sigma^2}{2\sigma^2} \right|$. The integral operators satisfy the following relationship:

$$\|\mathcal{I}u\| \leq C_I \|u\|, \quad (4.11)$$

where C_I is a constant that is contingent upon the parameter t , and $\|u\| := (\int_{\Omega} |u(x)|^2 dx)^{\frac{1}{2}}$. Meanwhile, we routinely apply Young's inequality and relevant algebraic identities:

$$(a, b) \leq \theta a^2 + \frac{1}{4\theta} b^2, \quad a, b \in \mathbb{R} \quad \text{and} \quad \theta > 0, \quad (4.12)$$

$$2(3a - 4b + c, a) = a^2 + (2a - b)^2 - b^2 - (2b - c)^2 + (a - 2b + c)^2. \quad (4.13)$$

Lemma 4.1. [36, 37] (Discrete Grönwall's lemma) Let y^n , h^n , g^n , and f^n be non-negative series such that

$$y^m + \Delta t \sum_{n=0}^m h^n \leq B + \Delta t \sum_{n=0}^m (g^n y^n + f^n), \quad \Delta t \sum_{n=0}^m g^n \leq c, \quad 0 \leq m \leq M = \left\lceil \frac{T}{\Delta t} \right\rceil.$$

In addition, assume that $g^n \Delta t < 1$ for every n . Define $\nu = \max_{0 \leq n \leq M} (1 - g^n \Delta t)^{-1}$. Then,

$$y^m + \Delta t \sum_{n=0}^m h^n \leq e^{\nu K} \left(B + \Delta t \sum_{n=0}^m f^n \right), \quad 0 \leq m \leq M.$$

Lemma 4.2. [38] Suppose $u \in H^k(\Omega)$ with an integer $k \geq 1$. If a Q -point Gauss–Legendre quadrature rule is applied to approximate the integral of $u(x)$ over Ω , then there exists a positive constant C independent of Q , such that the quadrature error satisfies

$$\left| \int_{\Omega} u(x) dx - \sum_{q=1}^Q u(x_q) \omega_q \right| \leq C Q^{-k} |u|_{H^{k,Q}(\Omega)},$$

where x_q and ω_q are the Gauss–Legendre nodes, and their corresponding weights map to the interval Ω , respectively, and $|u|_{H^{k,Q}(\Omega)}$ is the Sobolev semi-norm involving the derivatives of u .

Theorem 4.2. Suppose the exact solution $u(x, t)$ is sufficiently smooth, specifically $u(\cdot, t) \in C^{d+3}(\Omega) \cap H^k(\Omega)$ for $t \in (0, T]$. For sufficiently small Δt and h , there exists a positive constant C independent of Δt , h and Q such that

$$\|\mathbf{R}^{n+1}\|_{\infty} \leq C(\Delta t^2 + h^{d-1} + Q^{-k}),$$

where Q is the number of Gauss–Legendre quadrature points.

Proof. Let $u(x_j, t_{n+1})$ be the exact solution of Eq (4.1). To derive the local truncation error, we substitute $u(x_j, t_{n+1})$ into the fully discrete scheme. By substituting the barycentric rational interpolant $u(x, t) \approx \sum_{i=0}^{M_x} \varphi_i(x) u_i(t)$ into the equation and employing the property $\varphi_i(x_j) = \delta_{ij}$, the fully discrete residual at node x_j can be expressed as

$$R_j^{n+1} = \frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\Delta t} - (\mathcal{D}_h u_j^{n+1} + \lambda \mathcal{Q}_h(2u_j^n - u_j^{n-1}) + \lambda \mathcal{R}_j^n). \quad (4.14)$$

We decompose the residual as $R_j^{n+1} = R_{time} + R_{quad} + R_{space}$, where R_{time} denotes the temporal discretization error, including the extrapolation of the integral term; $R_{space} = \mathcal{D}u_j^{n+1} - \mathcal{D}_h u_j^{n+1}$ is the spatial interpolation error; and $R_{quad} = \mathcal{I}(\mathcal{E}u_j^{n+1}) - \mathcal{Q}_h(\mathcal{E}u_j^{n+1})$ is the quadrature error.

For the temporal discretization error, using a Taylor expansion at t_{n+1} , we have

$$\begin{aligned} |R_{time}| &\leq \left| \frac{\partial u}{\partial t}(x_j, t_n) - \frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\Delta t} \right| + \lambda \left| \mathcal{I}(u_j^n) - \mathcal{I}(2u_j^n - u_j^{n-1}) \right| \\ &\leq \frac{\Delta t^2}{3} \sup_{t \in [t_{n-1}, t_{n+1}]} \left| \frac{\partial^3 u}{\partial t^3}(x_j, t) \right| + \lambda \Delta t^2 \sup_{t \in [t_{n-1}, t_{n+1}]} \left| \mathcal{I}\left(\frac{\partial^2 u}{\partial t^2}(x_j, t)\right) \right|. \end{aligned} \quad (4.15)$$

That is to say, $R_{time} = O(\Delta t^2)$. As established in Lemma 3.1, by taking $\mu = 2$ for the second-order differential operator, it follows that

$$|\mathcal{D}u_j^{n+1} - \mathcal{D}_h u_j^{n+1}| \leq ch^{d-1}. \quad (4.16)$$

In other words, $R_{space} = O(h^{d-1})$. For the error of integral term, according to Lemma 4.2, if the integrand belongs to $H^k(\Omega)$, the error bound is satisfied as follows:

$$R_{quad} = \lambda \mathcal{I}(2u_j^n - u_j^{n-1}) - \lambda \mathcal{Q}_h(2u_j^n - u_j^{n-1}) \leq CQ^{-k} |u(\cdot, t)|_{H^k, \mathcal{Q}}(\Omega) = O(Q^{-k}). \quad (4.17)$$

Consequently, the residual expression is derived as follows:

$$R_j^{n+1} = C_1 \Delta t^2 + C_2 h^{d-1} + C_3 Q^{-k}. \quad (4.18)$$

We apply the infinity norm to the residuals at all nodes x_j and employ the triangle inequality to consolidate the associated constants, defining $C = \max\{C_1, C_2, C_3\}$. This yields the final bound on the local truncation error:

$$\|\mathbf{R}^{n+1}\|_\infty = C(\Delta t^2 + h^{d-1} + Q^{-k}). \quad (4.19)$$

□

Theorem 4.3. Suppose the exact solution $u(x, t)$ is sufficiently smooth, specifically $u(\cdot, t) \in C^{d+3}(\Omega) \cap H^k(\Omega)$ for $t \in (0, T]$. For sufficiently small h and Δt satisfying $1 - 2\Delta t(2\alpha + C_B + 1) > 0$, let $\mathbf{u}^n$ and $\mathbf{U}^n$ be the exact and numerical solutions at time t_n , respectively. Then, there exists a positive constant C independent of Δt , h , and Q such that

$$\|\mathbf{u}^n - \mathbf{U}^n\| \leq C(\Delta t^2 + h^{d-1} + Q^{-k}),$$

where Q is the number of Gauss–Legendre quadrature points.

Proof. Define the error vector as $\mathbf{E}^n = \mathbf{u}^n - \mathbf{U}^n$, where $\mathbf{U}^n = [U_0^n, U_1^n, \dots, U_M^n]^T$ and $\mathbf{u}^n = [u_0^n, u_1^n, \dots, u_M^n]^T$ represent the numerical solution vector and the exact solution vector at the n -th time step, respectively. By subtracting Eq (4.9) from Eq (4.8) and rearranging the resulting expression, the following error equation is derived:

$$\frac{3\mathbf{E}^{n+1} - 4\mathbf{E}^n + \mathbf{E}^{n-1}}{2\Delta t} = \mathbf{A}\mathbf{E}^{n+1} + \mathbf{B}(2\mathbf{E}^n - \mathbf{E}^{n-1}) + \mathbf{R}^{n+1}, \quad (4.20)$$

where $\mathbf{A} = \frac{\sigma^2}{2}\mathbf{D}^{(2)} + (r - \frac{\sigma^2}{2} - \lambda\xi)\mathbf{D}^{(1)} - (r + \lambda)\mathbf{I}$, $\mathbf{B} = [B_{ij}]_{i,j=0}^M$ is the integral operator matrix, and $B_{ij} = \sum_{q=1}^Q \omega_q \varphi_i(y(\delta_q))f(y(\delta_q) - x_j)$, $\mathbf{R}^{n+1} = [R_0^{n+1}, R_1^{n+1}, \dots, R_M^{n+1}]^T$. Equation (4.20) can be rewritten as follows:

$$3\mathbf{E}^{n+1} - 4\mathbf{E}^n + \mathbf{E}^{n-1} = 2\Delta t\mathbf{A}\mathbf{E}^{n+1} + 2\Delta t\mathbf{B}(2\mathbf{E}^n - \mathbf{E}^{n-1}) + 2\Delta t\mathbf{R}^{n+1}. \quad (4.21)$$

Computing the inner product of both sides of Eq (4.21) with $\mathbf{E}^{n+1}$, we derive the following expression:

$$\begin{aligned} \langle 3\mathbf{E}^{n+1} - 4\mathbf{E}^n + \mathbf{E}^{n-1}, \mathbf{E}^{n+1} \rangle &= 2\Delta t \langle \mathbf{A}\mathbf{E}^{n+1}, \mathbf{E}^{n+1} \rangle + 2\Delta t \langle \mathbf{B}(2\mathbf{E}^n - \mathbf{E}^{n-1}), \mathbf{E}^{n+1} \rangle \\ &\quad + 2\Delta t \langle \mathbf{R}^{n+1}, \mathbf{E}^{n+1} \rangle. \end{aligned} \quad (4.22)$$

By applying the identity (4.13) on the left-hand side of Eq (4.22), we obtain

$$\begin{aligned} \langle 3\mathbf{E}^{n+1} - 4\mathbf{E}^n + \mathbf{E}^{n-1}, \mathbf{E}^{n+1} \rangle &= \frac{1}{2} \left(\|\mathbf{E}^{n+1}\|^2 - \|\mathbf{E}^n\|^2 + \|2\mathbf{E}^{n+1} - \mathbf{E}^n\|^2 - \|2\mathbf{E}^n - \mathbf{E}^{n-1}\|^2 \right. \\ &\quad \left. + \|\mathbf{E}^{n+1} - 2\mathbf{E}^n + \mathbf{E}^{n-1}\|^2 \right). \end{aligned} \quad (4.23)$$

Following the estimation of the continuous operator shown in (4.10), the boundedness of the discrete matrix $\mathbf{A}$ can be established. Specifically, there exists a constant $\alpha \geq 0$, such that for any discrete vector $\mathbf{E}^{n+1}$, $\langle \mathbf{A}\mathbf{E}^{n+1}, \mathbf{E}^{n+1} \rangle \leq \alpha \|\mathbf{E}^{n+1}\|^2$. Then, the term satisfies

$$2\Delta t \langle \mathbf{A}\mathbf{E}^{n+1}, \mathbf{E}^{n+1} \rangle \leq 2\Delta t \alpha \|\mathbf{E}^{n+1}\|^2. \quad (4.24)$$

Given the boundedness of the continuous integral operator $\mathcal{I}$ in (4.11), its discrete matrix representation $\mathbf{B}$ preserves this mathematical property. It can be established that there exists a constant $C_B > 0$ such that $\|\mathbf{B}\mathbf{E}^{n+1}\| \leq C_B \|\mathbf{E}^{n+1}\|$ for the discrete vector $\|\mathbf{E}^{n+1}\|$. For the explicit integral terms, the application of the Cauchy–Schwarz inequality, boundedness of discrete integral operator matrices, and Young’s inequality yields the desired estimate:

$$\begin{aligned} 2\Delta t \langle \mathbf{B}(2\mathbf{E}^n - \mathbf{E}^{n-1}), \mathbf{E}^{n+1} \rangle &\leq 2\Delta t \|\mathbf{B}\| \|2\mathbf{E}^n - \mathbf{E}^{n-1}\| \|\mathbf{E}^{n+1}\| \\ &\leq 2\Delta t C_B \|2\mathbf{E}^n - \mathbf{E}^{n-1}\| \|\mathbf{E}^{n+1}\| \\ &\leq \Delta t C_B (\|2\mathbf{E}^n - \mathbf{E}^{n-1}\|^2 + \|\mathbf{E}^{n+1}\|^2). \end{aligned} \quad (4.25)$$

For $2\Delta t \langle \mathbf{R}^{n+1}, \mathbf{E}^{n+1} \rangle$, applying Young’s inequality yields

$$2\Delta t \langle \mathbf{R}^{n+1}, \mathbf{E}^{n+1} \rangle \leq \Delta t (\|\mathbf{R}^{n+1}\|^2 + \|\mathbf{E}^{n+1}\|^2).$$

Let $\tilde{\mathcal{E}}^n = \|\mathbf{E}^n\|^2 + \|2\mathbf{E}^n - \mathbf{E}^{n-1}\|^2$, and substitute it into (4.22)–(4.25). After simplification, we obtain

$$\frac{1}{2}(\tilde{\mathcal{E}}^{n+1} - \tilde{\mathcal{E}}^n) \leq 2\Delta t\alpha\|\mathbf{E}^{n+1}\|^2 + \Delta tC_B(\|2\mathbf{E}^n - \mathbf{E}^{n-1}\|^2 + \|\mathbf{E}^{n+1}\|^2) + \Delta t(\|\mathbf{R}^{n+1}\|^2 + \|\mathbf{E}^{n+1}\|^2).$$

By grouping the terms containing $\|\mathbf{E}^{n+1}\|^2$ and $\|2\mathbf{E}^n - \mathbf{E}^{n-1}\|^2$, we have

$$\frac{1}{2}(\tilde{\mathcal{E}}^{n+1} - \tilde{\mathcal{E}}^n) \leq \Delta t(2\alpha + C_B + 1)\|\mathbf{E}^{n+1}\|^2 + \Delta tC_B\|2\mathbf{E}^n - \mathbf{E}^{n-1}\|^2 + \Delta t\|\mathbf{R}^{n+1}\|^2.$$

It is evident that $\|2\mathbf{E}^n - \mathbf{E}^{n-1}\|^2 \leq \tilde{\mathcal{E}}^n$, and $\|\mathbf{E}^{n+1}\|^2 \leq \tilde{\mathcal{E}}^{n+1}$. Therefore,

$$\frac{1}{2}(\tilde{\mathcal{E}}^{n+1} - \tilde{\mathcal{E}}^n) \leq \Delta t(2\alpha + C_B + 1)\tilde{\mathcal{E}}^{n+1} + \Delta tC_B\tilde{\mathcal{E}}^n + \Delta t\|\mathbf{R}^{n+1}\|^2.$$

Multiplying both sides by 2 and rearranging the terms yields

$$\left(1 - 2\Delta t(2\alpha + C_B + 1)\right)\tilde{\mathcal{E}}^{n+1} \leq (1 + 2\Delta tC_B)\tilde{\mathcal{E}}^n + 2\Delta t\|\mathbf{R}^{n+1}\|^2.$$

For a sufficiently small time step Δt such that $2\Delta t(2\alpha + C_B + 1) \leq \frac{1}{2}$, we can divide both sides by $1 - 2\Delta t(2\alpha + C_B + 1)$. By utilizing the algebraic inequality $\frac{1}{1-y} \leq 1 + 2y$, which holds for $0 \leq y \leq \frac{1}{2}$, the coefficient of $\tilde{\mathcal{E}}^n$ can be rigorously bounded as follows:

$$\begin{aligned} \frac{1 + 2\Delta tC_B}{1 - 2\Delta t(2\alpha + C_B + 1)} &\leq (1 + 2\Delta tC_B)(1 + 4\Delta t(2\alpha + C_B + 1)) \\ &= 1 + \Delta t[2C_B + 4(2\alpha + C_B + 1) + 8\Delta tC_B(2\alpha + C_B + 1)]. \end{aligned}$$

Similarly, for the source term, we have

$$\frac{2\Delta t}{1 - 2\Delta t(2\alpha + C_B + 1)} \leq 2\Delta t[1 + 4\Delta t(2\alpha + C_B + 1)].$$

Because Δt is bounded, the terms inside the parentheses can be bounded by positive constants C_1 and C_2 , independent of Δt . Therefore, we have

$$\tilde{\mathcal{E}}^{n+1} \leq (1 + C_1\Delta t)\tilde{\mathcal{E}}^n + C_2\Delta t\|\mathbf{R}^{n+1}\|^2.$$

By expanding the recurrence formula using the discrete Grönwall lemma 4.1, we obtain

$$\tilde{\mathcal{E}}^{n+1} \leq \exp(C_1n\Delta t)\tilde{\mathcal{E}}^1 + C_2\Delta t \sum_{k=2}^{n+1} \exp\left(C_1(n+1-k)\Delta t\right)\|\mathbf{R}^k\|^2.$$

Due to $\tilde{\mathcal{E}}^1 = 0$, together with $\Delta t \sum_{k=2}^{n+1} \exp\left(C_1(n+1-k)\Delta t\right) \leq T \exp(C_1T)$ and Theorem 4.2,

$$\tilde{\mathcal{E}}^{n+1} \leq C_3 \max_k \|\mathbf{R}^k\|^2. \quad (4.26)$$

By taking the square root of both sides of the aforementioned inequality (4.26), we have $\sqrt{\tilde{\mathcal{E}}^{n+1}} \leq C \max_k \|\mathbf{R}^k\|$. Combining this with Theorem 4.2 and considering that $\|\mathbf{E}^{n+1}\| \leq \sqrt{\tilde{\mathcal{E}}^{n+1}}$, we finally obtain

$$\|\mathbf{E}^{n+1}\| \leq C(\Delta t^2 + h^{d-1} + Q^{-k}).$$

□

5. Numerical method for solving American options

5.1. Operator splitting method

This section presents a numerical solution to the LCP arising from American option pricing, leveraging the IMEX-BDF2 time-stepping scheme in conjunction with the operator splitting methodology introduced by Ikonen and Toivanen [39]. On the domain $\Omega \times (0, T]$, we introduce an auxiliary variable $\phi = u_t - \mathcal{L}u$. Using the idea of operator splitting and with the help of the additional variable ϕ , the original LCP is reformulated into an equivalent, computationally tractable system:

$$\begin{cases} \frac{\partial u}{\partial t} - \mathcal{L}u = \phi, \\ (u(x, t) - \mathcal{P}(Ke^x))\phi = 0, \\ u(x, t) - \mathcal{P}(Ke^x) \geq 0, \\ \phi \geq 0, \end{cases} \quad (5.1)$$

where $\mathcal{L} = \mathcal{D} + \lambda \mathcal{I}$. Discretize Eq (5.1) over the time interval $[0, T]$ using uniform grids with a fixed time step size Δt , and define the time grid points as $t_n = n\Delta t, n = 1 \cdots N$. Given the known values of (u^{n-1}, ϕ^{n-1}) at time t_{n-1} and (u^n, ϕ^n) at time t_n , the computation of u^{n+1} at the discrete time point t_{n+1} involves a two-step procedure. First, an intermediate approximation $\tilde{u}^{n+1}$ is obtained by solving the associated initial-boundary value problem.

$$\begin{cases} \frac{3\tilde{u}^{n+1} - 4u^n + u^{n-1}}{2\Delta t} = \mathcal{D}\tilde{u}^{n+1} + \lambda \mathcal{I}(\mathcal{E}u^n) + \phi^n, \\ \tilde{u}^{n+1}(x_L) = K, \quad \tilde{u}^{n+1}(x_R) = 0. \end{cases} \quad (5.2)$$

In the second step, we continue to project $\tilde{u}^{n+1}$ onto the constraint space to obtain u^{n+1} with the following correction terms:

$$\begin{cases} \frac{3}{2} \frac{u^{n+1} - \tilde{u}^{n+1}}{\Delta t} = \phi^{n+1} - \phi^n, \\ \phi^{n+1}(u^{n+1} - \mathcal{P}) = 0, \\ u^{n+1} \geq \mathcal{P}, \quad \phi^{n+1} \geq 0. \end{cases} \quad (5.3)$$

By solving problem Eq (5.3), the expression for the numerical pair (u^{n+1}, ϕ^{n+1}) is obtained as

$$(u^{n+1}, \phi^{n+1}) = \begin{cases} \left(\mathcal{P}, \phi^n + \frac{3}{2} \frac{\mathcal{P} - \tilde{u}^{n+1}}{\Delta t} \right), & \text{if } \tilde{u}^{n+1} - \frac{2\Delta t}{3} \phi^n \leq \mathcal{P}, \\ \left(\tilde{u}^{n+1} - \frac{2\Delta t}{3} \phi^n, 0 \right), & \text{otherwise.} \end{cases} \quad (5.4)$$

For computational tractability, the above expression can be reformulated equivalently as

$$u^{n+1} = \max \left(\tilde{u}^{n+1} - \frac{2\Delta t}{3} \phi^n, \mathcal{P} \right), \quad \phi^{n+1} = \phi^n + \frac{3}{2} \frac{u^{n+1} - \tilde{u}^{n+1}}{\Delta t}.$$

Therefore, the first step is implemented by solving the discrete equation (5.2), and the second step is performed using the correction formula (5.4). This implicit–explicit method employs a three-level

temporal discretization scheme, wherein the solution at the third time level is computed exclusively from data available at the first two time levels. The (u^0, ϕ^0) values at the zeroth time level can be obtained by using the initial conditions. At the initial time, the payoff condition gives $u^0 = \mathcal{P}(Ke^x)$, so the complementarity condition $(u^0 - \mathcal{P}(Ke^x))\phi^0 = 0$ is automatically satisfied for any $\phi^0 \geq 0$. For simplicity and to avoid introducing an artificial constraint force at the initial time, we choose $\phi^0 = 0$. The (u^1, ϕ^1) values at the first time level can be obtained using the IMEX-BDF1 method, that is,

$$(u^1, \phi^1) = \begin{cases} \left(\mathcal{P}, \phi^0 + \frac{\mathcal{P} - \tilde{u}^1}{\Delta t} \right), & \text{if } \tilde{u}^1 - \Delta t \phi^0 \leq \mathcal{P}, \\ \left(\tilde{u}^1 - \Delta t \phi^0, 0 \right), & \text{otherwise.} \end{cases} \quad (5.5)$$

To enhance the algorithm's implementability, the above expression is equivalently written as

$$u^1 = \max(\tilde{u}^1 - \Delta t \phi^0, \mathcal{P}), \quad \phi^1 = \phi^0 + \frac{u^1 - \tilde{u}^1}{\Delta t}.$$

5.2. Stability results and error estimates

Subsequently, we analyze the stability properties of the IMEX-BDF2 scheme applied to the time semi-discretization of the LCP. To streamline the analysis, we introduce the auxiliary sequences $\{Z^n\}_{n=0}^N$ and $\{\tilde{Z}^n\}_{n=0}^N$ defined as $Z^n = u^n - \mathcal{P}$ and $\tilde{Z}^n = \tilde{u}^n - \mathcal{P}$. Then, the LCP can be reformulated accordingly:

$$\begin{cases} \frac{\partial Z}{\partial t} - \mathcal{L}Z = \mathcal{L}\mathcal{P} + \phi, \\ Z\phi = 0, \quad Z \geq 0, \quad \phi \geq 0. \end{cases} \quad (5.6)$$

We rewrite the Eqs (5.2) and (5.3) as

$$\begin{cases} \frac{3\tilde{Z}^{n+1} - 4Z^n + Z^{n-1}}{2\Delta t} = \mathcal{D}\tilde{Z}^{n+1} + \lambda I(\mathcal{E}Z^n) + \mathcal{L}\mathcal{P} + \phi^n, \\ \tilde{Z}(x_L) = 0, \quad \tilde{Z}(x_R) = 0. \end{cases} \quad (5.7)$$

and

$$\frac{3(Z^{n+1} - \tilde{Z}^{n+1})}{2\Delta t} = \phi^{n+1} - \phi^n \quad (5.8a)$$

$$\phi^{n+1} \geq 0, \quad Z^{n+1} \geq 0, \quad Z^{n+1}\phi^{n+1} = 0. \quad (5.8b)$$

The proposed numerical scheme exhibits the following stability property. First, we define the following the notations used to represent discrete differences:

$$\delta\phi^{i+1} = \phi^{i+1} - \phi^i, \quad (5.9)$$

$$\delta^2\phi^{i+1} = \phi^{i+1} - 2\phi^i + \phi^{i-1}, \quad (5.10)$$

$$\chi^2\phi^{i+1} = 3\phi^{i+1} - 4\phi^i + \phi^{i-1}. \quad (5.11)$$

Theorem 5.1. Assuming that Δt is sufficiently small such that $\Delta t < \frac{1}{4(2\alpha+1+\lambda C_I)}$, the numerical schemes (5.7) with (5.8a) and (5.8b) are stable for all $1 \leq m \leq \frac{T}{\Delta t}$ in the sense that

$$\|Z^m\|^2 \leq C \left[\|Z^0\|^2 + \|Z^1\|^2 + \|\phi^1\|^2 + \Delta t \sum_{n=1}^{m-1} \|\mathcal{L}\mathcal{P}\|^2 \right], \quad (5.12)$$

where C is a constant that depends on the parameters C_I, r, σ, λ , and T .

Proof. First, by taking the inner product of Eq (5.7) with $4\Delta t \tilde{Z}^{n+1}$, we obtain

$$2(3\tilde{Z}^{n+1} - 4Z^n + Z^{n-1}, \tilde{Z}^{n+1}) = 4\Delta t(\mathcal{D}\tilde{Z}^{n+1}, \tilde{Z}^{n+1}) + 4\Delta t\lambda(\mathcal{I}(\mathcal{E}Z^n), \tilde{Z}^{n+1}) + 4\Delta t(\mathcal{L}\mathcal{P}, \tilde{Z}^{n+1}) + 4\Delta t(\phi^n, \tilde{Z}^{n+1}). \quad (5.13)$$

We express the left-hand side of Eq (5.13) in the form

$$\begin{aligned} 2(3\tilde{Z}^{n+1} - 4Z^n + Z^{n-1}, \tilde{Z}^{n+1}) &= 2(3Z^{n+1} - 4Z^n + Z^{n-1}, \tilde{Z}^{n+1}) + 6(\tilde{Z}^{n+1} - Z^{n+1}, \tilde{Z}^{n+1}) \\ &= 2(\chi^2 Z^{n+1}, Z^{n+1}) + 2(3Z^{n+1} - 4Z^n + Z^{n-1}, \tilde{Z}^{n+1} - Z^{n+1}) \\ &\quad + 6(\tilde{Z}^{n+1} - Z^{n+1}, \tilde{Z}^{n+1}). \end{aligned}$$

Using (4.13) on the term $2(\chi^2 Z^{n+1}, Z^{n+1})$, we obtain

$$2(\chi^2 Z^{n+1}, Z^{n+1}) = \|Z^{n+1}\|^2 - \|Z^n\|^2 + \|2Z^{n+1} - Z^n\|^2 - \|2Z^n - Z^{n-1}\|^2 + \|\delta^2 Z^{n+1}\|^2. \quad (5.14)$$

Next, we have

$$2(3Z^{n+1} - 4Z^n + Z^{n-1}, \tilde{Z}^{n+1} - Z^{n+1}) = 2(\delta^2 Z^{n+1}, \tilde{Z}^{n+1} - Z^{n+1}) + 4(Z^{n+1} - Z^n, \tilde{Z}^{n+1} - Z^{n+1}). \quad (5.15)$$

Applying the Cauchy–Schwarz inequality to the term $2(\delta^2 Z^{n+1}, \tilde{Z}^{n+1} - Z^{n+1})$ yields

$$|2(\delta^2 Z^{n+1}, \tilde{Z}^{n+1} - Z^{n+1})| \leq \|\delta^2 Z^{n+1}\|^2 + \|\tilde{Z}^{n+1} - Z^{n+1}\|^2. \quad (5.16)$$

Using (5.8a) and (5.8b) on the term $4(Z^{n+1} - Z^n, \tilde{Z}^{n+1} - Z^{n+1})$, we get

$$\begin{aligned} 4(Z^{n+1} - Z^n, \tilde{Z}^{n+1} - Z^{n+1}) &= -\frac{8\Delta t}{3}(Z^{n+1} - Z^n, \phi^{n+1} - \phi^n) \\ &= -\frac{8\Delta t}{3}((Z^{n+1}, \phi^{n+1}) + (Z^n, \phi^n) - (Z^{n+1}, \phi^n) - (Z^n, \phi^{n+1})) \\ &= \frac{8\Delta t}{3}((Z^{n+1}, \phi^n) + (Z^n, \phi^{n+1})) \geq 0, \end{aligned} \quad (5.17)$$

as $(Z^n, \phi^n) = 0$ and $\phi^n, Z^n \geq 0$ for all n . For the term $6(\tilde{Z}^{n+1} - Z^{n+1}, \tilde{Z}^{n+1})$, we have

$$6(\tilde{Z}^{n+1} - Z^{n+1}, \tilde{Z}^{n+1}) = 3(\|\tilde{Z}^{n+1}\|^2 - \|Z^{n+1}\|^2 + \|\tilde{Z}^{n+1} - Z^{n+1}\|^2). \quad (5.18)$$

Subsequently, (5.8a) can be rewritten as

$$3Z^{n+1} - 2\Delta t\phi^{n+1} = 3\tilde{Z}^{n+1} - 2\Delta t\phi^n. \quad (5.19)$$

By taking the inner product of each side the above equation with itself and dividing by 3, and because the cross-term vanishes owing to $(Z^{n+1}, \phi^{n+1}) = 0$, we have

$$3\|Z_{n+1}\|^2 + \frac{4}{3}\Delta t^2\|\phi^{n+1}\|^2 = 3\|\tilde{Z}^{n+1}\|^2 - 4\Delta t(\tilde{Z}^{n+1}, \phi^n) + \frac{4}{3}\Delta t^2\|\phi^n\|^2. \quad (5.20)$$

Combining the above relations, we obtain

$$\begin{aligned}
 & \|Z^{n+1}\|^2 - \|Z^n\|^2 + \|2Z^{n+1} - Z^n\|^2 - \|2Z^n - Z^{n-1}\|^2 + \frac{4\Delta t^2}{3}(\|\phi^{n+1}\|^2 - \|\phi^n\|^2) \\
 & + 2\|\tilde{Z}^{n+1} - Z^{n+1}\|^2 \leq 4\Delta t(\mathcal{D}\tilde{Z}^{n+1}, \tilde{Z}^{n+1}) + 4\Delta t\lambda(\mathcal{I}(\mathcal{E}Z^n), \tilde{Z}^{n+1}) + 4\Delta t(\mathcal{L}\mathcal{P}, \tilde{Z}^{n+1}) \\
 & \leq 2\Delta t(2\alpha + \lambda C_I + 1)\|\tilde{Z}^{n+1}\|^2 + 2\Delta t\lambda C_I\|\mathcal{E}Z^n\|^2 + 2\Delta t\|\mathcal{L}\mathcal{P}\|^2 \\
 & \leq 2\Delta t(2\alpha + \lambda C_I + 1)\|\tilde{Z}^{n+1}\|^2 + 16\Delta t\lambda C_I\|Z^n\|^2 + 4\Delta t\lambda C_I\|Z^{n-1}\|^2 + 2\Delta t\|\mathcal{L}\mathcal{P}\|^2.
 \end{aligned} \tag{5.21}$$

Summing up the above inequality (5.21) for $n = 1$ to $n = m - 1$, we get

$$\begin{aligned}
 & \|Z^m\|^2 - \|Z^1\|^2 + \|2Z^m - Z^{m-1}\|^2 - \|2Z^1 - Z^0\|^2 + \frac{4\Delta t^2}{3}(\|\phi^m\|^2 - \|\phi^1\|^2) + 2\sum_{n=1}^{m-1}\|\tilde{Z}^{n+1} - Z^{n+1}\|^2 \\
 & \leq 2\Delta t(2\alpha + \lambda C_I + 1)\sum_{n=1}^{m-1}\|\tilde{Z}^{n+1}\|^2 + 16\Delta t\lambda C_I\sum_{n=1}^{m-1}\|Z^n\|^2 + 4\Delta t\lambda C_I\sum_{n=1}^{m-1}\|Z^{n-1}\|^2 \\
 & + 2\Delta t\sum_{n=1}^{m-1}\|\mathcal{L}\mathcal{P}\|^2.
 \end{aligned} \tag{5.22}$$

Based on $\|\tilde{Z}^{n+1}\|^2 \leq 2\|\tilde{Z}^{n+1} - Z^{n+1}\|^2 + 2\|Z^{n+1}\|^2$ and assuming that Δt is sufficiently small such that $2 - 4\Delta t(2\alpha + \lambda C_I + 1) > 0$, rearranging (5.22) yields

$$\begin{aligned}
 & \|Z^m\|^2 - \|Z^1\|^2 + \|2Z^m - Z^{m-1}\|^2 - \|2Z^1 - Z^0\|^2 + \frac{4\Delta t^2}{3}(\|\phi^m\|^2 - \|\phi^1\|^2) \\
 & \leq 4\Delta t(2\alpha + \lambda C_I + 1)\|Z^m\|^2 + 4\Delta t\lambda C_I\|Z^0\|^2 + 20\Delta t\lambda C_I\|Z^1\|^2 \\
 & + (4\Delta t(2\alpha + \lambda C_I + 1) + 20\Delta t\lambda C_I)\sum_{n=2}^{m-1}\|Z^n\|^2 + 2\Delta t\sum_{n=1}^{m-1}\|\mathcal{L}\mathcal{P}\|^2.
 \end{aligned}$$

Now, imagine that the Δt is small enough such that $(1 - 4\Delta t(2\alpha + 1 + \lambda C_I)) > 0$, or $\Delta t < \frac{1}{4(2\alpha+1+\lambda C_I)}$. The above inequality then implies

$$\|Z^m\|^2 \leq C \left[\|Z^0\|^2 + \|Z^1\|^2 + \|\phi^1\|^2 + \Delta t\sum_{n=2}^{m-1}\|Z^n\|^2 + \Delta t\sum_{n=1}^{m-1}\|\mathcal{L}\mathcal{P}\|^2 \right],$$

where C is a constant that depends on the parameters C_I, r, σ, λ , and T . Subsequently, the discrete Grönwall lemma 4.1 is applied to derive the desired estimate. $\square$

With the stability result established in the previous theorem, we may assume that the exact solution is sufficiently smooth and thereby derive the corresponding error estimate. First, we introduce the error functions and corresponding error equations. Let $Z(\cdot, t_n)$ denote the exact solution $Z(x, t)$ evaluated at time $t = t_n$; analogous notation is adopted for other related variables:

$$\begin{aligned}
 e^n &= Z(\cdot, t_n) - Z^n, & \tilde{e}^n &= Z(\cdot, t_n) - \tilde{Z}^n, \\
 h^n &= \phi(\cdot, t_n) - \phi^n, & g^n &= h^n + \phi(\cdot, t_{n+1}) - \phi(\cdot, t_n),
 \end{aligned}$$

and assume that ϕ is sufficiently smooth such that

$$\|g^n\| \leq \|h^n\| + c\Delta t^2, \quad (5.23)$$

where c is independent of n . Based on the continuous system (5.1), we obtain

$$\frac{3Z(\cdot, t_{n+1}) - 4Z(\cdot, t_n) + Z(\cdot, t_{n-1}))}{2\Delta t} = \mathcal{D}Z(\cdot, t_{n+1}) + \lambda \mathcal{I}(\mathcal{E}(Z(\cdot, t_n))) + \mathcal{L}\mathcal{P} + \phi(\cdot, t_{n+1}) + \tilde{R}^{n+1}, \quad (5.24)$$

where $\tilde{R}^{n+1}$ is truncation error for given method. Subtracting (5.24) from (5.7), we obtain the error equation

$$\frac{3\tilde{e}^{n+1} - 4e^n + e^{n-1}}{2\Delta t} = \mathcal{D}\tilde{e}^{n+1} + \lambda \mathcal{I}(\mathcal{E}e^n) + g^n + \tilde{R}^{n+1}. \quad (5.25)$$

Another error equation can be obtained from (5.3) in the following form:

$$\frac{3(e^{n+1} - \tilde{e}^{n+1})}{2\Delta t} = h^{n+1} - g^n. \quad (5.26)$$

Theorem 5.2. Assuming that Δt is sufficiently small such that $\Delta t < \frac{1}{4(2\alpha+1+\lambda C_I)}$ for all $1 \leq m \leq \frac{T}{\Delta t}$, and that the exact solution (Z, ϕ) is sufficiently smooth, the following error estimates hold for the proposed method:

$$\|e^m\|^2 + \frac{4\Delta t^2}{3}\|h^m\|^2 \leq C\Delta t^4,$$

where C is a constant that depends on the parameters C_I, r, σ, λ , and T .

Proof. Taking the inner product of both sides of Eq (5.25) with $4\Delta t\tilde{e}^{n+1}$, we obtain

$$\begin{aligned} 2(3\tilde{e}^{n+1} - 4e^n + e^{n-1}, \tilde{e}^{n+1}) &= 4\Delta t(\mathcal{D}\tilde{e}^{n+1}, \tilde{e}^{n+1}) + 4\Delta t\lambda(\mathcal{I}(\mathcal{E}e^n), \tilde{e}^{n+1}) \\ &\quad + 4\Delta t(g^n, \tilde{e}^{n+1}) + 4\Delta t(\tilde{R}^{n+1}, \tilde{e}^{n+1}). \end{aligned} \quad (5.27)$$

We rewrite the left-hand side of Eq (5.27) as

$$\begin{aligned} 2(3\tilde{e}^{n+1} - 4e^n + e^{n-1}, \tilde{e}^{n+1}) &\geq \|e^{n+1}\|^2 - \|e^n\|^2 + \|2e^{n+1} - e^n\|^2 - \|2e^n - e^{n-1}\|^2 \\ &\quad + 2\|\tilde{e}^{n+1} - e^{n+1}\|^2 + 3\|\tilde{e}^{n+1}\|^2 - 3\|e^{n+1}\|^2. \end{aligned} \quad (5.28)$$

Based on Eq (5.26), it follows that

$$3e^{n+1} - 2\Delta t h^{n+1} = 3\tilde{e}^{n+1} - 2\Delta t g^n. \quad (5.29)$$

Taking the inner product of both sides of Eq (5.29) with themselves, and then dividing the resulting expression by 3 yields the following result:

$$3\|e^{n+1}\|^2 - 4\Delta t(e^{n+1}, h^{n+1}) + \frac{4}{3}\Delta t^2\|h^{n+1}\|^2 = 3\|\tilde{e}^{n+1}\|^2 - 4\Delta t(\tilde{e}^{n+1}, g^n) + \frac{4}{3}\Delta t^2\|g^n\|^2. \quad (5.30)$$

Combining relations (5.27), (5.28), and (5.30) and noting that $(e^{n+1}, h^{n+1}) \leq 0$, we obtain

$$\begin{aligned} & \|e^{n+1}\|^2 - \|e^n\|^2 + \|2e^{n+1} - e^n\|^2 - \|2e^n - e^{n-1}\|^2 + \frac{4\Delta t^2}{3} (\|h^{n+1}\|^2 - \|g^n\|^2) + 2\|\tilde{e}^{n+1} - e^{n+1}\|^2 \\ & \leq 4\Delta t(\mathcal{D}\tilde{e}^{n+1}, \tilde{e}^{n+1}) + 4\Delta t\lambda(\mathcal{I}(\mathcal{E}e^n), \tilde{e}^{n+1}) + 4\Delta t(\tilde{R}^{n+1}, \tilde{e}^{n+1}) \\ & \leq 2\Delta t(2\alpha + \lambda C_I + 1)\|\tilde{e}^{n+1}\|^2 + 16\Delta t\lambda C_I\|e^n\|^2 + 4\Delta t\lambda C_I\|e^{n-1}\|^2 + 2\Delta t\|\tilde{R}^{n+1}\|^2. \end{aligned} \quad (5.31)$$

We derive from (5.23) that $\|g^n\| \leq \|h^n\|^2 + c\Delta t^2\|h^n\| + c\Delta t^4$, and then summing up the above inequality from $n = 1$ to $n = m - 1$, we get

$$\begin{aligned} & \|e^m\|^2 - \|e^1\|^2 - \|2e^1 - e^0\|^2 + \frac{4\Delta t^2}{3}\|h^m\|^2 - \frac{4\Delta t^2}{3}\|h^1\|^2 + 2\sum_{n=1}^{m-1}\|\tilde{e}^{n+1} - e^{n+1}\|^2 \\ & \leq 2\Delta t(2\alpha + \lambda C_I + 1)\sum_{n=1}^{m-1}\|\tilde{e}^{n+1}\|^2 + 16\Delta t\lambda C_I\sum_{n=1}^{m-1}\|e^n\|^2 + 4\Delta t\lambda C_I\sum_{n=1}^{m-1}\|e^{n-1}\|^2 \\ & \quad + 2\Delta t\sum_{n=1}^{m-1}\|\tilde{R}^{n+1}\|^2 + \frac{4\Delta t^2}{3}\sum_{n=1}^{m-1}(c\Delta t^2\|h^n\|^2 + c\Delta t^4). \end{aligned} \quad (5.32)$$

Based on $\|\tilde{e}^{n+1}\|^2 \leq 2\|\tilde{e}^{n+1} - e^{n+1}\|^2 + 2\|e^{n+1}\|^2$ and assuming that Δt is sufficiently small such that $2 - 4\Delta t(2\alpha + \lambda C_I + 1) > 0$, rearranging (5.32) yields

$$\begin{aligned} \|E^m\|^2 & \leq \|e^1\|^2 + \|2e^1 - e^0\|^2 + \frac{4\Delta t^2}{3}\|h^1\|^2 + 4\Delta t(2\alpha + \lambda C_I + 1)\|e^m\|^2 + 4\Delta t\lambda C_I\|e^0\|^2 \\ & \quad + 20\Delta t\lambda C_I\|e^1\|^2 + (4\Delta t(2\alpha + \lambda C_I + 1) + 20\Delta t\lambda C_I)\sum_{n=2}^{m-1}\|e^n\|^2 \\ & \quad + 2\Delta t\sum_{n=1}^{m-1}\|\tilde{R}^{n+1}\|^2 + \frac{4\Delta t^2}{3}\sum_{n=1}^{m-1}(c\Delta t^2\|h^n\|^2 + c\Delta t^4), \end{aligned}$$

where $\|E^m\|^2 := \|e^m\|^2 + \frac{4\Delta t^2}{3}\|h^m\|^2$. Due to $\|e^m\|^2 \leq \|E^m\|^2$ and $\frac{4\Delta t^2}{3}\|h^n\|^2 \leq \|E^n\|^2$ and assuming that Δt is sufficiently small such that $(1 - 4\Delta t(2\alpha + 1 + \lambda C_I)) > 0$ or $\Delta t < \frac{1}{4(2\alpha+1+\lambda C_I)}$, the above inequality implies

$$\begin{aligned} \|E^m\|^2 & \leq C(\|e^0\|^2 + \|e^1\|^2 + \|2e^1 - e^0\|^2 + \|h^1\|^2 + \Delta t\sum_{n=2}^{m-1}\|E^n\|^2) \\ & \quad + 2\Delta t\sum_{n=1}^{m-1}\|R^{n+1}\|^2 + c\Delta t^2\sum_{n=1}^{m-1}\|E^n\|^2 + O(\Delta t^5). \end{aligned}$$

where C is a constant that depends on the parameters C_I, r, σ, λ , and T . Subsequently, the discrete Grönwall lemma 4.1 is applied to derive the desired estimate,

$$\|e^m\|^2 + \frac{4\Delta t^2}{3}\|h^m\|^2 \leq C\Delta t^4.$$

□

6. Numerical experiments

In this section, we numerically solve the pricing problems of European and American options under the Merton and Kou jump-diffusion models through numerical experiments. We employed an IMEX-BDF2 scheme for temporal semi-discretization and adopted BRI for spatial discretization. All numerical experiments were implemented using MATLAB 2025b. For the implementation, the rational interpolation parameter was chosen as $d = 3$. To address the discontinuity of the first-order derivative at the initial condition of the option, which is at $S = K$ (corresponding to $x^* = \ln(K/S_0)$ in the logarithmic space), we employed a specific coordinate transformation. This transformation maps the computational domain $[-X, X]$ to a uniform reference domain $\xi \in [-1, 1]$, thereby concentrating a dense cluster of grid points in the vicinity of the nonsmooth point x^* . The transformation formula is as follows:

$$x_i = x^* + X \cdot \frac{\sinh(\alpha \xi_i)}{\sinh(\alpha)}, \quad i = 2, \dots, M-1,$$

where $\xi_i = -1 + \frac{2(i-1)}{M-1}$, for $i = 1, 2, \dots, M$. To ensure the exactness of the computational boundaries, the endpoints were strictly fixed as $x_1 = -X$ and $x_M = X$. Here, $\alpha > 0$ serves as the scaling parameter that governs the extent of spatial grid aggregation. In the implementation, it is assigned a fixed value of $\alpha = 0.4$. Furthermore, because the initial nonsmoothness also degrades the temporal convergence order of the BDF2 scheme, we incorporated the Rannacher time-stepping technique. Specifically, the fully implicit Euler scheme is utilized for the first two time steps to provide necessary damping and smooth out the initial singularity, after which the scheme switches to the IMEX-BDF2 method.

To illustrate the advantages of our approach, a comparative analysis was conducted against several established numerical methods, which are primarily manifested by the absolute error and convergence order. The absolute error is defined as

$$e_{N,M} = |u_{N,M} - u_{ref}|.$$

To evaluate the convergence rate of the proposed method across varying asset prices, we employed the following formula:

$$Order_1 = \log_2 \left(\frac{e_{N,M}}{e_{2N,2M}} \right), \quad Order_2 = \log_2 \frac{\|u_{N,M} - u_{2N,2M}\|_{l^2}}{\|u_{2N,2M} - u_{4N,4M}\|_{l^2}},$$

where $u_{N,M}$ denotes the numerically computed price of option, obtained via discretization over N uniformly spaced time steps and M spatial grid points, and u_{ref} represents the reference solution of the option. We utilized piecewise cubic Hermite interpolation to estimate option prices at asset price levels that fall between discrete grid points.

In the financial market, the risk characteristics of options are usually measured by some Greek letters, such as delta (Δ) and gamma (Γ). The delta value measures the extent of change in the option price when the price of the underlying asset changes and is defined as

$$\Delta := \frac{\partial V}{\partial S}.$$

For call options, the delta value ranges between 0 and 1; for put options, it ranges between -1 and 0. The delta value serves not only as a key metric for investors to quantify the sensitivity of an option's price

to changes in the underlying asset's price but also plays a critical role in dynamic hedging strategies aimed at mitigating portfolio market risk. The gamma value measures the rate of change of the delta value when the price of the underlying contract changes, and is defined as

$$\Gamma := \frac{\partial^2 V}{\partial S^2}.$$

The value of gamma ranges from 0 to 1, reflecting the sensitivity of the delta value to the price fluctuations of the underlying asset. Therefore, it can assist investors in predicting the nonlinear changes in option prices. In the subsequent numerical experiments, we not only calculated the price of the option, but also presented the corresponding delta and gamma sensitivities.

6.1. Example 1: European put option under Merton's model

In this example, we chose the parameters in model as

$$\begin{aligned}\sigma &= 0.15, r = 0.05, \mu_j = -0.9, \sigma_j = 0.45, \lambda = 0.1, \\ T &= 0.25, K = 100, X_{\min} = -1.5, X_{\max} = 1.5, S_0 = 100.\end{aligned}$$

These parameters were also used by [40]. The reference values for European put option under Merton's model are 9.285418 at $S = 90$, 3.149026 at $S = 100$, and 1.401186 at $S = 110$. Table 1 provides a comparative analysis of the European put option pricing results under Merton's jump-diffusion model. To highlight the performance of the present method, we report the computed option values, absolute errors, and convergence rates under various spatial grids M and time steps N alongside the pricing errors obtained by the methods in [40] and [9]. The numerical results clearly indicate that under identical grid resolutions, the absolute errors of the present method are smaller than those of the competing schemes. The numerical values, absolute errors, and convergence rates of the Greeks (delta and gamma) for the European put option under Merton's model are documented in Table 2. It can be observed that as the mesh numbers M and N increase, the errors decrease significantly. The empirical convergence orders listed in the table show excellent agreement with the theoretical prediction of second-order accuracy, demonstrating that the method is robust for pricing and hedging under jump-diffusion environments. Figure 1 presents the evolution of delta and gamma at the maturity in $S \in [50, 150]$.

Remark 6.1: To verify the theoretical conditional stability established in Theorem 4.3, we check the time-step constraint using the model parameters in Example 1. With $\alpha = \left| \frac{(r - \frac{1}{2}\sigma^2 - \lambda\xi)^2 - 2(r + \lambda)\sigma^2}{2\sigma^2} \right| \approx 0.0453$ and the computed norm $C_B = \|\mathbf{B}\|_2 \approx 8.3638$ (evaluated with a fine spatial grid of $M = 4096$, which is independent of the temporal grid size N), the theoretical threshold requires

$$\Delta t < \frac{1}{2(2\alpha + C_B + 1)} \approx 0.0529.$$

In our numerical implementation, even the coarsest time resolution ($N = 25$ for $T = 0.25$, corresponding to $\Delta t = 0.01$) strictly satisfies this stability bound, which guarantees the stable performance reported in Table 1.

Table 1. Comparison of European put option pricing under Merton's model, **Example 1.**

		Present method			Method in [40]	Method in [9]
M	N	$Value$	$Error$	$Order_1$	$Error$	$Error$
$S = 90$						
128	25	9.274201	1.12e-02	-	-	1.26e-02
256	50	9.282612	2.80e-03	1.99	2.14e-03	8.40e-05
512	100	9.284718	6.99e-04	2.00	4.54e-04	6.30e-05
1024	200	9.285245	1.73e-04	2.01	1.19e-04	5.10e-05
2048	400	9.285377	4.14e-05	2.06	3.00e-05	2.20e-05
4096	800	9.285409	8.50e-06	2.28	8.00e-06	-
$S = 100$						
128	25	3.126981	2.20e-02	-	-	3.61e-02
256	50	3.143535	5.49e-03	2.00	2.66e-02	8.76e-03
512	100	3.147657	1.36e-03	2.00	6.70e-03	2.12e-03
1024	200	3.148687	3.39e-04	2.01	1.67e-03	5.08e-04
2048	400	3.148944	8.21e-05	2.04	4.17e-04	1.22e-04
4096	800	3.149008	1.78e-05	2.20	1.04e-04	-
$S = 110$						
128	25	1.394533	6.65e-03	-	-	5.63e-03
256	50	1.399519	1.66e-03	1.99	6.05e-03	8.04e-04
512	100	1.400773	4.12e-04	2.01	1.59e-03	5.40e-05
1024	200	1.401087	9.89e-05	2.06	4.02e-04	7.10e-05
2048	400	1.401165	2.05e-05	2.26	1.01e-04	3.00e-05
4096	800	1.401185	9.26e-07	4.46	2.50e-05	-

Table 2. Delta and gamma results for European put option under Merton's model, **Example 1.**

M	N	Delta			Gamma		
		$Value$	$Error$	$Order_1$	$Value$	$Error$	$Order_1$
$S = 90$							
128	25	-0.848927	2.21e-03	-	0.034796	6.44e-05	-
256	50	-0.847265	5.50e-04	2.01	0.034845	1.55e-05	2.05
512	100	-0.846852	1.37e-04	2.01	0.034856	3.85e-06	2.01
1024	200	-0.846749	3.41e-05	2.01	0.034859	9.60e-07	2.00
2048	400	-0.846724	8.43e-06	2.02	0.034860	2.38e-07	2.01
4096	800	-0.846717	2.01e-06	2.07	0.034860	5.72e-08	2.06
$S = 100$							
128	25	-0.354819	8.44e-04	-	0.049186	3.61e-04	-
256	50	-0.355455	2.09e-04	2.02	0.048915	8.97e-05	2.01
512	100	-0.355611	5.21e-05	2.00	0.048848	2.24e-05	2.00
1024	200	-0.355650	1.31e-05	1.99	0.048831	5.61e-06	2.00
2048	400	-0.355660	3.41e-06	1.95	0.048827	1.41e-06	2.00
4096	800	-0.355662	9.77e-07	1.80	0.048826	3.55e-07	1.99
$S = 110$							
128	25	-0.056793	1.31e-03	-	0.011970	1.59e-04	-
256	50	-0.057776	3.25e-04	2.01	0.012090	3.94e-05	2.02
512	100	-0.058020	8.14e-05	2.00	0.012120	9.83e-06	2.00
1024	200	-0.058081	2.05e-05	1.99	0.012127	2.45e-06	2.00
2048	400	-0.058096	5.27e-06	1.96	0.012129	6.10e-07	2.01
4096	800	-0.058100	1.47e-06	1.85	0.012129	1.49e-07	2.03

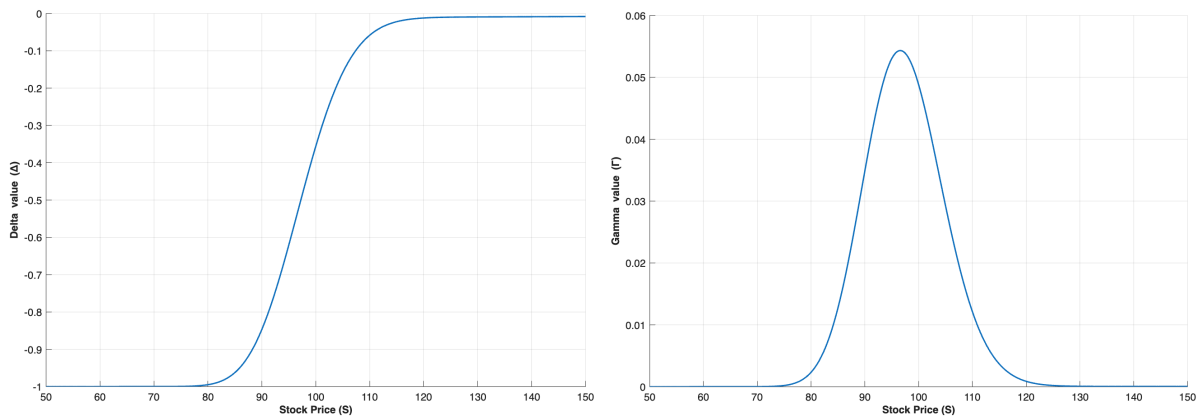


Figure 1. European put option delta and gamma under Merton's model, **Example 1**.

6.2. Example 2: European put option under Kou's model

In this example, we price a European put option under Kou's jump-diffusion model, with model parameters specified as follows:

$$\begin{aligned} \sigma &= 0.15, r = 0.05, \eta_1 = 3.0465, \eta_2 = 3.0775, \lambda = 0.1, p = 0.3445, \\ T &= 0.25, K = 100, X_{\min} = -1.5, X_{\max} = 1.5, S_0 = 100. \end{aligned}$$

These parameters are tested by [16, 41], and the reference prices are 9.430457 at $S = 90$, 2.731259 at $S = 100$, and 0.552363 at $S = 110$. Table 3 presents the approximate option prices and their corresponding absolute errors at asset prices of $S = 90$, $S = 100$, and $S = 110$. In addition, the absolute errors generated by the proposed methods are compared with those produced by the approaches reported in [9, 11]. A comparative analysis reveals that the proposed methodology outperforms existing approaches in both accuracy and computational efficiency. As shown in the table, the absolute errors consistently decreases with increasing values of M and N . Figure 2 presents the payoff function and the European put option price surface under the Kou's model at maturity, with the underlying asset price S ranging from 50 to 150.

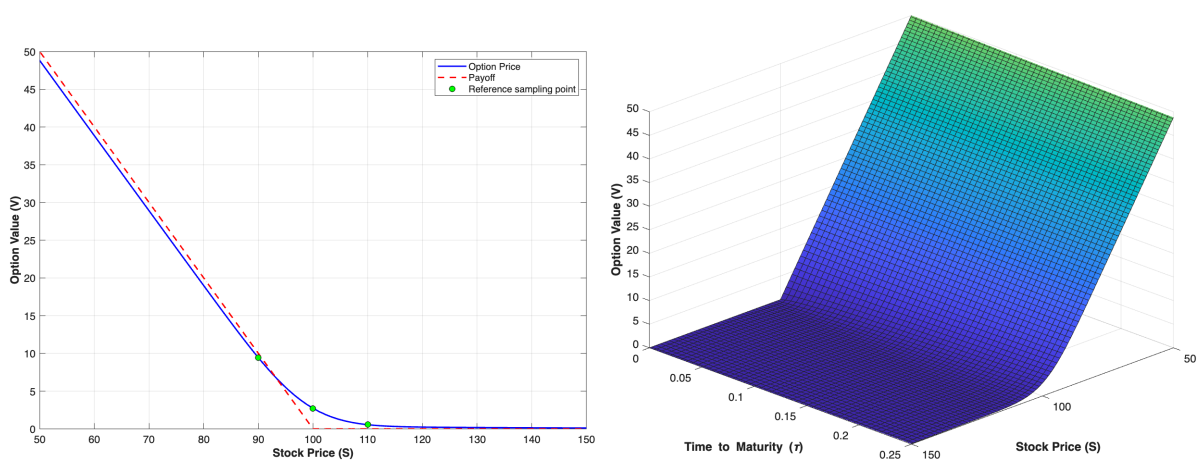


Figure 2. European put option price surface under Kou's model, **Example 2**.

Table 3. Comparison of European put option pricing under Kou's model, **Example 2.**

		Present method			Method in [11]	Method in [9]
M	N	$Value$	$Error$	$Order_1$	$Error$	$Error$
$S = 90$						
> 50	20	9.364786	6.57e-02	-	3.06e-01	3.79e-03
100	40	9.413778	1.67e-02	1.98	4.51e-02	8.70e-04
200	80	9.426610	3.85e-03	2.12	2.44e-03	2.63e-04
400	160	9.428841	1.62e-03	1.25	6.59e-04	9.40e-05
800	320	9.429988	4.69e-04	1.79	1.67e-04	3.80e-05
1600	640	9.430369	8.76e-05	2.42	4.10e-05	1.70e-05
$S = 100$						
50	20	2.579194	1.52e-01	-	4.45e-01	1.10e-01
100	40	2.694505	3.68e-02	2.05	1.11e-01	2.22e-02
200	80	2.722145	9.11e-03	2.01	2.64e-02	5.40e-03
400	160	2.729085	2.17e-03	2.07	6.55e-03	1.37e-03
800	320	2.730784	4.75e-04	2.19	1.63e-03	3.59e-04
1600	640	2.731203	5.63e-05	3.08	4.08e-04	9.90e-05
$S = 110$						
50	20	0.496382	5.59e-02	-	1.76e-01	5.49e-02
100	40	0.538475	1.39e-02	2.01	1.04e-02	4.77e-03
200	80	0.548893	3.47e-03	2.00	5.38e-03	2.11e-03
400	160	0.551525	8.38e-04	2.05	1.37e-03	3.67e-04
800	320	0.552178	1.85e-04	2.18	3.46e-04	1.43e-04
1600	640	0.552342	2.13e-05	3.11	8.60e-05	3.70e-05

6.3. Example 3: American put option under Merton's model

Merton's jump-diffusion model for the American put option is presented now, following the parameters

$$\sigma = 0.15, r = 0.05, \mu_j = -0.9, \sigma_j = 0.45, \lambda = 0.1,$$

$$T = 0.25, K = 100, X_{min} = -1.5, X_{max} = 1.5, S_0 = 100.$$

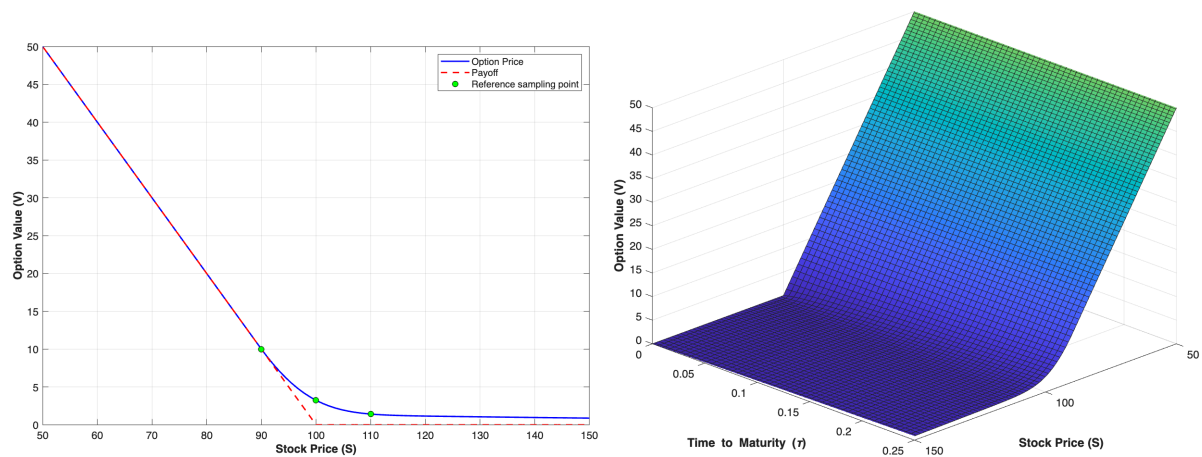
These parameters were adopted in Toivanen's studies [16, 42]. The reference prices are 10.003822 at $S = 90$, 3.241251 at $S = 100$, and 2.807879 at $S = 110$. To compute the errors reported in Table 4, we evaluate the successive price differences of an American put option across a grid of underlying stock prices. As shown in Table 4, the absolute errors of our scheme are lower than those reported in [11] across various asset prices, particularly on relatively coarse grids. Furthermore, Table 5 confirms that the convergence rate in the discrete l^2 norm remains stable at approximately 2. This aligns with the theoretical second-order accuracy expected from the BDF2-based full discretization, validating the efficiency of combining BDF2 with barycentric rational interpolation. Figure 3 depicts the payoff function and surface of American put option value in the Merton models at the maturity for the ranges $S \in [50, 150]$.

Table 4. Comparison of American put option pricing under Merton's model, **Example 3.**

M	N	$S = 90$		$S = 100$		$S = 110$	
		Error	Error [11]	Error	Error [11]	Error	Error [11]
50	20	1.88e-02	2.84e-01	1.20e-01	3.72e-01	3.81e-02	1.04e-01
100	40	6.41e-03	3.49e-02	2.72e-02	1.28e-01	8.94e-03	7.77e-04
200	80	2.52e-03	3.81e-03	6.83e-03	3.21e-02	2.25e-03	6.14e-03
400	160	9.85e-04	3.81e-03	1.44e-03	8.16e-03	4.89e-04	1.62e-03
800	320	2.88e-04	8.54e-04	2.30e-04	2.06e-03	7.91e-05	4.20e-04
1600	640	9.01e-05	2.84e-04	8.46e-06	5.06e-04	5.79e-06	1.04e-04

Table 5. The convergence rate of error estimate, **Example 3.**

M	N	$\ u_{N,M} - u_{2N,2M}\ _{l^2}$	$Order_2$
64	12	-	-
128	25	1.33e-02	-
256	50	3.16e-03	2.08
512	100	7.26e-04	2.12
1024	200	1.60e-04	2.18
2048	400	3.56e-05	2.17
4096	800	8.21e-06	2.12

**Figure 3.** American put option price surface under the Merton model, **Example 3.**

6.4. Example 4: American put option under Kou's model

We adopt Kou's model, with the option parameters specified as follows:

$$\sigma = 0.15, r = 0.05, \eta_1 = 3.0465, \eta_2 = 3.0775, \lambda = 0.1, p = 0.3445, \\ T = 0.25, K = 100, X_{\min} = -1.5, X_{\max} = 1.5, S_0 = 100.$$

These parameters are also used by [41] and the reference prices are 10.005071 at $S = 90$, 2.807879 at $S = 100$, and 0.561876 at $S = 110$. A comparative analysis in Table 6 reveals that the present method outperforms the existing scheme in [11], achieving superior accuracy with fewer grid points. The empirical convergence orders presented in Table 7 consistently hover around 2.1, providing strong evidence for the second-order convergence of the overall scheme. These results indicate that the proposed algorithm is highly effective for solving the integro-differential equations arising from option pricing with jumps. Figure 4 presents the delta values and gamma values under Kou's model. To demonstrate the computational efficiency of the proposed numerical methods, the CPU times for different experimental setups are documented in Table 8. The recorded CPU times show that although the computational workload grows as the mesh size increases, the overall execution time remains remarkably small. This highlights the high efficiency of our proposed algorithms in dealing with different types of options and jump structures.

Table 6. Comparison of American put option pricing under Kou's model, **Example 4**.

M	N	$S = 90$		$S = 100$		$S = 110$	
		Error	Error [11]	Error	Error [11]	Error	Error [11]
50	20	2.63e-02	2.43e-01	1.26e-01	4.27e-01	4.02e-02	1.75e-01
100	40	7.04e-03	2.98e-02	2.89e-02	1.20e-01	1.02e-02	1.06e-02
200	80	1.40e-03	5.07e-03	7.31e-03	2.98e-02	2.96e-03	5.76e-03
400	160	4.32e-04	4.26e-03	1.40e-03	7.62e-03	5.99e-04	1.54e-03
800	320	5.77e-05	3.12e-04	1.55e-04	1.96e-03	6.98e-05	4.12e-04

Remark 6.2: Regarding the conditional stability for American options (Theorems 5.1 and 5.2), we check the condition for Example 4. Substituting $\alpha \approx 0.1179$, $C_I = 1$, and $\lambda = 0.1$ yields the theoretical upper bound

$$\Delta t < \frac{1}{4(2\alpha + \lambda C_I + 1)} \approx 0.1872.$$

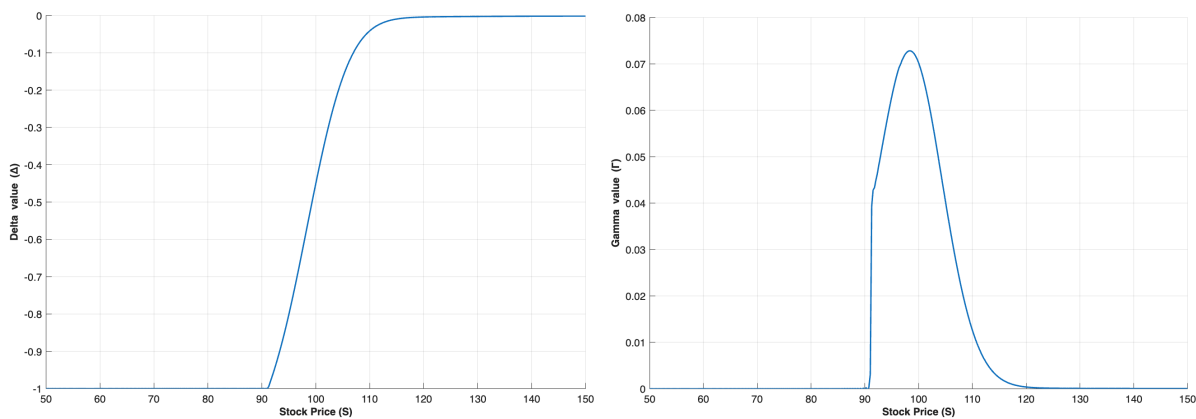
The time step used in our baseline test ($N = 20$ for $T = 0.25$, giving $\Delta t = 0.0125$) easily satisfies this restriction, ensuring the stable performance reported in Table 6.

Table 7. The convergence rate of error estimate, **Example 4**.

M	N	$\ u_{N,M} - u_{2N,2M}\ _{l^2}$	$Order_2$
64	12	-	-
128	25	1.35e-02	-
256	50	3.28e-03	2.03
512	100	7.56e-04	2.11
1024	200	1.71e-04	2.14
2048	400	4.00e-05	2.09
4096	800	8.36e-06	2.25

Table 8. CPU times of the proposed methods for different experiment setup.

M	N	Merton model		Kou model	
		European option	American option	European option	American option
50	20	0.0077	0.0131	0.0064	0.0014
100	40	0.0233	0.0144	0.0100	0.0031
200	80	0.0694	0.1207	0.0229	0.0198
400	160	0.1482	0.1364	0.0948	0.0636
800	320	0.5345	0.5629	0.6038	0.5009
1600	640	2.8889	3.0334	3.2048	2.8451

**Figure 4.** American put option delta and gamma under Kou's model, **Example 4**.

7. Conclusions

In this paper, we have proposed and rigorously analyzed an efficient, highly accurate numerical framework for pricing European and American options under Merton's and Kou's jump-diffusion models. The temporal evolution and the complex nonlocal integral term are effectively handled by combining the conditionally stable IMEX-BDF2 scheme with the Gauss–Legendre quadrature formula. For spatial discretization, the BRI collocation method is adopted to ensure high precision and excellent numerical stability. Furthermore, a comprehensive theoretical analysis is conducted, providing both convergence rates and error estimates. Specifically, we establish the stability of the operator splitting method based on the IMEX-BDF2 scheme for American option pricing. Numerical results demonstrate that the proposed framework approximates the option values with high efficiency while providing stable and reliable approximations for delta and gamma values. Although this paper primarily focuses on the theoretical analysis and computational efficiency of the proposed numerical method, exploring its empirical performance is a natural next step. In future work, we aim to calibrate the model parameters using real-world financial market data and further investigate its practical advantages in real trading and risk management environments.

Use of AI tools declaration

AI tools were used for language editing and readability improvements.

Conflict of interest

The authors declare there are no conflicts of interest.

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