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Research article

Time periodic solution to a two-species chemotaxis-Stokes system with porous medium diffusion

Hui Tang[1,*] **and Chengxin Du**[2,*]

[1] College of Mathematics, Changchun Normal University, Changchun 130032, China
[2] Information Engineering University, Zhengzhou 450001, China

* **Correspondence:** Email: tangtangth83@163.com, 1550650920@qq.com.

Abstract: In this paper, we considered a two-species chemotaxis-Stokes system with porous medium diffusion Δn_1^m in two-dimensional smooth bounded domains. We proved the existence of a time periodic solution for any $m \geq \frac{6}{5}$ and any large periodic sources $g_1(x,t)$ and $g_2(x,t)$.

Keywords: two-species chemotaxis-Stokes system; time periodic solution; double-level approximation

1. Introduction

In this paper, we are concerned with the following two-species chemotaxis-Stokes system:

$$\begin{cases} (n_1)_t + u\cdot\nabla n_1 = \Delta n_1^m - \chi_1 \nabla\cdot\left(n_1 \frac{1}{(1+c)^{m_1}}\nabla c\right) \\ \qquad\qquad + \mu_1 n_1(1-n_1-a_1 n_2) + g_1(x,t), \text{ in } Q, \\ (n_2)_t + u\cdot\nabla n_2 = \Delta n_2 - \chi_2 \nabla\cdot\left(n_2 \frac{1}{(1+c)^{m_2}}\nabla c\right) \\ \qquad\qquad + \mu_2 n_2(1-a_2 n_1-n_2) + g_2(x,t), \text{ in } Q, \\ c_t + u\cdot\nabla c = \Delta c - (\alpha_1 n_1 + \alpha_2 n_2)c + \beta n_1, \text{ in } Q, \\ u_t = \Delta u - \nabla\pi + (\gamma n_1 + \theta n_2)\nabla\varphi, \text{ in } Q, \\ \nabla\cdot u = 0, \text{ in } Q, \\ \frac{\partial n_1^m}{\partial\nu}\Big|_{\partial\Omega} = \frac{\partial n_2}{\partial\nu}\Big|_{\partial\Omega} = \frac{\partial c}{\partial\nu}\Big|_{\partial\Omega} = 0, \quad u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $m \geq \frac{6}{5}$, $Q = \Omega \times R^+$, $\Omega \subset R^2$ is a bounded domain with a smooth boundary, μ_1, μ_2, α_1, β, χ_1, χ_2, γ, θ, m_1, m_2 are positive constants, and $\alpha_2, a_1, a_2 \geq 0$ are constants. n_1, n_2 represent the population densities of the species, c denotes the concentration of the chemoattractant, u shows the fluid velocity field, π represents the associated pressure, and $g_1(x,t), g_2(x,t) \geq 0$ are source terms. The fluid couples to n_1, n_2, c through transport and gravitational forcing modeled by $\nabla\varphi$. Here, $\varphi(x,t)$, $g_1(x,t)$, $g_2(x,t)$ are time periodic functions with period T.

Problem (1.1) is a generalized system of the chemotaxis-fluid system, and the coupled chemotaxis-fluid equations have been studied by a number of authors ever since Tuval et al. [1] first introduced the concept. To elucidate the bio-ecological significance of periodic solutions in dynamical systems, please refer to [2]. In a two-dimensional bounded domain, Winkler [3] established the global existence and stability of classical solutions. However, in the three-dimensional case, the result on the global existence of classical solutions is not perfect. For the existence of weak solutions, eventual smoothness of solutions, and stability results in both two-dimensional and three-dimensional cases, we can see [3–5]. Xu et al. [6] studied the following problem:

$$\begin{cases} u_t = \Delta u^m - \chi \nabla \cdot \left(\frac{u}{(1+u)^\alpha} \nabla v \right) - \xi \nabla \cdot \left(\frac{u}{(1+u)^\beta} \nabla w \right) \\ \qquad + \mu u(1-u-w), \quad x \in \Omega, t > 0, \\ v_t - \Delta v + v = u, \quad x \in \Omega, t > 0, \\ w_t = -vw, \quad x \in \Omega, t > 0. \end{cases}$$

It is shown that under zero-flux boundary conditions, for any $m > 0$, the above problem admits a global bounded weak solution. What is more, they also discussed the large time behavior of solutions for the fast diffusion case, and showed that if $0 < m \leq 1$, for appropriately large μ, for any initial datum, the solution (u, v, w) goes to a steady $(1, 1, 0)$ as $t \to \infty$. See also [7].

Cao et al. [8] considered the two-species chemotaxis-Stokes system with competitive kinetics:

$$\begin{cases} (n_1)_t + u \cdot \nabla n_1 = \Delta n_1 - \chi_1 \nabla \cdot (n_1 \nabla c) \\ \qquad + \mu_1 n_1 (1 - n_1 - a_1 n_2), \quad x \in \Omega, \quad t > 0, \\ (n_2)_t + u \cdot \nabla n_2 = \Delta n_2 - \chi_2 \nabla \cdot (n_2 \nabla c) \\ \qquad + \mu_2 n_2 (1 - a_2 n_1 - n_2), \quad x \in \Omega, \quad t > 0, \\ c_t + u \cdot \nabla c = \Delta c - (\alpha n_1 + \beta n_2) c, \quad x \in \Omega, \quad t > 0, \\ u_t + \kappa (u \cdot \nabla) u = \Delta u + \nabla P + (\gamma n_1 + \delta n_2) \nabla \phi, \quad x \in \Omega, \quad t > 0, \\ \nabla \cdot u = 0, \quad x \in \Omega, \quad t > 0. \end{cases}$$

They proved the global existence, boundedness, and stabilization of solutions in the 3-dimensional case when $\frac{\max\{\chi_1,\chi_2\}}{\min\{\mu_1,\mu_2\}} \|c_0\|_{L^\infty(\Omega)}$ is sufficiently small. Han and Liu [9] considered a two-species chemotaxis-Navier-Stokes system with a p-Laplacian. They proved that for any $p \geq 2$, the problem admits a global weak solution.

Jin [10, 11] studied the time periodic problem to the coupled chemotaxis-fluid model:

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \cdot \nabla c) + \gamma n - \mu n^2 + g(x,t), \text{ in } Q, \\ c_t + u \cdot \nabla c = \Delta c - c + n, \text{ in } Q, \\ u_t + \kappa u \cdot \nabla u = \Delta u - \nabla \pi + n \nabla \varphi, \text{ in } Q, \\ \nabla \cdot u = 0, \text{ in } Q, \\ \frac{\partial n}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial c}{\partial \nu}\Big|_{\partial\Omega} = 0, \, u|_{\partial\Omega} = 0. \end{cases}$$

They proved the existence of a large time periodic solution in spatial dimension $N = 3, \kappa = 0$, or in spatial dimension $N = 2$, $\kappa = 1$. Recently, Huang and Jin [12] were concerned with the time periodic problem of a coupled chemotaxis-fluid model with porous medium diffusion Δn^m. They found the existence of a uniformly bounded time periodic solution for any $m \geq \frac{6}{5}$ and any large periodic source $g(x,t)$ in dimension 3. Liu and Li [13] proved the existence of time periodic solutions for the two-species chemotaxis-Navier-Stokes system (1.1) with $m = 1$. Later, the two-species chemotaxis-Stokes system with a p-Laplacian equation in two-dimensional smooth bounded domains was investigated in [14]. By employing methods such as regularization, a priori estimates, energy inequalities, and operator theories, the existence, uniqueness, and regularity of time periodic solutions for partial differential equations were proven, please refer to [15–19].

This paper focuses on a two-species chemotaxis-Stokes system with porous medium diffusion, and discusses the existence and regularity estimates of periodic solutions for this hydrodynamic model. First, a fourth-order regularization model is defined to approximate the original model. Then, each individual equation is linearized, and the existence and regularity estimates of periodic solutions for each linearized equation are obtained by using methods such as upper and lower solutions, induction, and energy estimates. Finally, the existence of periodic solutions for the fourth-order regularization model is established by applying the Leray-Schauder fixed point theorem. At this point, by letting the coefficient δ in front of the fourth-order term tend to zero and combining with the Aubin-Lions lemma, the first-layer approximation is obtained. In addition, in order to eliminate the influence of the term $\varepsilon \|n_1\|^s_{L^\infty}$ added in the fourth-order regularization model on the proof, it is necessary to obtain a better estimate of n_1. For this purpose, the $\|n_1\|_{L^\infty}$ estimate is derived by using the standard Stokes semigroup theory and Moser iteration method. Then, by letting ε tend to zero and combining with the Aubin-Lions lemma, the second-layer approximation is obtained. When both δ and C tend to zero, the existence and regularity estimates of periodic solutions for the original model in a bounded domain of two-dimensional space are obtained.

Different from the literatures [10, 11], and [20], this paper introduces porous medium diffusion, breaking through the limitations of single-species and linear diffusion. When proving the existence of periodic solutions for this model, the porous medium diffusion term brings great difficulties. Therefore, this paper mainly adopts a two-layer approximation method to overcome this difficulty. To solve this problem, a fourth-order regularization model is used to approximate the original model. In the process of energy estimation, the introduced fourth-order term is used to control the porous medium diffusion term. In addition, this paper skillfully obtains the boundedness of the solution through the upper and lower solutions method, induction, and Moser iteration method, and then applies the Leray-Schauder fixed point theorem to establish the existence of periodic solutions for the fourth-order regularization model and the corresponding regularity estimates, providing a new method for similar strongly nonlinear diffusion models.

Before giving the main result, let us first give some notations which will be used throughout this paper.

Notations:

(1) $Q_T = \Omega \times (0,T)$; $H^k_\sigma(\Omega) = \{u \in H^k(\Omega); \nabla \cdot u = 0\}$; $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\Omega)}$; $\|\cdot\|_{H^1} = \|\cdot\|_{H^1(\Omega)}$; $\|\cdot\|_{H^2} = \|\cdot\|_{H^2(\Omega)}$; $\|\cdot\|_{W^{1,\infty}} = \|\cdot\|_{W^{1,\infty}(\Omega)}$; $L^p_T(\mathbb{R}^+, X)$ denotes the T-time periodic function in $L^p(\mathbb{R}^+, X)$. For these time periodic solutions, $\sup\limits_t \|\cdot\| := \sup\limits_{t\in(0,T)} \|\cdot\| = \sup\limits_{t\in\mathbb{R}^+} \|\cdot\|$; $g_1 = g_1(x,t)$, $g_2 = g_2(x,t)$.

(2) If $q, r \geq 1$, we say that $u \in L^{q,r}(Q_T) \equiv L^r(0,T; L^q(\Omega))$. If u is measurable on Q_T, then for almost all $t \in (0,T)$, $u(\cdot,t) \in L^q(\Omega)$ and $\|u(\cdot,t)\|_{L^q(\Omega)} \in L^r(0,T)$, that is,

$$\|u\|_{L^{q,r};Q_T} = \left(\int_0^T \left(\int_\Omega |u|^q \, dx\right)^{\frac{r}{q}} dt\right)^{\frac{1}{r}} < \infty.$$

Theorem 1.1. *Assume $\Omega \subset R^2$, $m \geq \frac{6}{5}$, $0 \leq g_1, g_2 \in L^\infty_T(Q)$, $\nabla\varphi \in L^\infty_T(Q)$. Then problem (1.1) admits a time periodic solution (n_1, n_2, c, u, π) such that $n_1, n_2, c \geq 0$, and*

$$\begin{aligned}&\sup_t\{\|u\|^2_{H^1} + \|c\|^2_{H^1}\}\\ &+ \int_0^T (\|u\|^2_{H^2} + \|\pi\|^2_{H^1} + \|c\|^2_{H^2} + \|n_2\|^2_{H^1} + \|(u_t, c_t)\|^2_{L^2})dt \leq C,\end{aligned} \tag{1.2}$$

$$\sup_t\{\|u\|_{L^\infty} + \|n_1\|_{L^\infty} + \|n_2\|_{L^\infty} + \|c\|_{W^{1,\infty}}\} \leq C, \tag{1.3}$$

$$\sup_t \int_\Omega |\nabla n_1^m|^2 dx + \int_0^T \int_\Omega n_1^{m-1}|n_{1t}|^2 dxdt + \int_0^T \int_\Omega n_1^{m-2}|\nabla n_1|^2 dxdt \leq C. \tag{1.4}$$

Morever, if $\frac{6}{5} \leq m \leq 2$, we further have

$$\begin{aligned}&\sup_t\{\|u\|^2_{H^1} + \|c\|^2_{H^2} + \|n_2\|^2_{H^1}\}\\ &+ \int_0^T (\|u\|^2_{H^2} + \|\pi\|^2_{H^1} + \|c\|^2_{H^3} + \|n_1\|^2_{H^1} + \|n_2\|^2_{H^2} + \|(u_t, c_t)\|^2_{L^2})dt \leq C,\end{aligned} \tag{1.5}$$

$$\sup_t\{\|u\|_{L^\infty} + \|n_1\|_{L^\infty} + \|n_2\|_{L^\infty} + \|c\|_{W^{1,\infty}}\} \leq C, \tag{1.6}$$

$$\sup_t \int_\Omega |\nabla n_1^m|^2 dx + \int_0^T \int_\Omega n_1^{m-1}|n_{1t}|^2 dxdt + \int_0^T \int_\Omega n_1^{m-2}|\nabla n_1|^2 dxdt \leq C, \tag{1.7}$$

where C is only dependent on μ_1, μ_2, χ_1, χ_2, m_1, m_2, a_1, a_2, α_1, α_2, β, γ, θ, Ω, g, $\nabla\varphi$, and T.

2. Preliminaries

In this section, we list some lemmas which will be used throughout this paper.

Lemma 2.1. ([21]) *Assume that Ω is bounded and let $\omega \in C^2(\bar{\Omega})$ satisfy $\frac{\partial \omega}{\partial \nu}\Big|_{\partial\Omega} = 0$. Then we have*

$$\frac{\partial |\nabla \omega|^2}{\partial \nu} \le 2\kappa |\nabla \omega|^2, \ \text{on } \partial\Omega,$$

where $\kappa > 0$ is an upper bound for the curvatures of Ω.

By a direct calculation, we have:

Lemma 2.2. *If f satisfies*

$$f'(t) - g(t)f(t) \le h(t), \quad t \in R,$$

then for any $t > t_0$, we have

$$f(t) \le f(t_0)e^{\int_{t_0}^t g(s)ds} + \int_{t_0}^t h(\tau)e^{\int_\tau^t g(s)ds} d\tau.$$

Next, we review the relevant knowledge of the heat semigroup. When $0 < p_0 < p < p_1$, the Lorentz space $L^{p,q}(\Omega)$ can be controlled by the real interpolation spaces $L^{p_0}(\Omega)$ and $L^{p_1}(\Omega)$, that is,

$$L^{p,q}(\Omega) = (L^{p_0}(\Omega), L^{p_1}(\Omega))_{\theta,q},$$

where $1 \le q \le \infty$ and $0 < \theta < 1$ satisfy $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. For $1 < r < \infty$, let $A = -\Delta$ be the heat semigroup on $L^r(\Omega)$ with $D(A) = \{u \in W^{2,r}(\Omega) \mid \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0\}$. Then $-A$ generates a bounded analytic C_0-semigroup on $L^r(\Omega)$. For A^α with $0 \le \alpha \le 1$, by standard semigroup theory, we have

$$\|A^\alpha e^{-tA} a\|_r \le Ct^{-\alpha}\|a\|_r.$$

In addition, the boundary $\partial\Omega$ of $\Omega \subset \mathbb{R}^N$ is bounded, and the corresponding heat semigroup has a kernel function $(4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right)$. Therefore, through Young's inequality and the definition of the Lorentz space, we can easily obtain the following L^p-L^q estimate.

Lemma 2.3. ([22]) *Let $(e^{t\Delta})_{t\ge 0}$ be the Neumann heat semigroup in Ω, and $\lambda_1 > 0$ is the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there exists constant C depending on Ω such that:*
(i) If $1 \le q \le p \le \infty$, then

$$\|e^{t\Delta} w\|_{L^p(\Omega)} \le C(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}\|w\|_{L^q(\Omega)}, \ t > 0, \tag{2.1}$$

holds for all $w \in L^q(\Omega)$ satisfying $\int_\Omega w = 0$.
(ii) If $1 \le q \le p \le \infty$, then

$$\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \le C(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}\|w\|_{L^q(\Omega)}, \quad t > 0, \tag{2.2}$$

holds for each $w \in L^q(\Omega)$.
(iii) If $2 \le p < \infty$, then

$$\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \le Ce^{-\lambda_1 t}\|\nabla w\|_{L^p(\Omega)}, \quad t > 0, \tag{2.3}$$

holds for all $w \in W^{1,p}(\Omega)$.
(iv) If $1 < q \le p < \infty$, then

$$\|e^{t\Delta}\nabla \cdot w\|_{L^p(\Omega)} \le C(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}\|w\|_{L^q(\Omega)}, \quad t > 0, \tag{2.4}$$

holds for all $w \in (C_0^\infty(\Omega))^n$.

Lemma 2.4. (Leray-Schauder fixed point theorem [23]) *Let U be a Banach space, and $T(u,\sigma)$ be a mapping from $U\times[0,1]$ to U satisfying the following conditions:*
(i)T is a compact mapping;
(ii)$T(u,0)=0, \forall u\in U$;
(iii) There exists a constant $M>0$ such that for any $u\in U$, if $u=T(u,\sigma)$ for some $\sigma\in[0,1]$, then $\|u\|_U\le M$.
Then the mapping $T(\cdot,1)$ has a fixed point, that is, there exists $u\in U$, such that $u=T(u,1)$.

Lemma 2.5. (Gagliardo-Nirenberg inequality [24]) *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with a smooth boundary, and $u\in W^{m,p}(\Omega)$. Then, we have*

$$\|D^j u\|_{L^p}\le C_1\|D^m u\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha}+C_2\|u\|_{L^q},$$

where

$$\frac{j}{m}\le\alpha<1,\quad \frac{1}{p}=\frac{j}{n}+\alpha\left(\frac{1}{r}-\frac{m}{n}\right)+(1-\alpha)\frac{1}{q}.$$

Lemma 2.6. ([20]) *Let $T>0$, $a>0$, $\sigma>0$, and suppose that $f:\mathbb{R}^+\to[0,\infty)$ is absolutely continuous, f,g,h are time periodic functions with period T, and they satisfy*

$$f(t)-f(t_0)+a\int_{t_0}^{t}f^{1+\sigma}(s)\,\mathrm{d}s\leqslant\int_{t_0}^{t}g(s)f(s)\,\mathrm{d}s+\int_{t_0}^{t}h(s)\,\mathrm{d}s$$

for any $0\leqslant t_0<t$, where $g(t),h(t)\geqslant 0$ with $g,h\in L^1_T(\mathbb{R}^+)$ and

$$\int_0^T g(s)\,\mathrm{d}s\leqslant\alpha,\quad \int_0^T h(s)\,\mathrm{d}s\leqslant\beta.$$

Then we have

$$\sup_{t\in(0,T)}f(t)+a\int_0^T f^{1+\sigma}(t)\,\mathrm{d}t\leqslant C,$$

where C is a constant depending only on a,α,β,T. While, if $a=0$ in the above inequality, and

$$\int_0^T f(s)\,\mathrm{d}s\leqslant\gamma,$$

then we also have

$$\sup_{t\in(0,T)}f(t)\leqslant C,$$

where C is a constant depending only on γ,α,β,T.

3. A fourth-order regularized problem

In this paper, we use a double-level approximation scheme to show the existence of time periodic solutions. First, in order to obtain the compactness of the operator, we use a fourth-order regularized system as follows to approach the original system:

$$\begin{cases} (n_1)_t + u \cdot \nabla n_1 + \delta \Delta^2 n_1 - \Delta((n_1^2+\varepsilon)^{\frac{m-1}{2}} n_1) + \varepsilon |n_1|^s n_1 \\ = -\chi_1 \nabla \cdot (n_1 \frac{1}{(1+c)^{m_1}} \nabla c) + \mu_1 |n_1| (1 - n_1 - a_1 n_2) + g_1(x,t), \text{ in } Q, \\ (n_2)_t + u \cdot \nabla n_2 = \Delta n_2 - \chi_2 \nabla \cdot (n_2 \frac{1}{(1+c)^{m_2}} \nabla c) \\ \qquad + \mu_2 n_{2+} (1 - a_2 n_{1+} - n_2) + g_2(x,t), \text{ in } Q, \\ c_t + u \cdot \nabla c = \Delta c - (\alpha_1 n_{1+} + \alpha_2 n_{2+}) c + \beta n_{1+}, \text{ in } Q, \\ u_t = \Delta u - \nabla \pi + (\gamma n_1 + \theta n_2) \nabla \varphi, \text{ in } Q, \\ \nabla \cdot u = 0, \text{ in } Q, \\ \frac{\partial n_1}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial \Delta n_1}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial n_2}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial c}{\partial \nu}\Big|_{\partial\Omega} = 0, u|_{\partial\Omega} = 0. \end{cases} \tag{3.1}$$

To prove the existence of time periodic solutions for the problem, we linearize this problem. For u, consider the following linear problem:

$$\begin{cases} u_t - \Delta u + \nabla \pi = \rho(\gamma \hat{n}_1 \nabla \varphi + \theta \hat{n}_2 \nabla \varphi), \text{ in } Q, \\ \nabla \cdot u = 0, \text{ in } Q, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3.2}$$

where $\rho \in [0, 1]$ is a constant. Taking advantage of [10, 11], we have:

Lemma 3.1. *Assume that $\hat{n}_i \in L^2_T(R, L^2(\Omega))$, $i = 1, 2$, $\nabla\varphi \in L^\infty_T(Q)$. Then problem (3.2) admits a time periodic solution:*

$$u \in L^\infty((0,T), H^1_\sigma(\Omega)) \cap L^2((0,T), H^2_\sigma(\Omega)), \textit{ and } u_t \in L^2((0,T), L^2_\sigma(\Omega)),$$

where C is a constant depending on ρ, γ, θ, $\hat{n}_1$, $\hat{n}_2$, Ω, $\nabla\varphi$, T.

For the above obtained solution u, let us consider the following linear problem:

$$\begin{cases} c_t - \Delta c + u \cdot \nabla c = \rho(-(\alpha_1 \hat{n}_{1+} + \alpha_2 \hat{n}_{2+}) c + \beta \hat{n}_{1+}), \text{ in } Q, \\ \frac{\partial c}{\partial \nu}|_{\partial\Omega} = \frac{\partial \triangle c}{\partial \nu}|_{\partial\Omega} = 0. \end{cases} \tag{3.3}$$

The existence of time periodic solutions can be easily obtained by a fixed point method. That is, define a Poincaré map $\Psi : H^1(\Omega) \to H^1(\Omega)$, that is, $\Psi(c(x, 0)) = c(x, T)$. The time-periodic solution is then identified as a fixed point of the Poincaré map. We can further improve the regularity by using the classical theory of linear parabolic equations for the initial and boundary value problem, and we further have $c \in H^{2,1}_T(\Omega \times R)$ since $c(x, 0) \in H^1(\Omega)$. This means that there exists t_0 such that $c(x, t_0) \in H^2(\Omega)$ by the mean value theorem of integrals. The regularity can further be improved by using $c(x, t_0)$ as initial datum. In what follows, we omit the proof of the existence, and only give the regularity estimate.

Lemma 3.2. *Assume $\hat{n}_1 \in L_T^2(R, H^1(\Omega))$, $\hat{n}_2 \in L_T^2(R, H^1(\Omega))$. Let u be the time periodic solution of problem (3.2) and c be the time periodic solution of (3.3). Then $c \geq 0$ and*

$$\sup_t \|c\|_{H^2}^2 + \int_0^T (\|c\|_{H^3}^2 + \|c_t\|_{L^2}^2) dt \leq C,$$

where C is a constant depending on ρ, α_1, α_2, β, $\hat{n}_1$, $\hat{n}_2$, Ω, T.

Proof. We first prove the positivity of c. It is easy to see that $\frac{\beta}{\alpha_1}$ and 0 are the upper and lower periodic solutions of (3.3). Then by the example in [25], problem (3.3) has a time periodic solution c, and $0 \leq c \leq \frac{\beta}{\alpha_1}$.

Multiplying the first equation of (3.9) by c and integrating it over Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega c^2 dx + \int_\Omega |\nabla c|^2 dx + \rho\alpha_1 \int_\Omega \hat{n}_{1+} c^2 dx + \rho\alpha_2 \int_\Omega \hat{n}_{2+} c^2 dx \\
&\leq \rho\beta \|\hat{n}_1\|_{L^2} \|c\|_{L^2}.
\end{aligned}$$

Similar to the proof of Lemma 3.1, we obtain

$$\sup_t \|c\|_{L^2}^2 + \int_0^T \|c\|_{H^1}^2 dt \leq C\rho \int_0^T \|\hat{n}_1\|_{L^2}^2 dt. \tag{3.4}$$

Multiplying the first equation of (3.3) by Δc, integrating it over Ω, and combining with the Gagliardo-Nirenberg interpolation inequality, we see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c|^2 dx + \int_\Omega |\Delta c|^2 dx \\
&= \int_\Omega u \nabla c \Delta c dx + \rho\alpha_1 \int_\Omega \hat{n}_{1+} c \Delta c dx + \rho\alpha_2 \int_\Omega \hat{n}_{2+} c \Delta c dx - \rho\beta \int_\Omega \hat{n}_{1+} \Delta c dx \\
&\leq \|u\|_{L^4} \|\nabla c\|_{L^4} \|\Delta c\|_{L^2} + C\|\hat{n}_1\|_{L^2} \|c\|_{L^\infty} \|\Delta c\|_{L^2} + C\|\hat{n}_2\|_{L^2} \|c\|_{L^\infty} \|\Delta c\|_{L^2} \\
&\leq \|u\|_{L^4} \|\nabla c\|_{L^2}^{\frac{1}{2}} \|\Delta c\|_{L^2}^{\frac{3}{2}} + C\|\hat{n}_1\|_{L^2} \|\Delta c\|_{L^2} + C\|\hat{n}_2\|_{L^2} \|\Delta c\|_{L^2} \\
&\leq \frac{1}{2}\|\Delta c\|_{L^2}^2 + C\|u\|_{H^1}^4 \|\nabla c\|_{L^2}^2 + C\|\hat{n}_1\|_{L^2}^2 + C\|\hat{n}_2\|_{L^2}^2.
\end{aligned}$$

Similar to the proof of Lemma 3.1, we have

$$\sup_t \|\nabla c\|_{L^2}^2 + \int_0^T \|\nabla c\|_{H^1}^2 dt \leq C. \tag{3.5}$$

Multiplying the first equation of (3.3) by c_t, integrating it over Ω, and combining with the Gagliardo-Nirenberg interpolation inequality, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c|^2 dx + \int_\Omega |c_t|^2 dx \\
&= -\int_\Omega u \nabla c c_t dx - \rho\alpha_1 \int_\Omega \hat{n}_{1+} c c_t dx - \rho\alpha_2 \int_\Omega \hat{n}_{2+} c c_t dx + \rho\beta \int_\Omega \hat{n}_{1+} c_t dx \\
&\leq C(\|u\|_{L^4} \|\nabla c\|_{L^4} \|c_t\|_{L^2} + \|\hat{n}_1\|_{L^2} \|c\|_{L^\infty} \|c_t\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+\|\hat{n}_2\|_{L^2}\|c\|_{L^\infty}\|c_t\|_{L^2}+\|\hat{n}_1\|_{L^2}\|c_t\|_{L^2})\\
&\le C\|u\|_{H^1}\|\nabla c\|_{H^1}\|c_t\|_{L^2}+C\|\hat{n}_1\|_{L^2}\|c_t\|_{L^2}+C\|\hat{n}_2\|_{L^2}\|c_t\|_{L^2}\\
&\le \frac{1}{2}\|c_t\|_{L^2}^2+C\|u\|_{H^1}^2\|\nabla c\|_{H^1}^2+C\|\hat{n}_1\|_{L^2}^2+C\|\hat{n}_2\|_{L^2}^2.
\end{aligned}
$$

Integrating the above inequality from 0 to T gives

$$
\int_0^T \|c_t\|_{L^2}^2 dt \le C. \tag{3.6}
$$

Applying ∇ to the first equation of (3.9), multiplying it by $\nabla\Delta c$, integrating it over $\Omega\times(t_0,t)$ for any $t_0<t<t_0+T$, and using Lemma 2.6 , we see that

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega(|\Delta c(x,t)|^2-|\Delta c(x,t_0)|^2)dx+\int_{t_0}^t\int_\Omega|\nabla\Delta c|^2dxds\\
=&\int_{t_0}^t\int_\Omega\nabla(u\cdot\nabla c)\nabla\Delta c dxds+\rho\int_{t_0}^t\int_\Omega(\alpha_1\nabla(\hat{n}_{1+}c)\nabla\Delta c+\alpha_2\nabla(\hat{n}_{2+}c)\nabla\Delta c\\
&-\beta\nabla\hat{n}_{1+}\nabla\Delta c)dxds\\
\le&\int_{t_0}^t\|\nabla\Delta c\|_{L^2}(\|u\|_{L^4}\|\Delta c\|_{L^4}+\|\nabla u\|_{L^4}\|\nabla c\|_{L^4})ds+C\int_{t_0}^t\|\nabla\Delta c\|_{L^2}(\|\nabla\hat{n}_1c\|_{L^2}\\
&+\|\hat{n}_1\|_{L^4}\|\nabla c\|_{L^4}+\|\nabla\hat{n}_2c\|_{L^2}+\|\hat{n}_2\|_{L^4}\|\nabla c\|_{L^4})\,ds+C\int_{t_0}^t\|\nabla\hat{n}_1\|_{L^2}\|\nabla\Delta c\|_{L^2}ds\\
\le& C\int_{t_0}^t\|\nabla\Delta c\|_{L^2}(\|u\|_{H^1}\|\Delta c\|_{L^2}^{\frac12}\|\nabla\Delta c\|_{L^2}^{\frac12}+\|u\|_{H^1}\|\Delta c\|_{L^2}+\|\nabla u\|_{H^1}\|\Delta c\|_{L^2}\\
&+\|\nabla u\|_{H^1}\|\nabla c\|_{L^2})ds+C\int_{t_0}^t\|\nabla\Delta c\|_{L^2}(\|\nabla\hat{n}_1\|_{L^2}\|c\|_{L^\infty}+\|\hat{n}_1\|_{H^1}\|\Delta c\|_{L^2}\\
&+\|\nabla\hat{n}_2\|_{L^2}\|c\|_{L^\infty}+\|\hat{n}_2\|_{H^1}\|\Delta c\|_{L^2}+\|\hat{n}_1\|_{H^1}\|\nabla c\|_{L^2}+\|\nabla\hat{n}_1\|_{L^2})ds\\
\le& C\int_{t_0}^t\|\nabla\Delta c\|_{L^2}(\|u\|_{H^1}\|\Delta c\|_{L^2}^{\frac12}\|\nabla\Delta c\|_{L^2}^{\frac12}+\|u\|_{H^1}\|\Delta c\|_{L^2}+\|\nabla u\|_{H^1}\|\Delta c\|_{L^2}\\
&+\|\nabla u\|_{H^1}\|\nabla c\|_{L^2})ds+C\int_{t_0}^t\|\nabla\Delta c\|_{L^2}(\|\nabla\hat{n}_1\|_{L^2}\|\Delta c\|_{L^2}+\|\hat{n}_1\|_{H^1}\|\Delta c\|_{L^2}\\
&+\|\hat{n}_1\|_{H^1}\|\nabla c\|_{L^2}+\|\hat{n}_2\|_{H^1}\|\Delta c\|_{L^2}+\|\nabla\hat{n}_2\|_{L^2}+\|\nabla\hat{n}_1\|_{L^2})ds\\
\le&\frac{1}{2}\int_{t_0}^t\int_\Omega|\nabla\Delta c|^2dxds+C\int_{t_0}^t(\|\Delta c\|_{L^2}^2+\|\nabla u\|_{H^1}^2\|\Delta c\|_{L^2}^2+\|\nabla u\|_{H^1}^2\\
&+\|\hat{n}_1\|_{H^1}^2\|\Delta c\|_{L^2}^2+\|\hat{n}_1\|_{H^1}^2+\|\hat{n}_2\|_{H^1}^2\|\Delta c\|_{L^2}^2+\|\hat{n}_2\|_{H^1}^2)ds,
\end{aligned}
$$

which means

$$
\begin{aligned}
&\int_\Omega(|\Delta c(x,t)|^2-|\Delta c(x,t_0)|^2)dx+\int_{t_0}^t\int_\Omega|\nabla\Delta c|^2dxds\\
&\le C\int_{t_0}^t(\|\Delta c\|_{L^2}^2+\|\nabla u\|_{H^1}^2\|\Delta c\|_{L^2}^2+\|\nabla u\|_{H^1}^2
\end{aligned}
$$

$$+\|\hat{n}_1\|_{H^1}^2\|\Delta c\|_{L^2}^2+\|\hat{n}_1\|_{H^1}^2+\|\hat{n}_2\|_{H^1}^2\|\Delta c\|_{L^2}^2+\|\hat{n}_2\|_{H^1}^2)ds.$$

Using Lemma 2.6 yields

$$\sup_t \|\Delta c\|_{L^2}^2+\int_0^T\|\nabla\Delta c\|_{L^2}^2dt\le C. \tag{3.7}$$

By combining (3.5)–(3.7) with the definition of the norm in Sobolev spaces, we derive Lemma 3.2. The proof is complete. □

For the above obtained solutions u, c, we consider the following linear problem:

$$\begin{cases}(n_2)_t-\Delta n_2+u\cdot\nabla n_2+n_2+\mu_2\hat{n}_{2+}n_2\\ =-\rho\chi_2\nabla\cdot(n_2\frac{1}{(1+c)^{m_2}}\nabla c)+\rho(\mu_2+1)\hat{n}_{2+}-a_2\mu_2\hat{n}_{1+}n_2+\rho g_2,\ \text{in } Q,\\ \frac{\partial n_2}{\partial\nu}|_{\partial\Omega}=\frac{\partial\Delta n_2}{\partial\nu}|_{\partial\Omega}=0,\end{cases} \tag{3.8}$$

where $\rho\in[0,1]$. Similar to the above two problems, the existence of time periodic solutions can be easily obtained. We only give the regularity estimate.

Lemma 3.3. *Assume that*
(1) $\hat{n}_1\in L_T^2(R,H^1(\Omega))$, $\hat{n}_2\in L_T^2(R,H^1(\Omega))$*;*
(2) u, c, n_2 *are the periodic solutions of (3.2), (3.3), and (3.8), respectively;*
(3) $0\le g_2(x,t)\in L_T^\infty(Q)$.
Then the following results hold:
(1) $n_2\ge 0$*;*
(2) $n_2\in L^\infty((0,T),H^1(\Omega))\cap L^2((0,T),H^2(\Omega))$*;*
(3) $(n_2)_t\in L^2((0,T),L^2(\Omega))$.

Proof. We first prove the positivity of n_2. We examine the set $A(t)=\{x;n_2(x,t)<0\}$, and from (3.8), we have

$$\frac{d}{dt}\int_{A(t)}n_2dx-\int_{\partial A(t)}\frac{\partial n_2}{\partial\nu}ds+\int_{A(t)}n_2dx$$
$$=\rho(\mu_2+1)\int_{A(t)}\hat{n}_{2+}dx-\mu_2\int_{A(t)}\hat{n}_{2+}n_2dx-\mu_2a_2\int_{A(t)}\hat{n}_{1+}n_2dx+\rho\int_{A(t)}g_2dx.$$

Note that $\frac{\partial n_2}{\partial\nu}\ge 0$ on $\partial A(t)$, and the right-hand side is nonnegative. Then integrating the above equality from 0 to T gives

$$\int_0^T\int_{A(t)}n_2dxdt\ge 0.$$

On the other hand, it is obvious that

$$\int_0^T\int_{A(t)}n_2dxdt\le 0.$$

So

$$\int_0^T\int_{A(t)}n_2dxdt=0,$$

which means that $n_2 \geq 0$. Next, integrating the first equation of (3.8) over Ω, we get

$$\begin{aligned}&\frac{d}{dt}\int_\Omega n_2 dx + \int_\Omega n_2 dx + \mu_2 \int_\Omega \hat{n}_{2+} n_2 dx + a_2\mu_2 \int_\Omega \hat{n}_{1+} n_2 dx\\ &= \rho(\mu_2+1)\int_\Omega \hat{n}_{2+} dx + \rho \int_\Omega g_2 dx.\end{aligned}$$

Hence, we have

$$\sup_t \int_\Omega n_2 dx \leq C\rho(\int_0^T \|\hat{n}_2\|_{L^1} dt + 1). \tag{3.9}$$

Multiplying the first equation of (3.8) by n_2, integrating it over Ω, and combining with the fact that c is bounded, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega n_2^2 dx + \int_\Omega (|n_2|^2 + |\nabla n_2|^2) dx + \mu_2 \int_\Omega \hat{n}_2 n_2^2 dx + a_2\mu_2 \int_\Omega \hat{n}_{1+} n_2^2 dx\\ &= -\chi_2\rho \int_\Omega \nabla\cdot(n_2 \frac{1}{(1+c)^{m_2}}\nabla c) n_2 dx + \rho(\mu_2+1)\int_\Omega \hat{n}_2 n_2 dx + \rho\int_\Omega g_2 n_2 dx\\ &= -\frac{\chi_2\rho}{2}\int_\Omega n_2^2 \nabla\cdot(\frac{1}{(1+c)^{m_2}}\nabla c) dx + \rho(\mu_2+1)\int_\Omega \hat{n}_2 n_2 dx + \rho\int_\Omega g_2 n_2 dx\\ &\leq C\rho\|n_2\|_{L^4}^2(\|\Delta c\|_{L^2} + \|\nabla c\|_{L^4}^2) + C\rho\|\hat{n}_2\|_{L^2}\|n_2\|_{L^2}\\ &\leq C\rho\|n_2\|_{L^2}\|\nabla n_2\|_{L^2}(\|\Delta c\|_{L^2} + \|\nabla c\|_{L^2}\|\Delta c\|_{L^2}) + C\rho\|\hat{n}_2\|_{L^2}\|n_2\|_{L^2} + C\\ &\leq \frac{1}{2}\|n_2\|_{H^1}^2 + C(1 + \|\Delta c\|_{L^2}^2\|n_2\|_{L^2}^2 + \|\nabla c\|_{L^2}^2\|\Delta c\|_{L^2}^2\|n_2\|_{L^2}^2 + \|\hat{n}_2\|_{L^2}^2),\end{aligned}$$

and we further obtain

$$\sup_t \|n_2\|_{L^2}^2 + \int_0^T \|n_2\|_{H^1}^2 dt \leq C. \tag{3.10}$$

Multiplying the first equation of (3.8) by Δn_2, and integrating it over Ω, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla n_2|^2 dx + \int_\Omega |\Delta n_2|^2 dx + \int_\Omega |\nabla n_2|^2 dx\\ &= \int_\Omega u\nabla n_2 \Delta n_2 dx + \int_\Omega (\rho\chi_2 \nabla\cdot(n_2\frac{1}{(1+c)^{m_2}}\nabla c)\Delta n_2 - \rho(1+\mu_2)\hat{n}_{2+}\Delta n_2\\ &\quad + \mu_2\hat{n}_{2+}n_2\Delta n_2 + a_2\mu_2\hat{n}_{1+}n_2\Delta n_2 - g_2\Delta n_2) dx\\ &\leq C(\|u\|_{L^4}\|\nabla n_2\|_{L^4}\|\Delta n_2\|_{L^2} + \|n_2\|_{L^4}\|\Delta c\|_{L^4}\|\Delta n_2\|_{L^2} + \|\nabla n_2\|_{L^4}\|\nabla c\|_{L^4}\|\Delta n_2\|_{L^2}\\ &\quad + \|n_2\|_{L^4}\|\nabla c\|_{L^8}^2\|\Delta n_2\|_{L^2} + \|\hat{n}_2\|_{L^2}\|\Delta n_2\|_{L^2} + \|n_2\|_{L^4}\|\hat{n}_2\|_{L^4}\|\Delta n_2\|_{L^2}\\ &\quad + \|n_2\|_{L^4}\|\hat{n}_1\|_{L^4}\|\Delta n_2\|_{L^2} + \|g_2\|_{L^2}\|\Delta n_2\|_{L^2}) = \sum_{i=1}^{8} I_i.\end{aligned}$$

For I_1, by the Gagliardo-Nirenberg interpolation inequality, we get

$$I_1 \leq C\|u\|_{L^4}\|\nabla n_2\|_{L^2}^{\frac{1}{2}}\|\Delta n_2\|_{L^2}^{\frac{3}{2}} + C\|u\|_{L^4}\|\nabla n_2\|_{L^2}\|\Delta n_2\|_{L^2}$$

$$\leq C\|\nabla n_2\|_{L^2}^{\frac{1}{2}}\|\Delta n_2\|_{L^2}^{\frac{3}{2}} + C\|\nabla n_2\|_{L^2}\|\Delta n_2\|_{L^2}.$$

For I_2, we have

$$\begin{aligned}I_2 &\leq C\left(\|n_2\|_{L^2} + \|\nabla n_2\|_{L^2}\right)\|\Delta c\|_{H^1}\|\Delta n_2\|_{L^2}\\ &\leq C\|\Delta c\|_{H^1}\|\Delta n_2\|_{L^2} + C\|\nabla n_2\|_{L^2}\|\Delta c\|_{H^1}\|\Delta n_2\|_{L^2}.\end{aligned}$$

For I_3, by the Gagliardo-Nirenberg interpolation inequality, we have

$$\begin{aligned}I_3 &\leq C\|\nabla c\|_{H^1}\left(\|\nabla n_2\|_{L^2}^{\frac{1}{2}}\|\Delta n_2\|_{L^2}^{\frac{3}{2}} + \|\nabla n_2\|_{L^2}\|\Delta n_2\|_{L^2}\right)\\ &\leq C\left(\|\nabla n_2\|_{L^2}^{\frac{1}{2}}\|\Delta n_2\|_{L^2}^{\frac{3}{2}} + \|\nabla n_2\|_{L^2}\|\Delta n_2\|_{L^2}\right).\end{aligned}$$

For I_4, we obtain

$$I_4 \leq C\|\nabla c\|_{H^1}^2\|\nabla n_2\|_{L^2}\|\Delta n_2\|_{L^2} \leq C\|\nabla n_2\|_{L^2}\|\Delta n_2\|_{L^2}.$$

Further, for I_5, I_6, I_7, we see that

$$\begin{aligned}&I_5 + I_6 + I_7\\ &\leq C\|\hat{n}_2\|_{L^2}\|\Delta n_2\|_{L^2} + C(\|\nabla n_2\|_{L^2} + \|n_2\|_{L^2})(\|\hat{n}_2\|_{L^4}\|\Delta n_2\|_{L^2} + \|\hat{n}_1\|_{L^4}\|\Delta n_2\|_{L^2}).\end{aligned}$$

Substitute the estimates for $I_i (i = 1, 2..., 8)$ into the above inequality and we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla n_2|^2 dx + \int_\Omega (|\Delta n_2|^2 + |\nabla n_2|^2) dx\\ &\leq \frac{1}{2}\|\Delta n_2\|_{L^2}^2 + C(\|\nabla n_2\|_{L^2}^2 + \|\nabla n_2\|_{L^2}^2\|\Delta c\|_{H^1}^2 + \|\Delta c\|_{H^1}^2 + \|\hat{n}_1\|_{L^2}^2\\ &\quad + \|\hat{n}_2\|_{L^2}^2 + \|\nabla n_2\|_{L^2}^2\|\hat{n}_2\|_{L^4}^2 + \|\nabla n_2\|_{L^2}^2\|\hat{n}_1\|_{L^4}^2 + \|\hat{n}_2\|_{L^4}^2 + \|\hat{n}_1\|_{L^4}^2 + \|g_2\|_{L^2}^2).\end{aligned}$$

By Lemma 2.2 and combining with Lemma 3.1, we further have

$$\sup_t \|\nabla n_2\|_{L^2}^2 + \int_0^T \int_\Omega (|\Delta n_2|^2 + |\nabla n_2|^2) dx dt \leq C. \tag{3.11}$$

Multiplying the first equation of (3.8) by $(n_2)_t$, and integrating it over Ω, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla n_2|^2 + n_2^2) dx + \int_\Omega |(n_2)_t|^2 dx\\ &= -\int_\Omega u\cdot\nabla n_2 (n_2)_t dx + \int_\Omega (\rho(\mu_2+1)\hat{n}_2(n_2)_t - \rho\chi\nabla\cdot(n_2\frac{1}{(1+c)^{m_2}}\nabla c)(n_2)_t\\ &\quad - \mu_2\hat{n}_2 n_2 (n_2)_t - a_2\mu_2\hat{n}_1 n_2(n_2)_t - g_2(n_2)_t) dx\\ &\leq \frac{1}{2}\|(n_2)_t\|_{L^2}^2 + C(\|u\|_{L^4}^2\|\nabla n_2\|_{H^1}^2 + \|n_2\|_{L^4}^2\|\Delta c\|_{L^4}^2 + \|\nabla n_2\|_{L^4}^2\|\nabla c\|_{L^4}^2\\ &\quad + \|n_2\|_{L^4}^2\|\nabla c\|_{L^2}^2\|\Delta c\|_{L^2}^2 + \|\hat{n}_2\|_{L^2}^2 + \|n_2\|_{L^4}^2\|\hat{n}_1\|_{L^4}^2 + \|n_2\|_{L^4}^2\|\hat{n}_2\|_{L^4}^2 + \|g_2\|_{L^2}^2).\end{aligned}$$

Integrating the above inequality from 0 to T, we get

$$\begin{aligned}&\int_0^T \|(n_2)_t\|_{L^2}^2 dt\\ \leq{}& C\int_0^T (\|\nabla n_2\|_{H^1}^2 + \|\Delta c\|_{H^1}^2 + \|\hat{n}_2\|_{L^2}^2 + \|\hat{n}_1\|_{L^4}^2 + \|\hat{n}_2\|_{L^4}^2 + \|g_2\|_{L^2}^2)dt.\end{aligned} \tag{3.12}$$

The proof is complete. □

For the above obtained solutions u, c, n_2, we consider the following linear problem.

$$\begin{cases}(n_1)_t - \Delta((\hat{n}_1^2+\varepsilon)^{\frac{m-1}{2}} n_1) + \varepsilon|\hat{n}_1|^s n_1 + \delta\Delta^2 n_1 + u\cdot\nabla n_1 + a_1\mu_1 n_2 n_1\\ = -\rho\chi_1\nabla\cdot(\hat{n}_1\frac{1}{(1+c)^{m_1}}\nabla c) + \rho(\mu_1+A)|\hat{n}_1| - An_1 - \mu_1|\hat{n}_1|n_1 + \rho g_1,\\ \frac{\partial n_1}{\partial\nu}|_{\partial\Omega} = \frac{\partial\Delta n_1}{\partial\nu}|_{\partial\Omega} = 0.\end{cases} \tag{3.13}$$

For the above linear parabolic problem, when A is sufficiently large, the existence of time periodic solutions can be easily obtained by a fixed point method. That is, defining a Poincaré map from $n_1(x,0)$ to $n_1(x,T)$, the time periodic solution is then identified as a fixed point of this Poincaré map. We only give the regularity estimates.

Lemma 3.4. *Assume that $\hat{n}_1 \in L^\infty \cap L_T^\infty(R, H^1(\Omega)) \cap L_T^2(R, H^2(\Omega))$, $0 \leq g_1 \in L_T^\infty(Q)$, $\frac{\partial\hat{n}_1}{\partial\nu}|_{\partial\Omega} = 0$, and A is a sufficiently large constant. Let u, c, n_2, n_1 be the time periodic solutions of problems (3.2), (3.3), (3.8), and (3.13), respectively, and then*

$$n_1 \in L^\infty((0,T), H^2(\Omega)) \cap L^2((0,T), H^4(\Omega)),\ \textit{and}\ (n_1)_t \in L^2((0,T), L^2(\Omega)).$$

Proof. First, similar to the proof of Lemmas 3.2 and 3.3, multiplying the first equation of (3.13) by n_1, and integrating it over $\Omega\times(t_0,t)$ for any $t_0 < t \leq t_0+T$, when A is sufficiently large, we see that

$$\begin{aligned}&\frac{1}{2}\int_\Omega (|n_1(x,t)|^2 - |n_1(x,t_0)|^2)dx + \int_{t_0}^t\int_\Omega (\delta|\Delta n_1|^2 + \varepsilon|\hat{n}_1|^s n_1^2 + \mu_1|\hat{n}_1|n_1^2\\ &+ An_1^2 + a_1\mu_1 n_2 n_1^2)dxds\\ ={}& \int_{t_0}^t\int_\Omega \Delta n_1((\hat{n}_1^2+\varepsilon)^{\frac{m-1}{2}} n_1)dxds + \int_{t_0}^t\int_\Omega \rho\chi_1\hat{n}_1\frac{1}{(1+c)^{m_1}}\nabla c\nabla n_1 dxds\\ &+ \int_{t_0}^t\int_\Omega (\rho(\mu_1+A)|\hat{n}_1|n_1 + \rho g_1 n_1)dxds\\ \leq{}& \frac{\delta}{4}\int_{t_0}^t\int_\Omega |\Delta n_1|^2 dxds + C\int_{t_0}^t (\|n_1\|_{L^2}^2 + \|\hat{n}_1\|_{L^\infty}\|\nabla c\|_{L^2}\|\nabla n_1\|_{L^2} + \|n_1\|_{L^1})ds\\ \leq{}& \frac{\delta}{2}\int_{t_0}^t\int_\Omega |\Delta n_1|^2 dxds + \frac{A}{2}\int_{t_0}^t\int_\Omega n_1^2 dxds + C,\end{aligned}$$

which means

$$\int_\Omega (|n_1(x,t)|^2 - |n_1(x,t_0)|^2)dx + \int_{t_0}^t\int_\Omega (\delta|\Delta n_1|^2 + 2\varepsilon|\hat{n}_1|^s n_1^2 + 2\mu_1|\hat{n}_1|n_1^2$$

$$+ An_1^2 + 2a_1\mu_1 n_2 n_1^2)dxdt \le C.$$

By Lemma 2.6, it follows that

$$\sup_t \int_\Omega n_1^2 dx + \int_0^T \|n_1\|_{H^2}^2 dt + A\int_0^T \|n_1\|_{L^2} dt \le C. \tag{3.14}$$

Next, we multiply the first equation of (3.13) by Δn_1, and integrate it over $\Omega \times (t_0, t)$ for any $t_0 < t \le t_0 + T$, and we see that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (|\nabla n_1(x,t)|^2 - |\nabla n_1(x,t_0)|^2)dx + \int_{t_0}^t \int_\Omega (\delta|\nabla\Delta n_1|^2 + A|\nabla n_1|^2)dxds \\
&= \int_{t_0}^t \int_\Omega \nabla\Delta n_1 \nabla((\hat{n}_1^2+\varepsilon)^{\frac{m-1}{2}} n_1)dxds + \int_{t_0}^t \int_\Omega (u\cdot\nabla n_1 + \rho\nabla\cdot(\hat{n}_1 \frac{1}{(1+c)^{m_1}}\nabla c) \\
&\quad - \rho(A+\mu_1)|\hat{n}_1| + \mu_1|\hat{n}_1|n_1 + a_1\mu_1 n_2 n_1 + \varepsilon|\hat{n}_1|^s n_1 - \rho g_1)\Delta n_1 dxds \\
&= \int_{t_0}^t \int_\Omega \nabla\Delta n_1((\hat{n}_1^2+\varepsilon)^{\frac{m-1}{2}}\nabla n_1 + (m-1)(\hat{n}_1^2+\varepsilon)^{\frac{m-3}{2}}\hat{n}_1 n_1 \nabla\hat{n}_1)dxds \\
&\quad + \int_{t_0}^t \int_\Omega (u\cdot\nabla n_1 + \rho\nabla\cdot(\hat{n}_1\frac{1}{(1+c)^{m_1}}\nabla c) - \rho(A+\mu_1)|\hat{n}_1| + \mu_1|\hat{n}_1|n_1 \\
&\quad + a_1\mu_1 n_2 n_1 + \varepsilon|\hat{n}_1|^s n_1 - \rho g_1)\Delta n_1 dxds \\
&\le \frac{\delta}{2}\int_{t_0}^t \|\nabla\Delta n_1\|_{L^2}^2 ds + C\int_{t_0}^t \|\nabla n_1\|_{L^2}^2 \|(\hat{n}_1^2+\varepsilon)^{\frac{m-1}{2}}\|_{L^\infty} ds \\
&\quad + C\int_{t_0}^t \int_\Omega n_1^2(\hat{n}_1^2+\varepsilon)^{m-2}|\nabla\hat{n}_1|^2 dxds + \int_{t_0}^t \|u\|_{L^4}\|\nabla n_1\|_{L^4}\|\Delta n_1\|_{L^2} ds \\
&\quad + \int_{t_0}^t \|(\rho\nabla\cdot(\hat{n}_1\frac{1}{(1+c)^{m_1}}\nabla c) - \rho(A+\mu_1)|\hat{n}_1| + \mu_1|\hat{n}_1|n_1 + a_1\mu_1 n_2 n_1 \\
&\quad + \varepsilon|\hat{n}_1|^s n_1 - \rho g_1)\|_{L^2}\|\Delta n_1\|_{L^2} ds \\
&\le \frac{\delta}{2}\int_{t_0}^t \|\nabla\Delta n_1\|_{L^2}^2 ds + C\int_{t_0}^t \|\nabla n_1\|_{L^2}^2 \|(\hat{n}_1^2+\varepsilon)^{\frac{m-1}{2}}\|_{L^\infty} ds \\
&\quad + C\int_{t_0}^t \|n_1\|_{L^\infty}^2 \|(\hat{n}_1^2+\varepsilon)^{m-2}\|_{L^\infty}\|\nabla\hat{n}_1\|_{L^2}^2 ds + \int_{t_0}^t \|u\|_{H^1}\|\nabla n_1\|_{L^2}^{\frac{1}{2}}\|\Delta n_1\|_{L^2}^{\frac{3}{2}} ds \\
&\quad + \int_{t_0}^t \|\rho\nabla\cdot(\hat{n}_1\frac{1}{(1+c)^{m_1}}\nabla c) - \rho(A+\mu_1)|\hat{n}_1| + \mu_1|\hat{n}_1|n_1 + a_1\mu_1 n_2 n_1 \\
&\quad + \varepsilon|\hat{n}_1|^s n_1 - \rho g_1\|_{L^2}\|\Delta n_1\|_{L^2} ds \\
&\le \frac{\delta}{2}\int_{t_0}^t \|\nabla\Delta n_1\|_{L^2}^2 ds + C\int_{t_0}^t \|\nabla n_1\|_{H^1}^2 ds + C\int_{t_0}^t (\|n_1\|_{L^\infty}^2 + \|u\|_{H^1}^4\|\nabla n_1\|_{L^2}^2)ds \\
&\quad + C\int_{t_0}^t (\|\hat{n}_1\|_{L^\infty}^2\|\Delta c\|_{L^2}^2 + \|\nabla\hat{n}_1\|_{L^4}^2\|\nabla c\|_{L^4}^2 + \|\hat{n}_1\|_{L^\infty}^2\|\nabla c\|_{L^2}^2\|\Delta c\|_{L^2}^2)ds \\
&\quad + C\int_{t_0}^t (\|\hat{n}_1\|_{L^\infty}^2\|n_1\|_{L^2}^2 + \|n_2\|_{L^4}^4 + \|n_1\|_{L^4}^4 + \|\hat{n}_1\|_{L^\infty}^{2s}\|n_1\|_{L^2}^2 + \|g_1\|_{L^2}^2)ds \\
&\le \frac{\delta}{2}\int_{t_0}^t \|\nabla\Delta n_1\|_{L^2}^2 ds + C\int_{t_0}^t \|n_1\|_{H^2}^2 ds + C\int_{t_0}^t (\|\nabla\hat{n}_1\|_{L^4}^4 + \|\nabla c\|_{L^4}^4 + 1)ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_0}^{t}(\|n_1\|_{L^2}^2\|\nabla n_1\|_{L^2}^2+\|n_2\|_{L^2}^2\|\nabla n_2\|_{L^2}^2)ds\\
\leq\ &\frac{\delta}{2}\int_{t_0}^{t}\|\nabla\Delta n_1\|_{L^2}^2ds+C\int_{t_0}^{t}\|n_1\|_{H^2}^2ds+C\int_{t_0}^{t}(\|\hat{n}_1\|_{L^\infty}^2\|\Delta\hat{n}_1\|_{L^2}^2+\|c\|_{H^2}^4+1)ds,
\end{aligned}
$$

which means

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega(|\nabla n_1(x,t)|^2-|\nabla n_1(x,t_0)|^2)dx+\int_{t_0}^{t}\int_\Omega(\frac{\delta}{2}|\nabla\Delta n_1|^2+A|\nabla n_1|^2)dxds\\
\leq\ &C\int_{t_0}^{t}(\|n_1\|_{H^2}^2+\|c\|_{H^2}^4+\|\Delta\hat{n}_1\|_{L^2}^2+1)ds.
\end{aligned}
$$

Combining with Lemma 3.2, and using Lemma 2.6, we get that

$$
\sup_t\|n_1\|_{H^1}^2+\int_0^T\|n_1\|_{H^3}^2dt\leq C.
$$

Next, multiplying the first equation of (3.13) by $\Delta^2 n_1$, integrating it over $\Omega\times(t_0,t)$ for any $t_0<t\leq t_0+T$, and using Lemmas 3.1–3.3 and (3.14), we get

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega(|\Delta n_1(x,t)|^2-|\Delta n_1(x,t_0)|^2)dx+\int_{t_0}^{t}\int_\Omega(\delta|\Delta^2 n_1|^2+A|\Delta n_1|^2)dxds\\
=\ &\int_{t_0}^{t}\int_\Omega\Delta^2 n_1\Delta((\hat{n}_1^2+\varepsilon)^{\frac{m-1}{2}}n_1)dxds-\int_{t_0}^{t}\int_\Omega(u\cdot\nabla n_1)\Delta^2 n_1dxds\\
&+\int_{t_0}^{t}\int_\Omega(-\rho\nabla\cdot(\hat{n}_1\frac{1}{(1+c)^{m_1}}\nabla c)+\rho(A+\mu_1)|\hat{n}_1|-\mu_1|\hat{n}_1|n_1-a_1\mu_1n_2n_1\\
&-\varepsilon|\hat{n}_1|^s n_1+\rho g_1)\Delta^2 n_1dxds\\
\leq\ &\int_{t_0}^{t}\|\Delta^2 n_1\|_{L^2}\|\Delta((\hat{n}_1^2+\varepsilon)^{\frac{m-1}{2}}n_1)\|_{L^2}ds+\int_{t_0}^{t}\|u\|_{L^4}\|\nabla n_1\|_{L^4}\|\Delta^2 n_1\|_{L^2}ds\\
&+\int_{t_0}^{t}\|-\rho\nabla\cdot(\hat{n}_1\frac{1}{(1+c)^{m_1}}\nabla c)+\rho(A+\mu_1)|\hat{n}_1|-\mu_1|\hat{n}_1|n_1+a_1\mu_1n_2n_1\\
&-\varepsilon|\hat{n}_1|^s n_1+\rho g_1\|_{L^2}\|\Delta^2 n_1\|_{L^2}ds\\
\leq\ &\int_{t_0}^{t}\|\Delta^2 n_1\|_{L^2}\|(\hat{n}_1^2+\varepsilon)^{\frac{m-1}{2}}\Delta n_1+2\nabla(\hat{n}_1^2+\varepsilon)^{\frac{m-1}{2}}\nabla n_1\\
&+n_1\Delta(\hat{n}_1^2+\varepsilon)^{\frac{m-1}{2}}\|_{L^2}ds+\int_{t_0}^{t}\|u\|_{H^1}\|\nabla n_1\|_{L^2}^{\frac{1}{2}}\|\nabla n_1\|_{H^1}^{\frac{1}{2}}\|\Delta^2 n_1\|_{L^2}ds\\
&+C\int_{t_0}^{t}(\|\hat{n}_1\|_{L^\infty}\|\Delta c\|_{L^2}+\|\hat{n}_1\|_{L^\infty}\|\nabla c\|_{L^2}\|\Delta c\|_{L^2}+\|\hat{n}_1\|_{L^2}+\|\nabla\hat{n}_1\|_{L^4}\|\nabla c\|_{L^4}\\
&+\|n_1\|_{L^4}^2+\|n_2\|_{L^4}^2+\|\hat{n}_1\|_{L^\infty}^s\|n_1\|_{L^2}+\|g_1\|_{L^2})\|\Delta^2 n_1\|_{L^2}ds\\
\leq\ &C\int_{t_0}^{t}\|\Delta^2 n_1\|_{L^2}(\|\Delta n_1\|_{L^2}+\|2(m-1)(\hat{n}_1^2+\varepsilon)^{\frac{m-3}{2}}\hat{n}_1\nabla\hat{n}_1\nabla n_1\|_{L^2})ds\\
&+C\int_{t_0}^{t}\|\Delta^2 n_1\|_{L^2}\|n_1\|_{L^\infty}\|\Delta(\hat{n}_1^2+\varepsilon)^{\frac{m-1}{2}}\|_{L^2}ds+C\int_{t_0}^{t}\|\nabla n_1\|_{H^1}^{\frac{1}{2}}\|\Delta^2 n_1\|_{L^2}ds
\end{aligned}
$$

$$
\begin{aligned}
&+ C\int_{t_0}^{t}(1+\|\nabla\hat{n}_1\|_{H^1}+\|n_1\|_{H^1}^2+\|n_2\|_{H^1}^2)\|\Delta^2 n_1\|_{L^2}ds\\
\le{}& C\int_{t_0}^{t}\|\Delta^2 n_1\|_{L^2}(\|\Delta n_1\|_{L^2}+\|2(m-1)(\hat{n}_1^2+\varepsilon)^{\frac{m-2}{2}}\nabla\hat{n}_1\nabla n_1\|_{L^2}\\
&+(m-1)\|n_1\|_{L^\infty}\|(m-3)(\hat{n}_1^2+\varepsilon)^{\frac{m-5}{2}}|\hat{n}_1|^2|\nabla\hat{n}_1|^2+(\hat{n}_1^2+\varepsilon)^{\frac{m-3}{2}}|\nabla\hat{n}_1|^2\\
&+(\hat{n}_1^2+\varepsilon)^{\frac{m-3}{2}}\hat{n}_1\Delta\hat{n}_1\|_{L^2})ds+C\int_{t_0}^{t}\|\nabla n_1\|_{H^1}^{\frac{1}{2}}\|\Delta^2 n_1\|_{L^2}ds\\
&+C\int_{t_0}^{t}(1+\|\nabla\hat{n}_1\|_{H^1}+\|n_1\|_{H^1}^2+\|n_2\|_{H^1}^2)\|\Delta^2 n_1\|_{L^2}ds\\
\le{}& C\int_{t_0}^{t}\|\Delta^2 n_1\|_{L^2}(\|\Delta n_1\|_{L^2}+\|\nabla\hat{n}_1\|_{L^4}\|\nabla n_1\|_{L^4}\\
&+\|n_1\|_{L^\infty}(\|\nabla\hat{n}_1\|_{L^4}^2+\|\Delta\hat{n}_1\|_{L^2}))ds+C\int_{t_0}^{t}\|\nabla n_1\|_{H^1}^{\frac{1}{2}}\|\Delta^2 n_1\|_{L^2}ds\\
&+C\int_{t_0}^{t}(1+\|\nabla\hat{n}_1\|_{H^1}+\|n_1\|_{H^1}^2+\|n_2\|_{H^1}^2)\|\Delta^2 n_1\|_{L^2}ds\\
\le{}& C\int_{t_0}^{t}(1+\|n_1\|_{H^2}^2+\|\nabla\hat{n}_1\|_{L^4}^4+\|\nabla n_1\|_{L^4}^4+\|n_1\|_{L^\infty}^2\|\nabla\hat{n}_1\|_{L^4}^4+\|n_1\|_{L^\infty}^2\|\Delta\hat{n}_1\|_{L^2}^2\\
&+\|\nabla\hat{n}_1\|_{H^2}^2+\|n_1\|_{H^1}^4+\|n_2\|_{H^1}^4)ds+\frac{\delta}{4}\int_{t_0}^{t}\|\Delta^2 n_1\|_{L^2}^2ds\\
\le{}& C\int_{t_0}^{t}(1+\|n_1\|_{H^2}^2+\|\hat{n}_1\|_{L^\infty}^2\|\Delta\hat{n}_1\|_{L^2}^2+\|\nabla n_1\|_{L^2}^3\|\Delta^2 n_1\|_{L^2}+\|n_1\|_{H^1}^4+\|n_2\|_{H^1}^4\\
&+\|n_1\|_{H^1}\|\Delta n_1\|_{L^2}\|\hat{n}_1\|_{L^\infty}^2\|\Delta\hat{n}_1\|_{L^2}^2+\|n_1\|_{H^1}\|\Delta n_1\|_{L^2}\|\Delta\hat{n}_1\|_{L^2}^2+\|\nabla\hat{n}_1\|_{H^1}^2)ds\\
&+\frac{\delta}{4}\int_{t_0}^{t}\|\Delta^2 n_1\|_{L^2}^2ds\\
\le{}& \frac{\delta}{2}\int_{t_0}^{t}\|\Delta^2 n_1\|_{L^2}^2ds+C\int_{t_0}^{t}(1+\|\nabla\hat{n}_1\|_{H^1}^2+\|n_1\|_{H^2}^2+\|\Delta n_1\|_{L^2}\|\Delta\hat{n}_1\|_{L^2}^2)ds,
\end{aligned}
$$

which means

$$
\begin{aligned}
&\int_\Omega(|\Delta n_1(x,t)|^2-|\Delta n_1(x,t_0)|^2)dx+\delta\int_{t_0}^{t}\int_\Omega|\Delta^2 n_1|^2dxds+A\int_{t_0}^{t}\int_\Omega|\Delta n_1|^2dxds\\
\le{}& C\int_{t_0}^{t}(1+\|\nabla\hat{n}_1\|_{H^1}^2+\|n_1\|_{H^2}^2+\|\Delta n_1\|_{L^2}\|\Delta\hat{n}_1\|_{L^2}^2+1)ds.
\end{aligned}
$$

Using Lemma 2.6 yields

$$
\sup_t\int_\Omega|\Delta n_1|^2dx+\int_0^T\int_\Omega|\Delta^2 n_1|^2dxdt\le C. \tag{3.15}
$$

Combining everything above, we finally get

$$
\sup_t\int_\Omega(n_1^2+|\nabla n_1|^2+|\Delta n_1|^2)dx+\int_0^T\|n_1\|_{H^4}^2dt\le C. \tag{3.16}
$$

Multiplying the first equation of (3.13) by $(n_1)_t$ and integrating it over Ω, similar to the proof of the above lemma, we can easily get

$$\int_0^T \|(n_1)_t\|_{L^2}^2 dt \le C. \tag{3.17}$$

The proof is complete. $\square$

Now, we consider the existence of time periodic solutions to problem (3.1). Denote

$$X_1 = L^\infty \cap L_T^\infty(R, H^1(\Omega)) \cap L_T^2(R, H^2(\Omega)), \quad X_2 = L_T^2(R, H^1(\Omega)) \cap L_T^4(R, L^4(\Omega)).$$

Define a map

$$\begin{aligned}&\mathscr{F} : X_1 \times X_2 \times [0,1] \to X_1 \times X_2,\\ &\mathscr{F}(\hat{n}_1, \hat{n}_2, \rho) = (n_1, n_2),\end{aligned}$$

where n_1, n_2 are the solutions of problems (3.8), (3.13), respectively. Then, we use the Leray-Schauder fixed point theorem to prove the existence of time periodic solutions to problem (3.1). Therefore, we first give the following lemma.

Lemma 3.5. *Assume that $s > max\{2(m-1), 10\}$, $0 \le g_1, g_2 \in L_T^\infty(Q)$, $\nabla\varphi \in L_T^\infty(Q)$, and let $\mathscr{F}(n_1, n_2, \rho) = (n_1, n_2)$. Then there exists $C > 0$, such that*

$$\begin{aligned}&\sup_t(\|(u, n_1, n_2)\|_{H^1}^2 + \|c\|_{H^2}^2)\\ &+ \int_0^T (\|c\|_{H^3}^2 + \|(u, n_1, n_2)\|_{H^2}^2 + \|(c_t, u_t, (n_1)_t, (n_2)_t)\|_{L^2}^2) dt \le C,\end{aligned}$$

where C depends on ε, δ and is independent of A.

Proof. The nonnegativity of c, n_2 are direct results of Lemmas 3.2 and 3.3. Therefore, taking $\hat{n}_1 = n_1$ in (3.13), multiplying the first equation of (3.13) by n_1, and integrating it over $\Omega \times (t_0, t)$ for any $t_0 < t < t_0 + T$, we see that

$$\begin{aligned}&\frac{1}{2}\int_\Omega (|n_1(x,t)|^2 - |n_1(x,t_0)|^2)dx + \int_{t_0}^t \int_\Omega (\delta|\Delta n_1|^2 + An_1^2 + \varepsilon|n_1|^{s+2} + \mu_1|n_1|^3\\ &+ (n_1^2 + \varepsilon)^{\frac{(m-1)}{2}}|\nabla n_1|^2 + a_1\mu_1 n_2 n_1^2)dxds\\ =& -\frac{\rho\chi_1}{2}\int_{t_0}^t \int_\Omega n_1^2 \frac{1}{(1+c)^{m_1}}\Delta c dxds + \frac{\rho\chi_1 m_1}{2}\int_{t_0}^t \int_\Omega n_1^2 \frac{1}{(1+c)^{m_1+1}}|\nabla c|^2 dxds\\ &+ \rho(A+\mu_1)\int_{t_0}^t \int_\Omega n_1^2 dxds + \rho\int_{t_0}^t \int_\Omega g_1 n_1 dxds\\ \le& \frac{\rho\chi_1}{2}\int_{t_0}^t \int_\Omega n_1^2 \Delta c dxds + \rho(A+\mu_1)\int_{t_0}^t \int_\Omega n_1^2 dxds + \rho\int_{t_0}^t \int_\Omega g_1 n_1 dxds\\ \le& \int_{t_0}^t (\frac{\rho\chi_1}{4}\|n_1\|_{L^4}^4 + \frac{\rho\chi_1}{4}\|\Delta c\|_{L^2}^2 + \frac{\rho\chi_1 m_1}{4}\|n_1\|_{L^4}^4 + \frac{\rho\chi_1 m_1}{4}\|\nabla c\|_{L^2}^2\|\Delta c\|_{L^2}^2\end{aligned}$$

$$+ \rho A\|n_1\|_{L^2}^2 + \frac{\mu_1}{2}\|n_1\|_{L^3}^3 + \rho C)ds,$$

which implies that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (|n_1(x,t)|^2 - |n_1(x,t_0)|^2)dx + \int_{t_0}^t \int_\Omega (\delta|\Delta n_1|^2 + \varepsilon|n_1|^{s+2} + \frac{\mu_1}{2}|n_1|^3 \\
&+ (n_1^2+\varepsilon)^{\frac{(m-1)}{2}}|\nabla n_1|^2)dxds \\
\leq\ & \rho C\int_{t_0}^t (\|n_1\|_{L^4}^4 + \|\Delta c\|_{L^2}^2 + \|\nabla c\|_{L^2}^2\|\Delta c\|_{L^2}^2 + 1)ds. \qquad (3.18)
\end{aligned}$$

Taking $\hat{n}_1 = n_1$, $\hat{n}_2 = n_2$ in (3.8) and integrating the first equation of (3.8) over Ω, we see that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega n_2dx + \int_\Omega (n_2 + \mu_2 n_2^2 + a_2\mu_2 n_{1+}n_2)dx &= \rho(\mu_2+1)\int_\Omega n_2dx \\
&\leq \frac{\mu_2}{2}\int_\Omega n_2^2dx + \rho C,
\end{aligned}$$

which implies that

$$\sup_{t\in(0,T)}\int_\Omega n_2dx + \int_0^T\int_\Omega (n_2+n_2^2)dxdt \leq C. \qquad (3.19)$$

Taking $\hat{n}_1 = n_1$, $\hat{n}_2 = n_2$ in (3.2), multiplying the corresponding equation by $2u$, $2\Delta u$, respectively, and then combining the two inequalities, it is easy to see that

$$\begin{aligned}
&\int_\Omega (|u(x,t)|^2 + |\nabla u(x,t)|^2)dx - \int_\Omega (|u(x,t_0)|^2 + |\nabla u(x,t_0)|^2)dx \\
&+ \int_{t_0}^t\int_\Omega (|\nabla u|^2 + |\Delta u|^2)dxds \leq C\int_{t_0}^t\int_\Omega (n_1^2+n_2^2)dxds.
\end{aligned}$$

Combining with (3.19), this implies that

$$\sup_{t\in(0,T)}\int_\Omega (u^2+|\nabla u|^2)dx + \int_0^T\int_\Omega (|\nabla u|^2 + |\Delta u|^2)dxdt \leq C\int_0^T\int_\Omega n_1^2dxdt. \qquad (3.20)$$

Taking $\hat{n}_1 = n_1$, $\hat{n}_2 = n_2$ in (3.3), multiplying this equation by c, Δc, integrating them over $\Omega \times (t_0, t)$ for any $t \geq t_0$, and combining with the fact that c is bounded, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (|c(x,t)|^2 - |c(x,t_0)|^2)dx + \int_{t_0}^t\int_\Omega |\nabla c|^2dxds \\
\leq\ & \frac{1}{2}\int_{t_0}^t\int_\Omega |\nabla c|^2dxds + C\int_{t_0}^t\int_\Omega n_1^2dxds,
\end{aligned}$$

which implies that

$$\sup_{t\in(0,T)}\int_\Omega c^2dx + \int_0^T\int_\Omega |\nabla c|^2dxdt \leq C\int_0^T\int_\Omega n_1^2dxdt, \qquad (3.21)$$

and

$$\frac{1}{2}\int_\Omega(|\nabla c(x,t)|^2-|\nabla c(x,t_0)|^2)dx+\int_{t_0}^t\int_\Omega|\Delta c|^2dxds$$
$$\leq\int_{t_0}^t\int_\Omega u\cdot\nabla c\Delta cdxds+\rho\int_{t_0}^t\int_\Omega(\alpha_1n_1+\alpha_2n_2)c\Delta cdxds-\rho\beta\int_{t_0}^t\int_\Omega n_1\Delta cdxds$$
$$\leq C\int_{t_0}^t\|u\|_{L^4}\|\nabla c\|_{L^4}\|\Delta c\|_{L^2}ds+C\int_{t_0}^t\|n_1\|_{L^2}\|c\|_{L^\infty}\|\Delta c\|_{L^2}ds$$
$$+C\int_{t_0}^t\|n_2\|_{L^2}\|c\|_{L^\infty}\|\Delta c\|_{L^2}ds+C\int_{t_0}^t\|n_1\|_{L^2}\|\Delta c\|_{L^2}ds$$
$$\leq C\int_{t_0}^t(\|u\|_{H^1}\|\nabla c\|_{L^2}^{\frac{1}{2}}\|\Delta c\|_{L^2}^{\frac{3}{2}}+\|u\|_{H^1}\|\nabla c\|_{L^2}\|\Delta c\|_{L^2})ds$$
$$+C\int_{t_0}^t\|n_1\|_{L^2}\|\Delta c\|_{L^2}ds+C\int_{t_0}^t\|n_2\|_{L^2}\|\Delta c\|_{L^2}ds$$
$$\leq C\int_{t_0}^t(\|u\|_{H^1}^4\|\nabla c\|_{L^2}^2+\|u\|_{H^1}^2\|\nabla c\|_{L^2}^2)ds+\frac{1}{2}\int_{t_0}^t\int_\Omega|\Delta c|^2dxds+C\int_{t_0}^t\|n_1\|_{L^2}^2ds.$$

Combining with (3.20), (3.21), and using Lemma 2.6, we can see that

$$\sup_{t\in(0,T)}\int_\Omega|\nabla c|^2dx+\int_0^T\int_\Omega|\Delta c|^2dxdt$$
$$\leq C\left(\int_0^T\int_\Omega n_1^2dxdt+(\int_0^T\int_\Omega n_1^2dxdt)^3\right). \tag{3.22}$$

Recalling (3.18), using (3.22), and noticing that $s>10$, we get that

$$\sup_t\int_\Omega n_1^2dx+\int_{t_0}^t\int_\Omega(\delta|\Delta n_1|^2+\varepsilon|n_1|^{s+2}+\frac{\mu_1}{2}|n_1|^3+(n_1^2+\varepsilon)^{\frac{(m-1)}{2}}|\nabla n_1|^2)dxds$$
$$\leq C\int_0^T\int_\Omega|n_1|^4dxdt+C\int_0^T\int_\Omega|\Delta c|^2dxdt+\sup_t\int_\Omega|\nabla c|^2dx\int_0^T\int_\Omega|\Delta c|^2dxdt+C$$
$$\leq C\left(1+\int_0^T\int_\Omega\left(n_1^2+n_1^{12}\right)dxdt\right)$$
$$\leq\frac{\mu_1}{4}\int_0^T\int_\Omega|n_1|^3dxdt+\frac{\varepsilon}{2}\int_0^T\int_\Omega|n_1|^{s+2}dxdt+C_\varepsilon,$$

which implies

$$\sup_{t\in(0,T)}\int_\Omega n_1^2dx+\delta\int_0^T\int_\Omega|\Delta n_1|^2dxdt+\frac{\varepsilon}{2}\int_0^T\int_\Omega|n_1|^{s+2}dxdt$$
$$+\frac{\mu_1}{4}\int_0^T\int_\Omega|n_1|^3dxdt+\int_0^T\int_\Omega(n_1^2+\varepsilon)^{\frac{m-1}{2}}|\nabla n_1|^2dxdt\leq C_\varepsilon, \tag{3.23}$$

where C_ε is independent of δ and A, but depends on ε. Because s is appropriately large, we get $n_1\in L^\infty\cap L_T^2(R,H^1(\Omega))$. Therefore, according to (3.20)–(3.22), we can get that

$$\sup_t\{\|u\|_{H^1}^2+\|c\|_{H^1}^2\}+\int_0^T(\|u\|_{H^2}^2+\|c\|_{H^2}^2)dt\leq C_\varepsilon. \tag{3.24}$$

Then we give the estimate about $\|n_2\|_{L^\infty}$. Taking $\hat{n}_1 = n_1$, $\hat{n}_2 = n_2$ in (3.14), multiplying the first equation of (3.8) by n_2^r, and integrating it over Ω, for any $r \geq 1$, we see that

$$\begin{aligned}
&\frac{1}{r+1}\frac{d}{dt}\int_\Omega n_2^{r+1}dx + \frac{4r}{(r+1)^2}\int_\Omega |\nabla n_2^{\frac{r+1}{2}}|^2 dx + \int_\Omega n_2^{r+1}dx + \mu_2\int_\Omega n_2^{r+2}dx \\
=&-\rho\chi_2\frac{r}{r+1}\int_\Omega n_2^{r+1}\frac{1}{(1+c)^{m_2+1}}\Delta c dx + \rho\chi_2 m_2\frac{r}{r+1}\int_\Omega n_2^{r+1}\frac{1}{(1+c)^{m_2}}|\nabla c|^2 dx \\
&+\rho(1+\mu_2)\int_\Omega n_2^{r+1}dx - a_2\mu_2\int_\Omega n_{1+}n_2^{r+1}dx + \int_\Omega g_2 n_2^r dx \\
\leq & C\|n_2^{\frac{r+1}{2}}\|_{L^4}^2\|\Delta c\|_{L^2} + C\|n_2^{\frac{r+1}{2}}\|_{L^4}^2\|\nabla c\|_{L^4}^2 + \rho(1+\mu_2)\|n_2\|_{L^{r+1}}^{r+1} + C\|n_2\|_{L^{r+1}}^{r+1} \\
\leq & C\|n_2^{\frac{r+1}{2}}\|_{L^2}\|\nabla n_2^{\frac{r+1}{2}}\|_{L^2}\|\Delta c\|_{L^2} + C\|n_2\|_{L^1}^{r+1}\|\Delta c\|_{L^2} + \rho(1+\mu_2)\|n_2\|_{L^{r+1}}^{r+1} \\
&+ C\|n_2^{\frac{r+1}{2}}\|_{L^2}\|\nabla n_2^{\frac{r+1}{2}}\|_{L^2}\|\nabla c\|_{L^2}\|\Delta c\|_{L^2} + C\|n_2\|_{L^1}^{r+1}\|\nabla c\|_{L^2}\|\Delta c\|_{L^2} \\
\leq & \frac{2r}{(r+1)^2}\|\nabla n_2^{\frac{r+1}{2}}\|_{L^2}^2 + \frac{\mu_2}{2}\|n_2\|_{L^{r+2}}^{r+2} + C(1+\|\Delta c\|_{L^2}^2) + C\|n_2\|_{L^{r+1}}^{r+1} \\
&+ C\|n_2^{\frac{r+1}{2}}\|_{L^2}^2\|\Delta c\|_{L^2}^2(1+\|\nabla c\|_{L^2}^2),
\end{aligned}$$

which means that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega n_2^{r+1}dx + \frac{2r}{(r+1)}\int_\Omega |\nabla n_2^{\frac{r+1}{2}}|^2 dx + \frac{r+1}{2}\int_\Omega n_2^{r+1}dx + \frac{\mu_2(r+1)}{2}\int_\Omega n_2^{r+2}dx \\
\leq & C\|n_2^{\frac{r+1}{2}}\|_{L^2}^2\|\Delta c\|_{L^2}^2 + C\|n_2^{\frac{r+1}{2}}\|_{L^2}^2\|\nabla c\|_{L^2}^2\|\Delta c\|_{L^2}^2 + C(1+\|\Delta c\|_{L^2}^2).
\end{aligned} \tag{3.25}$$

Then integrating (3.25) from t_0 to t and combining with Lemma 2.6, we can see that if

$$n_2 \in L_T^{r+1}(R, L^{r+1}(\Omega)),$$

then we have

$$\sup_t \|n_2\|_{L^{r+1}}^{r+1} + \int_0^T\int_\Omega |\nabla n_2^{\frac{r+1}{2}}|^2 dxdt + \int_0^T\int_\Omega n_2^{r+2}dxdt \leq C. \tag{3.26}$$

Taking $r = 1$ and combining with (3.19), we can get

$$\sup_t \|n_2\|_{L^2}^2 + \int_0^T\int_\Omega |\nabla n_2|^2 dxdt + \int_0^T\int_\Omega n_2^3 dxdt \leq C. \tag{3.27}$$

By the recursion method, we further have

$$\|n_2\|_{L^\infty} \leq C. \tag{3.28}$$

Recalling Lemmas 3.2 and 3.3, we also have

$$\sup_{t\in(0,T)}\int_\Omega (|c|^2 + |\nabla c|^2 + |\Delta c|^2)dx + \int_0^T\int_\Omega |\nabla\Delta c|^2 dxdt \leq C_\varepsilon, \tag{3.29}$$

and

$$\sup_{t\in(0,T)}\int_\Omega(|n_2|^2+|\nabla n_2|^2)dx+\int_0^T\int_\Omega|\Delta n_2|^2dxdt\le C_\varepsilon. \tag{3.30}$$

Here C_ε is independent of δ and A, but depends on ε. Taking $\hat{n}_1 = n_1$ in (3.13), testing the first equation of (3.13) by $-\Delta n_1$, integrating it over $\Omega\times(t_0,t)$ for any $t_0 < t \le t_0+T$, using (3.20), (3.24), (3.26), and combining with $s > 2(m-1)$, then we get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega(|\nabla n_1(x,t)|^2-|\nabla n_1(x,t_0)|^2)dx+\int_{t_0}^t\int_\Omega(\delta|\nabla\Delta n_1|^2+(s+1)\varepsilon|n_1|^s|\nabla n_1|^2\\
&\quad+A|\nabla n_1|^2+2\mu_1|n_1||\nabla n_1|^2)dxds\\
&=\int_{t_0}^t\int_\Omega(\nabla\Delta n_1\nabla((n_1^2+\varepsilon)^{\frac{m-1}{2}}n_1)+u\cdot\nabla n_1\Delta n_1+\rho(n_1\frac{1}{(1+c)^{m_1}}\nabla c)\nabla\Delta n_1\\
&\quad-\rho(A+\mu_1)|n_1|\Delta n_1+a_1\mu_1n_1n_2\Delta n_1-\rho g_1\Delta n_1)dxds\\
&\le\frac{1}{2\delta}\int_{t_0}^t\int_\Omega|\nabla((n_1^2+\varepsilon)^{\frac{m-1}{2}}n_1)|^2dxds+\frac{\delta}{2}\int_{t_0}^t\int_\Omega|\nabla\Delta n_1|^2dxds\\
&\quad+\int_{t_0}^t\|u\|_{L^4}\|\nabla n_1\|_{L^4}\|\Delta n_1\|_{L^2}ds+C\int_{t_0}^t(\|\nabla c\|_{L^2}^2+\|\Delta n_1\|_{L^2}\|n_1n_2\|_{L^2})ds\\
&\quad+\rho(A+\mu_1)\int_{t_0}^t\int_\Omega\frac{n_1}{|n_1|}|\nabla n_1|^2dxds+\int_{t_0}^t\|g_1\|_{L^2}\|\Delta n_1\|_{L^2}ds\\
&\le\frac{1}{2\delta}\int_{t_0}^t\int_\Omega|(m-1)(n_1^2+\varepsilon)^{\frac{m-3}{2}}n_1^2\nabla n_1+(n_1^2+\varepsilon)^{\frac{m-1}{2}}\cdot\nabla n_1|^2dxds\\
&\quad+\frac{\delta}{2}\int_{t_0}^t\int_\Omega|\nabla\Delta n_1|^2dxds+\int_{t_0}^t\|u\|_{H^1}\|\nabla n_1\|_{L^4}\|\Delta n_1\|_{L^2}ds+C\int_{t_0}^t(\|\nabla c\|_{L^2}^2\\
&\quad+\|\Delta n_1\|_{L^2}\|n_1n_2\|_{L^2}+\rho(A+\mu_1)\|\nabla n_1\|_{L^2}^2+\|g_1\|_{L^2}\|\Delta n_1\|_{L^2})ds\\
&\le\frac{m^2}{2\delta}\int_{t_0}^t\int_\Omega(n_1^2+\varepsilon)^{m-1}|\nabla n_1|^2dxds+\frac{\delta}{2}\int_{t_0}^t\int_\Omega|\nabla\Delta n_1|^2dxds+C\\
&\quad+C\int_{t_0}^t\|\Delta n_1\|_{L^2}\|n_1n_2\|_{L^2}ds+C\int_{t_0}^t\|u\|_{H^1}\|\nabla n_1\|_{L^2}^{\frac{1}{2}}\|\Delta n_1\|_{L^2}^{\frac{3}{2}}ds\\
&\quad+C\int_{t_0}^t\|u\|_{H^1}\|\nabla n_1\|_{L^2}^2ds+C\int_{t_0}^t(\rho(A+\mu_1)\|\nabla n_1\|_{L^2}^2+\|g_1\|_{L^2}\|\Delta n_1\|_{L^2})ds\\
&\le\frac{C}{\delta}\int_{t_0}^t\int_\Omega n_1^{2(m-1)}|\nabla n_1|^2dxds+C_{\varepsilon\delta}\int_{t_0}^t\int_\Omega|\nabla n_1|^2dxds+\frac{\delta}{2}\int_{t_0}^t\int_\Omega|\nabla\Delta n_1|^2dxds\\
&\quad+C\int_{t_0}^t(\|\nabla n_1\|_{L^2}^2+\|\Delta n_1\|_{L^2}^2+\|n_1\|_{L^\infty}^2\|n_2\|_{L^2}^2+1)ds\\
&\quad+\int_{t_0}^t\rho(A+\mu_1)\|\nabla n_1\|_{L^2}^2+\|g_1\|_{L^2}\|\Delta n_1\|_{L^2}ds\\
&\le\frac{(s+1)\varepsilon}{2}\int_{t_0}^t\int_\Omega|n_1|^s|\nabla n_1|^2dxds+C_{\varepsilon\delta}\int_{t_0}^t\int_\Omega|\nabla n_1|^2dxds+\frac{\delta}{2}\int_{t_0}^t\int_\Omega|\nabla\Delta n_1|^2dxds
\end{aligned}$$

$$+\int_{t_0}^{t}(A\|\nabla n_1\|_{L^2}^2+C\|\Delta n_1\|_{L^2}^2+C_{\varepsilon\delta})ds.$$

Summing up the above inequality, we get

$$\begin{aligned}&\frac{1}{2}\int_\Omega(|\nabla n_1(x,t)|^2-|\nabla n_1(x,t_0)|^2)dx+\frac{\delta}{2}\int_{t_0}^{t}\int_\Omega|\nabla\Delta n_1|^2dxds\\&+\frac{s+1}{2}\varepsilon\int_{t_0}^{t}\int_\Omega|n_1|^s|\nabla n_1|^2dxds+2\mu_1\int_{t_0}^{t}\int_\Omega|n_1||\nabla n_1|^2dxds\\ \leq{}& C_{\varepsilon\delta}\int_{t_0}^{t}\|\nabla n_1\|_{H^1}^2ds+C_{\varepsilon\delta}.\end{aligned}$$

Therefore, combining with Lemma 2.6, (3.23), and (3.24), we can see that

$$\begin{aligned}&\sup_t\int_\Omega|\nabla n_1|^2dx+\delta\int_0^T\int_\Omega|\nabla\Delta n_1|^2dxdt+\varepsilon\int_0^T\int_\Omega|n_1|^s|\nabla n_1|^2dxdt\\&+\mu_1\int_0^T\int_\Omega|n_1||\nabla n_1|^2dxdt\leq C_{\varepsilon\delta}.\end{aligned}\tag{3.31}$$

By (3.29)–(3.31), we complete the proof. □

By Lemmas 3.2–3.4, we see that $\mathscr{F}$ is a compact operator by the embedding theorem. Furthermore, by (3.18) and (3.19), it is easy to see that $\mathscr{F}(\hat{n}_1,\hat{n}_2,0)=0$. By Lemma 3.5, we see that if $\mathscr{F}(n_1,n_2,\rho)=(n_1,n_2)$, then there exists a constant $M\geq 0$, such that

$$\|n_1\|_{X_1}+\|n_2\|_{X_2}\leq M,$$

where M depends on ε, δ. Then by Leray-Schauder's fixed point theorem, the mapping $\mathscr{F}(\cdot,1)$ has a fixed point, that is, there exists $n\in X$ such that $\mathscr{F}(n_1,n_2,1)=(n_1,n_2)$, that is, problem (3.1) admits a solution (n_1,n_2,c,u).

4. Existence of time periodic solutions

In this section, we use a double-level approximation scheme to prove the existence of time periodic solutions of problem (1.1). We consider the first level approximation by letting $\delta\to 0$. For this purpose, we first give some energy inequalities. Taking advantage of the former proof, we have:

Lemma 4.1. *Assume that $s>\max\{2(m-1),10\}$, $0\leq g_1,g_2\in L_T^\infty(Q)$, $\nabla\varphi\in L_T^\infty(Q)$, and let $(n_{1\varepsilon\delta},n_{2\varepsilon\delta},c_{\varepsilon\delta},u_{\varepsilon\delta},\pi_{\varepsilon\delta})$ be the periodic solution of problem (3.1). We see that*

$$\begin{aligned}&\sup_t\{\|u_{\varepsilon\delta}\|_{H^1}^2+\|c_{\varepsilon\delta}\|_{H^2}^2+\|n_{2\varepsilon\delta}\|_{H^1}^2\}+\int_0^T(\|u_{\varepsilon\delta}\|_{H^2}^2+\|\pi_{\varepsilon\delta}\|_{H^1}^2+\|c_{\varepsilon\delta}\|_{H^3}^2\\&+\|n_{2\varepsilon\delta}\|_{H^2}^2+\|(u_t,c_t)\|_{L^2}^2)dt\leq C,\end{aligned}\tag{4.1}$$

$$\sup_{t\in(0,T)}\int_\Omega n_{1\varepsilon\delta}^2dx+\delta\int_0^T\int_\Omega|\Delta n_{1\varepsilon\delta}|^2dxdt+\int_0^T\int_\Omega(\frac{\varepsilon}{2}|n_{1\varepsilon\delta}|^{s+2}+\frac{\mu}{4}|n_{1\varepsilon\delta}|^2)dxdt$$

$$+\int_0^T\int_\Omega((n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}|\nabla n_{1\varepsilon\delta}|^2)dxdt\leq C, \tag{4.2}$$

where C is a constant depending on α_1, α_2, β, γ, θ, χ_1, χ_2, a_1, a_2, μ_1, μ_2, m_1, m_2, Ω, T, g_1, g_2, $\nabla\varphi$, *and* ε, *but not depending on* δ.

Lemma 4.2. *Assume that* $s>\max\{2(m-1),10\}$, $0\leq g_1,g_2\in L_T^\infty(Q)$, $\nabla\varphi\in L_T^\infty(Q)$, *and let* $(n_{1\varepsilon\delta},n_{2\varepsilon\delta},c_{\varepsilon\delta},u_{\varepsilon\delta},\pi_{\varepsilon\delta})$ *be the periodic solution of problem (3.1). We see that*

$$\sup_t\int_\Omega(\delta|\nabla n_{1\varepsilon\delta}|^2+(n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m+1}{2}})dx+\int_0^T\int_\Omega(|\delta\nabla\Delta n_{1\varepsilon\delta}-\nabla(n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}n_{1\varepsilon\delta}|^2$$
$$+(n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}(\varepsilon|n_{1\varepsilon\delta}|^{s+2}+\mu|n_{1\varepsilon\delta}|^3))dxdt\leq C, \tag{4.3}$$

where C is a constant depending on Ω, T, g_1, g_2, $\nabla\varphi$, *and* ε, *but not depending on* δ.

Proof. Multiplying the first equation of (3.1) by $-\delta\Delta n_{1\varepsilon\delta}+(n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}n_{1\varepsilon\delta}$, and using (4.1) and (4.2), we obtain

$$\begin{aligned}
&\frac{\delta}{2}\int_\Omega(|\nabla n_{1\varepsilon\delta}(x,t)|^2-|\nabla n_{1\varepsilon\delta}(x,t_0)|^2)dx\\
&+\frac{1}{m+1}\int_\Omega((|n_{1\varepsilon\delta}(x,t)|^2+\varepsilon)^{\frac{m+1}{2}}-(|n_{1\varepsilon\delta}(x,t_0)|^2+\varepsilon)^{\frac{m+1}{2}})dx\\
&+\int_{t_0}^t\int_\Omega(|\delta\nabla\Delta n_{1\varepsilon\delta}-\nabla(n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}n_{1\varepsilon\delta}|^2+\varepsilon\delta(s+1)|n_{1\varepsilon\delta}|^s|\nabla n_{1\varepsilon\delta}|^2)dxds\\
&+\int_{t_0}^t\int_\Omega((n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}(\varepsilon|n_{1\varepsilon\delta}|^{s+2}+\mu_1|n_{1\varepsilon\delta}|^3+a_1\mu_1n_{1\varepsilon\delta}^2n_{2\varepsilon\delta})\\
&+2\mu_1\delta|n_{1\varepsilon\delta}||\nabla n_{1\varepsilon\delta}|^2)dxds\\
=\;&\delta\int_{t_0}^t\int_\Omega u_{\varepsilon\delta}\nabla n_{1\varepsilon\delta}\Delta n_{1\varepsilon\delta}dxds+\chi\int_{t_0}^t\int_\Omega(n_{1\varepsilon\delta}\frac{1}{(1+c_{\varepsilon\delta})^{m_1}}\nabla c_{\varepsilon\delta}(\delta\nabla\Delta n_{1\varepsilon\delta}\\
&-\nabla(n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}n_{1\varepsilon\delta}))dxds+\int_{t_0}^t\int_\Omega\delta a_1\mu_1n_{2\varepsilon\delta}n_{1\varepsilon\delta}\Delta n_{1\varepsilon\delta}dxds\\
&+\int_{t_0}^t\int_\Omega(\mu_1n_{1\varepsilon\delta}+g_1)(-\delta\Delta n_{1\varepsilon\delta}+(n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}n_{1\varepsilon\delta})dxds\\
\leq\;&\delta\int_{t_0}^t\|u_{\varepsilon\delta}\|_{L^4}\|\nabla n_{1\varepsilon\delta}\|_{L^4}\|\Delta n_{1\varepsilon\delta}\|_{L^2}ds+\int_{t_0}^t\int_\Omega|n_{1\varepsilon\delta}\nabla c_{\varepsilon\delta}|^2dxds\\
&+\frac14\int_{t_0}^t\int_\Omega|\delta\nabla\Delta n_{1\varepsilon\delta}-\nabla(n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}n_{1\varepsilon\delta}|^2dxds+\delta\int_{t_0}^t\int_\Omega|\Delta n_{1\varepsilon\delta}|^2dxds\\
&+\delta a_1\mu_1\int_{t_0}^t\int_\Omega\|n_{2\varepsilon\delta}\|_{L^\infty}\|n_{1\varepsilon\delta}\|_{L^2}\|\Delta n_{1\varepsilon\delta}\|_{L^2}dxds+\delta\int_{t_0}^t\int_\Omega(\mu_1|n_{1\varepsilon\delta}|+g_1)^2dxds\\
&+\frac{\mu}{2}\int_{t_0}^t\int_\Omega(n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}|n_{1\varepsilon\delta}|^3dxds+C\\
\leq\;&\int_{t_0}^t(\delta\|u_{\varepsilon\delta}\|_{H^1}\|n_{1\varepsilon\delta}\|_{L^2}^{\frac14}\|\Delta n_{1\varepsilon\delta}\|_{L^2}^{\frac74}+C\|n_{1\varepsilon\delta}\|_{L^4}^2\|\nabla c_{\varepsilon\delta}\|_{L^4}^2+C\|n_{1\varepsilon\delta}\|_{L^2}^2)ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\int_{t_0}^{t}\int_{\Omega}|\delta\nabla\Delta n_{1\varepsilon\delta}-\nabla(n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}n_{1\varepsilon\delta}|^2dxds\\
&+\delta\int_{t_0}^{t}\int_{\Omega}|\Delta n_{1\varepsilon\delta}|^2dxds+\frac{\mu_1}{2}\int_{t_0}^{t}\int_{\Omega}(n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}|n_{1\varepsilon\delta}|^3dxds+C\\
\le{}& 2\delta\int_{t_0}^{t}\|\Delta n_{1\varepsilon\delta}\|_{L^2}^2ds+\int_{t_0}^{t}(\frac{\mu_1}{4}\|n_{1\varepsilon\delta}\|_{L^3}^3+C\|\nabla c_{\varepsilon\delta}\|_{H^1}^6+\|n_{1\varepsilon\delta}\|_{L^2}^6)ds\\
&+\frac{\mu_1}{2}\int_{t_0}^{t}\int_{\Omega}(n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}|n_{1\varepsilon\delta}|^3dxds+C\\
&+\frac{1}{4}\int_{t_0}^{t}\int_{\Omega}|\delta\nabla\Delta n_{1\varepsilon\delta}-\nabla(n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}n_{1\varepsilon\delta}|^2dxds\\
\le{}&\int_{t_0}^{t}(2\delta\|\Delta n_{1\varepsilon\delta}\|_{L^2}^2+\|n_{1\varepsilon\delta}\|_{L^3}^3+\frac{1}{4}\int_{t_0}^{t}\int_{\Omega}|\delta\nabla\Delta n_{1\varepsilon\delta}-\nabla(n_{1\varepsilon\delta}+\varepsilon)^{\frac{m-1}{2}}n_{1\varepsilon\delta}|^2)dxds\\
&+\frac{\mu_1}{2}\int_{t_0}^{t}\int_{\Omega}(n_{1\varepsilon\delta}+\varepsilon)^{\frac{m-1}{2}}|n_{1\varepsilon\delta}|^3dxds+C.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{\delta}{2}\int_{\Omega}(|\nabla n_{1\varepsilon\delta}(x,t)|^2-|\nabla n_{1\varepsilon\delta}(x,t_0)|^2)dx+\frac{1}{m+1}\int_{\Omega}\Big((|n_{1\varepsilon\delta}(x,t)|^2+\varepsilon)^{\frac{m+1}{2}}\\
&-(|n_{1\varepsilon\delta}(x,t_0)|^2+\varepsilon)^{\frac{m+1}{2}}\Big)dx+\int_{t_0}^{t}\int_{\Omega}\Big(\frac{3}{4}|\delta\nabla\Delta n_{1\varepsilon\delta}-\nabla(n_{1\varepsilon\delta}+\varepsilon)^{\frac{m-1}{2}}n_{1\varepsilon\delta}|^2\\
&+\varepsilon\delta(s+1)|n_{1\varepsilon\delta}|^s|\nabla n_{1\varepsilon\delta}|^2\Big)dxds+\frac{1}{2}\int_{t_0}^{t}\int_{\Omega}((n_{1\varepsilon\delta}^2+\varepsilon)^{\frac{m-1}{2}}(\varepsilon|n_{1\varepsilon\delta}|^{s+2}+\mu_1|n_{1\varepsilon\delta}|^3\\
&+a_1\mu_1n_{1\varepsilon\delta}^2n_{2\varepsilon\delta}+2\mu_1\delta|n_{1\varepsilon\delta}||\nabla n_{1\varepsilon\delta}|^2)dxds\\
\le{}&2\delta\int_{t_0}^{t}\|\Delta n_{1\varepsilon\delta}\|_{L^2}^2ds+\frac{\mu_1}{4}\int_{t_0}^{t}\|n_{1\varepsilon\delta}\|_{L^3}^3ds+C_2.
\end{aligned}
$$

Then combining with (4.2) and using Lemma 2.6, this lemma is complete. □

By (4.2), we can get that

$$\|\delta\Delta n_{1\varepsilon\delta}\|_{L^2(Q_T)}^2\le C\delta\to 0.$$

Combined with the above three lemmas, and letting $\delta\to 0$, then we have that

$$
\begin{aligned}
&u_{\varepsilon\delta}\to u_\varepsilon,\ \text{in } L^4(Q_T),\\
&u_{\varepsilon\delta}\rightharpoonup u_\varepsilon,\ \text{in } W_2^{2,1}(Q_T),\\
&c_{\varepsilon\delta}\to c_\varepsilon,\ \text{in } C(\bar{Q}_T),\\
&c_{\varepsilon\delta}\rightharpoonup c_\varepsilon,\ \text{in } W_2^{2,1}(Q_T),\\
&n_{2\varepsilon\delta}\to n_{2\varepsilon},\ \text{in } L^4(Q_T),\\
&n_{2\varepsilon\delta}\rightharpoonup n_{2\varepsilon},\ \text{in } W_2^{2,1}(Q_T),\\
&n_{1\varepsilon\delta}\to n_{1\varepsilon},\ \text{in } L^p(Q_T),\ \text{for any } p<s+m+1,\\
&\nabla n_{1\varepsilon\delta}\rightharpoonup \nabla n_{1\varepsilon},\ \text{in } L^2(Q_T),
\end{aligned}
$$

$$\delta \Delta n_{1\varepsilon\delta} \to 0, \text{ in } L^2(Q_T).$$

Then we can easily get that $(n_{1\varepsilon}, n_{2\varepsilon}, u_\varepsilon, c_\varepsilon, \pi_\varepsilon)$ is the solution of the following problem:

$$\begin{cases} (n_1)_t + u \cdot \nabla n_1 - \Delta((n_1^2 + \varepsilon)^{\frac{m-1}{2}} n_1) + \varepsilon |n_1|^s n_1 \\ = -\chi_1 \nabla \cdot (n_1 \frac{1}{(1+c)^{m_1}} \nabla c) + \mu_1 |n_1|(1 - n_1 - a_1 n_2) + g_1(x,t), \text{ in } Q, \\ (n_2)_t + u \cdot \nabla n_2 \\ = \Delta n_2 - \chi_2 \nabla \cdot (n_2 \frac{1}{(1+c)^{m_2}} \nabla c) + \mu_2 n_2 (1 - a_2 n_1 - n_2) + g_2(x,t), \text{ in } Q, \\ c_t + u \cdot \nabla c = \Delta c - (\alpha_1 n_{1+} + \alpha_2 n_2) c + \beta n_{1+}, \text{ in } Q, \\ u_t = \Delta u - \nabla \pi + (\gamma n_1 + \theta n_2) \nabla \varphi, \text{ in } Q, \\ \nabla \cdot u = 0, \text{ in } Q, \\ \frac{\partial n}{\partial \nu}|_{\partial\Omega} = \frac{\partial c}{\partial \nu}|_{\partial\Omega} = 0, u|_{\partial\Omega} = 0, \end{cases} \tag{4.4}$$

and $(n_{1\varepsilon}, n_{2\varepsilon}, c_\varepsilon, u_\varepsilon, \pi_\varepsilon)$ satisfies (4.1)–(4.3). Next, we consider the second level approximation. First, we make some energy estimates independent of ε.

Proposition 4.1. *Assume that $m \geq \frac{6}{5}$, $g_1, g_2 \geq 0$ with $g_1, g_2, \nabla\varphi \in L_T^\infty(Q)$. Then problem (4.4) admits a time periodic solution $(n_{1\varepsilon}, n_{2\varepsilon}, c_\varepsilon, u_\varepsilon, \pi_\varepsilon)$, such that $n_{1\varepsilon}, n_{2\varepsilon}, c_\varepsilon \geq 0$, and*

$$\begin{aligned} &\sup_t \{\|u_\varepsilon\|_{H^1}^2 + \|c_\varepsilon\|_{H^1}^2\} \\ &+ \int_0^T (\|u_\varepsilon\|_{H^2}^2 + \|\pi_\varepsilon\|_{H^1}^2 + \|c_\varepsilon\|_{H^2}^2 + \|n_{2\varepsilon}\|_{H^1}^2 + \|(u_{\varepsilon t}, c_{\varepsilon t})\|_{L^2}^2) dt \leq C, \\ &\sup_t \{\|u_\varepsilon\|_{L^\infty} + \|n_{1\varepsilon}\|_{L^\infty} + \|n_{2\varepsilon}\|_{L^\infty} + \|c_\varepsilon\|_{W^{1,\infty}}\} \leq C, \\ &\sup_t \int_\Omega n_{1\varepsilon} \ln n_{1\varepsilon} dx + \int_0^T \int_\Omega (n_{1\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} |\nabla \sqrt{n}_{1\varepsilon}|^2 dx dt \leq C, \\ &\sup_t \int_\Omega |\nabla((n_{1\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} n_{1\varepsilon})|^2 dx + \int_0^T \int_\Omega (n_{1\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} \left|\frac{\partial n_{1\varepsilon}}{\partial t}\right|^2 dx dt \leq C, \end{aligned} \tag{4.5}$$

where C is independent of ε. For simplicity, we still denote the solution of (4.4) by (n_1, n_2, c, u, π).

Lemma 4.3. *Assume $g_1, g_2 \geq 0$ with $g_1, g_2, \nabla\varphi \in L_T^\infty(Q)$, and let (n_1, n_2, c, u, π) be a time periodic solution of (4.4). Then we have $n_1, n_2, c \geq 0$ and*

$$\begin{aligned} &\sup_t \{\|n_1\|_{L^1} + \|n_2\|_{L^\infty} + \|u\|_{H^1}^2 + \|c\|_{H^1}^2 + \|c\|_{L^\infty}\} \\ &+ \int_0^T (\|n_1\|_{L^2}^2 + \|n_2\|_{H^1}^2 + \|u\|_{H^2}^2 + \|c\|_{H^2}^2 + \|\pi\|_{H^1}^2 + \|(u_t, c_t)\|_{L^2}^2) dt \leq C. \end{aligned} \tag{4.6}$$

Proof. By Lemmas 3.2–3.4, we can get $c, n_1, n_2 \geq 0$. Then integrating the first and second equation of (4.4) over Ω, and combining with (3.19)–(3.22) and (3.28), the proof is complete. □

Lemma 4.4. *Assume $m > 1$, $g_1 \geq 0$ with $g_1, \nabla\varphi \in L_T^\infty(Q)$. Let (n_1, n_2, c, u, π) be a time periodic solution of (4.4). Then there exists $\rho \in (0,1)$ such that*

$$\sup_t \int_\Omega n_1^{\rho+1} dx + \int_0^T \int_\Omega n_1^{m+\rho-2} |\nabla n_1|^2 dx dt + \int_0^T \int_\Omega n_1^{\rho+2} dx dt \leq C, \tag{4.7}$$

where C is independent of ε.

Proof. Testing the first equation of (4.4) by n_1^q with $0 < q < 1$ and integrating it over $\Omega \times (t_0, t)$ for any $t_0 < t \le t_0 + T$, we get

$$\begin{aligned}
&\frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t)dx + q\int_{t_0}^t\int_\Omega (n_1^2+\varepsilon)^{\frac{m-1}{2}} n_1^{q-1}|\nabla n_1|^2 dxds \\
&+ \varepsilon\int_{t_0}^t\int_\Omega n_1^{s+1+q}dxds + q(m-1)\int_{t_0}^t\int_\Omega (n_1^2+\varepsilon)^{\frac{m-3}{2}} n_1^2 n_1^{q-1}|\nabla n_1|^2 dxds \\
&+ a_1\mu_1\int_{t_0}^t\int_\Omega n_1^{q+1}n_2 dxds + \mu_1\int_{t_0}^t\int_\Omega n_1^{2+q}dxds \\
\le\ & \frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t_0)dx + q\int_{t_0}^t\int_\Omega n_1^q \frac{1}{(1+c)^{m_1}}\nabla c\nabla n_1 dxds \\
&+ \int_{t_0}^t\int_\Omega (\mu_1 n_1^{q+1} + g_1 n_1^q)dxds \\
\le\ & \frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t_0)dx + \int_{t_0}^t\int_\Omega (\mu_1 n_1^{q+1} + g_1 n_1^q)dxds \\
&- \frac{q}{q+1}\int_{t_0}^t\int_\Omega n_1^{q+1}\frac{1}{(1+c)^{m_1}}\Delta c dxds + \frac{qm_1}{q+1}\int_{t_0}^t\int_\Omega n_1^{q+1}\frac{1}{(1+c)^{m_1+1}}|\nabla c|^2 dxds \\
\le\ & \frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t_0)dx + \frac{\mu_1}{2}\int_{t_0}^t\int_\Omega n_1^{q+2}dxds + \frac{Cq^{q+2}}{\mu_1^{q+1}}\int_{t_0}^t\int_\Omega |\Delta c|^{q+2}dxds \\
&+ \frac{Cq^{q+2}}{\mu_1^{q+1}}\int_{t_0}^t \|\nabla c\|_{L^{2(q+2)}}^{2(q+2)}ds + C \\
\le\ & \frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t_0)dx + \frac{\mu_1}{2}\int_{t_0}^t\int_\Omega n_1^{q+2}dxds + \frac{Cq^{q+2}}{\mu_1^{q+1}}\int_{t_0}^t\int_\Omega |\Delta c|^{q+2}dxds \\
&+ \frac{Cq^{q+2}}{\mu_1^{q+1}}\int_{t_0}^t \|c\|_{L^\infty}^{q+2}\|\Delta c\|_{L^{(q+2)}}^{(q+2)}ds + C \\
\le\ & \frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t_0)dx + \frac{\mu_1}{2}\int_{t_0}^t\int_\Omega n_1^{q+2}dxds + \frac{Cq^{q+2}}{\mu_1^{q+1}}\int_{t_0}^t\int_\Omega |\Delta c|^{q+2}dxds + C.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t)dx + q\int_{t_0}^t\int_\Omega (n_1^2+\varepsilon)^{\frac{m-1}{2}} n_1^{q-1}|\nabla n_1|^2 dxds + \frac{\mu_1}{2}\int_{t_0}^t\int_\Omega n_1^{2+q}dxds \\
&\le \frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t_0)dx + \frac{Cq^{q+2}}{\mu_1^{q+1}}\int_{t_0}^t\int_\Omega |\Delta c|^{q+2}dxds + C.
\end{aligned}$$

Because of Lemma 2.6, we get

$$\begin{aligned}
&\sup_t \int_\Omega n_1^{q+1}dx + q\int_0^T\int_\Omega n_1^{m+q-2}|\nabla n_1|^2 dxdt + \frac{\mu_1}{2}\int_0^T\int_\Omega n_1^{q+2}dxdt \\
&\le \frac{Cq^{q+2}}{\mu_1^{q+1}}\int_{t_0}^t\int_\Omega |\Delta c|^{q+2}dxds + C. \qquad (4.8)
\end{aligned}$$

We choose a proper small q, and use $c \in L^\infty(Q)$, $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \geq 1$. By Lemma 2.3 in [20], we arrive that

$$
\begin{aligned}
&\frac{Cq^{q+2}}{\mu_1^{q+1}} \int_0^T \int_\Omega |\Delta c|^{q+2} dxdt \leq \frac{Cq^{q+2}}{\mu_1^{q+1}} \int_0^T \int_\Omega (n_1^{q+2} + |u\nabla c|^{q+2} + (n_1 c)^{q+2}) dxdt \\
\leq\ & \frac{Cq^{q+2}}{\mu_1^{q+1}} \int_0^T (\|n_1\|_{L^{q+2}}^{q+2} + \|u\|_{L^{2(q+2)}}^{q+2} \|\nabla c\|_{L^{2(q+2)}}^{q+2}) dt \\
\leq\ & \frac{Cq^{q+2}}{\mu_1^{q+1}} \int_0^T (\|n_1\|_{L^{q+2}}^{q+2} + \|u\|_{H^1}^{q+2} \|\nabla c\|_{L^{2(q+2)}}^{q+2}) dt \\
\leq\ & \frac{Cq^{q+2}}{\mu_1^{q+1}} \int_0^T (\|n_1\|_{L^{q+2}}^{q+2} + \|\nabla c\|_{L^{2(q+2)}}^{q+2}) dt.
\end{aligned}
\tag{4.9}
$$

Thus, invoking the Gagliardo-Nirenberg inequality, we have

$$
\|\nabla c\|_{L^{2(q+2)}}^{q+2} \leq \|\nabla c\|_{L^2}^{(1-\alpha)(q+2)} \|\Delta c\|_{L^{q+2}}^{\alpha(q+2)} \leq \eta \|\Delta c\|_{L^{q+2}}^{q+2} + C_\eta,
\tag{4.10}
$$

where η is sufficiently small. Putting (4.10) into (4.9), we can see

$$
\frac{Cq^{q+2}}{\mu_1^{q+1}} \int_0^T \int_\Omega |\Delta c|^{q+2} dxdt \leq \frac{\tilde{C}q^{q+2}}{\mu_1^{q+1}} \int_0^T \int_\Omega |n_1|^{q+2} dxdt + C.
\tag{4.11}
$$

Recalling (4.8), we get

$$
\begin{aligned}
&\sup_t \int_\Omega n_1^{q+1} dx + q \int_0^T \int_\Omega n_1^{m+q-2} |\nabla n_1|^2 dxdt + \frac{\mu_1}{2} \int_0^T \int_\Omega n_1^{q+2} dxdt \\
\leq\ & \frac{\tilde{C}q^{q+2}}{\mu_1^{q+1}} \int_0^T \int_\Omega |n_1|^{q+2} dxdt + C,
\end{aligned}
\tag{4.12}
$$

where $\tilde{C}$ is independent of μ_1 and q. So there exists $\rho > 0$ such that $\frac{\tilde{C}\rho^{\rho+2}}{\mu_1^{\rho+1}} < \frac{\mu_1}{4}$, and this proof is complete. □

Lemma 4.5. *Let $m > 1$, $0 \leq g_1, \nabla\varphi \in L_T^\infty(Q)$. Assume that (n_1, n_2, c, u, π) is a time periodic solution of (4.4) with periodic T. If there exists $q > 2$, such that*

$$
\int_0^T \|n_1\|_{L^q}^q dt \leq C,
$$

then for any $r < \frac{3q}{(5-q)_+}$, we get

$$
\sup_t \|\nabla c\|_{L^r}^r + \int_0^T \int_\Omega |\nabla |\nabla c|^{\frac{r}{2}}|^2 dxdt \leq C,
\tag{4.13}
$$

and

$$
\int_\Omega \|\nabla c\|_{L^{\frac{5r}{3}}}^{\frac{5r}{3}} dt \leq C,
\tag{4.14}
$$

where C is independent of ε.

Proof. Applying ∇ to the third equation of (4.4), multiplying the equation by $|\nabla c|^{r-2}\nabla c\chi_{[t_0,t]}$ for any $r \ge 2$, $t_0 < t \le t_0 + T$, and combining with Lemma 2.1, Lemma 4.3, $\|c\|_{L^\infty} \le C$, and $\|n_2\|_{L^\infty} \le C$, we get

$$
\begin{aligned}
&\frac{1}{r}\int_\Omega |\nabla c(x,t)|^r dx + \int_{t_0}^t\int_\Omega (|\nabla c|^{r-2}|\Delta c|^2 + (r-2)|\nabla c|^{r-2}(\nabla|\nabla c|)^2)dxds \\
=& \frac{1}{r}\int_\Omega |\nabla c(x,t_0)|^r dx - \int_{t_0}^t\int_\Omega (u\cdot\nabla c)\nabla\cdot(|\nabla c|^{r-2}\nabla c)dxds \\
&+ \int_{t_0}^t\int_\Omega (\alpha_1 n_1 c + \alpha_2 n_2 c - \beta n_1)\nabla\cdot(|\nabla c|^{r-2}\nabla c)dxds \\
&+ \int_{t_0}^t ds\int_{\partial\Omega}\frac{\partial|\nabla c|}{\partial\mathbf{n}}|\nabla c|^{r-2}|\nabla c|dS \\
=& \frac{1}{r}\int_\Omega |\nabla c(x,t_0)|^r dx + \int_{t_0}^t\int_\Omega (\alpha_1 n_1 c + \alpha_2 n_2 c - \beta n_1 + u\cdot\nabla c)|\nabla c|^{r-2}\Delta c dxds \\
&+ \int_{t_0}^t\int_\Omega (\alpha_1 n_1 c + \alpha_2 n_2 c - \beta n_1 + u\cdot\nabla c)(r-2)|\nabla c|^{r-3}\nabla c\nabla|\nabla c|dxds \\
&+ \kappa\int_{t_0}^t ds\int_{\partial\Omega}|\nabla c|^r dS \\
\le& \frac{1}{r}\int_\Omega |\nabla c(x,t_0)|^r dx + \frac{1}{4}\int_{t_0}^t\int_\Omega (|\nabla c|^{r-2}|\Delta c|^2 + (r-2)|\nabla c|^{r-2}(\nabla|\nabla c|)^2)dxds \\
&+ C\int_{t_0}^t\int_\Omega |\nabla c|^{r-2}n_1^2 dxds + \int_{t_0}^t\int_\Omega u^2|\nabla c|^r dxds + C\int_{t_0}^t\int_\Omega |\nabla c|^{r-2}n_1^2c^2 dxds + C \\
\le& \frac{1}{r}\int_\Omega |\nabla c(x,t_0)|^r dx + \frac{1}{4}\int_{t_0}^t\int_\Omega (|\nabla c|^{r-2}|\Delta c|^2 + (r-2)|\nabla c|^{r-2}(\nabla|\nabla c|)^2)dxds \\
&+ C\int_{t_0}^t\int_\Omega |\nabla c|^{r-2}n_1^2 dxds + C\int_{t_0}^t \|u\|_{L^6}^2\|\nabla c\|_{L^{\frac{3r}{2}}}^r ds + C.
\end{aligned}
\tag{4.15}
$$

Using the Gagliardo-Nirenberg interpolation inequality and Lemma 4.3, we arrive at

$$
\begin{aligned}
\|\nabla c\|_{L^{\frac{3r}{2}}}^r = \||\nabla c|^{\frac{r}{2}}\|_{L^3}^2 &\le C\||\nabla c|^{\frac{r}{2}}\|_{L^{\frac{4}{r}}}^{\frac{8}{3r}}\|\nabla(|\nabla c|^{\frac{r}{2}})\|_{L^2}^{\frac{2(3r-4)}{3r}} + C\|\nabla c\|_{L^2}^r \\
&\le C\|\nabla(|\nabla c|^{\frac{r}{2}})\|_{L^2}^{\frac{2(3r-4)}{3r}} + C.
\end{aligned}
$$

Noting that $\frac{2(3r-4)}{3r} \le 2$, we further have

$$
C\|u\|_{L^6}^2\|\nabla c\|_{L^{\frac{3r}{2}}}^r \le C\|u\|_{H^1}^2\|\nabla c\|_{L^{\frac{3r}{2}}}^r \le \frac{r-2}{4}\int_\Omega |\nabla c|^{r-2}(\nabla|\nabla c|)^2 dx + C. \tag{4.16}
$$

Putting (4.16) into (4.15), we have

$$
\begin{aligned}
&\frac{1}{r}\int_\Omega |\nabla c(x,t)|^r dx + \frac{1}{2}\int_{t_0}^t\int_\Omega (|\nabla c|^{r-2}|\Delta c|^2 + (r-2)|\nabla c|^{r-2}(\nabla|\nabla c|)^2)dxds \\
&+ \int_{t_0}^t\int_\Omega |\nabla c|^r dxds
\end{aligned}
$$

$$\leq \frac{1}{r}\int_\Omega |\nabla c(x,t_0)|^r dx + C\int_{t_0}^t\int_\Omega |\nabla c|^{r-2}n_1^2 dxds + C$$
$$\leq \frac{1}{r}\int_\Omega |\nabla c(x,t_0)|^r dx + \frac{1}{2}\int_{t_0}^t \|\nabla c\|_{L^{R(r-2)}}^{R(r-2)} ds + C\int_{t_0}^t \|n_1\|_{L^{\frac{2R}{R-1}}}^{\frac{2R}{R-1}} ds + C. \tag{4.17}$$

First, we let $R = \frac{q}{q-2}$, $r = q$,

$$\frac{1}{r}\int_\Omega |\nabla c(x,t)|^q dx + \frac{1}{2}\int_{t_0}^t\int_\Omega (|\nabla c|^{q-2}|\Delta c|^2 + (r-2)|\nabla c|^{q-2}(\nabla|\nabla c|)^2) dxds$$
$$+\frac{1}{2}\int_{t_0}^t\int_\Omega |\nabla c|^q dxds \leq \frac{1}{r}\int_\Omega |\nabla c(x,t_0)|^q dx + C\int_{t_0}^t \|n_1\|_{L^q}^q ds + C.$$

Using Lemma 2.6, we can easily get

$$\sup_t \|\nabla c\|_{L^q}^q + \int_0^T\int_\Omega (\nabla(|\nabla c|^{\frac{q}{2}}))^2 dxdt \leq C. \tag{4.18}$$

Next, we will use a recurrence method to solve this lemma. If

$$\sup_t \|\nabla c\|_{L^{r_k}}^{r_k} + \int_0^T\int_\Omega (\nabla(|\nabla c|^{\frac{r_k}{2}}))^2 dxdt \leq C, \tag{4.19}$$

then

$$\sup_t \|\nabla c\|_{L^{r_{k+1}}}^{r_{k+1}} + \int_0^T\int_\Omega (\nabla(|\nabla c|^{\frac{r_{k+1}}{2}}))^2 dxdt \leq C, \tag{4.20}$$

where $r_{k+1} = \frac{5(q-2)}{3q}r_k + 2$. By the Gagliardo-Nirenberg interpolation inequality, we get

$$\||\nabla c|^{\frac{r_k}{2}}\|_{L^{2+\frac{4}{3}}}^{2+\frac{4}{3}} \leq \||\nabla c|^{\frac{r_k}{2}}\|_{L^2}^2 \|\nabla(|\nabla c|^{\frac{r_k}{2}})\|_{L^2}^{\frac{4}{3}} + \||\nabla c|^{\frac{r_k}{2}}\|_{L^2}^{2+\frac{4}{3}}$$
$$\leq C(1 + \|\nabla(|\nabla c|^{\frac{r_k}{2}})\|_{L^2}^2),$$

which implies that

$$\int_0^T \|\nabla c\|_{\frac{5r_k}{3}}^{\frac{5r_k}{3}} dt = \int_0^T \||\nabla c|^{\frac{r_k}{2}}\|_{L^{2+\frac{4}{3}}}^{2+\frac{4}{3}} dt \leq C(1 + \int_0^T \|\nabla(|\nabla c|^{\frac{r_k}{2}})\|_{L^2}^2 dt). \tag{4.21}$$

Taking $R = \frac{q}{q-2}$, $r = r_{k+1} = \frac{5(q-2)}{3q}r_k + 2$ in (4.17), we get

$$\frac{3q}{6q+5(q-2)r_k}\int_\Omega |\nabla c(x,t)|^{2+\frac{5(q-2)}{3q}r_k} dx + \frac{1}{2}\int_{t_0}^t\int_\Omega |\nabla c|^{\frac{5(q-2)}{3q}r_k}|\Delta c|^2 dxds$$
$$+\frac{5(q-2)}{6q}\int_{t_0}^t\int_\Omega |\nabla c|^{\frac{5(q-2)}{3q}r_k}(\nabla|\nabla c|)^2 dxds + \int_{t_0}^t\int_\Omega |\nabla c|^{2+\frac{5(q-2)}{3q}r_k} dxds$$
$$\leq \frac{3q}{6q+5(q-2)r_k}\int_\Omega |\nabla c(x,t_0)|^{2+\frac{5(q-2)}{3q}r_k} dx + \frac{1}{4}\int_{t_0}^t \|\nabla c\|_{\frac{5r_k}{3}}^{\frac{5r_k}{3}} ds$$

$$+ C\int_{t_0}^{t}\|n_1\|_{L^q}^q ds + \hat{C}. \tag{4.22}$$

So (4.20) holds. Next we consider the sequence $r_{k+1} = \frac{5(q-2)}{3q}r_k + 2$, with $r_1 = q > 2$. First, we can see $r_2 > r_1$, and then r_k is a monotonically increasing sequence. What is more, if $2 < q < 5$, we have $0 < \frac{5(q-2)}{3q} < 1$ and $r_1 \le \frac{3q}{5-q}$, and it is easy to show that r_k increases to $\frac{3q}{5-q}$. While if $q \ge 5$, then $\frac{5(q-2)}{3q} \ge 1$, and it implies that $r_k \to +\infty$. Therefore, (4.13) is correct. What is more, because of (4.21), it is easy to see that (4.14) is correct. □

Lemma 4.6. *Assume $m \ge \frac{6}{5}$, $g_1, g_2 \ge 0$ with $g_1, g_2, \nabla\varphi \in L_T^\infty(Q)$. Let (n_1, n_2, c, u, π) be a time periodic solution of (4.4). If for $q_i > 2$, we have*

$$\int_0^T\int_\Omega n^{q_i}dxds \le C_i,$$

then for any $q < \frac{m}{2}\frac{5q_i}{(5-q_i)_+} - 2$, we can get

$$\sup_t\int_\Omega n_1^{q+1}dx + \int_0^T\int_\Omega n_1^{m+q-2}|\nabla n_1|^2dxds + \int_0^T\int_\Omega n_1^{q+2}dxds \le C_{i+1}(q), \tag{4.23}$$

where $C_{i+1}(q)$ is independent of ε.

Proof. First, by Lemma 4.5, we have

$$\int_0^T\|\nabla c\|_{L^r}^r ds \le C_i(r), \tag{4.24}$$

for any $r < \frac{5q_i}{(5-q_i)_+}$. Testing the first equation of (4.4) by n_1^q and integrating it over $\Omega\times(t_0,t)$ for $t_0 < t \le t_0 + T$, we have

$$\begin{aligned}
&\frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t)dx + \int_{t_0}^t\int_\Omega(qn_1^{m+q-2}|\nabla n_1|^2 + \varepsilon n_1^{s+1+q} + \mu_1 n_1^{2+q}\\
&+ a_1\mu_1 n_2 n_1^{q+1})dxds\\
\le\ &\frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t_0)dx + \int_{t_0}^t\int_\Omega(qn_1^q\frac{1}{(1+c)^{m_1}}\nabla c\nabla n_1 + \mu n_1^{1+q} + g_1 n_1^q)dxds\\
\le\ &\frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t_0)dx\\
&+ \int_{t_0}^t\int_\Omega(\frac{q}{2}(n_1^{m+q-2}|\nabla n_1|^2 + n_1^{q+2-m}|\nabla c|^2) + \frac{\mu}{4}n_1^{q+2})dxds + C\\
\le\ &\frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t_0)dx + \frac{q}{2}\int_{t_0}^t\int_\Omega n_1^{m+q-2}|\nabla n_1|^2dxds + C\int_{t_0}^t\int_\Omega|\nabla c|^{\frac{2(q+2)}{m}}dxds\\
&+ \frac{\mu}{2}\int_{t_0}^t\int_\Omega n_1^{q+2}dxds + C,
\end{aligned}$$

which implies

$$\frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t)dx+\frac{q}{2}\int_{t_0}^t\int_\Omega n_1^{m+q-2}|\nabla n_1|^2dxds+\frac{\mu}{2}\int_{t_0}^t\int_\Omega n_1^{2+q}dxds$$
$$+\int_{t_0}^t\int_\Omega a_1\mu_1 n_2 n_1^{q+1}dxds\le\frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t_0)dx+C\int_{t_0}^t\int_\Omega|\nabla c|^{\frac{2(q+2)}{m}}dxds+C.$$

Because $\frac{2(q+2)}{m}<\frac{5q_i}{(5-q_i)_+}$, using Lemma 2.6, it is easy to get (4.23). □

Lemma 4.7. *Assume $m\ge\frac{6}{5}$, $g_1,g_2\ge 0$ with $g_1,g_2,\nabla\varphi\in L_T^\infty(Q)$, and (n_1,n_2,u,c,π) is a time periodic solution of (4.4). We have*

$$\sup_t\{\|u\|_{L^\infty}+\|n_1\|_{L^\infty}+\|c\|_{W^{1,\infty}}\}\le C,\tag{4.25}$$

where C is independent of ε.

Proof. Let $P_1=\rho+2$, $P_{i+1}=\frac{m}{2}\frac{5P_i}{(5-P_i)_+}$, where $\rho>0$ is defined in Lemma 4.4. We have

$$\frac{P_{i+1}}{P_i}>\frac{m}{2}\frac{5}{5-(\rho+2)}=\frac{m}{2}\frac{5}{3-\rho}\ge\frac{6}{6-2\rho}>1.$$

By Lemma 4.4, we have

$$\int_0^T\int_\Omega n_1^{\rho+2}\le C.$$

Thus there must exist a unique $K>0$ such that $P_{K-1}<5$ and $P_K\ge 5$. Using Lemma 4.6 repeatedly from $P_1=2+\rho$, we have

$$\int_0^T\int_\Omega n_1^4dxds\le C.$$

By Lemma 4.6 again, we have

$$\sup_t\int_\Omega n_1^{q+1}dx+\int_0^T\int_\Omega n_1^{m+q-2}|\nabla n_1|^2dxds+\int_0^T\int_\Omega n_1^{q+2}dxds\le C(q).\tag{4.26}$$

Using the standard smoothing properties of the Stokes semigroup, and combining Lemmas 2.3 and 4.3, there exists $\lambda>0$ such that

$$\begin{aligned}\|u\|_{L^\infty}&=\int_{-\infty}^t\|e^{-(t-s)A}P((\gamma n_1(s)+\theta n_2(s))\nabla\varphi(s)\|_{L^\infty}ds\\&\le\int_{-\infty}^t e^{-\lambda(t-s)}(t-s)^{-\frac{3}{4}}(\|(\gamma n_1(s)+\theta n_2(s))\nabla\varphi\|_{L^2}ds\\&\le C\int_{-\infty}^t e^{-\lambda(t-s)}(t-s)^{-\frac{3}{4}}(\|n_1(s)\|_{L^2}+\|n_2(s)\|_{L^2})\|\nabla\varphi\|_{L^\infty}ds\\&\le C.\end{aligned}\tag{4.27}$$

In the same way, combining Lemmas 2.3 and 4.3, (4.26), and (4.27), we have

$$\|\nabla c(\cdot,t)\|_{L^\infty}\le\int_{-\infty}^t e^{-(t-s)}(t-s)^{-\frac{3}{8}-\frac{1}{2}}\|c+\beta n_1-u\nabla c-\alpha_1 n_1 c-\alpha_2 n_2 c\|_{L^4}ds$$

$$
\begin{aligned}
&\leq \int_{-\infty}^{t} e^{-(t-s)}(t-s)^{-\frac{7}{8}}(\|n_1\|_{L^4}+\|u\nabla c\|_{L^4}+\|n_1 c\|_{L^4}+\|n_2 c\|_{L^4})ds\\
&\leq \int_{-\infty}^{t} e^{-(t-s)}(t-s)^{-\frac{7}{8}}(C\|n_1\|_{L^4}+\|u\|_{L^\infty}\|\nabla c\|_{L^4})ds\\
&\leq \sup_s(\|n_1\|_{L^4}+\|u\|_{L^\infty}\|\nabla c\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{L^\infty}^{\frac{1}{2}})\int_0^\infty e^{-s}s^{-\frac{7}{8}}ds\\
&\leq C(1+\sup_s\|\nabla c\|_{L^\infty}^{\frac{1}{2}}).
\end{aligned}
\tag{4.28}
$$

Therefore, we get

$$
\sup_t\|\nabla c(\cdot,t)\|_{L^\infty}\leq C. \tag{4.29}
$$

For n_1, we use the Morse iterative method to complete the proof. Multiplying the first equation of (4.4) by n_1^q, and integrating it over $\Omega\times(t_0,t)$, for any $t_0<t\leq t_0+T$, we get

$$
\begin{aligned}
&\frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t)dx+\int_{t_0}^t\int_\Omega(q(n_1^2+\varepsilon)^{\frac{m-1}{2}}n_1^{q-1}|\nabla n_1|^2\\
&+\mu_1 n_1^{q+2}+a_1\mu_1 n_2 n_1^{q+1}+n_1^{q+1})dxds\\
\leq{}&\frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t_0)dx+\int_{t_0}^t\int_\Omega\left(\frac{qn_1^q}{(1+c)^{m_1}}\nabla c\nabla n_1+(\mu_1+1)n_1^{1+q}+g_1 n_1^q\right)dxds\\
\leq{}&\frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t_0)dx+\frac{q}{2}\int_{t_0}^t\int_\Omega n_1^{m+q-2}|\nabla n_1|^2dxds+Cq\int_{t_0}^t\int_\Omega n_1^{q+2-m}dxds\\
&+(1+\mu_1)\int_{t_0}^t\int_\Omega n_1^{1+q}dxds+C\int_{t_0}^t\int_\Omega n_1^q dxds\\
\leq{}&\frac{1}{q+1}\int_\Omega n_1^{q+1}(x,t_0)dx+\frac{q}{2}\int_{t_0}^t\int_\Omega n_1^{m+q-2}|\nabla n_1|^2dxds+Cq\int_{t_0}^t\int_\Omega n_1^{q+2-m}dxds\\
&+\frac{\mu_1}{2}\int_{t_0}^t\int_\Omega n_1^{2+q}dxds+C\int_{t_0}^t\int_\Omega n_1^q dxds,
\end{aligned}
$$

which means

$$
\begin{aligned}
&\int_\Omega n_1^{q+1}(x,t)dx+\frac{4q(q+1)}{(m+q)^2}\int_{t_0}^t\int_\Omega|\nabla n_1^{\frac{m+q}{2}}|^2dxds+\int_{t_0}^t\int_\Omega n_1^{q+1}dxds\\
\leq{}&\int_\Omega n_1^{q+1}(x,t_0)dx+Cq^2\int_{t_0}^t\int_\Omega n_1^{q+2-m}dxds+Cq\int_{t_0}^t\int_\Omega n_1^q dxds\\
={}&I_1+I_2+I_3.
\end{aligned}
\tag{4.30}
$$

For I_2, we have

$$
\begin{aligned}
&Cq^2\int_\Omega n_1^{q+2-m}dx=Cq^2\|n_1^{\frac{m+q}{2}}\|_{L^{\frac{2(q+2-m)}{m+q}}}^{\frac{2(q+2-m)}{m+q}}\\
\leq{}&Cq^2\|\nabla n_1^{\frac{m+q}{2}}\|_{L^2}^{\frac{6(q-2m+3)}{6m+5q-1}}\|n_1^{\frac{m+q}{2}}\|_{L^{\frac{1+q}{m+q}}}^{\frac{4(q+1)(q+2m-1)}{(q+m)(6m+5q-1)}}+Cq^2\|n_1\|_{L^{\frac{1+q}{2}}}^{q+2-m}
\end{aligned}
$$

$$\leq \sigma\|\nabla n_1^{\frac{m+q}{2}}\|_{L^2}^2 + Cq^5\|n_1\|_{L^{\frac{1+q}{2}}}^{\frac{(q+1)(q+2m-1)}{6m+q-5}} + Cq^2\|n_1\|_{L^{\frac{1+q}{2}}}^{q+2-m},$$

where $\sigma > 0$ is small. For I_3, we have

$$\begin{aligned}
Cq\int_\Omega n_1^q dx &= Cq\|n_1^{\frac{m+q}{2}}\|_{L^{\frac{2q}{m+q}}}\\
&\leq Cq\|\nabla n_1^{\frac{m+q}{2}}\|_{L^2}^{\frac{6(q-1)}{5q+6m-1}}\|n_1^{\frac{m+q}{2}}\|_{L^{frac1+qm+q}}^{\frac{2(q+1)(2q+3m)}{(5q+6m-1)(q+m)}} + Cq\|n_1\|_{L^{\frac{1+q}{2}}}^q\\
&\leq \sigma\|\nabla n_1^{\frac{m+q}{2}}\|_{L^2}^2 + Cq^{\frac{5}{2}}\|n_1\|_{L^{\frac{1+q}{2}}}^{\frac{(q+1)(2q+3m)}{2q+6m+2}} + Cq\|n_1\|_{L^{\frac{1+q}{2}}}^q.
\end{aligned}$$

Putting I_2, I_3 in (4.30), and letting σ be appropriately small, we can get

$$\begin{aligned}
&\int_\Omega n_1^{q+1}(x,t)dx + \frac{2q(q+1)}{(m+q)^2}\int_{t_0}^t\int_\Omega|\nabla n_1^{\frac{m+q}{2}}|^2dxds + \int_{t_0}^t\int_\Omega n_1^{q+1}dxds\\
\leq &\int_\Omega n_1^{q+1}(x,t_0)dx + Cq^5\|n_1\|_{L^{\frac{1+q}{2}}}^{\frac{(q+1)(q+2m-1)}{6m+q-5}} + Cq^2\|n_1\|_{L^{\frac{1+q}{2}}}^{q+2-m}\\
&+ Cq^{\frac{5}{2}}\|n_1\|_{L^{\frac{1+q}{2}}}^{\frac{(q+1)(2q+3m)}{2q+6m+2}} + Cq\|n_1\|_{L^{\frac{1+q}{2}}}^q.
\end{aligned}$$

Because of $\frac{(q+1)(q+2m-1)}{6m+q-5}, q+2-m, \frac{(q+1)(2q+3m)}{2q+6m+2} \leq q+1$, we have

$$\begin{aligned}
&\int_\Omega n_1^{q+1}(x,t)dx + \frac{2q(q+1)}{(m+q)^2}\int_{t_0}^t\int_\Omega|\nabla n_1^{\frac{m+q}{2}}|^2dxds + \int_{t_0}^t\int_\Omega n_1^{q+1}dxds\\
\leq &\int_\Omega n_1^{q+1}(x,t_0)dx + Cq^5\int_{t_0}^t\|n_1\|_{L^{\frac{1+q}{2}}}^{q+1}ds + C.
\end{aligned}\tag{4.31}$$

Using Lemma 2.6, we have

$$\sup_t\|n_1\|_{L^{q+1}}^{q+1} \leq Cq^5\sup_t\|n_1\|_{L^{\frac{q+1}{2}}}^{q+1} + C.$$

Then let $p_0 = 2$, $p_i = 2p_{i-1} = 2^i p_0$, $M_i = \max\{1, \sup_t\|n_1\|_{L^{p_i}}\}$. We have

$$M_i \leq C^{\frac{1}{p_i}}p_i^{\frac{5}{p_i}}M_{i-1} + C^{\frac{1}{p_i}}.$$

Using the Morse iterative method, we have $M_i \leq C^*$. Then, let $i \to \infty$, and we get that

$$\|n_1\|_{L^\infty} \leq C^*,$$

where C^* is independent of ε. So the proof is complete. □

Proof of Proposition 4.1. Testing the first equation of (4.4) by $1 + \ln n_1$, and integrating it over $\Omega \times (t_0, t)$ for any $t_0 < t \leq t_0 + T$, we have

$$\int_\Omega n_1(x,t)\ln n_1(x,t)dx + \int_{t_0}^t\int_\Omega\frac{(n_1^2+\varepsilon)^{\frac{m-1}{2}}}{n_1}|\nabla n_1|^2dxds$$

$$= \int_\Omega n_1(x,t_0)\ln n_1(x,t_0)dx - \chi_1\int_{t_0}^t\int_\Omega\left(\frac{n_1}{(1+c)^{m_1}}\Delta c - \frac{m_1 n_1}{(1+c)^{m_1+1}}|\nabla c|^2\right)dxds$$
$$+\int_{t_0}^t\int_\Omega[g_1+\mu_1(n_1-n_1^2-a_1n_1n_2)-\varepsilon n_1^{s+1}](1+\ln n_1)dxds. \tag{4.32}$$

Next we talk about $x^\alpha \ln x$, for $x > 0$ and $\alpha \geq 1$. If $x \leq 1$, we have

$$x^\alpha \ln x < x^{\alpha+1}.$$

If $0 < x < 1$, we have

$$-\frac{1}{\alpha e} \leq x^\alpha \ln x < 0.$$

Therefore, $|x^\alpha \ln x| \leq \max\{\frac{1}{\alpha e}, x^{\alpha+1}\}$. Recalling (4.32) and combining with $\|n_1\|_{L^\infty} \leq C^*$, we have

$$\int_\Omega n_1(x,t)\ln n_1(x,t)dx + \int_{t_0}^t\int_\Omega\frac{(n_1^2+\varepsilon)^{\frac{m-1}{2}}}{n_1}|\nabla n_1|^2dxds$$
$$\leq \int_\Omega n_1(x,t_0)\ln n_1(x,t_0)dx + \int_{t_0}^t\int_\Omega|\Delta c|^2dxds + C\int_{t_0}^t\int_\Omega|\nabla c|^2dxds + C.$$

By Lemma 2.6, we derive

$$\sup_t\int_\Omega n_1\ln n_1dx + \int_0^T\int_\Omega\frac{(n_1^2+\varepsilon)^{\frac{m-1}{2}}}{n_1}|\nabla n_1|^2dxdt \leq C. \tag{4.33}$$

Multiplying the first equation of (4.4) by $\frac{\partial((n_1^2+\varepsilon)^{\frac{m-1}{2}}n)}{\partial t}\chi_{[t_0,t]}$, we have

$$\frac{1}{2}\int_\Omega|\nabla((n_1^2+\varepsilon)^{\frac{m-1}{2}}n_1)|^2(x,t)dx + \int_{t_0}^t\int_\Omega(n_1^2+\varepsilon)^{\frac{m-1}{2}}|\frac{\partial n_1}{\partial t}|^2dxds$$
$$\leq \frac{1}{2}\int_\Omega|\nabla((n_1^2+\varepsilon)^{\frac{m-1}{2}}n_1)|^2(x,t_0)dx - \int_{t_0}^t\int_\Omega(u\nabla n_1+\varepsilon n_1^4)\frac{\partial((n_1^2+\varepsilon)^{\frac{m-1}{2}}n_1)}{\partial t}dxds$$
$$-\int_{t_0}^t\int_\Omega\Bigg(\nabla\cdot(n_1\frac{1}{(1+c)^{m_1}}\nabla c)\frac{\partial((n_1^2+\varepsilon)^{\frac{m-1}{2}}n_1)}{\partial t}$$
$$-(\mu_1n_1-\mu_1n_1^2-a_1\mu_1n_1n_2+g_1)\frac{\partial((n_1^2+\varepsilon)^{\frac{m-1}{2}}n_1)}{\partial t}\Bigg)dxds$$
$$\leq \frac{1}{2}\int_\Omega|\nabla((n_1^2+\varepsilon)^{\frac{m-1}{2}}n_1)|^2(x,t_0)dx + \frac{1}{2}\int_{t_0}^t\int_\Omega(n_1^2+\varepsilon)^{\frac{m-1}{2}}|\frac{\partial n_1}{\partial t}|^2dxds + C$$
$$+C(\int_{t_0}^t\int_\Omega(n_1^2+\varepsilon)^{\frac{m-1}{2}}|\nabla n_1|^2dxds + \int_{t_0}^t\int_\Omega|\Delta c|^2dxds + \int_{t_0}^t\|\nabla c\|_{L^2}^2\|\Delta c\|_{L^2}^2ds).$$

Using Lemma 2.6, we finally have

$$\sup_t\int_\Omega|\nabla((n_1^2+\varepsilon)^{\frac{m-1}{2}}n_1)|^2dx + \int_0^T\int_\Omega(n_1^2+\varepsilon)^{\frac{m-1}{2}}|\frac{\partial n_1}{\partial t}|^2dxdt \leq C. \tag{4.34}$$

Then we talk about n_2. Multiplying the second equation of (4.4) by n_2, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega n_2^2 dx + \int_\Omega |\nabla n_2|^2 dx + \mu_2 \int_\Omega (a_2 n_1 n_2 + n_2^2) dx \\
=& \chi_2 \int_\Omega n_2 \nabla n_2 \nabla c dx + \mu_2 \int_\Omega n_2^2 dx + \int_\Omega g_2 n_2 dx \\
\leq& C\|n_2\|_{L^2}\|\nabla n_2\|_{L^2}\|\nabla c\|_{L^\infty} + C\|n_2\|_{L^2}^2 + C \leq \frac{1}{2}\|\nabla n_2\|_{L^2}^2 + C\|n_2\|_{L^2}^2 + C,
\end{aligned}$$

so we can easily obtain

$$\sup_t \|n_2\|_{L^2}^2 + \int_0^T \|n_2\|_{H^1}^2 dt \leq C. \tag{4.35}$$

Summing up, the proof of Proposition 4.1 is complete. □

Proof of Theorem 1.1. Letting $\varepsilon \to 0$ (if necessary, we can choose a subsequence) and combining with Proposition 4.1, we get

$$\begin{aligned}
& u_\varepsilon \to u, \text{ in } L^p(Q_T), \text{ for any } p > 1, \quad u_\varepsilon \rightharpoonup u, \text{ in } W_2^{2,1}(Q_T), \\
& c_\varepsilon \to c, \text{ in } L^q(Q_T), \text{ for any } p > 1, \quad c_\varepsilon \rightharpoonup c, \text{ in } W_2^{2,1}(Q_T), \\
& n_{2\varepsilon} \to n_2, \text{ in } L^p(Q_T), \text{ for any } p > 1, \quad n_{2\varepsilon} \rightharpoonup n_2, \text{ in } W_2^{2,0}(Q_T), \\
& \pi_\varepsilon \rightharpoonup \pi, \text{ in } W_2^{1,0}(Q_T), \\
& n_{1\varepsilon} \to n_1, \text{ in } L^p(Q_T), \text{ for any } p > 1, \\
& \varepsilon n_\varepsilon^{s+1} \to 0, \text{ in } L^\infty(Q_T), \\
& (n_{1\varepsilon}^2 + \varepsilon)^{\frac{m-1}{2}} n_{1\varepsilon} \to n_1^m \text{ in } L^p(Q_T), \text{ for any } p > 1.
\end{aligned}$$

So from (4.5), we can easily get that (1.2)–(1.4) hold. Furthermore, if $\frac{6}{5} \leq m \leq 2$ and combining with (1.2)–(1.4), we have

$$\begin{aligned}
\int_0^T \int_\Omega |\nabla n_1|^2 dxdt &= \int_0^T \int_\Omega |\nabla n_1|^2 n_1^{m-2} n_1^{2-m} dxdt \\
&\leq \int_0^T \int_\Omega n_1^{m-2} |\nabla n_1|^2 dxdt \leq C,
\end{aligned}$$

which means

$$\int_0^T \int_\Omega |\nabla n_1|^2 dxdt \leq C.$$

Recalling Lemmas 3.2 and 3.3, and taking $\hat{n}_1 = n_1, \hat{n}_2 = n_2$, we have

$$\sup_t \{\|c\|_{H^2}^2 + \|n_2\|_{H^1}^2\} + \int_0^T (\|c\|_{H^3}^2 + \|n_2\|_{H^2}^2) dt \leq C.$$

So the proof of Theorem 1.1 holds. □

5. Results

This paper investigates the two-species chemotaxis-Stokes system with porous medium diffusion in two-dimensional smooth bounded domains. Under certain approximate conditions, the authors establish the existence of time periodic solutions.

Notable are the key differences between two-dimensional and three-dimensional cases. In two dimensions, the complete chemotaxis-Navier-Stokes system with diffusion terms is tractable, allowing the derivation of classical solutions without strict parameter constraints. Conversely, the complexity of three-dimensional Navier-Stokes equations precludes general results for strong solutions.

Regarding solution regularity, two-dimensional solutions inherently attain higher regularity due to favorable Sobolev embeddings, which simplify the estimation of nonlinear terms. In contrast, three-dimensional solutions are constrained by stricter embeddings and require more rigorous conditions to ensure boundedness. The challenges in three dimensions are more pronounced, and whether the results for three-dimensional bounded domains and the restrictions on m (in Lemma 4.7) can be relaxed remains a direction for future research by the authors.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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