Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 33(8): 5085–5099.
DOI: 10.3934/era.2025228
Received: 17 June 2025
Revised: 18 August 2025
Accepted: 20 August 2025
Published: 29 August 2025

Research article

A class of hyperbolic fractional Kirchhoff equations involving viscoelastic and dissipative terms

Lijun Zhou and Ning Pan*

Department of Mathematics, Northeast Forestry University, Harbin 150040, China

* **Correspondence:** Email: hljpning@163.com.

Abstract: This article investigates a class of hyperbolic equations of the fractional Kirchhoff type with viscoelastic and nonlinear terms:

$$\begin{cases} u_{tt} + M([u]_s^2)(-\Delta)^s u - \int_0^t g(t-\tau)(-\Delta)^s u(\tau)d\tau + |u_t|^{a-2}u_t + u_t + u = |u|^{b-2}u, & \\ & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases}$$

where $[u]_s$ is the Gagliardo semi-norm of u, $\Omega \subset \mathbb{R}^N$ is a confined area featuring a smooth boundary, $(-\Delta)^s$ is the fractional Laplacian with $s \in (0,1)$, $2 < a < 2\gamma < b < 2_s^*$, u_0 and u_1 are the initial function. First, we obtain the existence of global solutions by combining the potential wells with the Galerkin method. Moreover, employing the perturbed energy approach, we systematically study the asymptotic behavior of solutions.

Keywords: hyperbolic; fractional Kirchhoff type; viscoelastic; global existence; asymptotic behavior

1. Introduction

In this paper, we deal with the following initial boundary value problem for a fractional hyperbolic equation involving the Kirchhoff term, the viscoelastic term, and the nonlinear dissipative term:

$$\begin{cases} u_{tt} + M([u]_s^2)(-\Delta)^s u - \int_0^t g(t-\tau)(-\Delta)^s u(\tau)d\tau + g(u_t) + u = f(u), & \\ & \text{in } \Omega \times (0,\infty), \\ u(\cdot,t) = 0, & \text{in } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0, \quad u_t(\cdot,0) = u_1, & \text{in } \Omega, \end{cases} \tag{1.1}$$

where the definition of the Gagliardo semi-norm $[u]_s$ is given by

$$[u]_s = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x,t) - u(y,t)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}}.$$

And for $s \in (0,1)$ and $\forall \varphi \in C_0^\infty(\mathbb{R}^N)$, give insight into the traits of the fractional Laplacian $(-\Delta)^s$

$$(-\Delta)^s \varphi(x) = 2 \lim_{\theta_1 \to 0^+} \int_{\mathbb{R}^N \setminus B_{\theta_1(x)}} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} dy,$$

and it goes up to a normalized constant, $B_{\theta_1(x)}$ here means that the sphere in $\mathbb{R}^N$ where radius $\theta_1 > 0$. Besides, it is imperative to point out that the Kirchhoff function $M(m) = 1 + m^{\gamma-1}$ is a positive C^1−function for all $m \geq 0, \gamma > 1$, the damping term $g(u_t) = |u_t|^{a-2}u_t + u_t$, and the source term $f(u) = |u|^{b-2}u$.

Considerable work has been devoted to analyzing the following wave equation:

$$u_{tt} - \Delta u + h(u_t) = f(u),$$

with appropriate initial values and boundary constraints, leading to established theories concerning solution existence and asymptotic trends, more details can be see [1, 2]. Physically, h accounts for friction or damping effects, whereas f corresponds to the external source. The dynamics of an elastic string undergoing nonlinear vibrations can be modeled through the Kirchhoff equation

$$u_{tt} + M(\|A^{1/2}u\|^2)Au + |u_t|^\beta u_t = f(u), \tag{1.2}$$

where $A \equiv -\Delta$ is the Laplace operator. In [3], *Ono* established comprehensive results for problem (1.2), including global existence, solution decay rates, and finite-time blowup criteria, provided that appropriate assumptions are made regarding $f(u)$. On a more fundamental level, *Esquivel-Avila* examined the case where the nonlinear dissipative term $|u_t|^\beta u_t$ in (1.2) is replaced by $\delta|u_t|^{a-2}u_t$, with parameters $\delta \geq 0,\ a > 2$, while adopting the power-type source term $f(u) = \mu|u|^{b-2}u$, $\mu > 0, b > 2$ in the fourth part of [4]. Consequently, the author demonstrated that the system (1.2) augmented by a nonlinear source term admits both global and non-global solutions, employing the potential wells method for the proof. In [5], the global solvability and blowup of solutions to the nonlinear damped and perturbed Klein–Gordon equation were analyzed by *Aassila* via the potential well theory. A broader discussion of related results is available in references [6, 7]. Pan et al. [8] treatment of the fractionally damped Kirchhoff equation considered weak damping with nonlinear perturbations

$$u_{tt} + [u]_s^{2\theta-2}(-\Delta)^s u + |u_t|^{a-2}u_t + u = |u|^{b-2}u.$$

Their application of potential wells theory yielded complete characterization of solution behavior, including global existence, vacuum isolation, long-term dynamics, and blow-up phenomena. In subsequent developments, the potential well theory has become a standard tool for analyzing solution existence in evolution equations, as demonstrated by numerous studies, for example, [9, 10].

Before proceeding further, the following nonlinear integro-differential equation

$$u_{tt} + M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \Delta u_t = |u|^{p-2}u$$

describes the motion of deformable solids while accounting for hereditary effects. In [11], Wu established that subject to specific constraints on g, the system admits global solutions with exponential energy decay. As a natural extension, Wu [12] considered the dissipative term $|u_t|^{p-1}u_t$ instead of Δu_t, providing a comprehensive study on the long-time behavior and blow-up of solutions for this system. Besides, Boumaza and Gheraibia [13] considered a nonlinear degenerate viscoelastic equation of Kirchhoff type with source term and established the global existence of solutions, general decay properties, and finite-time blow-up of solutions with negative initial energy. More recently, in [14], through various analytical approaches, the authors had obtained solutions to the fractional viscoelastic hyperbolic equation with the Kirchhoff term, accounting for both strong damping effects and nonlinearity with variable coefficients

$$u_{tt} + M([u]_{\alpha,2}^2)(-\Delta)^\alpha u - \int_0^t g(t-\tau)(-\Delta)^\alpha u(\tau)d\tau + (-\Delta)^s u_t = \lambda|u|^{b-2}u,$$

where $0 < s \leq \alpha < 1, 1 < b < \infty, \lambda > 0$. Depending on the parameter values of b and λ, they employed the Galerkin method to obtain both local and global solutions, respectively. Furthermore, they established global nonexistence results through blow-up analysis.

Partial differential equations serve as powerful tools for modeling complex phenomena across various fields such as biomechanics, fluid dynamics, and financial analysis. Among these, Kirchhoff-type equations—originally developed to describe vibrations of elastic plates—have been extensively investigated in their classical form, which employs a local Laplacian operator and does not account for memory effects. However, such models fail to capture critical features of modern applications, such as anomalous diffusion in porous media, viscoelastic damping in polymers, or memory-dependent stiffness in biological tissues. To address these limitations, we investigate a fractional Kirchhoff equation incorporating dissipative terms and viscoelastic terms. From a mathematical perspective, the fractional Laplacian introduces memory effects, which are essential for modeling complex media with nonlocal diffusion or dispersion (e.g., porous materials, turbulent flows). The viscoelastic term (e.g., $\int_0^t g(t-\tau)(-\Delta)^s u(\tau)d\tau$) accounts for time-dependent material relaxation, requiring delicate analysis of integro-differential operators and energy decay rates. Physically, such equations model, e.g., viscoelastic plates with fractional damping or biological tissues with memory-dependent stiffness, where classical PDEs fail to capture multiscale behaviors. In addition, the existence of solutions for certain related problems possessing nonlocal characteristics could be considered, as addressed in [15–17].

Motivated by the above works, in this paper we are mainly interested in the effects of the Kirchhoff function, the memory kernel, and the damping term on the evolution behavior of solutions to viscoelastic Kirchhoff equations with fractional Laplacian. Note that the joint presence of the Kirchhoff function, the viscoelastic term and the damping term introduces challenges; we choose the potential wells and the Galerkin method to study the global existence of the solutions. Of course, in studying the asymptotic behavior of solutions, we repeatedly employ Hölder's inequality, Young's inequality and Cauchy's inequality to overcome the analytical challenges posed by the viscoelastic and dissipative terms. The subsequent content consists of five parts: In Section 2, we give the relevant definitions about fractional Sobolev some necessary notation and lemmas. In addition, we present the formal definition of a potential well and systematically investigate its mathematical properties. Section 3, we construct approximate solutions by using Galerkin method, thus discussing the existence of global solutions. Section 4: Through an application of the potential well theory in conjunction with the perturbed energy technique [18–20], we derive the asymptotic properties of global solutions. Section 5, we present a comprehensive summary of the main results established in this article.

2. Preliminaries

The following introduces key concepts and notation for fractional Sobolev spaces. Additional background on fractional-order operators is available in the literature [21, 22]. In what follows, $s \in (0,1)$ and $N > 2s$ are considered. The critical exponent in the fractional setting, denoted 2_s^*, is

$$2_s^* = \frac{2N}{N-2s}.$$

The Lebesgue space $L^m(\Omega)$ for $m \geq 1$ consists of measurable functions u with finite norm

$$\|u\|_m = \left(\int_\Omega |u|^m dx\right)^{\frac{1}{m}}.$$

Denote $Q = \mathbb{R}^N \backslash \mathscr{D}$, $\mathscr{D} = (\mathbb{R}^N \backslash \Omega) \times (\mathbb{R}^N \backslash \Omega) \subset \mathbb{R}^{2N}$. Write X as a linear space, and it consists of, Lebesgue measurable function u: $\mathbb{R}^n \to \mathbb{R}$, ensuring that $\forall u \in L^2(\Omega)$ holds for the restriction to Ω in X and

$$\iint_Q |u(x)-u(y)|^2 K(x-y)dxdy < \infty.$$

The canonical norm of X is

$$\|u\|_X = \|u\|_{L^2(\Omega)} + \left(\iint_Q |u(x)-u(y)|^2 K(x-y)dxdy\right)^{\frac{1}{2}}.$$

And we introduce the following closed linear subspace of X:

$$X_0 = \left\{u \in X | u(x) = 0 \;\; a.e. \;\text{ in } \mathbb{R}^N \backslash \Omega\right\},$$

and the norm

$$\|u\|_{X_0} = \left(\iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{\frac{1}{2}}.$$

Besides, we restrict our consideration to the canonical kernel $K(x-y) = |x-y|^{-(N+2s)}$, which captures the essential features of the general case. The foundational theory for such kernels is developed in [23].

Now, we begin by stating the general assumptions imposed on the memory kernel g:

$(A1)$ $g(t) : [0,+\infty) \to [0,+\infty)$ *is a* C^1 *function and satisfies*

$$g'(t) \leq 0, k(t) = 1 - \int_0^t g(\tau)d\tau \geq 1 - \int_0^{+\infty} g(\tau)d\tau = k > 0.$$

$(A2)$ $\exists \varrho > 0$ *such that* $g'(t) \leq -\varrho g(t)$ *holds for all* $t > 0$.

From now on, the specific expression of the total energy functional is given below

$$E(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2\gamma}\|u\|_{X_0}^{2\gamma}+\frac{1}{2}\left(1-\int_0^t g(\tau)d\tau\right)\|u\|_{X_0}^2+\frac{1}{2}(g\circ u)(t)-\frac{1}{b}\|u\|_b^b+\frac{1}{2}\|u\|_2^2 \tag{2.1}$$

for $u\in X_0$, where

$$(g\circ u)(t)=\int_0^t g(t-\tau)\|u(\tau)-u(t)\|_{X_o}^2 d\tau.$$

The potential well W is given by

$$W=\left\{u\in X_0\middle|\|u\|_{X_0}<\left(\frac{2b}{(b-2)k}d\right)^{\frac{1}{2}}\right\}, \tag{2.2}$$

whose boundary ∂W is given by

$$\partial W=\left\{u\in X_0\middle|\|u\|_{X_0}=\left(\frac{2b}{(b-2)k}d\right)^{\frac{1}{2}}\right\}, \tag{2.3}$$

with d being the potential well depth

$$d=\frac{b-2}{2b}k^{\frac{b}{b-2}}\mathscr{E}_1^{-\frac{2b}{b-2}}, \tag{2.4}$$

and $\mathscr{E}_1$ is the best Sobolev constant for the embedding $X_0\hookrightarrow L^b(\Omega)$, i.e.,

$$\mathscr{E}_1=\sup_{u\in X_0\setminus\{0\}}\frac{\|u\|_b}{\|u\|_{X_0}}.$$

Lemma 2.1. (1) *There exists $\alpha=\alpha(N,\upsilon,s)$, where $\upsilon\in[1,2_s^*]$, such that, for all $v\in X_0$*

$$\|v\|_{L^{\upsilon}(\Omega)}^2\le\alpha\iint_{\Omega\times\Omega}\frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}dxdy\le\frac{\alpha}{\beta}\iint_Q|v(x)-v(y)|^2K(x-y)dxdy.$$

(2) *There exists $\widetilde{\alpha}=\widetilde{\alpha}(N,s,\beta,\Omega)>0$ such that, for arbitrary $v\in X_0$,*

$$\iint_Q|v(x)-v(y)|^2K(x-y)dxdy\le\|v\|_X^2\le\widetilde{\alpha}\iint_Q|v(x)-v(y)|^2K(x-y)dxdy.$$

(3) *There exists $v\in L^{\upsilon}(\mathbb{R}^{\mathbb{N}})$ such that up to a subsequence, where $\{v_j\}\in X_0$ is a bounded sequence and $\forall\upsilon\in[1,2_s^*)$,*

$$v_j\to v\quad\text{strongly in } L^{\upsilon}(\Omega)\text{ as } j\to\infty.$$

Lemma 2.2. *Let* (*A*1) *and* (*A*2) *be fulfilled. This leads to the following propositions:*
(1) *If $u\in W$ and $\|u\|_{X_0}\neq 0$, then $k\|u\|_{X_0}^2>\|u\|_b^b$.*
(2) *If $u\in\partial W$, then $k\|u\|_{X_0}^2\ge\|u\|_b^b$.*

Proof. (1) The condition $u \in W$ coupled with (2.2) yields

$$\|u\|_{X_0} < \left(\frac{2b}{(b-2)k}d\right)^{\frac{1}{2}}.$$

An application of (2.4) leads to the conclusion that

$$\|u\|_{X_0} < k^{\frac{1}{b-2}}\mathscr{E}_1^{-\frac{b}{b-2}}.$$

Observing that $\|u\|_{X_0}$ is nonvanishing, we derive

$$\mathscr{E}_1^b\|u\|_{X_0}^b < k\|u\|_{X_0}^2.$$

Hence, we attain

$$\|u\|_b^b < k\|u\|_{X_0}^2.$$

(2) The condition $u \in \partial W$ coupled with (2.3) yields

$$\|u\|_{X_0} = \left(\frac{2b}{(b-2)k}d\right)^{\frac{1}{2}}.$$

By employing reasoning parallel to that used in establishing (1), we can straightforwardly demonstrate the inequality $k\|u\|_{X_0}^2 \geq \|u\|_b^b$. □

Overall, Section 3 and 4 contain the proofs of our main results.

3. Global existence of the solutions

Definition 3.1. *Function $u = u(t) \in L^\infty(0,\infty;X_0)$ is called a weak solution of problem (1.1), in the event of $u_t \in L^\infty(0,\infty;L^2(\Omega))$ and for $\forall\varphi \in X_0$ satisfy*

$$\begin{aligned}&(u_t(\cdot,t),\varphi) + \int_0^t M([u(\cdot,\tau)]_s^2)(u(\cdot,\tau),\varphi)_{X_0}d\tau - \int_0^t\int_0^s g(s-\tau)(u(\cdot,\tau),\varphi)_{X_0}d\tau ds\\&+\int_0^t(|u_t(\cdot,\tau)|^{a-2}u_t(\cdot,\tau),\varphi)d\tau + (u(\cdot,t),\varphi) + \int_0^t(u(\cdot,\tau),\varphi)d\tau\\&=(u_1,\varphi)+(u_0,\varphi)+\int_0^t(|u(\cdot,\tau)|^{b-2}u(\cdot,\tau),\varphi)d\tau,\end{aligned} \tag{3.1}$$

where

$$(u(\cdot,t),\varphi)_{X_0} = \iint_Q [u(x,t)-u(y,t)][\varphi(x)-\varphi(y)]K(x-y)dxdy.$$

Theorem 3.1. *Let (A1) and (A2) be fulfilled. For initial conditions $u_0 \in W, u_1 \in L^2(\Omega)$, and $E(0) < d$, the problem (1.1) admits a global solution $u(t) \in \overline{W} := W \cup \partial W$ for all $t \in (0,\infty)$.*

Proof. To start with, consider $\{\omega_j\} \subset C_0^\infty$, the eigenfunctions of the fractional Laplace operator $(-\Delta)^s$, which form an orthogonal basis for both X_0 and $L_2(\Omega)$. The approximate solution u_n of problem (1.1) can be formulated as:

$$u_n(x,t) = \sum_{j=1}^{n} \xi_{jn}(t)\omega_j(x), \quad n = 1, 2, \cdots, \tag{3.2}$$

meet with

$$\begin{aligned}(u_{ntt}(\cdot,t),\omega_j) + M([u]_s^2)(u_n(\cdot,t),\omega_j)_{X_0} - \int_0^t g(t-\tau)(u_n(\cdot,\tau),\omega_j)_{X_0}d\tau \\ + (|u_{nt}(\cdot,t)|^{a-2}u_{nt}(\cdot,t),\omega_j) + (u_{nt}(\cdot,t),\omega_j) + (u_n(\cdot,t),\omega_j) \\ = (|u_n(\cdot,t)|^{b-2}u_n(\cdot,t),\omega_j), \quad j = 1, 2, \cdots, n,\end{aligned} \tag{3.3}$$

$$u_n(\cdot,0) = \sum_{j=1}^{n} \xi_{jn}(0)\omega_j(x) \to u_0(x) \text{ in } X_0, \tag{3.4}$$

$$u_{nt}(\cdot,0) = \sum_{j=1}^{n} \xi'_{jn}(0)\omega_j(x) \to u_1(x) \text{ in } L^2(\Omega). \tag{3.5}$$

Equation (3.3), when multiplied by $\xi'_{jn}(t)$ and summed over j, produces

$$\begin{aligned}(u_{ntt}(\cdot,t),u_{nt}(\cdot,t)) + M([u(\cdot,t)]_s^2)(u_n(\cdot,t),u_{nt}(\cdot,t))_{X_0} - \int_0^t g(t-\tau)(u_n(\cdot,\tau),u_{nt}(\cdot,t))_{X_0}d\tau \\ + (|u_{nt}(\cdot,t)|^{a-2}u_{nt}(\cdot,t),u_{nt}(\cdot,t)) + (u_{nt}(\cdot,t),u_{nt}(\cdot,t)) + (u_n(\cdot,t),u_{nt}(\cdot,t)) \\ = (|u_n(\cdot,t)|^{b-2}u_n(\cdot,t),u_{nt}(\cdot,t)).\end{aligned} \tag{3.6}$$

It should be noted that

$$\begin{aligned}&\int_0^t g(t-\tau)(u_n(\cdot,\tau),u_{nt}(\cdot,t))_{X_0}d\tau \\ &= \int_0^t g(t-\tau)(u_n(\cdot,\tau)-u_n(\cdot,t),u_{nt}(\cdot,t))_{X_0}d\tau + \int_0^t g(t-\tau)(u_n(\cdot,t),u_{nt}(\cdot,t))_{X_0}d\tau \\ &= -\frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}\|u_n(\cdot,\tau)-u_n(\cdot,t))\|_{X_0}^2d\tau + \frac{1}{2}\int_0^t g(t-\tau)\frac{d}{dt}\|u_n(\cdot,t)\|_{X_0}^2d\tau \\ &= -\frac{1}{2}\frac{d}{dt}\left((g\circ u_n)(\cdot,t) - \int_0^t g(\tau)d\tau\|u_n(\cdot,t)\|_{X_0}^2\right) + \frac{1}{2}(g'\circ u_n)(\cdot,t) - \frac{1}{2}g(t)\|u_n(\cdot,t)\|_{X_0}^2.\end{aligned}$$

With this in (3.6), integrating over t, implies

$$E_n(t) + \int_0^t \left(\|u_{nt}(\cdot,\tau)\|_a^a + \|u_{nt}(\cdot,\tau)\|_2^2 - \frac{1}{2}(g'\circ u_n)(\cdot,\tau) + \frac{1}{2}g(\tau)\|u_n(\cdot,\tau)\|_{X_0}^2\right)d\tau = E_n(0) \tag{3.7}$$

for all $t \in [0,T]$, where

$$\begin{aligned}E_n(t) =& \frac{1}{2}\|u_{nt}(\cdot,t)\|_2^2 + \frac{1}{2\gamma}\|u_n(\cdot,t)\|_{X_0}^{2\gamma} + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right)\|u_n(\cdot,t)\|_{X_0}^2 \\ &+ \frac{1}{2}(g\circ u_n)(\cdot,t) - \frac{1}{b}\|u_n(\cdot,t)\|_b^b + \frac{1}{2}\|u_n(\cdot,t)\|_2^2.\end{aligned} \tag{3.8}$$

In light of (3.4)–(3.5), the established results show for a sufficiently large n, both $E_n(0) < d$ and $u_n(\cdot, 0) \in W$ are satisfied. Next, we verify that

$$u_n(\cdot, t) \in W. \tag{3.9}$$

Postulate that there exists some $t \in (0, T)$ such that $u_n(\cdot, t) \notin W$, i.e., by continuity, we attain $u_n(\cdot, t_0) \in \partial W$ for $0 < t_0 < T$ as well as $u_n(\cdot, t) \in W$ for $0 \le t < t_0$. This implies

$$\|u_n(\cdot, t_0)\|_{X_0} = \left(\frac{2b}{(b-2)k} d\right)^{\frac{1}{2}}.$$

Through (3.8), Lemma 2.2 (2) and $(A1)$, we obtain

$$\begin{aligned} E_n(t_0) &\ge \frac{1}{2}(1 - \int_0^t g(\tau)d\tau)\|u_n(\cdot, t_0)\|_{X_0}^2 - \frac{1}{b}\|u_n(\cdot, t_0)\|_b^b \\ &\ge \frac{1}{2}k\|u_n(\cdot, t_0)\|_{X_0}^2 - \frac{1}{b}\|u_n(\cdot, t_0)\|_b^b \\ &= (\frac{1}{2} - \frac{1}{b})k\|u_n(\cdot, t_0)\|_{X_0}^2 + \frac{1}{b}(k\|u_n(\cdot, t_0)\|_{X_0}^2 - \|u_n(\cdot, t_0)\|_b^b) \\ &\ge \frac{b-2}{2b}k\|u_n(\cdot, t_0)\|_{X_0}^2 \\ &= d, \end{aligned}$$

this in contradiction with $E_n(0) < d$ specified in (3.7). Hence, it is clearly seen that $u_n(\cdot, t) \in W$.

For one thing, from (3.8), (3.9), and (1) in Lemma 2.2, we attain

$$\begin{aligned} E_n(t) \ge& \frac{1}{2}\|u_{nt}(\cdot, t)\|_2^2 + \frac{1}{2\gamma}\|u_n(\cdot, t)\|_{X_0}^{2\gamma} + \frac{1}{2}(1 - \int_0^t g(\tau)d\tau)\|u_n(\cdot, t)\|_{X_0}^2 \\ &- \frac{1}{b}\|u_n(\cdot, t)\|_b^b + \frac{1}{2}\|u_n(\cdot, t)\|_2^2 \\ \ge& \frac{1}{2}\|u_{nt}(\cdot, t)\|_2^2 + \frac{1}{2\gamma}\|u_n(\cdot, t)\|_{X_0}^{2\gamma} + \frac{1}{2}k\|u_n(\cdot, t)\|_{X_0}^2 \\ &- \frac{1}{b}\|u_n(\cdot, t)\|_b^b + \frac{1}{2}\|u_n(\cdot, t)\|_2^2 \\ \ge& \frac{1}{2}\|u_{nt}(\cdot, t)\|_2^2 + \frac{1}{2\gamma}\|u_n(\cdot, t)\|_{X_0}^{2\gamma} + \frac{b-2}{2b}k\|u_n(\cdot, t)\|_{X_0}^2 \\ &+ \frac{1}{b}\left(k\|u_n(\cdot, t)\|_{X_0}^2 - \|u_n(\cdot, t)\|_b^b\right) + \frac{1}{2}\|u_n(\cdot, t)\|_2^2 \\ \ge& \frac{1}{2}\|u_{nt}(\cdot, t)\|_2^2 + \frac{1}{2\gamma}\|u_n(\cdot, t)\|_{X_0}^{2\gamma} + \frac{b-2}{2b}k\|u_n(\cdot, t)\|_{X_0}^2 + \frac{1}{2}\|u_n(\cdot, t)\|_2^2, \end{aligned} \tag{3.10}$$

this, along with (3.7), implies

$$\frac{1}{2}\|u_{nt}(\cdot, t)\|_2^2 + \frac{1}{2\gamma}\|u_n(\cdot, t)\|_{X_0}^{2\gamma} + \frac{b-2}{2b}k\|u_n(\cdot, t)\|_{X_0}^2 + \frac{1}{2}\|u_n(\cdot, t)\|_2^2 < d$$

for all $t \in [0, T]$.

For another, a direct consequence of (3.7) and (3.8)

$$\int_0^t \|u_{nt}(\cdot,\tau)\|_a^a d\tau \le E_n(0) < d.$$

To put it another way,

$$\|u_{nt}(\cdot,t)\|_2^2 < 2d, \tag{3.11}$$

$$\|u_n(\cdot,t)\|_{X_0}^2 < \frac{2b}{(b-2)k}d, \tag{3.12}$$

$$\int_0^t \|u_{nt}(\cdot,\tau)\|_a^a d\tau < d, \tag{3.13}$$

$$\|u_n(\cdot,t)\|_2^2 < 2d. \tag{3.14}$$

.

Besides, Lemma 2.1 gives

$$\|u_n(\cdot,t)\|_b^b \le \mathscr{E}_1^b \|u_n(\cdot,t)\|_{X_0}^b \le \mathscr{E}_1^b \left(\frac{2b}{(b-2)k}d\right)^{\frac{b}{2}}. \tag{3.15}$$

Therefore, combing (3.11)–(3.15), there exists $\varkappa$, ς, u and a subsequence of $\{u_n\}_{n=1}^{\infty}$, still denoted by $\{u_n\}_{n=1}^{\infty}$, such that

$$u_n \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,\infty;X_0), \tag{3.16}$$

$$u_{nt} \overset{*}{\rightharpoonup} u_t \text{ in } L^\infty(0,\infty;L^2(\Omega)), \tag{3.17}$$

$$|u_n|^{b-2}u_n \overset{*}{\rightharpoonup} \varkappa \text{ in } L^\infty(0,\infty;L^{\frac{b}{b-1}}(\Omega)), \tag{3.18}$$

$$|u_{nt}|^{a-2}u_{nt} \rightharpoonup \varsigma \text{ in } L^\infty(0,\infty;L^{\frac{a}{a-1}}(\Omega)), \tag{3.19}$$

as $n \to \infty$.

Performing time-integration on (3.3) results in

$$\begin{aligned}
&(u_{nt}(\cdot,t),\omega_j) + \int_0^t M([u_n(\cdot,\tau)]_s^2)(u_n(\cdot,\tau),\omega_j)_{X_0} d\tau - \int_0^t \int_0^s g(s-\tau)(u_n(\cdot,\tau),\omega_j)_{X_0} d\tau ds \\
&\quad + \int_0^t (|u_{nt}(\cdot,\tau)|^{a-2}u_{nt}(\cdot,\tau),\omega_j) d\tau + (u_n(\cdot,t),\omega_j) + \int_0^t (u_n(\cdot,\tau),\omega_j) d\tau \\
&= (u_{nt}(\cdot,0),\omega_j) + (u_n(\cdot,0),\omega_j) + \int_0^t (|u_n(\cdot,\tau)|^{b-2}u_n(\cdot,\tau),\omega_j) d\tau.
\end{aligned}$$

For fixed j, letting $n \to \infty$, from this we deduce the additional result

$$\begin{aligned}
&(u_t(\cdot,t),\omega_j) + \int_0^t M([u\cdot,\tau]_s^2)(u(\cdot,\tau),\omega_j)_{X_0} d\tau - \int_0^t \int_0^s g(s-\tau)(u(\cdot,\tau),\omega_j)_{X_0} d\tau ds \\
&\quad + \int_0^t (\varsigma,\omega_j) d\tau + (u(\cdot,t),\omega_j) + \int_0^t (u(\cdot,\tau),\omega_j) d\tau \\
&= (u_1,\omega_j) + (u_0,\omega_j) + \int_0^t (\varkappa,\omega_j) d\tau.
\end{aligned}$$

Therefore, since C_0^∞ is dense in X_0, as is shown [22], and the fact that $\{\omega_j\} \subset C_0^\infty$ is an orthonormal basis of $L^2(\Omega)$, we get for all $\rho \in X_0$

$$(u_t(\cdot,t),\rho) + \int_0^t M([u(\cdot,\tau)]_s^2)(u(\cdot,\tau),\rho)_{X_0}d\tau - \int_0^t\int_0^s g(s-\tau)(u(\cdot,\tau),\rho)_{X_0}d\tau ds$$
$$+ \int_0^t (\varsigma,\rho)d\tau + (u(\cdot,t),\rho) + \int_0^t (u(\cdot,\tau),\rho)d\tau = (u_1,\rho) + (u_0,\rho) + \int_0^t (\varkappa,\rho)d\tau.$$

Using the method in [24], we have therefore shown that $\varkappa = |u|^{b-2}u$ and $\varsigma = |u_t|^{a-2}u_t$. By virtue of (3.4)–(3.5), $u(\cdot,0) = u_0$ in X_0 is obtained, and $u_t(\cdot,0) = u_1$ in $L^2(\Omega)$ as well. Besides, setting $\rho(x) = \varphi(\cdot,t)$, fixing t here, as well as integrating with respect to t for $\forall\varphi \in L^1(0,\infty;X_0)$. By reason of the foregoing, there exists a consequence that $u(\cdot,t)$ serves as a global solution to question (1.1).

In addition, via the norm's weak lower semi-continuity, this leads to

$$\|u(\cdot,t)\|_{X_0} \le \lim_{n\to\infty}\inf \|u_n(\cdot,t)\|_{X_0},$$

combining this with (3.12), tells us that

$$\|u(\cdot,t)\|_{X_0} \le \left(\frac{2b}{(b-2)k}d\right)^{\frac{1}{2}}$$

i.e., $u(\cdot,t) \in \overline{W}$ for all $t \in (0,\infty)$. □

4. Asymptotic behavior of the solutions

Theorem 4.1. *Under the hypotheses of Theorem 3.1. Impose that for all $t \in [0,\infty)$ and two positive constants p, q, we have*

$$\|u(t)\|_{X_0}^2 + \|u_t(t)\|_2^2 \le pe^{-qt}, \quad \forall t \in [0,\infty).$$

Proof. We construct

$$\mathscr{L}(t) = E_n(t) + \delta\mathscr{Q}(t), \quad \forall t \in [0,\infty), \tag{4.1}$$

where $\mathscr{Q}(t) = \int_\Omega u_n(t)u_{nt}(t)dx$ and $\delta > 0$ is a constant to be determined later.

It can be shown that there exist two numbers $\iota_i > 0 (i = 1,2)$, such that

$$\iota_1 E_n(t) \ge \mathscr{L}(t) \ge \iota_2 E_n(t), \quad \forall t \in [0,\infty). \tag{4.2}$$

In reality, by application of Hölder's inequality and Young's inequality, we find

$$|\mathscr{Q}(t)| \le \frac{1}{2}\|u_n(t)\|_2^2 + \frac{1}{2}\|u_{nt}(t)\|_2^2,$$

and thus

$$|\mathscr{Q}(t)| \le \frac{\mathscr{E}_2^2}{2}\|u_n(t)\|_{X_0}^2 + \frac{1}{2}\|u_{nt}(t)\|_2^2, \tag{4.3}$$

gives the constant $\mathscr{E}_2$ for which the Sobolev embedding $X_0 \hookrightarrow L^2(\Omega)$ remains valid. By combining (3.10) and (4.3), we get a conclusion that $|\mathscr{Q}(t)| \le C_1E_n(t)$ with $C_1 > 0$. The validity of (4.2) is consequently established via (4.1).

The results indicate that

$$E_n'(t) = \frac{1}{2}\left(g' \circ u_n\right)(t) - \frac{1}{2}g(t)\|u_n(t)\|_{X_0}^2 - \|u_{nt}(t)\|_a^a - \|u_{nt}(t)\|_2^2.$$

By performing the calculation explicitly in the next, we attain

$$\begin{aligned}\mathscr{L}'(t) =&\frac{1}{2}\left(g' \circ u_n\right)(t) - \frac{1}{2}g(t)\|u_n(t)\|_{X_0}^2 - \|u_{nt}(t)\|_a^a - \|u_{nt}(t)\|_2^2 + \delta\|u_{nt}(t)\|_2^2\\ &- \delta\|u_n(t)\|_{X_0}^{2\gamma} - \delta\|u_n(t)\|_{X_0}^2 + \delta\int_0^t g(t-\tau)(u_n(\tau), u_n(t))_{X_0}d\tau\\ &- \delta(|u_{nt}(t)|^{a-2}u_{nt}(t), u_n(t)) - \delta(u_{nt}(t), u_n(t)) - \delta\|u_n(t)\|_2^2 + \delta\|u_n(t)\|_b^b.\end{aligned} \tag{4.4}$$

Through simple mathematical manipulations of these three terms, we first find that it follows from Cauchy's inequality with $\sigma_1 > 0$ that

$$\begin{aligned}&\int_0^t g(t-\tau)(u_n(\tau), u_n(t))_{X_0}d\tau\\ =&\int_0^t g(t-\tau)\|u_n(t))\|_{X_0}^2 d\tau + \int_0^t g(t-\tau)(u_n(\tau) - u_n(t), u_n(t))_{X_0}d\tau\\ \leq&\int_0^t g(\tau)d\tau\|u_n(t))\|_{X_0}^2 + \sigma_1\int_0^t g(\tau)d\tau\|u_n(t))\|_{X_0}^2 + \frac{1}{4\sigma_1}(g \circ u_n)(t)\\ \leq&(1-k)\|u_n(t))\|_{X_0}^2 + \sigma_1(1-k)\|u_n(t))\|_{X_0}^2 + \frac{1}{4\sigma_1}(g \circ u_n)(t).\end{aligned}$$

Next follows from Young's inequality with $\sigma_2 > 0$ that

$$-(|u_{nt}(t)|^{a-2}u_{nt}(t), u_n(t)) \leq \sigma_2\|u_n(t))\|_a^a + C(\sigma_2)\|u_{nt}(t))\|_a^a.$$

As a final step, an application of Cauchy's inequality with $\sigma_3 > 0$ yields

$$\begin{aligned}-(u_n(t), u_{nt}(t)) &\leq \sigma_3\|u_n(t))\|_2^2 + \frac{1}{4\sigma_3}\|u_{nt}(t))\|_2^2\\ &\leq \sigma_3\mathscr{E}_2^2\|u_n(t))\|_{X_0}^2 + \frac{1}{4\sigma_3}\|u_{nt}(t))\|_2^2.\end{aligned}$$

From these considerations we attain

$$\begin{aligned}\mathscr{L}'(t) \leq&\left(\delta + \frac{\delta}{4\sigma_3} - 1\right)\|u_{nt}(t))\|_2^2 - \delta\|u_n(t)\|_{X_0}^{2\gamma}\\ &+ \delta(\sigma_1(1-k) + \sigma_3\mathscr{E}_2^2 - k)\|u_n(t)\|_{X_0}^2 - \delta\|u_n(t)\|_2^2\\ &+ \left(\frac{\delta}{4\sigma_1} - \frac{\varrho}{2}\right)(g \circ u_n)(t) + \delta\|u_n(t)\|_b^b\\ &+ \delta\sigma_2\|u_n(t))\|_a^a + (\delta C(\sigma_2) - 1)\|u_{nt}(t))\|_a^a,\end{aligned}$$

and so,

$$\begin{aligned}\mathscr{L}'(t) \leq &-\delta\lambda E_n(t) + \left(\delta + \frac{\delta}{4\sigma_3} + \frac{\delta\lambda}{2} - 1\right)\|u_{nt}(t))\|_2^2 + \delta\left(\frac{\lambda}{2\gamma} - 1\right)\|u_n(t)\|_{X_0}^{2\gamma} \\ &+ \delta\left(\sigma_1(1-k) + \sigma_2\mathscr{E}_2^2 + \frac{\lambda}{2} - k\right)\|u_n(t)\|_{X_0}^2 + \delta\left(\frac{\lambda}{2} - 1\right)\|u_n(t)\|_2^2 \\ &+ \left(\frac{\delta}{4\sigma_1} + \frac{\delta\lambda}{2} - \frac{\varrho}{2}\right)(g \circ u_n)(t) + \delta\left(1 - \frac{\lambda}{b}\right)\|u_n(t)\|_b^b + \delta\sigma_2\|u_n(t)\|_a^a \\ &+ (\delta C(\sigma_2) - 1)\|u_{nt}(t))\|_a^a,\end{aligned} \tag{4.5}$$

where the parameter $\lambda > 0$ will be chosen appropriately subsequently. It follows from (3.7) and (3.8) that

$$\frac{b-2}{2b}k\|u_n(t)\|_{X_0}^2 \leq E_n(t) \leq E_n(0),$$

which leads to

$$\|u_n(t)\|_{X_0} \leq \left(\frac{2b}{(b-2)k}E_n(0)\right)^{\frac{1}{2}}.$$

Hence, we have

$$\|u_n(t)\|_b^b \leq \mathscr{E}_1^b\|u_n(t)\|_{X_0}^{b-2}\|u_n(t)\|_{X_0}^2 \leq \mathscr{E}_1^b\left(\frac{2b}{(b-2)k}E_n(0)\right)^{\frac{b-2}{2}}\|u_n(t)\|_{X_0}^2,$$

$$\|u_n(t)\|_a^a \leq \mathscr{E}_a^a\|u_n(t)\|_{X_0}^{a-2}\|u_n(t)\|_{X_0}^2 \leq \mathscr{E}_a^a\left(\frac{2b}{(b-2)k}E_n(0)\right)^{\frac{a-2}{2}}\frac{2b}{(b-2)k}E_n(t) \equiv RE_n(t),$$

where the best Sobolev constant $\mathscr{E}_a$ is achieved for the embedding $X_0 \hookrightarrow L^a(\Omega)$.

When this inequality is substituted into (4.5), the result is

$$\begin{aligned}\mathscr{L}'(t) \leq &-\delta(\lambda - \sigma_2 R)E_n(t) + \left(\delta + \frac{\delta}{4\sigma_3} + \frac{\delta\lambda}{2} - 1\right)\|u_{nt}(t))\|_2^2 + \delta\left(\frac{\lambda}{2\gamma} - 1\right)\|u_n(t)\|_{X_0}^{2\gamma} \\ &+ \delta\left(\sigma_1(1-k) + \sigma_2\mathscr{E}_2^2 + \frac{\lambda}{2} + \mathscr{E}_1{}^b\left(\frac{2b}{(b-2)k}E_n(0)\right)^{\frac{b-2}{2}} - k\right)\|u_n(t)\|_{X_0}^2 \\ &+ \delta\left(\frac{\lambda}{2} - 1\right)\|u_n(t)\|_2^2 + \left(\frac{\delta}{4\sigma_1} + \frac{\delta\lambda}{2} - \frac{\varrho}{2}\right)(g \circ u_n)(t) + (\delta C(\sigma_2) - 1)\|u_{nt}(t))\|_a^a.\end{aligned}$$

Combined with $E_n(0) < d$, we claim that

$$\mathscr{E}_1{}^b\left(\frac{2b}{(b-2)k}E_n(0)\right)^{\frac{b-2}{2}} < \mathscr{E}_1{}^b\left(\frac{2b}{(b-2)k}d\right)^{\frac{b-2}{2}} = k.$$

Let

$$\mu = k - \mathscr{E}_1{}^b\left(\frac{2b}{(b-2)k}E_n(0)\right)^{\frac{b-2}{2}} - \sigma_1(1-k) - \sigma_2\mathscr{E}_2^2.$$

We select the sufficiently small parameter $\sigma_i (i = 1, 2, 3)$ to ensure that

$$\sigma_2 R < \lambda < \min\{2\mu, 2\}.$$

Accordingly, for fixed $\sigma_i (i = 1, 2, 3)$ and λ, we settle on

$$\delta < \min\left\{\frac{1}{C(\sigma_2)}, \frac{4\sigma_3}{4\sigma_3 + 2\lambda\sigma_3 + 1}, \frac{2\varrho\sigma_1}{1 + 2\lambda\sigma_1}\right\}$$

such that

$$\mathscr{L}'(t) \le -\delta(\lambda - \sigma_2 R)E_n(t)$$

which, applying the second inequality in (4.2) simultaneously, yields

$$\mathscr{L}'(t) \le -\frac{\delta(\lambda - \sigma_2 R)}{\iota_2}\mathscr{L}(t).$$

Therefore, the argument establishes the existence of a constant $C_2 > 0$ with the property that

$$\mathscr{L}(t) \le C_2 e^{-\frac{\delta(\lambda-\sigma_2 R)}{\iota_2}t}, \quad \forall t \in [0, \infty).$$

An additional conclusion can be drawn from the first inequality in (4.2), namely that

$$E_n(t) \le \frac{C_2}{\iota_1} e^{-\frac{\delta(\lambda-\sigma_2 R)}{\iota_2}t}, \quad \forall t \in [0, \infty). \tag{4.6}$$

Given that the norm is weakly lower semicontinuous, it follows that

$$\|u(t)\|_{X_0}^2 + \|u_t(t)\|_2^2 \le \liminf_{n\to\infty}\left(\|u_n(t)\|_{X_0}^2 + \|u_{nt}(t)\|_2^2\right),$$

which, combing with (3.8) and (4.6), gives

$$\liminf_{n\to\infty}\left(\|u_n(t)\|_{X_0}^2 + \|u_{nt}(t)\|_2^2\right) \le \liminf_{n\to\infty} C_3 E_n(t) \le \frac{C}{\iota_1} e^{-\frac{\delta(\lambda-\sigma_2 R)}{\iota_2}t}.$$

Letting $p = C/\iota_1$ and $q = \delta(\lambda - \sigma_2 R)/\iota_2$, the conclusion of Theorem (4.1) is valid. □

5. Conclusions

This work examines well-posedness issues for fractional viscoelastic Kirchhoff equations with a nonlinear dissipative term and a nonlinear source term. Firstly, we introduced fractional Sobolev spaces; in addition, the correlation function $E(t)$ and some necessary lemmas were introduced. Based on these, we combined the Galerkin method and potential wells to prove the global existence of the solutions. Then, using the perturbed energy technique and Gronwall's inequality, the solutions decay exponentially in time was proven.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by the Fundamental Research Funds for the Central Universities (No. 2572021BC01).

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. R. Ikehata, Decay estimates of solutions for the wave equations with strong damping terms in unbounded domains, *Math. Methods Appl. Sci.*, **24** (2001), 659–670. https://doi.org/10.1002/mma.235
2. T. Kawakami, Y. Ueda, Asymptotic profiles to the solutions for a nonlinear damped wave equation, *Differ. Int. Equations*, **26** (2013), 781–814. https://doi.org/10.57262/die/1369057817
3. K. Ono, On global solutions and blow-up solutions of nonlinear Kirchhoff strings with nonlinear dissipation, *J. Math. Anal. Appl.*, **216** (1997), 321–342. https://doi.org/10.1006/jmaa.1997.5697
4. J. A. Esquivel-Avila, A characterization of global and nonglobal solutions of nonlinear wave and Kirchhoff equations, *Nonlinear Anal.*, **52** (2003), 1111–1127. https://doi.org/10.1016/S0362-546X(02)00155-4
5. M. Aassila, Global existence and global nonexistence of solutions to a wave equation with nonlinear damping and source terms, *Asymptot. Anal.*, **30** (2002), 301–311. https://doi.org/10.3233/ASY-2002-507
6. G. Autuori, P. Pucci, M. C. Salvatori, Global nonexistence for nonlinear Kirchhoff systems, *Arch. Ration. Mech. Anal.*, **196** (2010), 489–516. https://doi.org/10.1007/s00205-009-0241-x
7. H. A. Levine, P. Pucci, J. Serrin, Some remarks on global nonexistence for nonautonomous abstract evolution equations, *Contemp. Math.*, **208** (1997), 253–263. https://doi.org/10.1090/conm/208/02743
8. N. Pan, P. Pucci, R. Z. Xu, B. L. Zhang, Degenerate Kirchhoff-type wave problems involving the fractional Laplacian with nonlinear damping and source terms, *J. Evol. Equations*, **19** (2019), 615–643. https://doi.org/10.1007/s00028-019-00489-6
9. M. S. T. Brahim, N. Doudi, R. Guefaifia, R. Jan, R. Alharbi, S. Boulaaras, Global existence and general decay of solutions for a wave equation with memory, fractional boundary damping terms and logarithmic non-linearity, *Partial Differ. Equations Appl. Math.*, **10** (2024), 100742. https://doi.org/10.1016/j.padiff.2024.100742
10. N. Irkıl, E. Pişkin, P. Agarwal, Global existence and decay of solutions for a system of viscoelastic wave equations of Kirchhoff type with logarithmic nonlinearity, *Math. Methods Appl. Sci.*, **45** (2022), 2921–2948. https://doi.org/10.1002/mma.7964
11. S. T. Wu, Exponential energy decay of solutions for an integro-differential equation with strong damping, *J. Math. Anal. Appl.,* **364** (2010), 609–617. https://doi.org/10.1016/j.jmaa.2009.11.046
12. S. T. Wu, On decay and blow-up of solutions for a system of nonlinear wave equations, *J. Math. Anal. Appl.*, **394** (2012), 360–377. https://doi.org/10.1016/j.jmaa.2012.04.054

13. N. Boumaza, B. Gheraibia, General decay and blowup of solutions for a degenerate viscoelastic equation of Kirchhoff type with source term, *J. Math. Anal. Appl.*, **489** (2020), 124185. https://doi.org/10.1016/j.jmaa.2020.124185

14. M. Q. Xiang, D. Hu, Existence and blow-up of solutions for fractional wave equations of Kirchhoff type with viscoelasticity, *Discrete Contin. Dyn. Syst. Ser.*, **14** (2021), 4609–4629. https://doi.org/10.3934/dcdss.2021125

15. N. S. Papageorgiou, J. Zhang, W. Zhang, Multiple solutions with sign information for Robin equations with indefinite potential, *Bull. Math. Sci.*, **15** (2025), 2450013. https://doi.org/10.1142/S1664360724500139

16. Q. Li, V. D. Rădulescu, J. Zhang, W. Zhang, Concentration of normalized solutions for non-autonomous fractional Schrödinger equations, *Z. Angew. Math. Phys.*, **76** (2025), 132. https://doi.org/10.1007/s00033-025-02510-0

17. W. Zhang, J. Zhang, Planar Hénon-type equation with Trudinger-Moser critical growth, *Discrete Contin. Dyn. Syst.*, **45** (20), 4529–4553. https://doi.org/10.3934/dcds.2025066

18. A. Haraux, E, Zuazua, Decay estimates for some semilinear damped hyperbolic problems, *Arch. Ration. Mech. Anal.*, **100** (1988), 191–206. https://doi.org/10.1007/BF00282203

19. Y. Liu, Global attractors for a nonlinear plate equation modeling the oscillations of suspension bridges, *Commun. Anal. Mech.*, **15** (2023), 436–456. https://doi.org/10.3934/cam.2023021

20. Y. Liu, B. Moon, V. Rădulescu, R. Z. Xu, C. Yang, Qualitative properties of solution to a viscoelastic Kirchhoff-like plate equation, *J. Math. Phys.*, **64** (2023), 051511. https://doi.org/10.1063/5.0149240

21. G. M. Bisci, V. D. Rădulescu, R. Servadei, *Variational Methods for Nonlocal Fractional Problems*, Cambridge University Press, Cambridge, 2016.

22. A. Fiscella, R. Servadei, E. Valdinoci, Density properties for fractional Sobolev spaces, *Ann. Acad. Sci. Fenn. Math.*, **40** (2015), 235–253. https://doi.org/10.5186/aasfm.2015.4009

23. R. Servadei, E. Valdinoci, Mountain pass solutions for non-local elliptic operators, *J. Math. Anal. Appl.*, **389** (2012), 887–898. https://doi.org/10.1016/j.jmaa.2011.12.032

24. J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non-Linéaires*, Dunod, Gauthier-Villars, Paris, 1969.